



# Reliability-based structural optimization using improved two-point adaptive nonlinear approximations

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## Abstract

The objective of this paper is to conduct reliability-based structural optimization in a multidisciplinary environment. An efficient reliability analysis is developed by approximating the limit-state functions using two-point adaptive nonlinear approximation. The nonlinear approximation is constructed by using the function values and the first-order gradients at two points of the limit-state function, and its nonlinearity index is automatically changed for different problems. The reduction in computational cost realized in safety index calculation and optimization are demonstrated through two structural problems. This paper presents the safety index computation, analytical sensitivity analysis of the reliability constraints and optimization using frame and turbine blade examples. © 1998 Elsevier Science B.V. All rights reserved.

**Keywords:** Reliability; Optimization; Approximations; Probability

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## 1. Introduction

Traditional structural design procedures utilize deterministic information of the problem. Suitable geometry, material properties, and loads are assumed, and an analysis is then performed to provide a detailed behavior of the structure. However, fluctuations of the loads, variability of the material properties, and uncertainties regarding the analytical models all contribute to the probability that the structure does not perform as intended. To address this concern, analysis methods have been developed to deal with the statistical nature of input information. Over the last ten years there has been an increasing trend for analyzing structures using probabilistic information of loads, geometry, material properties, and boundary conditions. As the structures are becoming more complex (e.g. space shuttle main engine components, space structures, advanced

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tactical fighter, etc.) and the performance requirements are becoming more ambitious, the need for analyzing the influence of uncertainties and computation of probabilities has been growing. Also, these methods are rapidly finding application in multidisciplinary structural optimization because of the stringent performance requirements and narrow margins of safety of the optimal solutions. In a structural design problem involving uncertainties, a structure designed by using a deterministic approach has a greater probability of failure than a structure of the same weight designed by using a probabilistic approach that accounts for the uncertainties.

Probabilistic structural analysis is inherently a computational intensive procedure, and it requires multiple, sometimes hundreds, of deterministic analyses. Each deterministic solution may need a detailed finite-element analysis. Thus, there is a strong motivation to develop efficient techniques for reducing the computer time. The objective of this work is to address the modern approximation concepts and their utility in structural reliability estimation and optimization. Emphasis is placed on building the high-quality approximations using intervening variables to closely represent the nonlinear limit state functions for safety index calculation and optimization using approximate representation of constraints.

In the last few years, use of approximations in structural reliability analysis and optimization to reduce the computer time were presented in several research papers. Belegundu [1] presented the formulation of the probabilistic constraints using the Hasofer–Lind method [2], in which he linearized the limit-state functions at the most probable failure point (MPP). Nikolaidis and Burdisso [3], also used the Hasofer–Lind method to optimize a simplified aircraft wing model for system reliability. Thanedar and Kodiyalam [4] used the HL–RF method to evaluate the probability of failure and optimized several simple examples with nonlinear, nonnormal limit-state functions that are explicit in terms of design variables. The HL–RF method is an extended Hasofer–Lind method including random variable distribution information presented by Rackwitz and Fiessler [5]. Reddy, et al. [6] used a simplified safety index calculation procedure and optimized plate structures using the NEWSUMT-A program. Chandu and Grandhi [7] used NESSUS [8] to estimate the structural reliability, and approximated the constraints using the conservative approximation and solved the approximate problem using DOT [9]. Wang and Grandhi [10] used an efficient safety index algorithm to perform structural reliability optimization. The safety index algorithm was presented in Ref. [11] by using intervening variables for limit-state function approximations. The optimization algorithm [12] was developed by using a multi-point spline approximation to approximate the reliability constraints and then the approximate optimization model was solved by using the sequential quadratic programming and dual methods.

In this paper, first, an improved two-point adaptive nonlinear approximation (TANA2) developed in Ref. [13] is applied to compute the structural reliability. This approach calculates the safety index by using the intervening variables for the limit-state function approximation and the HL–RF or DOT program [9] for reaching the constraint boundary. The method is applicable to complicated problems with arbitrary performance functions and different distributions because of its adaptive nonlinear approximation of the performance function. Unlike the approximation used in Ref. [11] for the safety index calculation, this improved approximation reproduces the exact function values and derivatives at the previous and current design points and the nonlinear indices of the function can be different for different variables. Second, the DOT program with TANA2 approximation is applied for approximating the reliability constraints of structural optimization. Finally, analytical sensitivity analysis is used to obtain the first-order derivatives. Two examples of

structural optimization are presented to demonstrate the efficiency and robustness of this reliability analysis and design approach.

