Homework 1

Solutions

1

a) $3e^{j\pi/3} + 4e^{-j\pi/6} = 5e^{j0.12} = 4.9641 + j0.5981$

b) $(1-j)^2 = 2e^{-j\pi/2}$

c) $(\sqrt{3}-j3)^{10} = (\sqrt{12}e^{-j\pi/3})^{10} = 248,832\,\underbrace{e^{-j10\pi/3}}_{e^{j2\pi/3}} = -124,416 + j215,494.83$

d) $(\sqrt{2}+j\sqrt{2})/(1+j\sqrt{3}) = e^{-j\pi/12}$

e) $Re\{je^{-j\pi/3}\} = R\{e^{j\pi/2}e^{-j\pi/3}\} = Re\{e^{j\pi/6}\} = \cos(\pi/6) = \sqrt{3}/2 = 0.866$

f) $j(1-j) = e^{j\pi/4}$

g) $(\sqrt{3}-j3)^{-1} = (\sqrt{12}\,e^{-j\pi/3})^{-1} = (1/\sqrt{12})e^{+j\pi/3} = 0.2887e^{+j\pi/3} = 0.14434 + j0.25$

2 a) $s_i(t) = Im\{5e^{j\pi/3}e^{j10\pi t}\} = Im\{5e^{j(10\pi t+\pi/3)}\}$

$\Rightarrow s_i(t) = 5\sin(10\pi t + \pi/3)$

We can convert to the cosine form: $\boxed{\sin\theta = \cos(\theta - \pi/2)}$

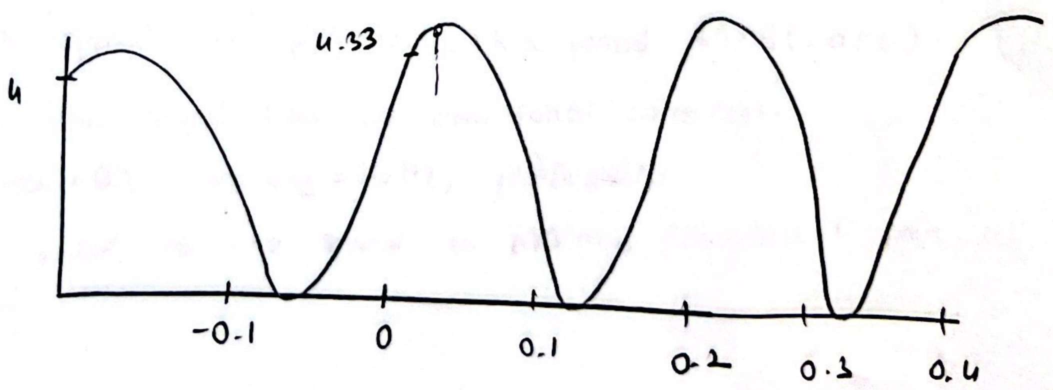
$s_i(t) = 5\cos(10\pi t - \pi/6)$

Period of $s_i(t)$ is $T = 1/5$ $\left(\frac{2\pi}{T} = 10\pi\right)$

Value at $t=0$ is $s_i(0) = 5\cos(-\pi/6) = \frac{5\sqrt{3}}{2} = 4.33$

Peak at $t = t_1$ where

$10\pi t_1 - \pi/6 = 0 \rightarrow t_1 = 1/60$



b) $q(t) = Im\{\dot{s}(t)\} = Im\{(5e^{j\pi/3})(j10\pi e^{j10\pi t})\}$

$$= Im\{50\pi e^{j(10\pi t + \pi/3 + \pi/2)}\}$$

$$= 50\pi \sin(10\pi t + 5\pi/6)$$

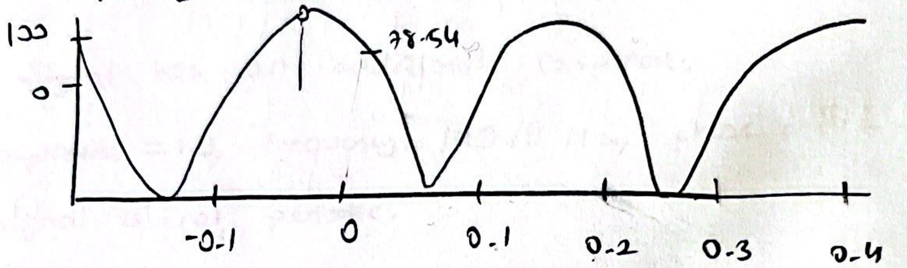
$$= 50\pi \cos(10\pi t + \pi/3)$$

Period of $q(t)$ is also $T = 1/5$.

