A Most Probable Point-Based Method for Reliability Analysis, Sensitivity Analysis and Design Optimization

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Abstract

A major step in a most probable point (MPP)-based method for reliability analysis is to determine the MPP. This is usually accomplished by using an optimization search algorithm. The optimal solutions associated with the MPP provide measurements related to safety probability. This study focuses on two commonly used approximate probability integration methods; i.e., the Reliability Index Approach (RIA) and the Performance Measurement Approach (PMA). Their reliability sensitivity equations are first derived in this paper, based on the derivatives of their respective optimal solutions. Examples are then provided to demonstrate the use of these derivatives for better reliability analysis and Reliability-Based Design Optimization (RBDO).

Introduction

The Most Probable Point (MPP)-based method plays a key role in the approximate integration approaches for reliability analysis. Many works in reliability engineering, particularly in Reliability-Based Design Optimization (RBDO), have given special attention to the reliability derivatives. Since the MPP can be obtained as a result of an optimization process, the probabilistic derivatives can be viewed as the derivatives of the optimal solution or optimum sensitivity derivatives. In other words, the computation of such derivatives should involve not only the function of the limit-state equation but also the Kuhn-Tucker Necessary Conditions at the MPP [1-6]. The main goal of this work is thus to present a procedure that computes the probabilistic derivatives as derivatives of an optimal solution. Examples are used to demonstrate the application of such optimum sensitivity derivatives to form better procedures for reliability analysis and RBDO. First, we derive the sensitivities for two approximate integration methods; i.e., RIA and PMA. Sensitivity analysis of the PMA leads to the development of a new RIA, called PMA-based RIA (PRIA). Both convex and concave algebraic examples [7-9] and a multidisciplinary flexible wing example [10] are used to demonstrate the use of the newly devised method for reliability analysis. An initial attempt is also made in devising a RBDO procedure in which the derived reliability derivatives are used to approximate the constraints and to screen off non-active constraints.

Reliability Analysis and Sensitivity Analysis

Given a response condition, $G(\boldsymbol{X})$, of random variables, $\boldsymbol{X}$, reliability analysis is interested in finding the probability of failure, $P_f = P(G(\boldsymbol{X})>0)$. The corresponding reliability is given by $R = 1 - P_f$. One approach to compute the probability of failure or the reliability of the limit-state equation, $G(\boldsymbol{X})=0$, is the approximate probability integration method. The goal of the method is to find a rotationally invariant reliability measurement of the given limit-state equation. Many variations of this method exist; however, most of them may be classified into two groups; RIA and PMA.

In RIA, the objective is to find the first-order safety reliability index, β, which is equal to the shortest distance between the origin of the $\boldsymbol{U}$-space and the failure surface. The contact point on the failure surface is the MPP. The components of the reduced variable vector, $\boldsymbol{u}$, are defined by $u_i = (X_i - \mu_i)/\sigma_i$ where μ_i and σ_i are the mean and the standard deviation of the corresponding X_i. The limit-state equation is thus rewritten as $G(\boldsymbol{u}, \sigma, \mu) = 0$. Mathematically, the reliability index can be viewed as the objective of an optimization problem with the limit-state equation as the equality constraint. At its optimal solution, $\boldsymbol{u}^*$, one finds the reliability index, $\beta = \|\boldsymbol{u}^*\|$, which yields the first-order probability of failure as $P_f = \Phi(-\beta)$.

In PMA, the objective is to compute the first-order probabilistic performance measure, G^*_p. It is defined as the offset of the performance, $G(\boldsymbol{X})$, so that the shortest distance between the limit-state equation, $G(\boldsymbol{X}) - G^*_p = 0$ and the origin of the $\boldsymbol{U}$-space is equal to a given target reliability service, $\Phi(-\beta_o)$. G^*_p can also be found as the smallest value of G that is tangent to the targeted reliability surface, represented by a sphere constraint, $\|\boldsymbol{u}\| = \beta_o$. Mathematically, the first-order probabilistic performance measure is obtained as the objective of an optimization problem in which the solution is required to achieve the targeted reliability index.

