EEE482: Homework 1

EFE ACER

Question 1:

a)

After performing the elementary row operations below, we obtain the reduced row echelon form ($RREF$) of $\boldsymbol{A}$, denoted as $RREF(\boldsymbol{A})$:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & -1 & 5 \\ 3 & 3 & 0 & 9 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_2]{} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = RREF(\boldsymbol{A})$$

Using the $RREF(\boldsymbol{A})$, the linear system can be rewritten as:

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{matrix} x_1 - x_3 + 2x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \end{matrix} \Longrightarrow \begin{matrix} x_1 = x_3 - 2x_4 \\ x_2 = -x_3 - x_4 \end{matrix}$$

Here, x_1 and x_2 are pivot variables; whereas x_3 and x_4 are free variables. Hence, the solution should be written in terms of some arbitrary values of x_3 and x_4:

Let $x_3 = \alpha$ and $x_4 = \beta$, where $\alpha, \beta \in \mathbb{R}$

Then,

$$\boldsymbol{x_n} = \begin{bmatrix} \alpha - 2\beta \\ -\alpha - \beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \text{ for } \forall\, \alpha, \beta \in \mathbb{R}\text{, is the general solution}$$

b)

Now, we construct the augmented matrix, $[\boldsymbol{A}|\boldsymbol{b}]$, and compute $RREF([\boldsymbol{A}|\boldsymbol{b}])$:

$$[\boldsymbol{A}|\boldsymbol{b}] = \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ 2 & 1 & -1 & 5 & 4 \\ 3 & 3 & 0 & 9 & 9 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 3 & 3 & 3 & 6 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_2]{}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = RREF([\boldsymbol{A}|\boldsymbol{b}])$$

Using $RREF([\boldsymbol{A}|\boldsymbol{b}])$, we can rewrite the system as:

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \Longrightarrow \begin{array}{c} x_1 - x_3 + 2x_4 = 1 \\ x_2 + x_3 + x_4 = 2 \end{array} \Longrightarrow \begin{array}{c} x_1 = x_3 - 2x_4 + 1 \\ x_2 = -x_3 - x_4 + 2 \end{array}$$

The values $x_1 = 4,\ x_2 = 2,\ x_3 = 1$ and $x_4 = -1$ satisfies the simplified system. Hence, a particular solution to the system is:

$$\boldsymbol{x_p} = \begin{bmatrix} 4 \\ 2 \\ 1 \\ -1 \end{bmatrix}$$

c)

Proceeding from the simplified system that we obtained in part b, we can construct a general solution in terms of arbitrary values of the free variables x_3 and x_4:

Again, let $x_3 = \alpha$ and $x_4 = \beta$, where $\alpha, \beta \in \mathbb{R}$

Then, we can write our solution set, say S, as:

$$S = \left\{ \begin{bmatrix} \alpha - 2\beta + 1 \\ -\alpha - \beta + 2 \\ \alpha \\ \beta \end{bmatrix},\ \alpha, \beta \in \mathbb{R} \right\} = \left\{ \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix},\ \alpha, \beta \in \mathbb{R} \right\}$$

d)

The pseudo-inverse of $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^+$. It is a generalization of the inverse matrix. When $\boldsymbol{A}$ has linearly independent column vectors, meaning that $\boldsymbol{A}^T\boldsymbol{A}$ is invertible, $\boldsymbol{A}^+$ is given by:

$$\boldsymbol{A}^+ = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T$$

We compute $\boldsymbol{A}^T$ by simply switching row and column indices of $\boldsymbol{A}$:

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ -1 & -1 & 0 \\ 2 & 5 & 9 \end{bmatrix}$$

Then, we compute $\boldsymbol{A}^T\boldsymbol{A}$ by performing a matrix multiplication:

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ -1 & -1 & 0 \\ 2 & 5 & 9 \end{bmatrix}_{4x3} . \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & -1 & 5 \\ 3 & 3 & 0 & 9 \end{bmatrix}_{3x4} = \begin{bmatrix} 14 & 11 & -3 & 39 \\ 11 & 10 & -1 & 32 \\ -3 & -1 & 2 & -7 \\ 39 & 32 & -7 & 110 \end{bmatrix}_{4x4}$$