## 2. Structural reliability analysis

### 2.1. Improved two-point adaptive nonlinear approximation (TANA2) [13]

The important details of this adaptive approximation are presented below and further details can be found in Ref. [13]. The physical variables are transformed to the intervening variables using the relation

$$y_i = x_i^{p_i}, \quad i = 1, 2, \dots, n, \quad (1)$$

where the exponents  $p_i$  represent the nonlinearity indices, which can be different for different variables. Information at two design points, namely, the previous design point ( $X_1$ ) and the current design point ( $X_2$ ), is used in building the approximation. The approximation is obtained by expanding the function at the current point  $X_2$  as

$$\tilde{g}(X) = g(X_2) + \sum_{i=1}^n \frac{\partial g(X_2)}{\partial x_i} \frac{x_{i,2}^{1-p_i}}{p_i} (x_i^{p_i} - x_{i,2}^{p_i}) + \frac{1}{2} \varepsilon_2 \sum_{i=1}^n (x_i^{p_i} - x_{i,2}^{p_i})^2. \quad (2)$$

This equation is a second-order Taylor series expansion in terms of the intervening variables  $y_i$  ( $y_i = x_i^{p_i}$ ), in which the Hessian matrix has only diagonal elements of the same value  $\varepsilon_2$ . Therefore, this approximation does not need the calculation of the second-order derivatives. Unlike the original second-order approximation, this approximation is expanded in terms of the intervening variables  $y_i$ , so the error from the approximate Hessian matrix is partially corrected by adjusting the nonlinearity index  $p_i$ . In contrast to the true quadratic approximation, this approximation is closer to the actual function for highly nonlinear problems because of its adaptability. Eq. (2) has  $n + 1$  unknown constants, so  $n + 1$  equations are required. Differentiating Eq. (2),  $n$  equations are obtained by matching the derivatives with the previous point,  $X_1$  values

$$\frac{\partial g(X_1)}{\partial x_i} = \left( \frac{x_{i,1}}{x_{i,2}} \right)^{p_i-1} \frac{\partial g(X_2)}{\partial x_i} + \varepsilon_2 (x_{i,1}^{p_i} - x_{i,2}^{p_i}) x_{i,1}^{p_i-1} p_i, \quad i = 1, 2, \dots, n. \quad (3)$$

Another equation is obtained by matching the exact and approximate function values with the previous point  $X_1$ , that is

$$g(X_1) = g(X_2) + \sum_{i=1}^n \frac{\partial g(X_2)}{\partial x_i} \frac{x_{i,2}^{1-p_i}}{p_i} (x_{i,1}^{p_i} - x_{i,2}^{p_i}) + \frac{1}{2} \varepsilon_2 \sum_{i=1}^n (x_{i,1}^{p_i} - x_{i,2}^{p_i})^2. \quad (4)$$

There are many algorithms to solve these  $n + 1$  equations as simultaneous equations. A simple adaptive search technique can be used to solve them [13].

In this approximation, both function and derivative values at two points were used to construct the approximation. The exact function and derivative values are equal to the approximate function and derivative values, respectively, at the previous and current points. This improved two-point approximation has been proved to have the best accuracy compared with other

two-point approximations by using several explicit functions in Ref. [13] and structural shape optimization problems in Ref. [14]. In this paper, the validity and accuracy of the approximation is checked by applying the method for the safety index calculation and reliability-based optimization. The analysis and design using probabilistic structural mechanics results in highly nonlinear limit-state functions. Hence, these problems become tough bench mark cases for the newly developed approximations.

## 2.2. Safety index calculation using TANA2

In the analysis of structural reliability, a “performance function” or “limit-state function”  $G(X)$  is defined in terms of a vector of basic random variables  $X = (x_1, x_2, \dots, x_n)^T$ . The limit state that separates the design space into “failure” and “safe” regions is  $G(X) = 0$ . Accordingly, the probability of structural failure in the specified mode is

$$P_f = \int_{\Omega} f_X(X) dX, \quad (5)$$

where  $f_X(X)$  is the joint probability density function of  $X$ , and  $\Omega$  is the failure region ( $G(X) > 0$ ).

The complex failure probability given in Eq. (5) can be approximately calculated by using the first-order second-moment method [15], that is, finding the minimum distance from the limit-state surface to the origin, i.e. the safety index  $\beta$ , by solving the following constrained optimization problem in the normal space,  $U$ :

$$\text{Minimize } F(U) = U^T U, \quad (6a)$$

$$\text{Subject to } G(U) = 0, \quad (6b)$$

where  $U$  are standard, uncorrelated and normal random variables. Then, the first-order approximation of the failure probability  $P_f$  can be calculated by using the first-order reliability method (FORM) [15]

$$P_f = \Phi(-\beta), \quad (7)$$

where  $\Phi(\cdot)$  is the standard normal distribution function. The second-order approximation of  $P_f$  is given in terms of  $\beta$  and the curvatures defined the quadratic surface [15].

