EEE 391 - Homework 1

EFE ACER
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Q1

a

$$x(t) = 5\sin(6\pi t - \pi/3)$$

$$= 5\cos(6\pi t - \pi/3 - \pi/2) = 5\cos(\underbrace{6\pi t}_{2\pi f_0 = 6\pi \rightarrow \boxed{f_0 = 3\text{ Hz}}} - 5\pi/6) = 5\cos\left(\frac{2\pi}{1/3}\left(t - \frac{5}{36}\right)\right)$$

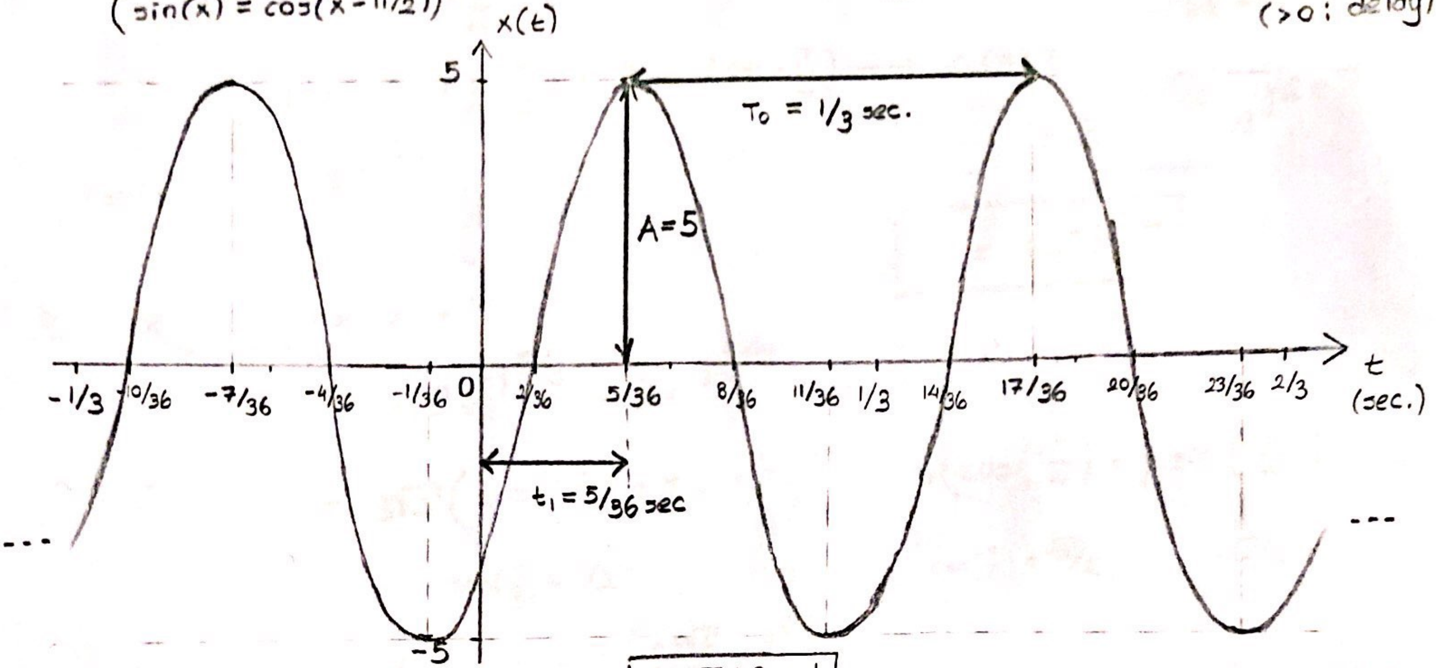
$(\sin(x) = \cos(x - \pi/2))$

$\boxed{A = 5}$

$T_0 = 1/f_0 \rightarrow \boxed{T_0 = 1/3 \text{ sec.}}$

$\boxed{\phi = -5\pi/6 \text{ rad}}$

$\boxed{t_1 = 5/36 \text{ sec.}}$ $(>0:$ delay$)$



b

$$x(t) = -6\cos(5t + 2)$$

$$= 6\cos(5t + \underbrace{2 + \pi}) = 6\cos\left(\frac{2\pi}{2\pi/5}\left(t - \left(\frac{-(\pi+2)}{5}\right)\right)\right)$$

$(-\cos(x) = \cos(x + \pi))$

$\boxed{\phi = \pi + 2 \text{ rad}}$

$\boxed{A = 6}$

$2\pi f_0 = 5 \rightarrow \boxed{f_0 = 5/2\pi \text{ Hz}}$

$T_0 = 1/f_0 \rightarrow \boxed{T_0 = 2\pi/5 \text{ sec.} \approx 1.257 \text{ sec.}}$

