EEE 431: Telecommunications 1

Quiz 3

Dec. 19, 2021, 13:30

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Section: ______________________________

Prob. 1: ________ / 48

Prob. 2: ________ / 52

Total: ________ / 100

Problem 1. Let $s(t)$ denote a rectangular pulse defined as follows: $s(t) = 2$ if $t \in [0, 1]$ and $s(t) = 0$ otherwise. We define a random process as $X(t) = As(t) + W(t)$, where A is equal to 3 or -3 with equal probabilities (i.e., $1/2$ each), and $W(t)$ is a zero-mean white Gaussian noise process with a power spectral density of $S_W(f) = 0.5$ for all f. It is assumed that A and $W(t)$ are independent for all t.

(a) Find the autocorrelation function of $X(t)$, that is, $R_X(t_1, t_2)$, in its simplest form.

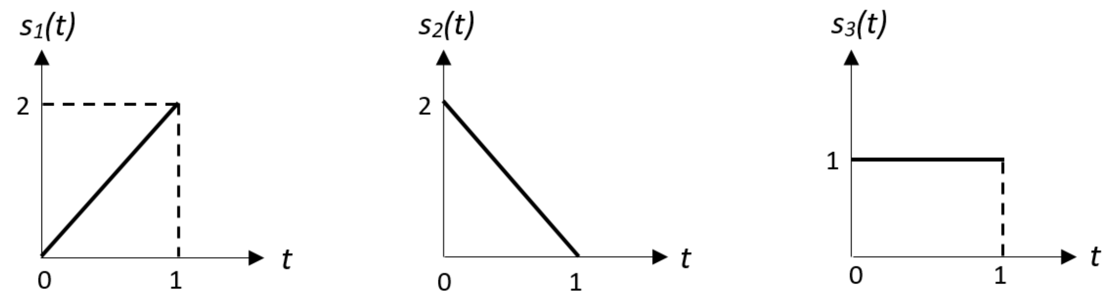
(b) Is $X(t)$ WSS? Why or why not? (please justify your answer)

(c) Calculate the mean and variance of $X(0.5)$.

(d) Calculate covariance of $X(0.5)$ and $X(0.75)$, i.e., $\mathrm{Cov}(X(0.5), X(0.75))$.

(e) Let Y be defined as $Y = \int_0^1 As(t - 0.25)X(t)dt$. Specify (completely) the probability distribution of Y.

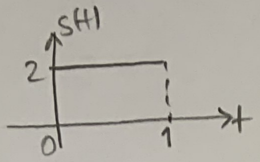
Problem 2. (a) For the following signals, find a set of orthonormal basis functions (both write down their mathematical expressions and plot them), and express each signal as a vector in the corresponding signal space.



(b) Suppose that these signals are equally likely and sent towards a receiver over an additive white Gaussian noise channel with spectral density level of $N_0/2$. Show the structure of the optimal receiver with all the details, and provide the mathematical expression for the maximum likelihood (ML) decision rule at this receiver in its simplest form.

(c) Find the exact probability of error for the optimal receiver in Part b).

(d) Calculate the union bound on the probability of error.

① 

$X(t) = A s(t) + W(t)$ $\quad$ $A \to \mp 3$ w/ probs 1/2

$W(t) \to$ AWGN, $S_W(f) = 0.5 \ \forall f$

10 ⓐ $R_X(t_1,t_2) = E[(As(t_1)+W(t_1))(As(t_2)+W(t_2))]$

$= E[A^2]\, s(t_1)s(t_2) + E[W(t_1)W(t_2)] = \boxed{9\, s(t_1)s(t_2) + 0.5\,\delta(t_1-t_2)}$

2 ⓑ Not WSS, $R_X(t_1,t_2)$ does not only depend on t_1-t_2

8 ⓒ $X(0.5) = A s(0.5) + W(0.5) = 2A + \underbrace{W(0.5)}_{N(0,\infty)}$ $\to$ $E[X(0.5)] = \boxed{0}$ ✓ (3) $\quad$ 36.5 → (2) (−3)

$Var(X(0.5)) = \boxed{\infty}$ (5)

18 ⓔ $Y = \int_0^1 A\, s(t-0.25)\,(As(t)+W(t))\,dt$

(4) $= \underbrace{A^2}_{9} \underbrace{\int_0^1 s(t-0.25)s(t)\,dt}_{3} + \underbrace{\int_0^1 \overbrace{A s(t-0.25)}^{h(t)}\, W(t)\,dt}_{N\left(0,\ 0.5 \underbrace{\int_0^1 A^2 s^2(t-0.25)\,dt}_{27}\right)}$

$\Rightarrow Y \sim N(27, 13.5)$

(4) (3) $\boxed{10}$

10 ⓓ $X(0.75) = A s(0.75) + W(0.75) = 2A + W(0.75)$

$E[X(0.75)X(0.5)] = E[(2A+W(0.5))(2A+W(0.75))]$

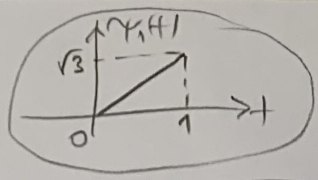
(4) $= \underbrace{E[4A^2]}_{(3)} + \underbrace{E[W(0.5)W(0.75)]}_{\frac{N_0}{2}\underbrace{\delta(0.75-0.5)}_{0}\ (3)} = \boxed{36}$ ✓

covariance since zero-mean

$X(0.5), X(0.75)$

(2)