The optimization formulation of the RIA is very similar to that of the PMA. In fact, at their respective optimal solutions, $\|\boldsymbol{u}^*\| = \beta^*$ in RIA and $\|\boldsymbol{u}^*\| = \beta_o$ in PMA, the Kuhn-Tucker Necessary Conditions yield convenient means to compute the respective Lagrange multipliers as

$$\lambda_r = \frac{-\beta^*}{\left(\frac{\partial G}{\partial \boldsymbol{u}}\right)^T \boldsymbol{u}^*} \qquad \text{and} \qquad \lambda_p = \frac{-\left(\frac{\partial G}{\partial \boldsymbol{u}}\right)^T \boldsymbol{u}^*}{\beta_0}$$

It is noted that their Lagrange multipliers can be either positive or negative, since they correspond to equality constraints. Furthermore, their Lagrange multipliers exhibit a reciprocal relationship if $\boldsymbol{u}^*$ is the same for both problems. However, their optimal solutions, $\boldsymbol{u}^*$, are usually different unless the targeted reliability index in PMA is exactly the same as the optimal reliability index obtained in RIA.

Since RIA or PMA are formulated as constrained optimization problems, derivations given by [5-6] can be directly applied here to calculate their optimum sensitivity derivatives. Note that in RIA and PMA, the reduced variables, $\boldsymbol{u}$, are treated as the design variables, whereas the standard deviations $\boldsymbol{\sigma}$, the mean values $\boldsymbol{\mu}$ of the random variables and the targeted β_0 in PMA can be treated as the problem parameters. Furthermore, the optimum sensitivity derivatives of RIA and PMA are readily available, as they are the by-products of their respective reliability analysis.

The sensitivity of the result of RIA, the reliability index, β^*, with respect to a problem parameter can be obtained from as

$$\frac{d\beta^*}{dp} = \frac{\partial \beta^*}{\partial p} + \lambda_r \frac{\partial G}{\partial p} = \lambda_r \frac{\partial G}{\partial p} \tag{1}$$

where $\partial\beta^*/\partial p = 0$ since β^* is a function of $\boldsymbol{u^*}$ only. On the other hand, the derivatives of the result of PMA, the probabilistic performance measure, G^*_p. with respect to the problem parameters are given as

$$\frac{dG_p^*}{dp} = \frac{\partial G_p^*}{\partial p} + \lambda_p \frac{\partial h}{\partial p} \tag{2}$$

In particular, if the standard deviations and the mean values of random variables are considered as the problem parameters, then Eq. (2) is simplified as

$$\frac{dG_p^*}{dp} = \frac{\partial G_p^*}{\partial p} \tag{3}$$

as the constraint, h, is a function of $\boldsymbol{u}$ and β_0 only. On the other hand, if the target β_0 is selected as the problem parameter, the first term, $\partial G_p^* / \partial p$, becomes zero and the second term, $\partial h/\partial p$, equals -1. Consequently, Eq. (3) is reduced to

$$\frac{dG_p^*}{d\beta_0} = -\lambda_p \tag{4}$$

Applications

The optimum sensitivity derivatives derived above provide means to support the development of a better algorithm for reliability analysis as well as RBDO. In this study, RIA employs the classical HL-RF method and PMA employs the hybrid method [8, 9].

Gradients for RBDO

A RBDO problem usually involves random variables as design variables and reliability index or performance measure, depending upon its formulation, as its objective or constraints. The randomness of a design variable is usually represented by its mean and standard deviation. Therefore, the mean and the standard deviation of the random variables can be directly modeled as the design variables. In this case, Eqs. (1), and (3) provide necessary derivatives with respect to the mean or the standard deviation of a random variable to support any RBDO algorithm.

PRIA – PMA-Based Algorithm for RIA

The key motivation of this new algorithm is the observation that the target β_0 of PMA is identical to the reliability index, β^*, of RIA, if the performance measure, G^*_p, in PMA reaches zero value. To achieve a zero G^*_p, the new algorithm repeats the PMA procedure with a newly updated β_0 in each PMA run. The update in β_0 is given by $\Delta\beta = G_p^* / \lambda_p$, where ΔG^* has been replaced by $-G^*_p$. The updated reliability index, $\beta + \Delta\beta$, will yield a new G^*_p that is closer to zero, at least in the first order sense. Repeated use of $\Delta\beta$ update can lead the PMA search to $G^*_p = 0$. The detail of PRIA is discussed in [7].