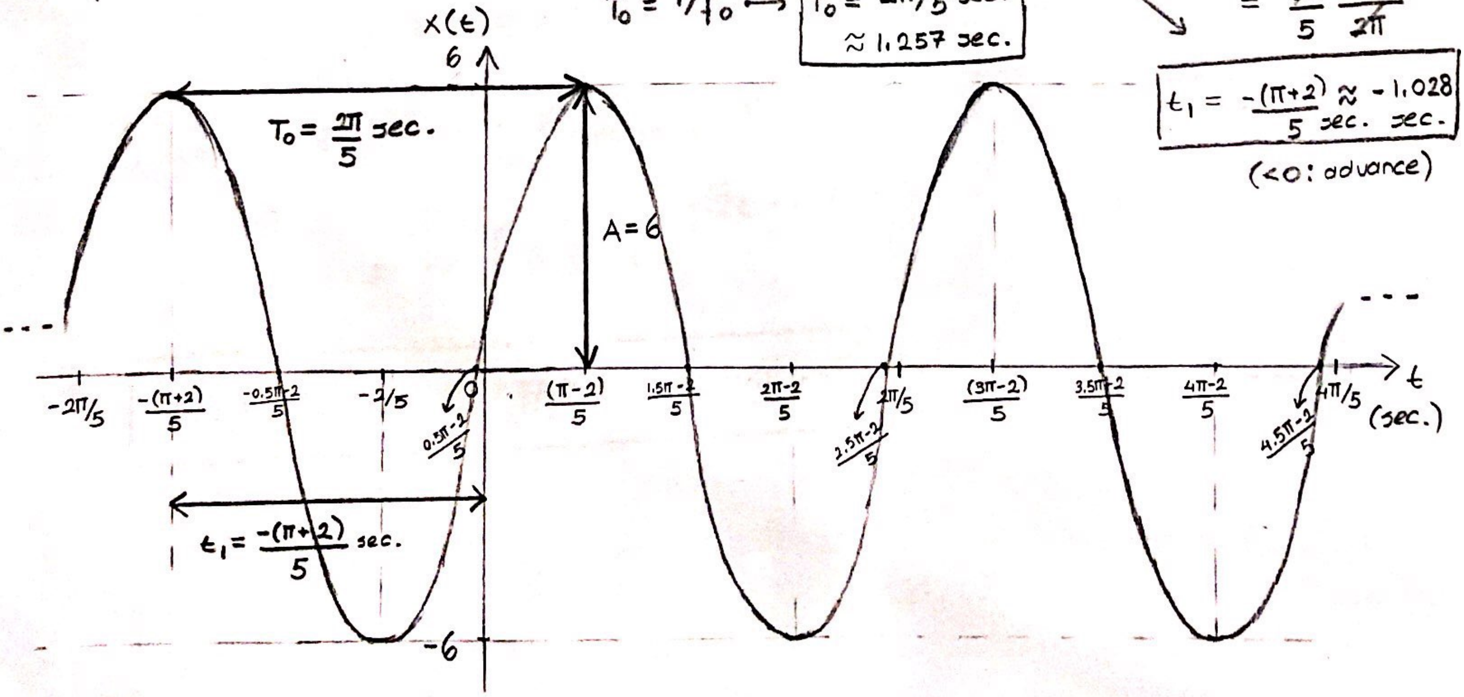
$$\phi = -2\pi\left(\frac{t_1}{T_0}\right)$$

$$t_1 = -T_0\phi/2\pi$$

$$= \frac{-\cancel{2\pi}}{5}\frac{(\pi+2)}{\cancel{2\pi}}$$

$\boxed{t_1 = \frac{-(\pi+2)}{5} \text{ sec.} \approx -1.028 \text{ sec.}}$

$(<0:$ advance$)$



Q2

a

$$x(t) = 3\sqrt{2}\cos(\underset{\omega_0}{2}t - \frac{\pi}{3}) \longrightarrow z_x(t) = 3\sqrt{2}\, e^{j(2t-\pi/3)}$$

$$= \underbrace{3\sqrt{2}\, e^{-j\pi/3}} \cdot e^{j2t}$$

$$\boxed{X = 3\sqrt{2}\, e^{-j\pi/3}}$$

$$y(t) = 3\sin(\overset{\omega_0}{2}t + \frac{\pi}{4})$$

$$= 3\cos(2t + \frac{\pi}{4} - \frac{\pi}{2}) = 3\cos(2t - \frac{\pi}{4}) \longrightarrow z_y(t) = 3e^{j(2t-\pi/4)}$$

