EEE 431: Telecommunications 1

MIDTERM 2

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Prob. 1: ________ / 25

Prob. 2: ________ / 15

Prob. 3: ________ / 30

Prob. 4: ________ / 30

Total: ________ / 100

- You are allowed to use a one-page (self prepared, two sided) cheat-sheet.

Problem 1 Let $N(t)$ denote a zero-mean white process with a spectral density level of 1 Watt/Hz. Also, let $X(t)$ represent a process with the following autocorrelation function:

$$R_X(\tau) = 6000\,\mathrm{sinc}(3000\tau)\,.$$

Suppose that $X(t)$ and $N(t)$ are independent processes. Then, define a new random process as

$$Y(t) = 3X(t) - 2N(t)\,.$$

(a) Find the autocorrelation function of $Y(t)$.

(b) Find the power spectral density of $Y(t)$. Also plot it.

(c) Calculate the average power of $Y(t)$.

(d) Let $Z(t)$ represent the output of a linear time invariant filter when the input is $Y(t)$ and the impulse response of the filter is $h(t) = 8000\,\mathrm{sinc}(4000t)$. Find the autocorrelation function of $Z(t)$.

Problem 2 Consider the two constellations as defined below (the signals are equally likely in each case):

$$\text{Constellation 1:}\ \ \mathbf{s}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} A \\ A \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} -A \\ A \end{bmatrix}, \quad \mathbf{s}_4 = \begin{bmatrix} 0 \\ -2A \end{bmatrix}$$

$$\text{Constellation 2:}\ \ \mathbf{s}_1 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} B/2 \\ \sqrt{3}B/2 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} B/2 \\ -\sqrt{3}B/2 \end{bmatrix}, \quad \mathbf{s}_4 = \begin{bmatrix} -2B/3 \\ 0 \end{bmatrix}$$

where $A > 0$ and $B > 0$.

(a) Calculate the average energy of each constellation.

(b) Find a relation between A and B such that the average probability of error is approximately equal at high signal-to-noise ratios for the two constellations.

(c) Considering the relation in Part (b), which constellation is more energy efficient at high signal-to-noise ratios?

Problem 3 Consider a ternary ($M = 3$) digital communication system with the following transmitted signals

$$s_1(t) = p(t) \ , \quad s_2(t) = p(t)\cos(2\pi f_c t) \ , \quad s_3(t) = p(t)\left(\cos(\pi f_c t)\right)^2$$

for $t \in [0, T_s]$, where $p(t)$ is a rectangular pulse which is equal to $\sqrt{2/T_s}$ for $t \in [0, T_s]$ and equal to zero otherwise. The system is operating over an additive white Gaussian noise (AWGN) channel with a power spectral density of $N_0/2$, the signals are equally likely, and $f_c T_s$ is an integer.

(a) Express the signals as vectors in a signal space by finding orthonormal basis function(s).

(b) Design the optimal coherent receiver for this system, present its block diagram, and provide mathematical expressions (simplify as much as possible) for the operations at the receiver.

(c) Calculate the exact average probability of error, P_e, and simplify it as much as possible.

Problem 4 In this problem, the aim is to design a digital communication system with $M = 4$. The system operates over an additive white Gaussian noise (AWGN) channel with a power spectral density of $N_0/2$, and the signals are equally likely. The signals must satisfy the following conditions:

$$s_3(t) = -2s_1(t), \quad s_4(t) = -2s_2(t), \quad E_{s_1} = E_{s_2} = A^2, \quad s_1(t) \text{ is orthogonal to } s_2(t)$$

where E_{s_i} denotes the energy of $s_i(t)$. In addition, all the signals are around a carrier frequency of f_c and they are zero outside the interval $[0, T_s]$, where $f_c T_s$ is an integer.

(a) Provide expressions for $s_1(t), s_2(t), s_3(t), s_4(t)$ that satisfy all the conditions above.

(b) Find a set of orthonormal basis functions for the signals in Part (a), and express the signals as vectors in the corresponding signal space.