There are many algorithms available to solve Eqs. (6a) and (6b), such as mathematical optimization schemes, HL-RF algorithm, the iterative algorithm given in Ref. [13], etc. In this paper, an improved two-point adaptive nonlinear approximation (TANA2) with different nonlinearity indices for each variable is used to approximate the limit-state function given in Eq. (6b) to perform more efficient reliability analysis.

Based on the above approximation model given in Eqs. (6a) and (6b), an efficient approach with adaptability is developed for the safety index calculation. The main steps are summarized as follows:

1. Obtain a linear approximation of the performance function by using the first-order Taylor's series expansion about the mean values,

2. Compute the most probable failure point  $X_k$  and safety index  $\beta_k$  using the HL-RF method,
3. Determine the exponents  $p_k$  ( $k = 1, 2, \dots, N$ ) by solving the Eqs. (3) and (4) based on the information of the current and previous points (when  $k$  equals 1, the previous design point is the mean value);
4. Obtain the nonlinear approximation of the performance function

$$\tilde{Z}(X) = g(X_2) + \sum_{k=1}^n \frac{\partial g(X_2)}{\partial x_k} \frac{x_{k,2}^{1-p_k}}{p_k} (x_k^{p_k} - x_{k,2}^{p_k}) + \frac{1}{2} \varepsilon_2 \sum_{k=1}^n (x_k^{p_k} - x_{k,2}^{p_k})^2, \quad (8)$$

5. Find the most probable failure point  $X_{k+1}$  of the nonlinear approximate function  $Z(X)$  and the safety index  $\beta_{k+1}$  using the HL-RF method or the DOT program and denote  $X_{k+1}$  as the current point and  $X_k$  as the previous point;
6. Compute  $\varepsilon = |(\beta_{k+1} - \beta_k)/\beta_k|$ , repeat steps (3)–(5) until  $\varepsilon$  is less than the allowable value.

In step (5), the safety index  $\beta$  is iteratively computed for the explicit approximate function  $Z(X)$  given in Eq. (8). Any iterative algorithm can be used for finding the MPP. The computation of the exact performance function  $Z(X)$  is not required; therefore, the computer time is greatly reduced for problems involving complex and implicit performance functions, particularly with finite-element models.

### 3. Safety index sensitivity analysis

An important task in optimal design is to obtain sensitivity derivatives, which are used for studying the effect of parametric modifications, calculating the search directions for finding an optimum and constructing the function approximations. An analytical approach for the sensitivity analysis of the safety index was presented in Ref. [15], and the formula is given as

$$\frac{\partial \beta}{\partial b} = \lambda^T \nabla_b G(b^0, U^*) = \lambda^T \frac{\partial G}{\partial U} \frac{\partial U}{\partial b} \quad (9)$$

and

$$\lambda = \frac{1}{\|\nabla_U G(b^0, U^*)\|}, \quad (10)$$

where  $U^*$  and  $b^0$  are the optimum solutions of Eqs. (6a) and (6b), and  $\lambda^T \nabla_b G(b^0, U^*)$  are the gradients of the failure mode  $G(U, b)$  in terms of the design variables,  $b$ . From Eqs. (9) and (10), the exact derivatives of the safety index  $\beta$  with respect to design variable  $b$  can be obtained when  $U^*$  is a real optimal solution of the minimization problem (6), that is, if  $\beta$  is exact.

In many practical problems, the design variables  $b$  usually are modeled with uncertain means and uncertain (or fixed) standard deviations. For this case,  $\partial U/\partial b$  in Eq. (9) can be derived easily since  $U$  is a function of means and standard deviation, and the  $\lambda$  and  $\partial g/\partial U$  in Eq. (9) are the by-product from the reliability analysis. Thus, no additional analysis is needed for the sensitivity calculation of the safety index.

#### 4. Reliability-based optimization

A robust structural system must not only satisfy the performance/weight/cost constraints, but also the reliability constraints due to uncertainty. In general, the design problem for this robust structural system can be defined as

$$\text{Minimize } F(b) \quad (11a)$$

$$\text{Subject to } P[G_j(b, X) \geq 0] \geq P_j, \quad j = 1, 2, \dots, J, \quad (11b)$$