$(\sin(x) = \cos(x - \pi/2))$

$$= \underbrace{3e^{-j\pi/4}} \cdot e^{j2t}$$

$$\boxed{Y = 3e^{-j\pi/4}}$$

b

$$X_{resultant} = X + Y$$

$$= 3\sqrt{2}\, e^{-j\pi/3} + 3e^{-j\pi/4}$$

$$= 3\sqrt{2}\left(\cos\left(-\frac{\pi}{3}\right) + j\sin\left(-\frac{\pi}{3}\right)\right) + 3\left(\cos\left(-\frac{\pi}{4}\right) + j\sin\left(-\frac{\pi}{4}\right)\right)$$

$\cos(\frac{\pi}{3}) = 1/2$, $-\sqrt{3}/2$, $\cos(\frac{\pi}{4}) = \sqrt{2}/2$, $-\sqrt{2}/2$

$$= \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} + j\left(\frac{-3\sqrt{6}}{2} - \frac{3\sqrt{2}}{2}\right)$$

$$= \sqrt{(3\sqrt{2})^2 + \left(\frac{-3}{2}(\sqrt{6}+\sqrt{2})\right)^2}\; e^{j\,\mathrm{atan2}\left(\frac{-3}{2}(\sqrt{6}+\sqrt{2}),\, 3\sqrt{2}\right)}$$

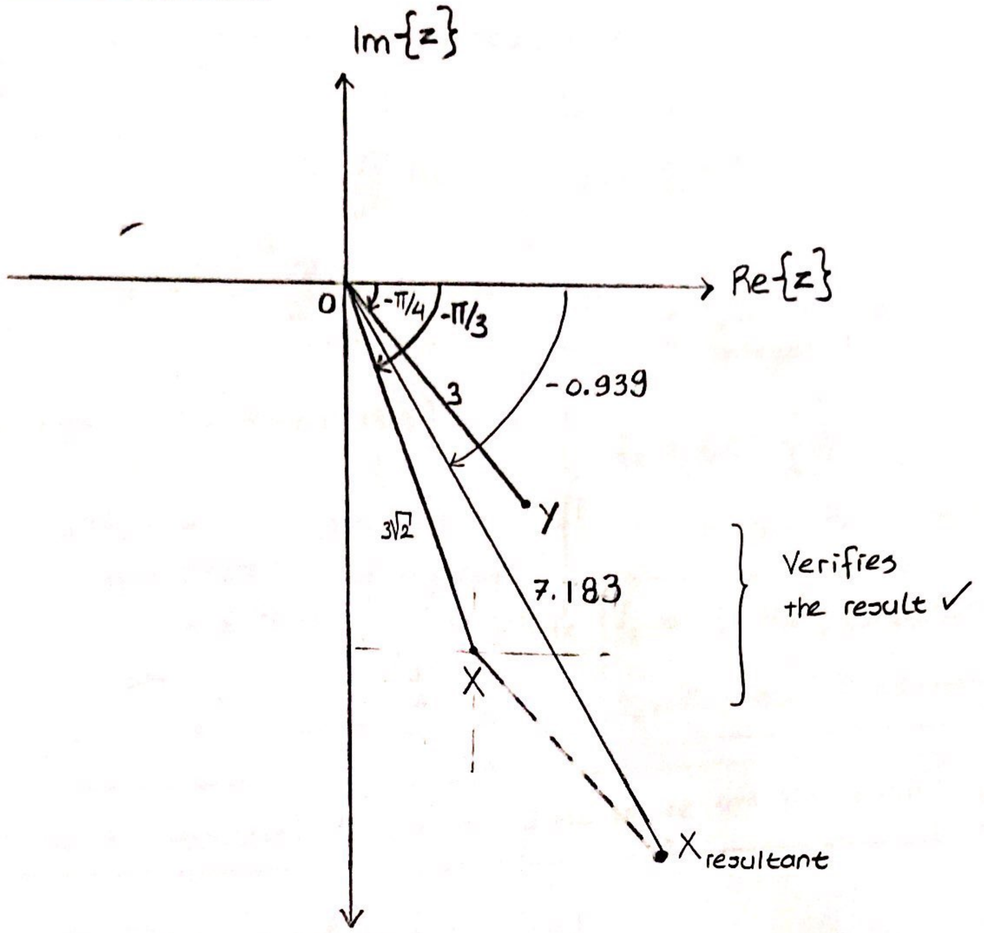
$$\approx 7.183\, e^{-j0.939}$$

$$x(t) + y(t) = \mathrm{Re}\left\{X_{resultant}\, e^{j\omega_0 t}\right\}$$

$$\approx \mathrm{Re}\left\{7.183\, e^{j(2t-0.939)}\right\}$$

$$\boxed{x(t) + y(t) \approx 7.183\cos(2t - 0.939)}$$

Phasor Addition Diagram:



