EEE 431: Telecommunications 1

FINAL

Date and Time: Friday, January 3, 2013, 18:30-21:00.

Instructors: Sinan Gezici and Tolga M. Duman

Name: SOLUTIONS.

Signature: ____________________

Bilkent ID: ____________________

Section: __________

Prob. 1: ______ / 25

Prob. 2: ______ / 10

Prob. 3: ______ / 13

Prob. 4: ______ / 12

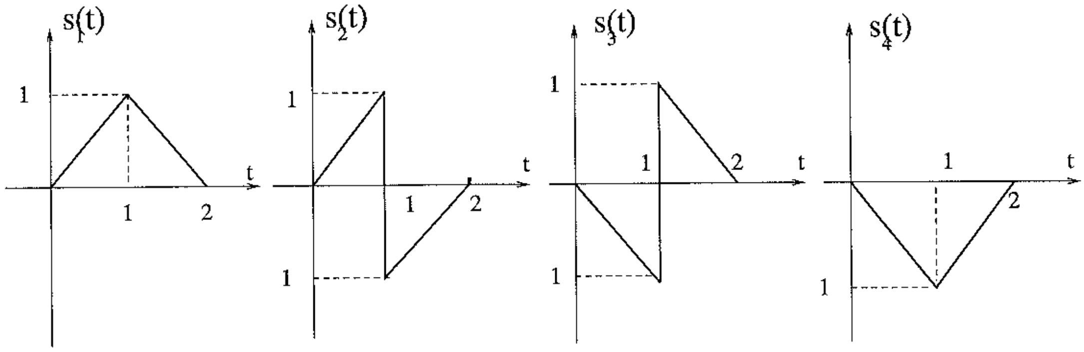
Prob. 5: ______ / 20

Prob. 6: ______ / 20

Total: ______ / 100

- You are allowed to use two sheets of formulas, both sides OK.
- Calculators are permitted (no cell phones).

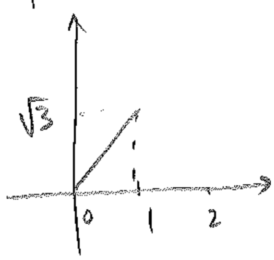
Prb. 1 The four signals shown in the figure are used to transmit four different messages (where the symbol period is $T = 2$). Assume that the symbols are equally likely.



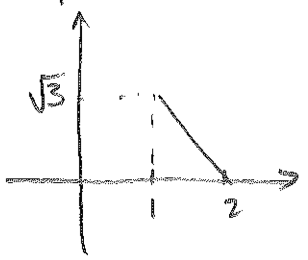
a) [4 Points] Find an orthonormal set of basis functions for this signal set.

By inspection.