Numerical Examples

Numerical studies are conducted in this section to verify the equations derived in this paper and to demonstrate their applications to reliability analysis, reliability sensitivity analysis and RBDO.

PRIA

The new RIA first uses the hybrid method [8] to find the performance measure, G^*_p, for the targeted β_0. If the value of G^*_p does not converge to zero, the hybrid method is then restarted to find G^*_p at the newly updated β_0. The process continues until G^*_p reaches zero within a tolerance. The example limit-state equations used in this study are the ones studied in [8, 9]. The first example is a convex function. The results show that the new PRIA took 7 iterations to find the performance measure, $G^*_p = -0.357$, for a β_0 of 3.0 and it took additional 5 iterations to reach the converged reliability index, β^*=2.878. The probability of failure was then calculated as 0.201%, which is in good agreement with the results of Monte Carlo simulation. Note that a single iteration involves a function and a gradient evaluation. The second example is a concave function. The targeted β_0 was set to be 3.0. The new algorithm, PRIA, took 9 iterations to reach the converged G^*_p of 0.204. Additional 7 iterations were needed for the method to find the converged reliability index, $\beta^* = 3.803$, which gives the probability of failure at .00716%.

Reliability Sensitivity Analysis

This section presents reliability sensitivity analysis results of a 3-D flexible wing at robust design points [10]. The four parameters chosen as uncertain design variables were the root airfoil section maximum thickness, t_r, the root airfoil section maximum camber, z_r, and the structural sizing factors for the two inboard regions, Γ_1 and Γ_2, as shown in Fig. 1. A coefficient of variation, 0.001, was chosen for all design variables. Two solution-dependent constraints were selected here as limit-state equations, $g(L\text{-}W)$ and $g(C_m)$. The former requires the total lift to be greater than the weight of the structure within a given limit. The later requires a lower limit on pitching moment, C_m. The lift and the pitching moment are the outcomes of a coupled fluid-structure solution based upon Euler CFD and FEM structural simulations. Note that at the wing surface where aerodynamic load and structural deflection information are interchanged, surface nodes of the FEM structural model were a subset of the CFD aerodynamic surface mesh points.

The reliability sensitivity analysis was conducted by using RIA, PMA and PRIA. Sample reliability sensitivity derivative comparisons are shown in Table 1. The first column, labeled "DV", indicates the variable with respect to which the function is differentiated. The second and third columns identify the reliability sensitivity case by the constraint and desired reliability. The sensitivity results for three reliability analysis methods are in the subsequent columns. The tables show the analytic sensitivity derivative of β^*, in the case of RIA and PRIA methods, and G^*_p in the case of PMA. The table also shows the ratio of the analytic result to the derivative obtained by finite differencing. Derivative results obtained by using the three methods are found to agree very well, i.e., the ratio is close to 1.0. The dagger symbols in the table indicate that a few of the finite difference values could not be obtained for comparison for RIA because at the perturbed point it failed to converge for at least one side of the central difference. As for the computational efficiency, RIA and PMA are equally efficient, compared to PRIA This is expected, as PRIA is essentially a sequence of PMA analyses.

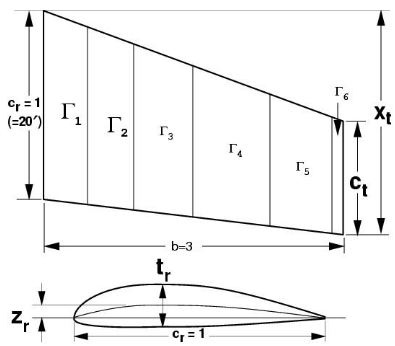

Fig. 1 Wing geometry and sizing parameterization

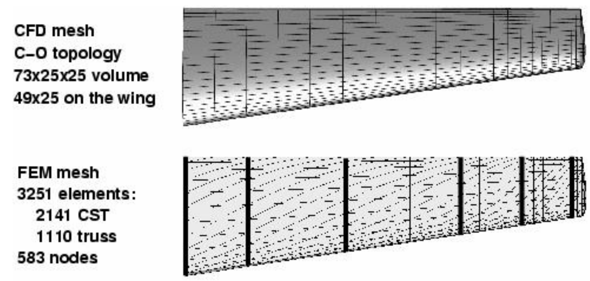

Fig. 2 CFD and FEM computational meshes.