Q3

a

$$x(t) = 3\sin(3t)\cos\left(\tfrac{\pi}{5}t + \tfrac{\pi}{3}\right) - 1$$

$$\underset{\substack{\downarrow \\ \text{(inverse Euler formulas)}}}{=} \underbrace{3\left(\frac{e^{j3t} - e^{-j3t}}{2j}\right)\left(\frac{e^{j(\pi/5 t + \pi/3)} + e^{-j(\pi/5 t + \pi/3)}}{2}\right)} - 1$$

$$\left[\frac{1}{4}\left(\underbrace{e^{j(3+\pi/5)t} \cdot e^{-j\pi/6}}_{(\star)} + \underbrace{e^{j(3-\pi/5)t} \cdot e^{-j5\pi/6}}_{(\bullet)} - \underbrace{e^{-j(3-\pi/5)t} \cdot e^{-j\pi/6}}_{(\bullet)} - \underbrace{e^{-j(3+\pi/5)t} \cdot e^{-j5\pi/6}}_{(\star)}\right)\right]$$

Multiplication cannot be writeen as a summation, thus the frequency spectrum of the signal cannot be defined. We say that the signal is $\boxed{\text{not periodic.}}$

EA

b

$$x(t) = 7\sin(\sqrt{3}t + 5) + 2\cos(\pi t)$$
$$= 7\cos(\underbrace{\sqrt{3}t}_{2\pi f_1 = \sqrt{3}} + 5 - \pi/2) + 2\cos(\underbrace{\pi t}_{2\pi f_2 = \pi})$$

$(\sin(x) = \cos(x - \pi/2))$

$f_1 = \frac{\sqrt{3}}{2\pi}$ Hz $\qquad f_2 = 1/2$ Hz

$f_0 = GCD\{f_1, f_2\}$
$= GCD\left\{\frac{\sqrt{3}}{2\pi}, \frac{1}{2}\right\}$
$= 1/2$

$f_1 = \overset{1/2}{f_0} \cdot \frac{\sqrt{3}}{\pi}$ $\qquad f_2 = \overset{1/2}{f_0} \cdot 1$

$\frac{\sqrt{3}}{\pi} \rightarrow$ k is not an integer $\Rightarrow$ The function is $\boxed{\text{not periodic.}}$

c

$$x(t) = \cos(\underbrace{12t}_{2\pi f_3 = 12}) - 3\cos(\underbrace{16t}_{2\pi f_4 = 16})$$

$f_3 = 12/2\pi = 6/\pi$ Hz $\qquad f_4 = 16/2\pi = 8/\pi$ Hz

$f_0 = GCD\{f_3, f_4\}$
$= GCD\{6/\pi, 8/\pi\}$

$\boxed{f_0 = \frac{2}{\pi} \text{ Hz, fundamental frequency}}$
$\boxed{T_0 = \pi/2 \text{ sec., fundamental period}}$

$f_3 = \overset{2/\pi}{f_0} \cdot 3$, $\quad f_4 = \overset{2/\pi}{f_0} \cdot 4$

$\boxed{k = 3,\ \frac{6}{\pi} \text{ Hz is the 3rd Harmonic}}$ $\qquad$ $\boxed{k = 4,\ \frac{8}{\pi} \text{ Hz is the 4th Harmonic}}$

$\Rightarrow$ The function is $\boxed{\text{periodic.}}$

d

$$x(t) = \sin^2(60\sqrt{2}t) + \cos^2(24\sqrt{2}t + \pi/4)$$
$$= \left(\frac{e^{j60\sqrt{2}t} - e^{-j60\sqrt{2}t}}{2j}\right)^2 + \left(\frac{e^{j(24\sqrt{2}t + \pi/4)} + e^{-j(24\sqrt{2}t + \pi/4)}}{2}\right)^2$$
$$= \frac{-1}{4}\left(e^{j120\sqrt{2}t} - 2\overset{1}{\cancel{e^0}} + e^{-j120\sqrt{2}t}\right) + \frac{1}{4}\left(e^{j(48\sqrt{2}t + \pi/2)} + 2\overset{1}{\cancel{e^0}} + e^{-j(48\sqrt{2}t + \pi/2)}\right)$$
$$= \frac{-1}{2}\cos(120\sqrt{2}t) + \frac{1}{2} + \frac{1}{2}\cos(48\sqrt{2}t + \pi/2) + \frac{1}{2}$$
$$= \frac{1}{2}\cos(\underbrace{48\sqrt{2}t}_{2\pi f_2 = 48\sqrt{2}} + \pi/2) - \frac{1}{2}\cos(\underbrace{120\sqrt{2}t}_{2\pi f_5 = 120\sqrt{2}}) + 1$$