$\psi_1(t)$



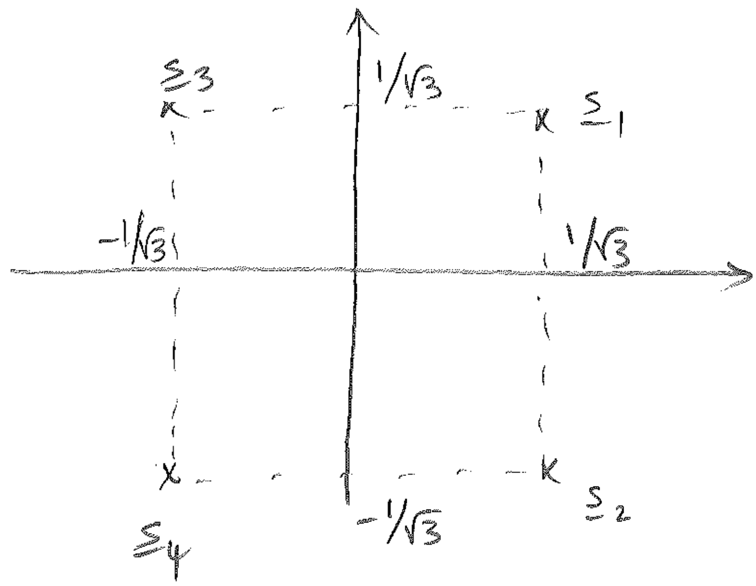
$\psi_2(t)$



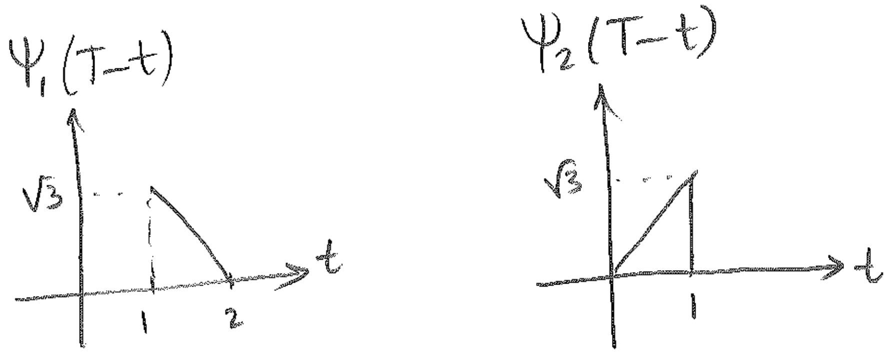
will work.

b) [3 Points] Plot the signal constellation.

$\underline{s}_1 = \left[1/\sqrt{3} \quad 1/\sqrt{3} \right]$ $\underline{s}_2 = \left[1/\sqrt{3} \quad -1/\sqrt{3} \right]$ $\underline{s}_3 = \left[-\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \right]$ $\underline{s}_4 = \left[-\frac{1}{\sqrt{3}} \quad -\frac{1}{\sqrt{3}} \right]$

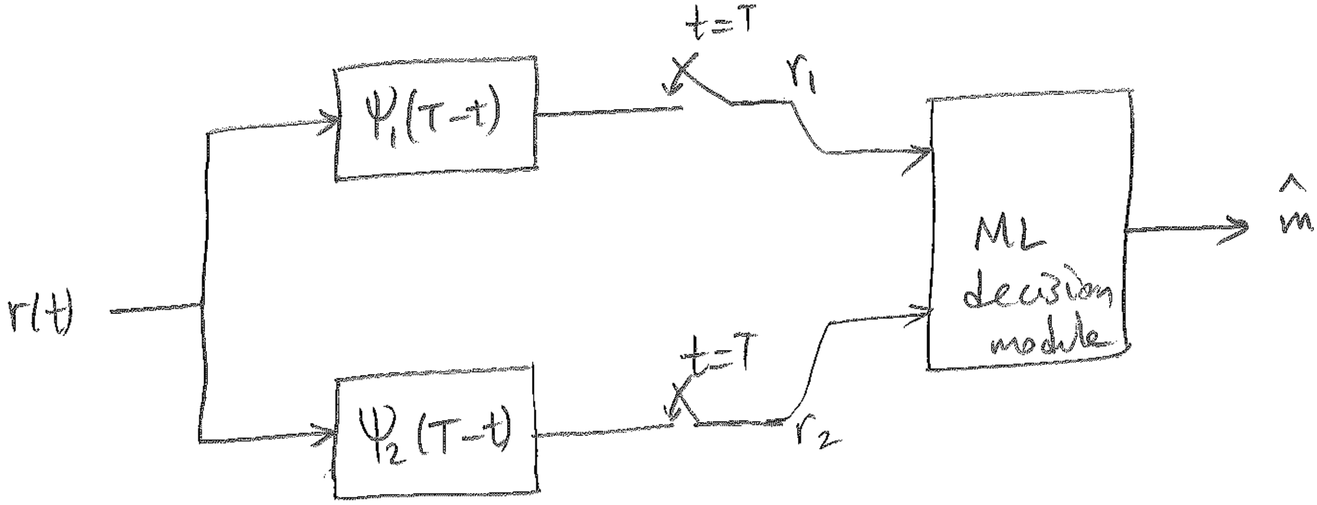


c) [4 Points] What are the impulse responses of the filters matched to the basis functions in part a?



d) [6 Points] This system is being used over an AWGN channel. Describe the structure of the optimal receiver.