$$b_i^L \leq b_i \leq b_i^U, \quad i = 1, 2, \dots, N, \quad (11c)$$

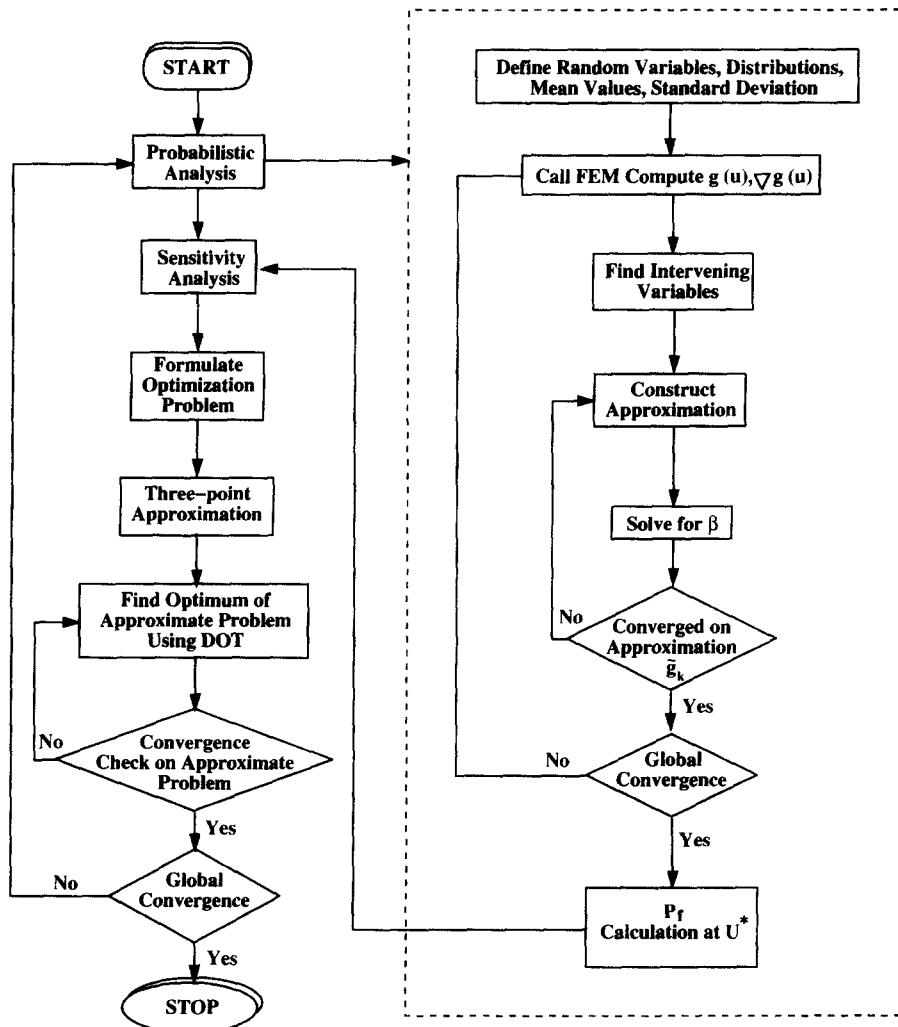


Fig. 1. Flow-chart of probabilistic analysis and optimization.

where  $X$  is a vector of random variables,  $b$  is a vector of design variables,  $P[.]$  denotes the probability of the event  $[.]$ , and  $P_j$  is the required probability of survival for the  $j$ th constraint or failure mode,  $G_j(b, X)$ .

In the above probabilistic optimization problem, some or all design variables  $b$  can also be random. In this paper, all design variables are assumed to be random, and their mean values are treated as the design variables, which are subjected to upper and lower bound constraints. The constraints defined in (11b) are assumed to be independent and thus no correlation exists. In general, it is almost impossible to calculate the probability of Eq. (11b) by a multiple integration, thus, the above probabilistic optimization problem is converted into the following equivalent deterministic one by using a first-order second-moment method.

$$\text{Minimize } F(b) \quad (12a)$$

$$\text{Subject to } g_j(b) = \Phi^{-1}(P_j) - \beta_j \leq 0, \quad j = 1, 2, \dots, J, \quad (12b)$$

$$b_i^L \leq b_i \leq b_i^U, \quad i = 1, 2, \dots, N, \quad (12c)$$

where  $J$  represents the number of limit state functions. This is an equivalent deterministic optimization problem, and the reliability constraints given in Eq. (12b) are approximated by the two-point adaptive nonlinear approximation. Comparison [14] of various types of two-point approximations demonstrated that TANA-2 is efficient and stable for a wide class of problems. TANA-2 adaptively approximates the second-order effects using a hessian matrix. The approximate optimization model is solved by using DOT [9]. Fig. 1 presents the nested optimization sequence and approximations in reliability analysis.

## 5. Numerical examples and discussions

A 313-member frame and a gas turbine blade with reliability constraints are selected to demonstrate the method. The required reliability of survival is 0.999. The structural responses include displacements and natural frequencies. The mean values are treated as the design variables during the optimization process.

**Example 1. (313-member frame).** The structure shown in Fig. 2 is modeled with 313 frame elements with an  $I$ -section. The area moment of inertia  $I_z$  is expressed as an explicit nonlinear function of  $A$  in the form  $I_z = 0.2072 A^3$ . The vertical loads at nodes 15, 16, 88, 89 are  $-26$ ,  $-30$ ,  $-18$ ,  $-20$  kips, respectively, and the horizontal loads at nodes 6, 11, 17, through 65 by 3, 68 through 82 by 7, and 90 through 175 by 5 are 4 kips, the horizontal load at node 1 is 2 kips.