$f_2 = \frac{24\sqrt{2}}{\pi}$ Hz $\qquad f_5 = \frac{60\sqrt{2}}{\pi}$ Hz

$f_0 = GCD\{f_2, f_5\}$
$= GCD\left\{\frac{24\sqrt{2}}{\pi}, \frac{60\sqrt{2}}{\pi}\right\}$

$\boxed{f_0 = \frac{12\sqrt{2}}{\pi} \text{ Hz, fundamental frequency}}$
$\boxed{T_0 = \frac{\pi}{12\sqrt{2}} \text{ sec., fundamental period}}$

$f_2 = \overset{12\sqrt{2}/\pi}{f_0} \cdot 2$, $\quad f_5 = \overset{12\sqrt{2}/\pi}{f_0} \cdot 5$

$\boxed{k = 2,\ \frac{24\sqrt{2}}{\pi} \text{ Hz is the 2nd Harmonic}}$ $\qquad$ $\boxed{k = 5,\ \frac{60\sqrt{2}}{\pi} \text{ Hz is the 5th Harmonic}}$

$\Rightarrow$ The function is $\boxed{\text{periodic.}}$

EA

Q4 $A_k \cos(\omega_k t + \phi_k) = A_k \dfrac{e^{j(\omega_k t+\phi_k)} + e^{-j(\omega_k t+\phi_k)}}{2}$

(inverse Euler's formulas)

(for real-valued functions of time) $\Rightarrow$ $= \underbrace{\frac{A_k}{2} e^{j\phi_k}}_{a_k} \cdot e^{j\omega_k t} + \underbrace{\frac{A_k}{2} e^{-j\phi_k}}_{a_{-k}} \cdot e^{j\omega_k t}$, $\rightarrow (a_{-k} = a_k^*)$

$\left(\omega_k = \frac{2\pi}{T_0} k\right)$

- $a_0 = \boxed{A_0 = -4}$ (average value)
- $a_1 = \frac{A_1}{2} e^{j\phi_1} = -j = e^{-j\pi/2} \longrightarrow \boxed{A_1 = 2}$ $\boxed{\phi_1 = -\pi/2 \text{ rad}}$ $\boxed{\omega_1 = \frac{2\pi}{5} \text{ rad/s}}$
- $a_5 = \frac{A_5}{2} e^{j\phi_5} = -4 e^{-j\pi/3} \longrightarrow \boxed{A_5 = -8}$ $\boxed{\phi_5 = -\pi/3 \text{ rad}}$ $\omega_5 = \frac{2\pi}{5} \cdot 5 \rightarrow \boxed{\omega_5 = 2\pi \frac{\text{rad}}{s}}$
- $a_8 = \frac{A_8}{2} e^{j\phi_8} = 2e^{j\pi/3} \longrightarrow \boxed{A_8 = 4}$ $\boxed{\phi_8 = \pi/3 \text{ rad}}$ $\omega_8 = \frac{2\pi}{5} \cdot 8 \rightarrow \boxed{\omega_8 = \frac{16\pi}{5} \frac{\text{rad}}{s}}$

$$\boxed{x(t) = -4 + 2\cos\left(\frac{2\pi}{5}t - \frac{\pi}{2}\right) - 8\cos\left(2\pi t - \frac{\pi}{3}\right) + 4\cos\left(\frac{16\pi}{5}t + \frac{\pi}{3}\right)}$$

Q5

$$x(t) = 5 - 2e^{-jt} + 3\left[\frac{e^{j(-t+\pi/4)} - e^{-j(-t+\pi/4)}}{2j}\right] - 4\left[\frac{e^{j(3t+5)} + e^{-j(3t+5)}}{2}\right]$$

$$+ 3\left[\left(\frac{e^{j4t} + e^{-j4t}}{2}\right)\left(\frac{e^{j(5t+\pi/2)} + e^{-j(5t+\pi/2)}}{2}\right)\right]$$

$$= 5 - 2e^{-jt} + \frac{3}{2}e^{j(-t-\pi/4)} - \frac{3}{2}e^{-j(-t+3\pi/4)} - 2e^{j(3t+5)} - 2e^{-j(3t+5)} + \frac{3}{4}e^{j(9t+\pi/2)} + \frac{3}{4}e^{-j(t+\pi/2)} + \frac{3}{4}e^{j(t+\frac{\pi}{2})} + \frac{3}{4}e^{-j(9t+\pi/2)}$$

$$\left(+\frac{3}{4}e^{jt}\cdot e^{j\pi/2}\right)$$

$$= 5 + \underbrace{e^{-jt}}_{(-f_0 = \frac{-1}{2\pi}Hz)}\left(-2 + \frac{3}{2}e^{-j\pi/4} + \frac{3}{4}e^{-j\pi/2}\right) - \frac{3}{2}\underbrace{e^{jt}}_{(f_0 = \frac{1}{2\pi}Hz)}\cdot e^{-j\frac{3\pi}{4}} - 2\underbrace{e^{j3t}}_{(f_3 = \frac{3}{2\pi}Hz)}e^{j5} - 2\underbrace{e^{-j3t}}_{(-f_3 = \frac{-3}{2\pi}Hz)}e^{-j5}$$

$$+ \frac{3}{4}\underbrace{e^{j9t}}_{(f_9 = \frac{9}{2\pi}Hz)}\cdot e^{j\pi/2} + \frac{3}{4}\underbrace{e^{-j9t}}_{(-f_9 = \frac{-9}{2\pi}Hz)}\cdot e^{-j\pi/2}$$