$q(0) = 50\pi \cos(\pi/3) = 25\pi = 78.54$

max value of $q(t)$ is at t_1 which solves:

$10\pi t_1 + \pi/3 = 0 \Rightarrow t_1 = -1/30$

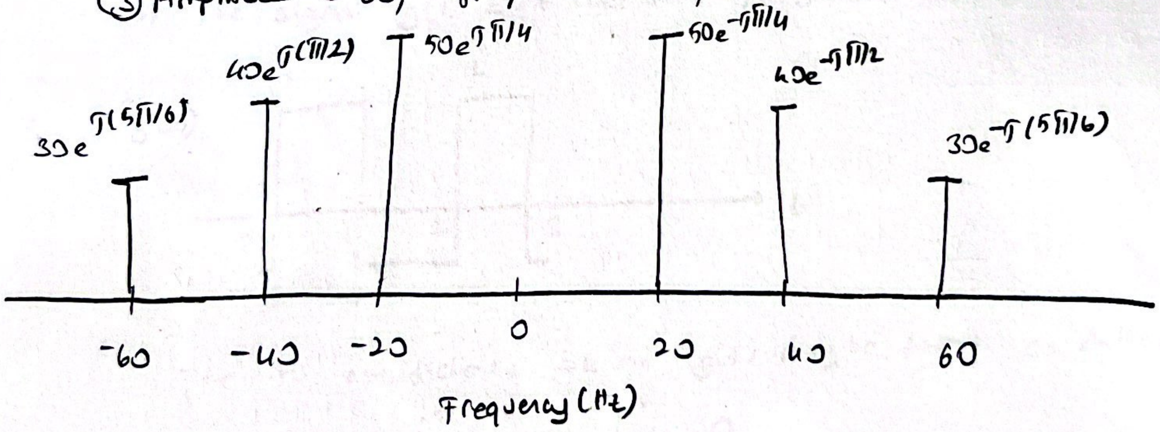


(3) x(t) has the following three components:

a) (1) Amplitude = 100, freq. = 20 Hz, phase = $-\pi/4$

(2) Amplitude = 80, freq = 40 Hz, phase = $-\pi/2$

(3) Amplitude = 60, freq = 60 Hz, phase = $-5\pi/6$

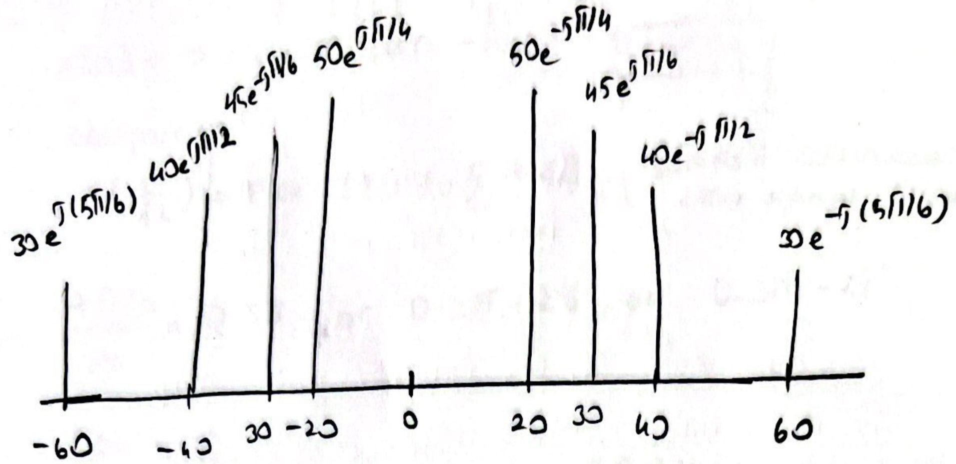


b) This signal is periodic, with a period 50 ms (20 Hz.)

c) This new signal has an additional component-

Amplitude = 90, frequency = 30 Hz, phase = $\pi/6$

New period of this signal is 100 ms; fundamental freq. of 10 Hz.



d) Signal has an additional component:

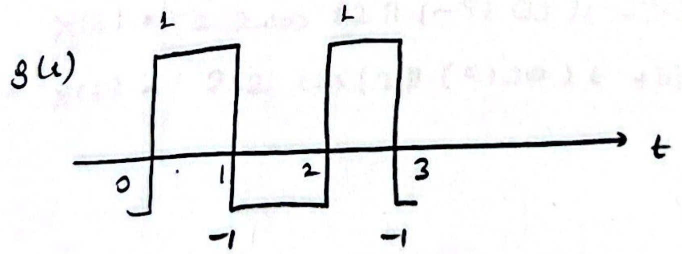
Amplitude $= 10$, frequency $= 140/\pi$ Hz, phase $= \pi/2$.

Signal is not periodic.

4 a) We have

$$a_0 = \frac{1}{2}\int_0^1 t\,dt + \frac{1}{2}\int_1^2 (2-t)\,dt = 1/2$$

b) The signal $g(t) = dx(t)/dt$ is shown below.