In order to demonstrate the importance of reliability-based optimization, two cases are considered and their results are compared in this example. In case 1, all the element areas are considered as the design variables, and the total number of linked variables is 86. These variables are assumed as deterministic and a total of 358 displacement constraints are considered. The allowable displacements in both directions of  $x$  and  $y$  are 4.0 in.

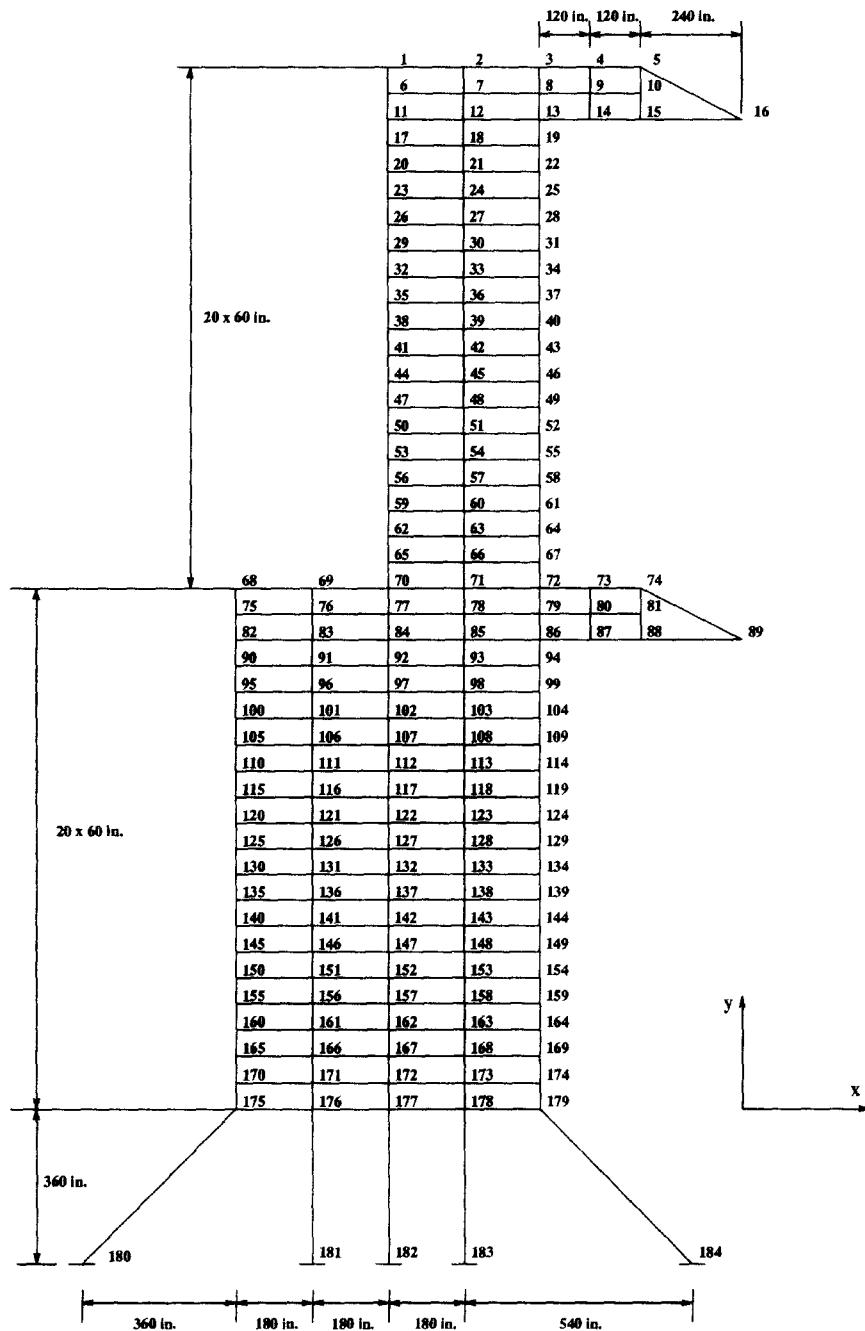


Fig. 2. 313 member frame.

In case 2, the cross-sectional areas of all members are selected as the random variables with normal distributions. The probabilistic optimization is conducted by assuming that the system failure occurs when in one or more than one of the constraints, failure occurs. Therefore, the system failure probability equals the largest probability of failure among the above constraints. Active

displacements,  $d_j$ , of the total 358 displacements are taken as the limit-state functions, and each of them is an implicit function of the random variables and the constraints are written as

$$g_j(X) = 1 - d_j/d_{\text{lim}},$$

where  $d_{\text{lim}}$  is the displacement limit of 4.0 in. All the element areas have a mean value of 28.0 in<sup>2</sup>, with a coefficient of variation (COV) of 0.05. All the mean values of the random variables are considered as the design variables of optimization, which make a total of 86 design variables. The reliability goal is 0.999 ( $\beta = 3.09$ ). The move limit on design variables is 70%.

