EEE 431 Homework 1 Solutions
Spring 2021

1) Problem 5.9.

1.

$$x < -1 \implies F_X(x) = 0 \tag{1}$$
$$-1 \leq x \leq 0 \implies F_X(x) = \int_{-1}^{x} (v+1)\, dv = \frac{1}{2}x^2 + x + \frac{1}{2} \tag{2}$$
$$0 \leq x \leq 1 \implies F_X(x) = \int_{-1}^{0} (v+1)\, dv + \int_{0}^{x} (-v+1)\, dv = -\frac{1}{2}x^2 + x + \frac{1}{2} \tag{3}$$
$$1 \leq x \implies F_X(x) = 1 \tag{4}$$

2.

$$P(X > \frac{1}{2}) = 1 - F_X(\frac{1}{2}) = \frac{1}{8} \tag{5}$$

and

$$P(X > 0 \mid X < \frac{1}{2}) = \frac{P(X > 0, X < \frac{1}{2})}{P(X < \frac{1}{2})} = \frac{F_X(\frac{1}{2}) - F_X(0)}{F_X(\frac{1}{2})} = \frac{3}{7} \tag{6}$$

3.

$$F_X(x \mid X > \frac{1}{2}) = P(X \leq x \mid X > \frac{1}{2}) = \frac{P(X \leq x, X > \frac{1}{2})}{P(X > \frac{1}{2})} \tag{7}$$

If $x \leq 1/2$, then $P(X \leq x, X > \frac{1}{2}) = 0$.
If $x > 1/2$, then

$$F_X(x \mid X > \frac{1}{2}) = \frac{F_X(x) - F_X(1/2)}{1 - F_X(1/2)} \tag{8}$$

Hence, $f_X(x \mid X > \frac{1}{2})$ is given by

$$f_X(x \mid X > \frac{1}{2}) = \begin{cases} \frac{f_X(x)}{1 - F_X(1/2)}, & \text{if } x > \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

4.

$$E[X \mid X > 1/2] = \int_{1/2}^{\infty} x \frac{f_X(x)}{1 - F_X(1/2)}\, dx = \tag{10}$$

$$= 8 \int_{1/2}^{1} x(-x+1)\, dx = \frac{2}{3}. \tag{11}$$

2) Problem 5.11

1. $X \sim \mathcal{N}(0, 10^{-8})$. Thus $P(X > 10^{-4}) = Q(10^{-4}/10^{-4}) = Q(1) = 0.159$.

$$P(X > 4 \times 10^{-4}) = Q(\frac{4 \times 10^{-4}}{10^{-4}}) = Q(4) = 3.17 \times 10^{-5} \tag{12}$$

$$P(-2 \times 10^{-4} < X \leq 10^{-4}) = 1 - Q(1) - Q(2) = 0.8182 \tag{13}$$

2.

$$P(X > 10^{-4} \mid X > 0) = \frac{P(X > 10^{-4}, X > 0)}{P(X > 0)} = \frac{P(X > 10^{-4})}{P(X > 0)} = \frac{0.159}{0.5} = 0.318 \tag{14}$$

3. Let $y = g(x) = xu(x)$. Clearly, $f_Y(y) = 0$ and $F_Y(y) = 0$ for $y < 0$
If $y > 0$, then the equation $y = xu(x)$ has a unique solution $x_1 = y$.
Hence $F_Y(y) = F_X(y)$ and $f_Y(y) = f_X(y)$ for $y > 0$.
$F_Y(y)$ is discontinuous at $y = 0$, and this jump is equal to $F_X(0)$.

$$F_Y(0^+) - F_Y(0^-) = F_X(0) = \frac{1}{2} \tag{15}$$

Hence, $f_Y(y)$ equals

$$f_Y(y) = f_X(y)u(y) + \frac{1}{2}\delta(y) \tag{16}$$

The general expression for finding $f_Y(y)$ cannot be used, $g(x)$ is constant for some interval. There is an uncountable number of solutions for x in this interval.