The FS coefficients b_k of $g(t)$ may be found as follows:

$$b_0 = \frac{1}{2}\int_0^1 dt - 1/2\int_1^2 dt = 0$$

and

$$b_k = \frac{1}{2}\int_0^1 e^{-j\pi kt}\,dt - \frac{1}{2}\int_1^2 e^{-j\pi kt}\,dt = \frac{1}{j\pi k}\left[1 - e^{-j\pi k}\right]$$

c) $g(t) = \dfrac{dx(t)}{dt} \xleftrightarrow{FS} b_k = jk\pi a_k$.

Therefore;

$$a_k = \frac{1}{jk\pi} b_k = -\frac{1}{\pi^2 k^2}\{1 - e^{-j\pi k}\}$$

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$x[n] = 2.2\cos(0.3\pi n - \pi/3)$ $\boxed{fs = 6000}$

compare to

$x\left(\frac{n}{fs}\right) = A\cos\left(2\pi f_0 \frac{n}{fs} + \phi\right)$ (sampled continuous time signal)

$\rightarrow \frac{2\pi f_0}{fs} = 0.3\pi$, or $0.3\pi + 2\pi$, or $0.3\pi - 2\pi$

Solve:

$\frac{2\pi f_0}{fs} = 0.3\pi \Rightarrow f_0 = fs\left(\frac{0.3}{2}\right) = 6000 \times 0.15$

$f_0 = 900\,Hz$

Note: difference is fs

$\rightarrow x(t) = 2.2\cos(1800\pi t - \pi/3)$

Then $\frac{2\pi f_0}{fs} = 2.3\pi \Rightarrow f_0 = fs\left(\frac{2.3}{2}\right) = 6900\,Hz$

$\rightarrow x(t) = 2.2\cos(2\pi(6900)t - \pi/3)$

Finally,

$\frac{2\pi f_0}{fs} = -1.7\pi \Rightarrow f_0 = fs\left(\frac{-1.7}{2}\right) = -5100\,Hz$

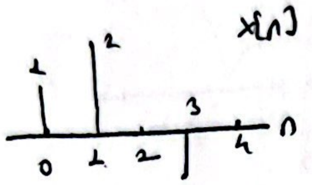
$x(t) = 2.2\cos(2\pi(-5100)t - \pi/3)$

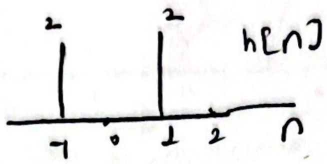
$\rightarrow x(t) = 2.2\cos(2\pi(5100)t + \pi/3)$

a)

6) we know that

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$





$$y_1[n] = h[-1]\,x[n+1] + h[1]\,x[n-1]$$
$$= 2x[n+1] + 2x[n-1]$$

This gives:

$$y_1[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$$

b) $$y_2[n] = x[n+2] * h[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n+2-k]$$

$$y_2[n] = y_1[n+2]$$

c) $$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

$$y_3[n] = x[n] * h[n+2] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n+2-k]$$

$$y_3[n] = y_1[n+2]$$

7) a) $$H(\hat{\omega}) = \sum_{k=0}^{M} h[k] e^{-j\omega k}$$

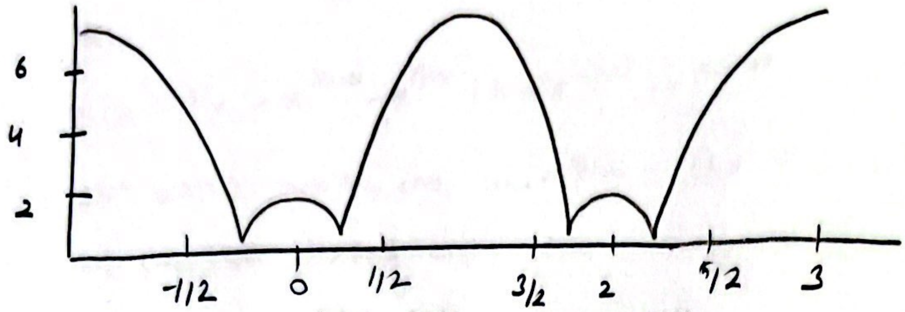
$$= 2e^{-j\omega 0} - 3e^{-j\omega 1} + 2e^{-j\omega 2}$$
$$= 2 - 3e^{-j\omega} + 2e^{-j\omega 2}$$
$$= e^{-j\omega}\left(2e^{-j\omega} - 3 + 2e^{-j\omega}\right)$$
$$= e^{-j\omega}\left(-3 + 4\cos(\hat{\omega})\right)$$