(c) Design the optimal coherent receiver for this system, present its block diagram, and provide mathematical expressions (simplify as much as possible) for the operations at the receiver.

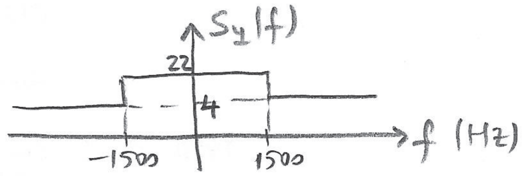
(d) Obtain an exact expression for the probability of error when message 1 is transmitted ($P_{e,1}$), and try to simply it as much as possible. (Try to obtain an expression that involves a single integral of some functions including Q function(s).)

①

a) $R_Y(\tau) = E[Y(t+\tau)Y(t)] = E\left[\left(3X(t+\tau) - 2N(t+\tau)\right)\left(3X(t) - 2N(t)\right)\right]$

$$= 9R_X(\tau) - 0 - 0 + 4R_N(\tau) = \boxed{54000\,\mathrm{sinc}(3000\tau) + 4\,\delta(\tau)}$$

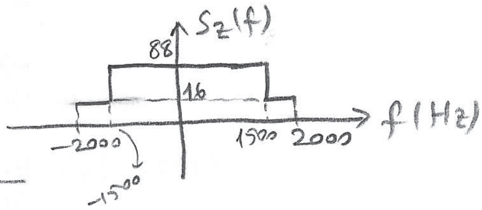
b) $S_Y(f) = \mathcal{F}\{R_Y(\tau)\} = \overset{18}{\cancel{54000}}\,\frac{1}{\cancel{3000}}\,\Pi\left(\frac{f}{3000}\right) + 4$



c) $P_Y = \int_{-\infty}^{\infty} S_Y(f)\,df = \boxed{\infty}$

d) $S_Z(f) = S_Y(f)\,|H(f)|^2 = S_Y(f)\left|\frac{8000}{4000}\,\Pi\left(\frac{f}{4000}\right)\right|^2$

$$\boxed{R_Z(\tau) = 16(4000)\,\mathrm{sinc}(4000\tau) + 72(3000)\,\mathrm{sinc}(3000\tau)}$$



②

a) $E_{av,1} = \frac{1}{4}\left(0 + 2A^2 + 2A^2 + 4A^2\right) = 2A^2$

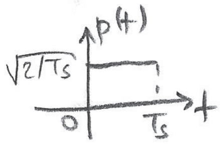
$E_{av,2} = \frac{1}{4}\left(B^2 + B^2 + B^2 + \frac{4B^2}{9}\right) = \frac{31}{36}B^2$

b) $d_{min,1} = \sqrt{2}A$ $\quad d_{min,2} = B$

For similar performance at high SNRs: $d_{min,1} = d_{min,2} \Rightarrow B = \sqrt{2}A$

c) $E_{av,2} = \frac{31}{36}B^2 = \frac{31}{36}2A^2 = \frac{31}{18}A^2 < E_{av,1} = 2A^2 \Rightarrow$ Constellation 2 is more energy efficient at high SNRs.

③ a) $s_1(t) = p(t)$ $\quad s_2(t) = p(t)\cos(2\pi f_c t)$ $\quad s_3(t) = \frac{1}{2}p(t) + \frac{1}{2}\cos(2\pi f_c t)\,p(t)$



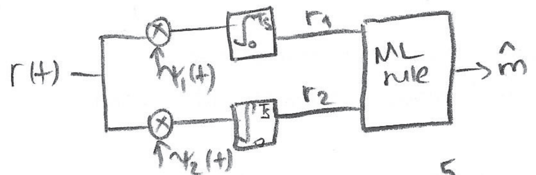
$\int p^2(t)\,dt = 2$ $\qquad \boxed{\psi_1(t) = \frac{p(t)}{\sqrt{2}}}$

$\int s_1(t)s_2(t)\,dt = \int p^2(t)\cos(2\pi f_c t)\,dt = \frac{2}{T_s}\int_0^{T_s}\cos(2\pi f_c t)\,dt = 0 \Rightarrow \boxed{\psi_2(t) = p(t)\cos(2\pi f_c t)}$