4.

$$E[Y] = \int_{-\infty}^{\infty} y[f_X(y)u(y) + \frac{1}{2}\delta(y)]\, dy \tag{17}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} y e^{-\frac{y^2}{2\sigma^2}} = \frac{\sigma}{\sqrt{2\pi}} \tag{18}$$

5. For $y > 0$

$$f_Y(y) = \frac{f_X(y)}{|sgn(y)|} + \frac{f_X(-y)}{|sgn(-y)|} = f_X(y) + f_X(-y) \tag{19}$$

$$= \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} \tag{20}$$

For $y < 0$, it is clear that $f_Y(y) = 0$. Hence,

$$E[Y] = \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} y e^{-\frac{y^2}{2\sigma^2}}\, dy = \frac{2\sigma}{\sqrt{2\pi}} \tag{21}$$

3) Problem 5.22.

1.

$$\int_0^{\infty} \int_y^{\infty} K e^{-x-y}\, dx\, dy = K \int_0^{\infty} e^{-y} \int_y^{\infty} e^{-x}\, dx\, dy \tag{22}$$

$$= K \int_0^{\infty} e^{-2y}\, dy = K\frac{1}{2} = 1 \tag{23}$$

Hence $K = 2$.

2.

$$f_X(x) = \int_0^{x} 2e^{-x-y}\, dy = 2e^{-x}(-e^{-y})\Big|_0^x = 2e^{-x}(1 - e^{-x}) \tag{24}$$

$$f_Y(y) = \int_y^{\infty} 2e^{-x-y}\, dx = 2e^{-y}(-e^{-x})\Big|_y^{\infty} = 2e^{-2y} \tag{25}$$

3.Since $f_X(x)f_Y(y) \neq f_{X,Y}(x,y)$, X and Y are not indepedent.

4.If $x < y$, then $f_{X|Y}(x \mid y) = 0$. If $x \geq y$, we obtain

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = e^{-x+y} \tag{26}$$

5. $E[X \mid Y = y]$ is given by

$$E[X \mid Y = y] = \int_y^{\infty} x e^{-x+y}\, dx = e^y \int_y^{\infty} x e^{-x}\, dx = e^y e^{-y}(y+1) = y + 1. \tag{27}$$

6. $COV(X,Y) = E[XY] - E[X]E[Y]$.

$$E[X] = \int_0^{\infty} 2e^{-x}(1 - e^{-x}) x\, dx = \frac{3}{2} \tag{28}$$

$$E[Y] = \int_0^{\infty} 2y e^{-2y}\, dy = \frac{1}{2} \tag{29}$$

$$E[XY] = \int_0^{\infty} \int_y^{\infty} 2xy e^{-x-y}\, dx\, dy = 1 \tag{30}$$

Then $COV(X,Y) = 1 - \frac{3}{4} = \frac{1}{4}$.

$$E[X^2] = \int_0^\infty 2x^2 e^{-x}(1-e^{-x})\,dx = \frac{7}{2} \tag{31}$$

$$E[Y^2] = \int_0^\infty 2y^2 e^{-2y}\,dy = \frac{1}{2} \tag{32}$$

Hence $Var(X) = \frac{7}{2} - \frac{9}{4} = \frac{5}{4}$, $Var(Y) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.
Then,

$$\rho_{X,Y} = \frac{\frac{1}{4}}{\sqrt{\frac{1}{4}\frac{5}{4}}} = \frac{1}{\sqrt{5}} \tag{33}$$

4) Problem 5.26. Mean value of Y can be written as

$$E[Y] = \frac{1}{2\pi}\int_{-\pi}^{\pi} cos(x)\,dx = \frac{1}{2\pi} sin(x)\Big|_0^{2\pi} = 0. \tag{34}$$

5)

a) Let $g(U) = \sqrt{\ln U}$. Then, probability density function of Y can be given by

$$f_Y(y) = f_U(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| \tag{35}$$

As $g^{-1}(y) = e^{y^2}$, (35) can be expressed as

$$f_Y(y) = f_U(e^{y^2})2ye^{y^2} = \begin{cases} 2ye^{y^2}/4, & \text{if } 0 \le y \le \sqrt{\ln 5}, \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

b) We will compute $\mathbb{P}(Z \le z)$ for any $z \in \mathbb{R}$. It is clear that

$$\mathbb{P}(Z \le z) = \begin{cases} 0, & \text{if } z < 3, \\ 1 & \text{if } z > 5. \end{cases} \tag{37}$$