$$|H(\hat{\omega})| = |-3 + 4\cos(\hat{\omega})|$$

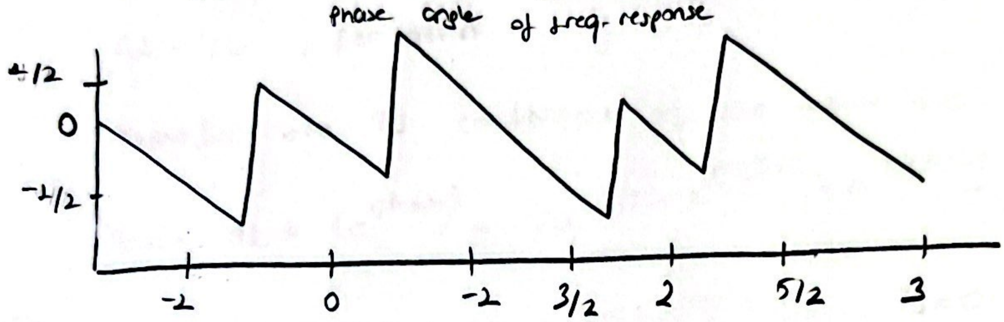
$$\angle H(\hat{\omega}) = -\omega$$

b) $H(\hat{\omega})$ has a period of 2π.

c) mgn. of freq. response



phase angle of freq. response



d) $-3+4\cos(\hat{\omega})=0$

$\cos(\hat{\omega}) = 3/4$

$\hat{\omega} = \cos^{-1}(3/4)$

$\approx \pm 0.7227 \pm 2\pi k$ radians ; k integer

e) since $x[n] = \sin\left(\frac{\pi}{13}n\right)$ we need only evaluate the frequency response $H(\hat{\omega})$ at $\hat{\omega} = \pi/13$:

$$H\left(\frac{\pi}{13}\right) = e^{-j\pi/13}\left(-3+4\cos\left(\frac{\pi}{13}\right)\right) \approx 0.8838\, e^{-j\pi/13}$$

The mgn. is $\left|H\left(\frac{\pi}{13}\right)\right| \approx 0.8838$ and phase is $\angle\left|H\left(\frac{\pi}{13}\right)\right| = -\frac{\pi}{13}$,

so the output $y[n]$ is:

$$y[n] = 0.8838 \sin\left(\frac{\pi}{13}n - \frac{\pi}{13}\right)$$

$$= 0.8838 \cos\left(\frac{\pi}{13}n - \frac{\pi}{13} - \frac{\pi}{2}\right)$$

$$= 0.8838 \cos\left(\frac{\pi}{13}(n-1) - \pi/2\right)$$

$$= 0.8838 \cos\left(\frac{\pi}{13}n - \frac{15\pi}{26}\right)$$

8. The frequency response of system may be evaluated as

$$H(e^{j\omega}) = -e^{2j\omega} - e^{j\omega} + 1 + e^{-j\omega} + e^{-2j\omega}$$

for $x[n]$, $N=4$, and $\omega_0 = \pi/2$. The FS coefficients of the input $x[n]$ are

$a_k = 1/4$, for all n.

Therefore the FS coefficients of the output are

$$b_k = a_k H(e^{jk\omega_0}) = 1/4\,[1 - e^{jk\pi/2} + e^{-jk\pi/2}]$$

9. a) $x[n] = x(nT_s) = 10\cos(880\pi n T_s + \phi)$ $\qquad T_s = 0.0001$

$880 T_s = 880 \times 10^{-4} = 0.088 = 11/125$

To find the number of samples within one period of the continuous cosine $x(t)$, find the largest integer satisfying $880\, n T_s \leq 2\pi$

$$n \leq \frac{2}{0.088} = \frac{250}{11} = 22.73$$

There are 23 samples in one period, because samples $n = 0, 1, 2, \ldots, 22$ are within one period.

<u>Note:</u> The period of $x[n]$ is not 23; it is actually 250.

b) $y[n] = 10\cos(\omega_0 n T_s + \phi)$

To get the same samples for $x[n]$ & $y[n]$ we solve:

$\omega_0 n T_s = 880 n T_s + 2\pi \ell n$ $\qquad \ell =$ integer

$\Rightarrow \omega_0 = 880\pi + \frac{2\pi \ell}{T_s}$ $\qquad \frac{2\pi}{T_s} = 20{,}000\,\pi$

Take $\ell = 1$

$$\omega_0 = 20{,}880\pi$$

c) Find largest integer satisfying

$(20{,}880\pi)\, n T_s \leq 2\pi$

$n \leq \frac{2}{2.088}$ which is less than one!

$\rightarrow$ only one sample per period is taken

10)

a) Linear

b) Time invariant, linear, causal

c) Linear

d) Linear, causal

e) Linear