In both cases, the initial design starts from the element areas of 28.0 in<sup>2</sup>, with an initial weight of 292829.7 lb. Comparison of the deterministic and probabilistic optimization results is shown in Table 1. For the deterministic design, 6 iterations are needed to find the optimum weight of 203397.8 lb. The active constraints at the optimum point are the y-direction displacements at the nodes 16 and 89. The corresponding constraint values ( $g(X) = 1 - d/4.0$ ) are  $3.6228 \times 10^{-4}$  and  $-3.0042 \times 10^{-2}$ , respectively. A probabilistic analysis is conducted at the deterministic optimum. The active constraints have reliabilities of 0.5054 ( $\beta = 0.0136$ ) and 0.8856 ( $\beta = 1.2033$ ), respectively. The results indicate that the deterministic design can significantly reduce the structural weight, but its ability to meet the design requirements for reliability under uncertainties is quite low. To obtain a more reliable design by considering uncertainties during the optimization process, probabilistic design is needed to meet the reliability goal of 0.999. Table 1 presents the iteration history and shows that the safety index  $\beta$  search could yield local minima or multiple minima. In iteration 4, there are two  $\beta$  values, but the optimization is based on the shortest distance.

For the probabilistic design, 7 optimization iterations with a total of 20 structural analyses are needed. The reliability at the optimum point is equal to 0.999 ( $\beta = 3.09$ ). The final weight is 230750.8 lb. Compared with the deterministic design, the final weight of the probabilistic design increases from 203397.8 to 230750.8 lb, an increase of 13.4%. However, the probability is greatly increased and meets the required level of 0.999.

Fig. 3 shows that the areas of the elements with nodes 17 through 67, 90 through 174, 175 and 180, which are located at the locations where the displacements are small, increase to meet the

Table 1  
Iteration history of 313 member frame

OPT. Iteration	Deterministic design Weight (lb)	Probabilistic design Weight (lb)	Safety Index $\beta^a$	Reliability $p$
1	292829.7	292829.7	5.2359(3)	1.0000
2	207541.4	252678.6	2.6824(2)	0.9963
3	204311.0	239597.5	3.0886(2)	0.9989
4 <sup>b</sup>	201215.3	229413.7	1.3909(2), 2.2713(5)	0.9179, 0.9884
5	203474.0	231647.6	3.5274(2)	0.9979
6	203397.8	230760.9	3.0444(2)	0.9988
7		230750.8	3.9000(2)	0.9990
Final reliability	0.5054	0.9990		

<sup>a</sup>The numbers in brackets are the number of iterations needed for computing  $\beta$ .

<sup>b</sup>In this iteration, there are two active  $\beta$  constraints.