From the expression above:

$$a_k = \begin{cases} 5 & \text{for } k=0, \\ \left(-2 + \frac{3}{2}e^{-j\pi/4} + \frac{3}{4}e^{-j\pi/2}\right) & \text{for } k=-1, \\ -\frac{3}{2}e^{-j\frac{3\pi}{4}} + \frac{3}{4}e^{j\pi/2} & \text{for } k=1 \\ -2e^{-j5} & \text{for } k=-3, \\ -2e^{j5} & \text{for } k=3, \\ \frac{3}{4}e^{-j\pi/2} & \text{for } k=-9, \\ \frac{3}{4}e^{j\pi/2} & \text{for } k=9 \\ 0 & \text{else} \end{cases}$$

a $\boxed{a_0 = 5}$ is the average value

b $f_0 = \frac{1}{2\pi}Hz \rightarrow$ fundamental frequency

$\Updownarrow$

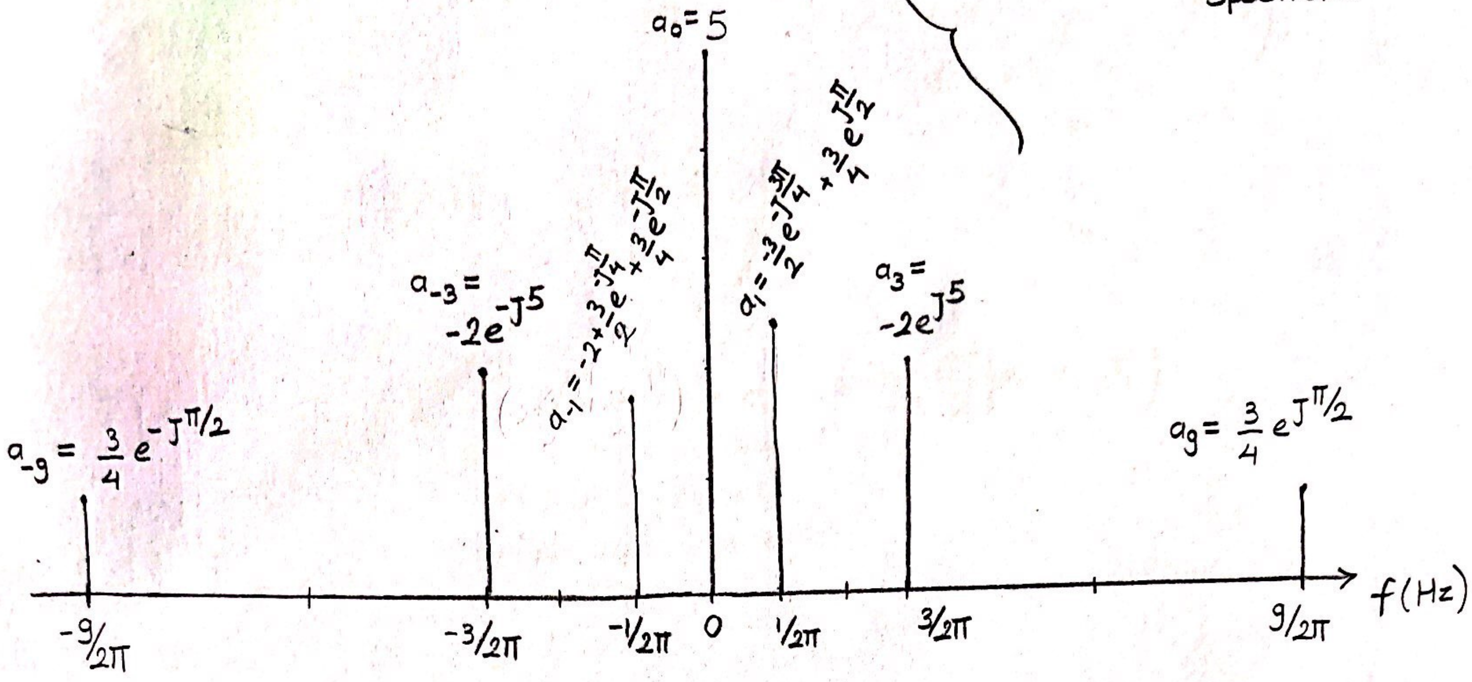
$\frac{1}{f_0} = \boxed{T_0 = 2\pi \text{ sec.}} \rightarrow$ fundamental period

c No, x(t) is $\boxed{\text{not a real signal}}$, since

$$a_{-1} \neq a_1^*$$

d. Frequency Spectrum of $x(t)$:

$\boxed{\text{Fundamental, 3rd and 9th}}$ Harmonics exist in the spectrum.