Now we concentrate on the region $3 \le z \le 5$. Then, the following equations must hold:

$$\begin{aligned} \mathbb{P}(Z \le z) &= \mathbb{P}(U \ge 3, Z \le z) + \mathbb{P}(U < 3, Z \le z) && (38)\\ &= \mathbb{P}(U \ge 3, U \le z) + \mathbb{P}(U < 3) && (39)\\ &= \mathbb{P}(3 \le U \le z) + \frac{1}{2} && (40)\\ &= \frac{z-3}{4} + \frac{1}{2} = \frac{z-1}{4}. && (41) \end{aligned}$$

Note that $\mathbb{P}(z \le z)$ has a discontinuity at $z = 3$.
By differentiating $\mathbb{P}(z \le z)$ and using the fact that $\mathbb{P}(Z = 3) = \frac{1}{2}$, we obtain that

$$f_Z(z) = \begin{cases} \frac{1}{2}\delta(z-3) + \frac{1}{4}, & \text{if } 3 \le z \le 5, \\ 0 & \text{otherwise.} \end{cases} \tag{42}$$

6)

a) Note that probability density function of X is given by

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2}, & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

Hence, one can obtain that

$$\mathbb{P}(X > 3) = \int_3^\infty \frac{1}{2}e^{-x/2}\,dx = e^{-3/2} \tag{44}$$

b) Via Bayes Rule and using the result in a),

$$\mathbb{P}(X > 5 \mid X > 3) = \frac{\mathbb{P}(X > 5, X > 3)}{\mathbb{P}(X > 3)} = \frac{\mathbb{P}(X > 5)}{\mathbb{P}(X > 3)} = \frac{e^{-5/2}}{e^{-3/2}} = \frac{1}{e} \tag{45}$$

c) Take any $t \geq 3$,

$$\mathbb{P}(X \leq t \mid X > 3) = \frac{\mathbb{P}(X \leq t, X > 3)}{\mathbb{P}(X > 3)} = \frac{e^{-3/2} - e^{-t/2}}{e^{-3/2}} \tag{46}$$

Then, the conditional probability density function of X is given by

$$f_{X|X>3}(t) = \begin{cases} \frac{e^{3/2}}{2}e^{-t/2}, & \text{if } t \geq 3, \\ 0 & \text{otherwise.} \end{cases} \tag{47}$$

Then, $\mathbb{E}[X \mid X > 3]$ is given by

$$\begin{aligned} \mathbb{E}[X \mid X > 3] &= \frac{e^{3/2}}{2}\int_3^\infty te^{-t/2}\,dt && (48) \\ &= \frac{1}{2}\int_0^\infty (x+3)e^{-x/2}\,dx && (49) \\ &= \underbrace{\frac{1}{2}\int_0^\infty xe^{-x/2}\,dx}_{2} + 3\underbrace{\int_0^\infty \frac{1}{2}e^{-x/2}\,dx}_{1} = 5. && (50) \end{aligned}$$

This result is a consequence of memoryless property of exponential random variables.

d) Take any t satisfying $3 \leq t \leq 5$,

$$\mathbb{P}(X \leq t \mid 3 < X < 5) = \frac{\mathbb{P}(3 < X \leq t)}{\mathbb{P}(3 < X < 5)} = \frac{e^{-3/2} - e^{-t/2}}{e^{-3/2} - e^{-5/2}} \tag{51}$$

Then, the conditional probability density function of X is given by

$$f_{X|3<X<5}(t) = \begin{cases} \frac{1}{2(e^{-3/2} - e^{-5/2})}e^{-t/2}, & \text{if } 3 \leq t \leq 5, \\ 0 & \text{otherwise.} \end{cases} \tag{52}$$