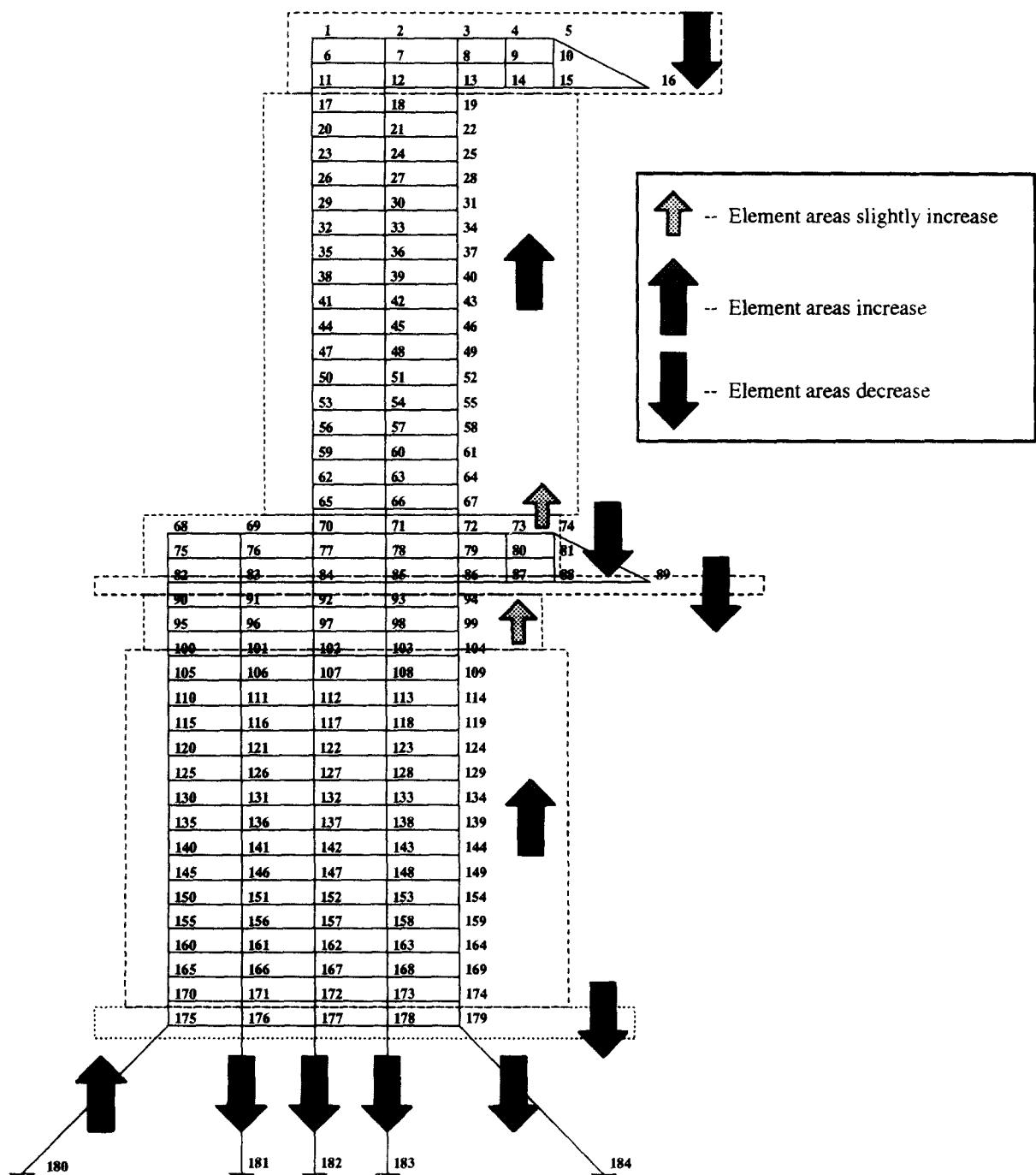


Fig. 3. Change in probabilistic optimization solution compared with the deterministic optimum.

reliability goal. Meanwhile, the areas of other elements, located at the locations where the displacements are critical or close to critical, decrease to reduce the structural weight. The comparison of deterministic and probabilistic designs indicates that the probabilistic optimization obtains more uniform material distribution, whereas the deterministic optimization reduces the structural weight significantly in uncritical regions.

**Example 2. (Twisted Gas Turbine Blade).** This example is a gas turbine blade (Fig. 4) with a  $2 \times 2$  in size with a  $45^\circ$  twist. The twist angle and the twist parameter are related to the shape of the blade. The blade is modeled with 80 quadrilateral plate bending elements with 99 nodes. All the degrees of freedom along the hub are fixed.

In the deterministic design, all the element thicknesses are considered as design variables, which make a total of eighty variables. The constraint is on the fundamental natural frequency of the blade with a lower limit of 8200.0 rad/s. The initial weight is  $1.069 \times 10^{-3}$  lb. Seven optimization iterations are needed to find the optimum weight of  $3.2746 \times 10^{-4}$  lb. The critical constraint value at the optimum point is  $-5.4799 \times 10^{-4}$ . The reliability of this final design is 0.5021 ( $\beta = 0.0053$ ) under the presence of uncertainties. The result indicates that the reliability at the deterministic optimum is quite low and needs to be improved by considering the probabilistic design.

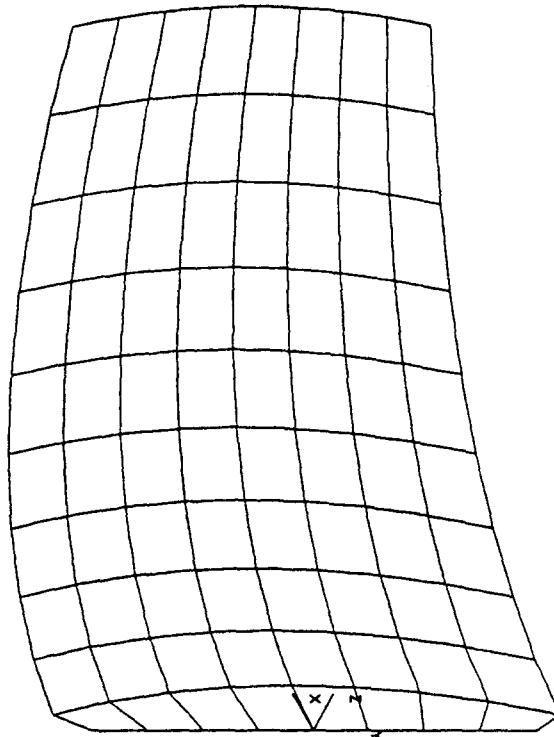


Fig. 4. Finite element model of a twisted gas turbine blade.