Table 1 4DV Reliability Assessment Sensitivity Derivative Comparisons

			$\frac{\partial \beta}{\partial DV}$, RIA		$\frac{\partial \beta}{\partial DV}$, PRIA		$\frac{\partial G}{\partial DV}$, PMA	
DV	Constraint	k	Analytic	Ratio	analytic	ratio	Analytic	ratio
Γ_1	$g(L-W)$	1	-6.530	1.042	-6.530	1.004	.06952	0.992
Γ_1	$g(L-W)$	2	-6.528	0.940	-6.527	1.008	.06917	0.998
Γ_1	$g(L-W)$	3	06.512	1.072	-6.511	1.011	.06881	0.998
Γ_1	$g(C_m)$	1	09.457	0.942			.05650	1.026
Γ_1	$g(C_m)$	2	-9.778	†	-9.779	1.000	.05799	0.941
Γ_1	$g(C_m)$	3	-9.984	†	-9.984		.05949	1.036
t_r	$g(C_m)$	3	-471.5		-471.5		2.802	1.008
z_r	$g(C_m)$	2	-818.4	†	-818.5	0.962	4.835	0.998
z_r	$g(C_m)$	3	-816.3		-816.3	0.990	4.846	1.000
Γ_2	$g(C_m)$	2	-22.47	1.099	-22.47	0.958	.1332	0.999
Γ_2	$g(C_m)$	3	-23.02	1.018	-23.02	1.003	.1372	0.991

†nonconvergent RIA for at least one side of finite differencing

Reliability-Based Design Optimization

Three approaches are considered here for RBDO; i.e., the PRIA-based, the PMA-based and the PMA/RIA-mixed RBDO. A Sequential Quadratic Programming Technique, called Linearization Method [11] is employed here to support all RBDO procedures. The method uses a linearized subproblem and a line search algorithm to determine the search direction and the step size, respectively. The method also uses the active constraint strategy. The specific test problem studied here is also taken from [8, 9]. The design variables are the means of two statistically independent and normally distributed variables. The objective and the constraints of the problem are respectively given as $f(\boldsymbol{\mu}) = 3\,\mu_1^2 - 2\,\mu_1\,\mu_2 + 3\,\mu_2^2$, subject to the constraints, $P_f\,(\,G_i(\boldsymbol{X}) < 0\,) \le P_{i0}$; $i = 1, 2$.

Both the PMA-based and the PRIA-based RBDO methods drove the L_2-norm of the search direction to an acceptable level and arrived at similar local minima. As expected, the PMA was more efficient than the PRIA for reliability analysis. The PMA-based method took 131 and 39 function and gradient evaluations and the PRIA-based method, 155 and

111, respectively, for reliability analysis of limit-state equations, G_1 and G_2. However, the PMA-based method took a lot more function and gradient evaluations for line search than the PRIA-based method did. Therefore, the line search factor makes the PRIA-based method a better method for this particular RBDO example. Note that in the current study, the converged random variables, $\boldsymbol{u}^*$, of the current line search step are kept as the initial values to start the next line search step; and the final β^* and $\boldsymbol{u}^*$ at the end of the current optimization iteration are also kept as the initial values to start the next optimization iteration. Furthermore, the concept of active-in-reliability is implemented here to reduce the unnecessary reliability analysis required by line search. If the performance measurement of a constraint is less than a small number, the constraint is called active-in-reliability. An active-in-reliability constraint must be included in the design search algorithm. The strategy can be monitored efficiently with PRIA. With this new implementation, the improvement of the PMA/RIA-based RBDO is quite evident [7].

Conclusions

The reliability sensitivity analysis equations for the popular RIA and PMA methods are derived in this paper. These equations lead to the development of a new RIA method, called PRIA. The numerical experience shows that the method is robust and accurate. The PRIA method is later used for RBDO. The application of this RBDO procedure to a simple example is very encouraging. However, further investigation is needed to validate the procedure on more practical examples.

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