EEE 431: Telecommunications 1

Quiz 2

March 10, 2018, 9:40-10:55

Instructor: Sinan Gezici

Name: ______________________________

Signature: ______________________________

Bilkent ID: ______________________________

Prob. 1: ________ / 22

Prob. 2: ________ / 15

Prob. 3: ________ / 28

Prob. 4: ________ / 35

Total: ________ / 100

Some trigonometric identities: $\sin(2x) = 2\sin(x)\cos(x)$
$\cos(2x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
$\sin(x)\cos(y) = 0.5\sin(x+y) + 0.5\sin(x-y)$
$\cos(x)\cos(y) = 0.5\cos(x+y) + 0.5\cos(x-y)$
$\sin(x)\sin(y) = 0.5\cos(x-y) - 0.5\cos(x+y)$.

Problem 1 Suppose that we have two analog messages $m_1(t)$ and $m_2(t)$, and we insert them into a signal $x(t)$ as follows: $x(t) = (m_1(t) + m_2(t))\cos(2\pi f_c t) + (m_1(t) - m_2(t))\sin(2\pi f_c t)$. Assume that the bandwidths of the analog messages are much lower than f_c. Design a receiver to extract both $m_1(t)$ and $m_2(t)$ from $x(t)$ by using only <u>one</u> local oscillator (which generates a sinusoidal signal at frequency f_c) and any other components that might be needed. Specify all the parameters at the receiver. The final outputs of the receiver must be $m_1(t)$ and $m_2(t)$. [<u>Hint:</u> A *phase shifter* (with a suitable value of phase shift) can be used to generate a cosine from a sine, or vice versa.]

Problem 2 Prove or disprove the following statement: "*If a random process $X(t)$ is stationary in the strict sense, then $E\big[(X(t))^3\big]$ does not depend on t, where E denotes the expectation operator.*"

Problem 3 Let X denote a random variable with the following probability density function (PDF):

$$f_X(x) = \begin{cases} 2(x+2)/9, & \text{if } -2 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

Suppose that X is input to a 2-level (1-bit) quantizer with the decision boundary at zero and the reconstruction (quantization) levels of a and b, where $a < b$. (That is, if the input is negative, the reconstruction level is a, and if the input is non-negative, the reconstruction level is b.)

a) Find the probability that the quantizer output is strictly less than b, that is, find $P(Q(X) < b)$.

b) Find the (locally) optimal values of a and b that minimize the mean-squared error distortion D, where $D = E[(X - Q(X))^2]$, with $Q(X)$ denoting the quantizer output.

Problem 4 Consider a random process $Y(t)$, which is expressed as $Y(t) = (2\sin(200\pi t + \theta) - A)^2$, where θ is a uniform random variable in the interval $[0, 2\pi)$, and A is a uniform random variable in the interval $[-1, 1]$. In addition, θ and A are independent. Calculate the mean and autocorrelation function of $Y(t)$. Is $Y(t)$ wide-sense stationary (WSS)? Why or why not? In addition, find $E[(Y(10))^2]$.

(2) If $X(t)$ is SSS, $f_{X(t+\tau)}(x) = f_{X(t)}(x)$ $\forall t, \tau$. So $f_{X(t)}(x)$ does not depend on t. Therefore,

$$E[X^3(t)] = \int_{-\infty}^{\infty} x^3 \underbrace{f_{X(t)}(x)}_{\text{not depend on } t} dx$$ does not depend on t. ✓

(4) $$Y(t) = 4\sin^2(200t\pi + \theta) - 4A\sin(200\pi t + \theta) + A^2$$

$$= 2 - 2\cos(400\pi t + 2\theta) - 4A\sin(200\pi t + \theta) + A^2$$

$$E[Y(t)] = 2 + E[A^2] = 2 + \int_{-1}^{1} a^2 \frac{1}{2}\, da = 2 + \frac{2}{6} = \boxed{\frac{7}{3}}$$

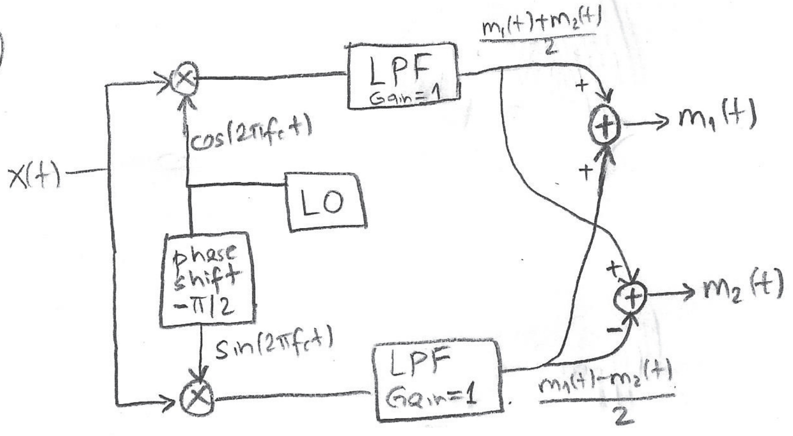
$$R_Y(t_1, t_2) = E[Y(t_1)Y(t_2)]$$

$$= 4 + 2\underbrace{E[A^2]}_{7/3} + 2\cos(400\pi(t_1 - t_2)) + 8\underbrace{E[A^2]}_{7/3}\cos(200\pi(t_1 - t_2)) + 2\underbrace{E[A^2]}_{7/3} + \overbrace{\underbrace{E[A^4]}_{1/5}}^{\int_{-1}^{1} a^4 \frac{1}{2} da}$$

$$= \boxed{4 + \frac{28}{3} + \frac{1}{5} + 2\cos(400\pi(t_1 - t_2)) + \frac{56}{3}\cos(200\pi(t_1 - t_2))}$$

WSS. $$E[Y^2(10)] = R_Y(0) = 4 + \frac{28}{3} + \frac{1}{5} + 2 + \frac{56}{3} = \boxed{34.2}$$

(1)



(3) a) $$P(Q(X) < b) = P(Q(X) = a) = \int_{-2}^{0} \frac{2(x+2)}{3} dx = \boxed{\frac{4}{3}}$$

b) $$D = E[(X - Q(X))^2] = \int_{-2}^{0} (x-a)^2 \frac{2(x+2)}{9} dx + \int_{0}^{1} (x-b)^2 \frac{2(x+2)}{9} dx$$

$$\frac{\partial D}{\partial a} = -\int_{-2}^{0} (x-a)\frac{4}{9}(x+2)\, dx = 0 \Rightarrow \int_{-2}^{0} (x^2 + 2x)\, dx = a\int_{-2}^{0} (x+2)\, dx \Rightarrow \boxed{a = -\frac{2}{3}}$$

$$\frac{\partial D}{\partial b} = -\int_{0}^{1} (x-b)\frac{4}{9}(x+2)\, dx = 0 \Rightarrow \int_{0}^{1} (x^2 + 2x)\, dx = b\int_{0}^{1} (x+2)\, dx \Rightarrow \boxed{b = \frac{8}{15}}$$