Q6

a

$$a_k' = \frac{1}{T_0}\int_0^{T_0} c\,x(t-t_0)\,e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt = \frac{c}{T_0}\int_0^{T_0} x(t-t_0)\,e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt$$

$$= \frac{c}{T_0}\int_{-t_0}^{T_0-t_0} x(u)\,e^{-j\left(\frac{2\pi}{T_0}\right)k(u+t_0)}\,dt = \frac{c}{T_0}e^{-j\left(\frac{2\pi}{T_0}\right)kt_0}\int_{-t_0}^{T_0-t_0} x(u)\,e^{-j\left(\frac{2\pi}{T_0}\right)ku}\,du$$

$\left(\begin{matrix} u = t - t_0 \\ du = dt \end{matrix}\right)$

$$= c\,.\,e^{-j\left(\frac{2\pi}{T_0}\right)kt_0}\underbrace{\frac{1}{T_0}\int_0^{T_0} x(u)e^{-j\left(\frac{2\pi}{T_0}\right)ku}\,du}_{a_k} = \boxed{a_k\,.\,c\,.\,e^{-j\left(\frac{2\pi}{T_0}\right)kt_0}}$$

(x(u) is periodic, we can redifine the limits)

b

$$a_k' = \frac{1}{T_0}\int_0^{T_0}\left(\frac{dx(t)}{dt}\right)e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt$$

$$\underset{\substack{\downarrow}}{=} \frac{1}{T_0}\left(\left[e^{-j\left(\frac{2\pi}{T_0}\right)kt}\cdot x(t)\right]_0^{T_0} + j\left(\frac{2\pi}{T_0}\right)k\underbrace{\int_0^{T_0} x(t)\cdot e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt}_{a_k\cdot T_0}\right)$$

Integration by parts

$$\left(\begin{array}{ll} u = e^{-j\left(\frac{2\pi}{T_0}\right)kt}, & dv = dx(t) \\ du = -j\left(\frac{2\pi}{T_0}\right)k\cdot e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt, & v = x(t) \end{array}\right)$$

$$= \frac{1}{T_0}\left(\underbrace{\cancel{e^{-j2\pi k}}^{\,1}\cdot x(T_0) - \cancel{e^{0}}^{\,1}\cdot x(0)}_{\substack{x(T_0)-x(0)=0 \\ \text{(since } x(t) \text{ is periodic)}}} + j\left(\frac{2\pi}{\cancel{T_0}}\right)k\cdot a_k\cdot\cancel{T_0}\right)$$

$$= \boxed{j\left(\frac{2\pi}{T_0}\right)k\cdot a_k}$$

Q7

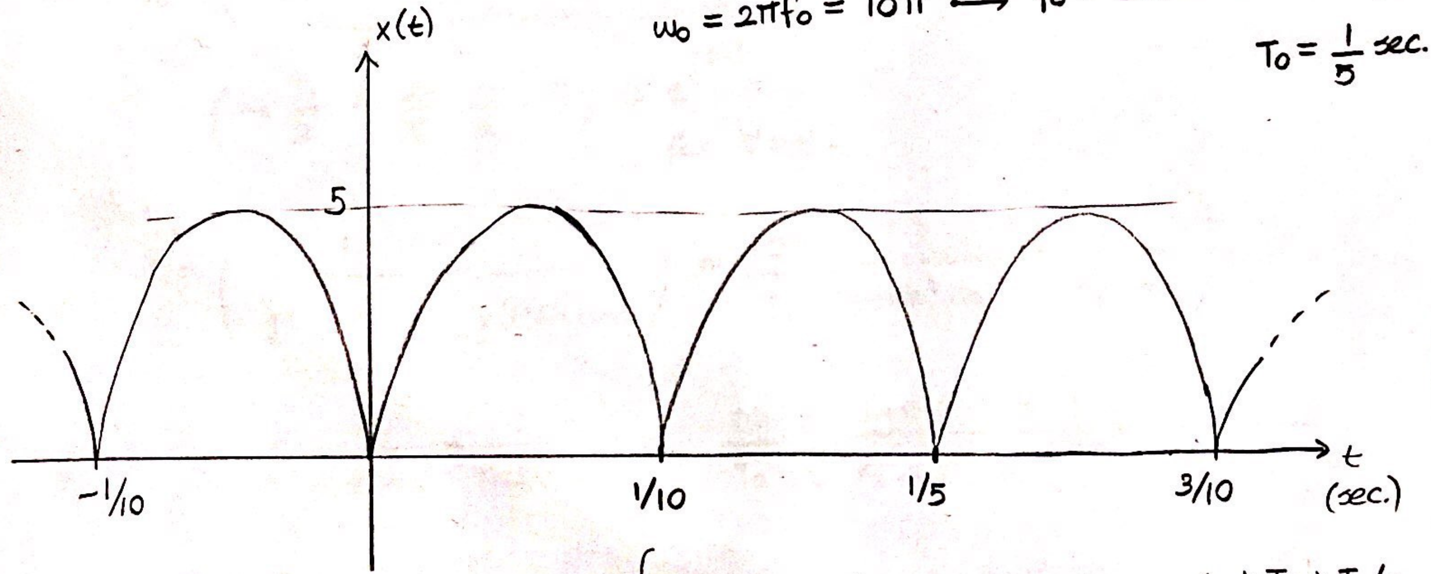
The waveform:

