Finding the constant

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We have

$$\inf_{\Pi} th(X) + \bar{t}h(Z) + \lambda I(X;Z) - dh(X+Z) \leq C(c,d,\lambda)$$

which comes from rearranging

$$\inf_{\Pi}(c_1 h(\frac{X}{c_1}) + c_2 h(\frac{Z}{c_2}) - dh(X+Z) - \lambda h(X,Z) \leq D(\{c_1,c_2\},\{1,\lambda\},\{[1,1],[1,0;0,1]\})$$

We note that we need to satisfy $c_1 + c_2 = d + 2\lambda$ to be a valid datum.

We will pick $d = 1$ case for now. We note that due to the forms we will choose, the final coefficients t and $\bar{t}$ will be partitioning d, so I don't think much is lost when we pick this specific case.

where we note that

$$D = D_g := \frac{1}{2}\sup(\log|u_1| + \lambda \log\det U_2 - c_1 \log|v_1| - c_2 \log|v_2|)$$

where U_j, V_i satisfy:

$$(c_1x_1 + c_2x_2)^2 u_1 + \lambda[c_1x_1, c_2x_2]U_2[c_1x_1, c_2x_2] \leq c_1^2 v_1 x_1^2 + c_2^2 v_2 x_2^2$$

Finding the maximizer here seems hard to compute.

So we are instead going to assume that there is a Gaussian coupling that achieves this bound, and just going to try to find the particular Gaussian with the given marginals. Let's start with the case where X has variance 1, Z has variance σ^2, and X and Z havecovariance c. We will try to maximize the entropy bound over all possible values of c. We note that $X + Z$ has variance $1 + \sigma^2 - c$. We want to maximize

$$\sum \lambda h(X,Z) + h(X+Z)$$

which gives us

$$\frac{\lambda}{2}\log((2\pi e)^2(\sigma^2 - c^2)) + \frac{1}{2}\log(2\pi e(1+\sigma^2 - c))$$

For now, let us specialize to the case $\sigma^2 = 1$. We definitely need to do the general case and then optimize over the σ to get the right bound. (Note that WLOG we can choose $\sigma^2 \leq 1$ since scaling should be invariant by the dimension condition and the problem is essentially symmetric in the two variables, so if there is a maximizer $(\alpha, 1)$, there has to be a maximizer $(1, \alpha)$.)

which, when optimized over c, gives us

$$c = \frac{-4\lambda \pm \sqrt{16\lambda^2 + 4(2\lambda+1)}}{-2(2\lambda+1)}$$

To pick the sign in front of the square root, we need to be somewhat careful, but a heuristic analysis shows that for $\lambda \geq 0$, we pick the positive root (We would normally want to differentiate the original expression a second time to test this) The heuristic we are considering here is mostly taking the limit of $\lambda \to \infty$, for which we know that we want the c to be 0. We note that this c value is equal to 1 when $\lambda = 0$ if we pick the positive square root, which is also what we want.

So, the net constant we get on the right hand side for D is

$$(c_1+\lambda)\frac{1}{2}\log\frac{2\pi e}{|c_1|} + (c_2+\lambda)\frac{1}{2}\log\frac{2\pi e}{|c_2|} - \frac{\lambda}{2}\log((2\pi e)^2(\sigma^2 - c^2)) - \frac{1}{2}\log(2\pi e(1+\sigma^2 - c))$$

where c was chosen before.

We note that all the factors of $(2\pi e)2$ actually cancel, and the first two terms will actually completely cancel in the fully correctly scaled version, so we will be left with just the last two terms

$$\inf_{\Pi}(c_1 h(\frac{X}{c_1}) + c_2 h(\frac{Z}{c_2}) - h(X+Z) - \lambda h(X,Z) \leq D(\{c_1,c_2\},\{1,\lambda\},\{[1,1],[1,0;0,1]\})$$

We will pick $c_1, c_2 = t+\lambda, \bar{t}+\lambda$

We can also move the denominators of the first two entropies to the constant side since $h(aX) = \log|a| + h(X)$, which gives us the form we want.