$\left(\int s_2^2(t)\,dt = 1\right)$

So, $\underline{s_1} = \begin{bmatrix}\sqrt{2}\\ 0\end{bmatrix}$ $\quad \underline{s_2} = \begin{bmatrix}0\\ 1\end{bmatrix}$ $\quad \underline{s_3} = \begin{bmatrix}\sqrt{2}/2\\ 1/2\end{bmatrix}$

b)



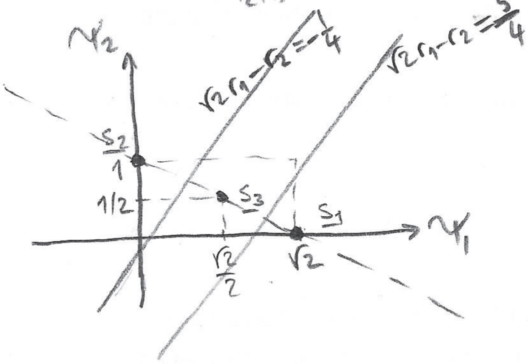
- $\underline{\hat{m} = 1}$ if $\|\underline{r} - \underline{s}_1\| \le \|\underline{r} - \underline{s}_2\|$ & $\|\underline{r} - \underline{s}_1\| \le \|\underline{r} - \underline{s}_3\|$

$-2\underline{r}^T\underline{s}_1 + \|\underline{s}_1\|^2 \le -2\underline{r}^T\underline{s}_2 + \|\underline{s}_2\|^2$

$-2\sqrt{2}\,r_1 + 2 \le -2r_2 + 1$ $\qquad$ &

$\boxed{\sqrt{2}r_1 - r_2 \ge \frac{1}{2}}$ & $\boxed{\sqrt{2}r_1 - r_2 \ge \frac{5}{4}}$

$\sqrt{2}r_1 - r_2 \ge \frac{5}{4}$



- $\underline{\hat{m} = 2}$ if $\sqrt{2}r_1 - r_2 < -\frac{1}{4}$

- $\underline{\hat{m} = 3}$ otherwise

c) $P_{e,1} = 1 - P_{c,1} = 1 - P\left(\sqrt{2}r_1 - r_2 \ge \frac{5}{4} \,\middle|\, m = 1\right) = 1 - P\left(\sqrt{2}(\sqrt{2} + n_1) - (0 + n_2) \ge \frac{5}{4}\right)$

$= 1 - P\big(\underbrace{\sqrt{2}n_1 - n_2}_{\mathcal{N}(0,\,3N_0/2)} \ge -\frac{3}{4}\big) = 1 - Q\left(\frac{-3/4}{\sqrt{3N_0/2}}\right) = \boxed{Q\left(\frac{3/4}{\sqrt{3N_0/2}}\right)}$

$P_{e,2} = P_{e,1}$ due to symmetry

$P_{e,3} = 1 - P_{c,3} = 1 - P\left(\sqrt{2}r_1 - r_2 < \frac{5}{4} \;\&\; \sqrt{2}r_1 - r_2 > -\frac{1}{4} \,\middle|\, m = 3\right)$

$= 1 - P\left(\sqrt{2}\left(\frac{\sqrt{2}}{2} + n_1\right) - \left(\frac{1}{2} + n_2\right) \in \left(-\frac{1}{4}, \frac{5}{4}\right)\right)$

$= 1 - P\left(\sqrt{2}n_1 - n_2 \in \left(-\frac{3}{4}, \frac{3}{4}\right)\right) = \boxed{2Q\left(\frac{3/4}{\sqrt{3N_0/2}}\right)}$