(a) $\varphi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{2t}{\sqrt{\frac{4}{3}}} = \boxed{\sqrt{3}\,t}$ (3) $\quad E_1 = \int_0^1 4t^2\,dt = \frac{4}{3}$

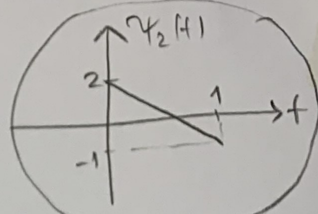


(14)

$s_{21} = \int_0^1 (2-2t)\sqrt{3}\,t\,dt = 2\sqrt{3}\left(\frac{1}{2}-\frac{1}{3}\right) = \boxed{\frac{\sqrt{3}}{3}}$

(5)

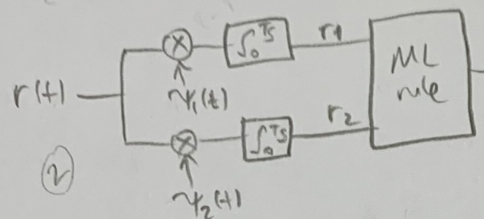
$d_2(t) = s_2(t) - \frac{\sqrt{3}}{3}\sqrt{3}\,t = 2-2t-t = \boxed{2-3t} \Rightarrow \varphi_2(t) = \boxed{2-3t}$



$\int_0^1 d_2^2(t)\,dt = \int_0^1 (4-12t+9t^2)\,dt = 4-6+3 = \boxed{1}$

$\underline{s}_1 = \begin{bmatrix} 2\sqrt{3}/3 \\ 0 \end{bmatrix}$ (2) $\quad \underline{s}_2 = \begin{bmatrix} \sqrt{3}/3 \\ 1 \end{bmatrix}$ (2) $\quad \underline{s}_3 = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$ (2)

(17) (b)



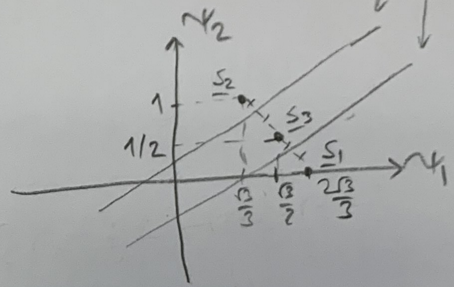
(2)

$\hat{m} = \begin{cases} 1, & \cancel{\frac{r_1}{\sqrt{3}} - r_2 \geq 0} \ \& \ \frac{r_1}{\sqrt{3}} - r_2 \geq \frac{1}{3} \\ 2, & \cancel{\frac{r_1}{\sqrt{3}} - r_2 < 0} \ \& \ \frac{r_1}{\sqrt{3}} - r_2 < -\frac{1}{3} \\ 3, & \text{o.w.} \end{cases}$ (15)

$-2\underline{r}^T\underline{s}_1 + \|\underline{s}_1\|^2 = -2\,\frac{2\sqrt{3}}{3} r_1 + \frac{4}{3}$

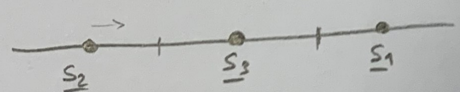
$-2\underline{r}^T\underline{s}_2 + \|\underline{s}_2\|^2 = -2\left(\frac{\sqrt{3}}{3} r_1 + r_2\right) + \frac{4}{3}$

$-2\underline{r}^T\underline{s}_3 + \|\underline{s}_3\|^2 = -2\left(\frac{\sqrt{3}}{2} r_1 + \frac{1}{2} r_2\right) + 1$



(15) (c)

$\|\underline{s}_1 - \underline{s}_2\| = \frac{2}{\sqrt{3}} \qquad \|\underline{s}_1 - \underline{s}_3\| = \|\underline{s}_2 - \underline{s}_3\| = \frac{1}{\sqrt{3}}$



$P_{e,1} = P_{e,2} = P\left(n_1 > \frac{1}{2\sqrt{3}}\right) = \boxed{Q\left(\frac{1}{2\sqrt{3}\sqrt{N_0/2}}\right)}$

$n_1 \sim N\left(0, \frac{N_0}{2}\right)$

$P_{e,3} = P\left(n_1 > \frac{1}{2\sqrt{3}} \text{ or } n_1 < -\frac{1}{2\sqrt{3}}\right) = \boxed{2\,Q\left(\frac{1}{2\sqrt{3}\sqrt{N_0/2}}\right)}$

$P_e = \frac{4}{3}\,Q\left(\frac{1}{\sqrt{6N_0}}\right)$

(6) (d) Union bound:

$P_e \leq \underbrace{\frac{1}{3}\,2\cdot Q\left(\frac{1/\sqrt{3}}{\sqrt{2N_0}}\right)}_{4} + \underbrace{\frac{1}{3}\,Q\left(\frac{2/\sqrt{3}}{\sqrt{2N_0}}\right)}_{2} = \boxed{\frac{2}{3}\,Q\left(\frac{1}{\sqrt{6N_0}}\right) + \frac{1}{3}\,Q\left(\frac{2}{\sqrt{6N_0}}\right)}$