$\omega_0 = 10\pi$ rad/sec.

$\downarrow$

$\omega_0 = 2\pi f_0 = 10\pi \longrightarrow f_0 = 5\,\text{Hz} \rightarrow T_0 = 1/f_0$

$T_0 = \frac{1}{5}$ sec.



Note that $x(t)$ is periodic with $\boxed{T_0' = T_0/2 \Leftrightarrow \omega_0' = 2\omega_0}$ $\Leftarrow$

$$x(t) = \begin{cases} 5\sin(\omega_0 t), & \text{for } kT_0 \le t \le kT_0 + T_0/2 \\ -5\sin(\omega_0 t), & \text{for } kT_0 + T_0/2 \le t \le kT_0 + T_0 \end{cases}$$

for $\forall\ k = 0, \pm 2, \pm 4, \pm 6, \ldots$

$$a_k = \frac{1}{T_0'}\int_0^{T_0'} x(t)\, e^{-j\omega_0' k t}\, dt$$

$$= \frac{2}{T_0}\int_0^{T_0/2} 5\sin(\omega_0 t)\, e^{-j2\omega_0 k t}\, dt$$

$$= \frac{10}{T_0}\int_0^{T_0/2} \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right) e^{-j2\omega_0 k t}\, dt$$

(inverse Euler's formulas)

$$= \frac{5}{jT_0}\int_0^{T_0/2} \left(e^{j(1-2k)\omega_0 t} - e^{-j(1+2k)\omega_0 t}\right) dt$$

$$= \frac{-5j}{T_0}\left[\frac{e^{j(1-2k)\omega_0 t}}{j(1-2k)\omega_0} + \frac{e^{-j(1+2k)\omega_0 t}}{j(1+2k)\omega_0}\right]_0^{T_0/2}$$

$$= \frac{-5j}{T_0}\left(\underbrace{\frac{\cancel{e^{j(1-2k)\omega_0 \frac{T_0}{2}}}^{-1}}{j(1-2k)\omega_0} + \frac{\cancel{e^{-j(1+2k)\omega_0 \frac{T_0}{2}}}^{-1}}{j(1+2k)\omega_0}}_{} - \frac{\cancel{e^0}^{1}}{j(1-2k)\omega_0} - \frac{\cancel{e^0}^{1}}{j(1+2k)\omega_0}\right)$$

$$\left(\omega_0 \frac{T_0}{2} = \frac{2\pi}{\cancel{T_0}} \cdot \frac{\cancel{T_0}}{2} = \pi\right) \Rightarrow e^{jc\pi} = -1 \text{ for } \forall c \in I$$

$$= \frac{10j}{T_0}\left(\frac{1}{j(1-2k)\omega_0} + \frac{1}{j(1+2k)\omega_0}\right) = \frac{10j}{T_0}\left(\frac{-j}{(1-2k)\omega_0} - \frac{j}{(1+2k)\omega_0}\right)$$

$$= \frac{-10\cancel{j^2}^{-1}}{T_0}\left(\frac{1}{(1-2k)\omega_0} + \frac{1}{(1+2k)\omega_0}\right) = \frac{10}{T_0}\left(\frac{1+\cancel{k}+1-\cancel{k}}{(1-4k^2)\omega_0}\right) = \frac{20}{\cancel{T_0}(1-4k^2)\omega_0}$$

$$\omega_0 = \frac{2\pi}{\cancel{T_0}}$$

$$= \boxed{\frac{10}{\pi(1-4k^2)}}$$