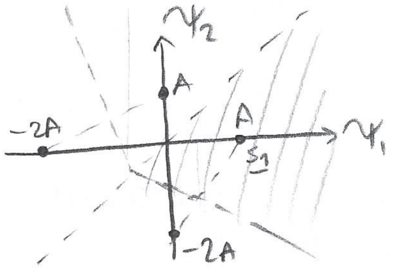
Similar to 3-PAM

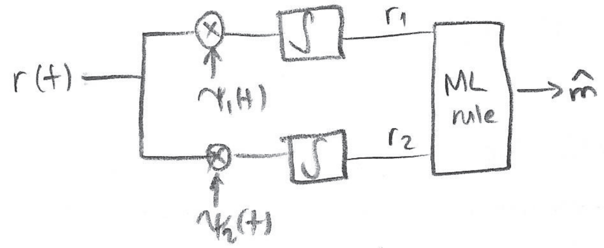
$$\boxed{P_e = \frac{4}{3}Q\left(\frac{3/4}{\sqrt{3N_0/2}}\right)}$$

④ $s_1(t) = A\underbrace{\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t)}_{\psi_1(t)}$ $\quad s_2(t) = A\underbrace{\sqrt{\frac{2}{T_s}}\sin(2\pi f_c t)}_{\psi_2(t)}$ $\quad t \in [0, T_s]$

(zero otherwise)

$\underline{s}_1 = \begin{bmatrix} A \\ 0 \end{bmatrix} \quad \underline{s}_2 = \begin{bmatrix} 0 \\ A \end{bmatrix} \quad \underline{s}_3 = \begin{bmatrix} -2A \\ 0 \end{bmatrix} \quad \underline{s}_4 = \begin{bmatrix} 0 \\ -2A \end{bmatrix}$





$\underline{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

$\underline{\hat{m} = 1:}$ If $-2\underline{r}^T\underline{s}_1 + \|\underline{s}_1\|^2 \le -2\underline{r}^T\underline{s}_2 + \|\underline{s}_2\|^2$ & $-2\underline{r}^T\underline{s}_1 + \|\underline{s}_1\|^2 \le -2\underline{r}^T\underline{s}_3 + \|\underline{s}_3\|^2$ & $-2\underline{r}^T\underline{s}_1 + \|\underline{s}_1\|^2 \le -2\underline{r}^T\underline{s}_4 + \|\underline{s}_4\|^2$

$-2Ar_1 + A^2 \le -2r_2A + A^2$

$\underbrace{\underbrace{r_1 \ge r_2}_{} \quad \& \quad \underbrace{r_1 \ge -\frac{A}{2}}_{} \quad \& \quad \underbrace{r_1 + 2r_2 \ge -\frac{3}{2}A}_{}}_{r_1 \ge r_2 \;\&\; r_1 + 2r_2 \ge -\frac{3A}{2}}$

$\underline{\hat{m} = 2:}$ If $\underbrace{r_1 < r_2}_{}$ & $\underbrace{2r_1 + r_2 > -\frac{3A}{2}}_{}$ & $\underbrace{r_2 > -\frac{A}{2}}_{\text{no need}}$

$\underline{\hat{m} = 3:}$ If $\underbrace{r_1 < -\frac{A}{2}}_{\text{no need}}$ & $2r_1 + r_2 \le -\frac{3A}{2}$ & $r_1 < r_2$

$\underline{\hat{m} = 4:}$ If $r_1 + 2r_2 < -\frac{3A}{2}$ & $\underbrace{r_2 \le -\frac{A}{2}}_{\text{no need}}$ & $r_1 \ge r_2$

$$P_{e,1} = 1 - P_{c,1} = 1 - P\left(r_1 \ge r_2 \;\&\; r_1 + 2r_2 \ge -\frac{3A}{2} \,\middle|\, m = 1\right)$$

$$= 1 - P\left(A + n_1 \ge n_2 \;\&\; A + n_1 + 2n_2 \ge -\frac{3A}{2}\right)$$

$$= 1 - \int_{-\infty}^{\infty} p_{n_2}(x)\, P\left(n_1 \ge x - A \;\&\; n_1 \ge -2x - \frac{5A}{2}\right) dx$$

$$= 1 - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} e^{-\frac{x^2}{N_0}} Q\left(\frac{\max\{x - A, -2x - 5A/2\}}{\sqrt{N_0/2}}\right) dx$$