Then, $\mathbb{E}[X^2 \mid X > 3]$ is given by

$$\mathbb{E}[X^2 \mid X > 3] = \frac{1}{2(e^{-3/2} - e^{-5/2})} \int_3^5 t^2 e^{-t/2}\, dt \tag{53}$$

$$= \frac{1}{2(e^{-3/2} - e^{-5/2})} e^{-x/2}(-2x^2 - 8x - 16)\Big|_3^5 \tag{54}$$

$$= \frac{29e - 53}{e - 1} \tag{55}$$

7)
a) Take any $z \geq 0$. Then,

$$\mathbb{P}(Z \leq z) = 1 - \mathbb{P}(Z > z) \tag{56}$$

$$= 1 - P(\min(X, Y) > z) = 1 - P(X > z)P(Y > z) \tag{57}$$

$$= 1 - e^{-2z}e^{-3z} = 1 - e^{-5z} \tag{58}$$

Hence, the probability density function of Z is given by

$$f_Z(z) = \begin{cases} 5e^{-5z}, & \text{if } z \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{59}$$

In other words, Z is exponentially distributed with mean $1/5$.
b) Take any $w \geq 0$. Then,

$$\mathbb{P}(W \leq w) = \mathbb{P}(\max(X, Y) \leq w) \tag{60}$$

$$= P(X \leq w)P(Y \leq w) \tag{61}$$

$$= (1 - e^{-2w})(1 - e^{-3w}) \tag{62}$$

Hence, the probability density function of W is given by

$$f_W(w) = \begin{cases} 2e^{-2w} + 3e^{-3w} - 5e^{-5w} & \text{if } w \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{63}$$

c) Take any $w, z \geq 0$. Then,

$$\mathbb{P}(W \leq w, Z \leq z) = \mathbb{P}(W \leq w) - \mathbb{P}(W \leq w, Z > z) \tag{64}$$

$$= (1 - e^{-2w})(1 - e^{-3w}) - \mathbb{P}(z < X \leq w)\mathbb{P}(z < Y \leq w) \tag{65}$$

$$= (1 - e^{-2w})(1 - e^{-3w}) - (e^{-2z} - e^{-2w})(e^{-3z} - e^{-3w}) \tag{66}$$

Then, the joint probability density function of (Z, W) is given by

$$f_{W,Z}(w, z) = \begin{cases} \frac{\partial^2 \mathbb{P}(W \leq w, Z \leq z)}{\partial w \partial z} = 6(e^{-2z}e^{-3w} + e^{-3z}e^{-2w}), & \text{if } w, z \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{67}$$

8)

a) Since $\mathbb{E}[X] = 1$ and $\mathbb{E}[Z] = 1$, then the mean vector of $\begin{bmatrix} X \\ Z \end{bmatrix}$ is equal to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
From the covariance matrix $\mathbf{C}$,

$$E[(X - E[X])^2] = 2, E[(X - E[X])(Z - E[Z])] = 1, E[(Z - E[Z])^2] = 2 \tag{68}$$

Hence, the covariance matrix of the random vector $\begin{bmatrix} X \\ Z \end{bmatrix}$ is equal to $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

b) Let's compute the mean and covariance matrix of $\begin{bmatrix} W \\ V \end{bmatrix}$.

$\mathbb{E}[W] = 2\mathbb{E}[X] + \mathbb{E}[Z] = 3$, and $\mathbb{E}[V] = -\mathbb{E}[Y] + 2\mathbb{E}[Z] = 4$. The mean vector is $\mu = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.
Note that

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{69}$$

Then, the covariance matrix of $\begin{bmatrix} W \\ Z \end{bmatrix}$ is given by

$$\Sigma = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{C} \begin{bmatrix} 2 & 0 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 11 \\ 11 & 15 \end{bmatrix} \tag{70}$$

Then, the joint probability density function of $x = \begin{bmatrix} W \\ V \end{bmatrix}$ is given by

$$f_{W,Z}(w, z) = \frac{\exp(-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu))}{2\pi \sqrt{|\Sigma|}} \tag{71}$$