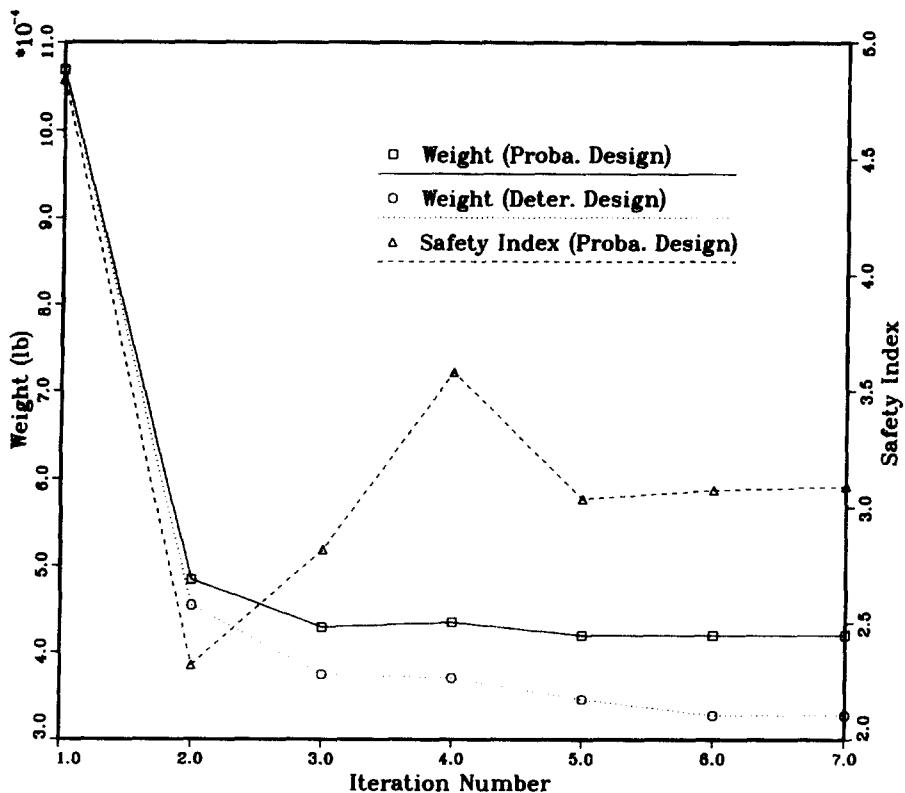


Fig. 5. Iteration history of turbine blade design.

In the probabilistic optimization, the geometric and material properties of the turbine blade are considered as random variables with normal distributions. The fundamental natural frequency of the blade is taken as the limit-state function, which is an implicit function of random variables. The mean value of Young's modulus  $E$  is  $2.9 \times 10^7$  lb/in<sup>2</sup>, with COV = 0.15; and the mean value of density  $\rho$  is  $7.51 \times 10^{-4}$  lb-s<sup>2</sup>/in<sup>4</sup>, with COV = 0.15. All the element thicknesses have a mean of 0.35 in, with COV = 0.10. The total number of random variables is 82. The limit on the safety index is 3.09. The thicknesses of the elements are considered as the design variables of optimization, which make a total of 80 design variables. The move limit on design variables is 70%. The initial design starts from the mean values, with an initial weight of  $1.069 \times 10^{-3}$  lb. The iteration history of optimization and the safety index  $\beta$  are shown in Fig. 5. The final weight is  $4.201 \times 10^{-4}$  lb, and seven optimization iterations with a total of 21 structural analyses are needed. The reliability at the optimum point is equal to 0.999 ( $\beta = 3.09$ ).

The results comparison shows that the blade weight of the probabilistic design increases 28.04% to meet the reliability goal of 0.999. The thicknesses of the elements near the blade root (close to the rotational axis) are increased, while the element thicknesses near the tip are the same as the deterministic optimum, which are the lower limits of the design variables.

## 6. Summary remarks

An efficient probabilistic analysis and optimal design methodology are presented for successful design of complex structures requiring FEM analysis. In this methodology, a safety index algorithm with an adaptive nonlinearity is used for the reliability analysis, and the DOT program minimized the structural weight with reliability constraints. The computational results show that the two-point adaptive nonlinear approximation, TANA2, is quite efficient and robust for the reliability analysis and structural optimization, which significantly reduces the number of FEM analyses in reliability analysis and optimization.

In the examples, the failure criteria were assumed to be independent, and the system failure probability was approximately calculated based on the most critical failure mode. However, the use of a single-failure mode in most situations may provide an inaccurate estimate of system reliability, particularly when one single-failure mode does not have a relatively much higher probability of occurrence than all the other failure modes. Therefore, the future research will concentrate on system reliability analysis and design.

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