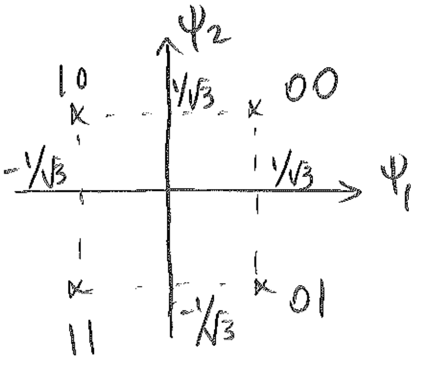
From the signal constellation, the decision regions are easily identified. Hence we have:



ML module:

$$\hat{m} = \begin{cases} 1 & \text{if } r_1, r_2 > 0 \\ 2 & \text{if } r_1 > 0, r_2 < 0 \\ 3 & \text{if } r_1 < 0, r_2 > 0 \\ 4 & \text{if } r_1, r_2 < 0. \end{cases}$$

e) [8 Points] Assume that mapping from two bits to the signals are as follows: 00 is transmitted by $s_1(t)$, 01 by $s_2(t)$, 10 by $s_3(t)$ and 11 by $s_4(t)$, and the system is used over an AWGN channel with power spectral density of $\frac{N_0}{2}$. What is the resulting (exact) bit error probability?



First bit gets affected by the noise in the direction of ψ_1 only (n_1)

& second bit gets affected by noise in the direction of ψ_2 only (n_2)

& n_1 & n_2 are independent.

$n_1, n_2 \sim \mathcal{N}(0, \frac{N_0}{2})$.

Hence the bit error prob:
(due to complete symmetry)

$$P_b = P(n_1 > 1/\sqrt{3})$$

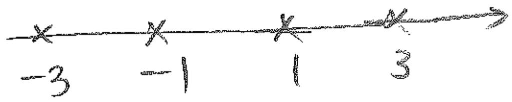
$$= Q\left(\frac{1/\sqrt{3}}{\sqrt{N_0/2}}\right)$$

$$\Rightarrow \boxed{P_b = Q\left(\sqrt{\frac{2}{3N_0}}\right)}$$

Prb. 2 [10 Points] Which constellation is more power efficient 4 PAM or 4 PSK (assuming that equally likely signaling is used)? That is, which one is better in terms of the error rates (at high signal to noise ratios)? By how much (expressed in dBs)?

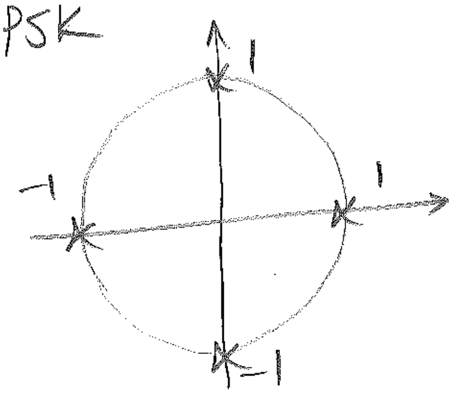
We need to compare $\frac{d_{min}^2}{E_{AV}}$ for the two signal constellations.

4 PAM:



$$\left.\begin{aligned} d_{min}^2 &= 4 \\ E_{AV} &= 5 \end{aligned}\right\} \frac{d_{min}^2}{E_{AV}} = \frac{4}{5}$$

4 PSK



$$\left.\begin{aligned} d_{min}^2 &= 2 \\ E_{AV} &= 1 \end{aligned}\right\} \frac{d_{min}^2}{E_{AV}} = 2.$$

4 PSK is better by $10 \log_{10}\left(\frac{2}{4/5}\right) \cong \underline{4\,dB}.$

Prb. 3 Consider a (7,3) (binary) linear block code (i.e., $n = 7$, $k = 3$). Denote the message bits by x_1, x_2, x_3, and the codeword bits by $c_1, c_2, \ldots, c_7$. Assume that the coded bits are obtained as follows:

$$\begin{aligned} c_1 &= x_1, \\ c_2 &= x_2, \\ c_3 &= x_3, \\ c_4 &= x_1 + x_2, \\ c_5 &= x_1 + x_2 + x_3, \\ c_6 &= x_2 + x_3, \\ c_7 &= x_1 + x_3. \end{aligned}$$

a) [1 Point] Is this a systematic code?

Yes. The message bits are present (unaltered) in the codeword.

b) [2 Points] What is the codeword corresponding to the message $(x_1, x_2, x_3) = (1, 1, 0)$?

Using the rules given:

$\underline{c} = [1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1]$

c) [3 Points] Find the generator matrix of the code.

$$\underline{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

d) [3 Points] Find the parity check matrix of the code.

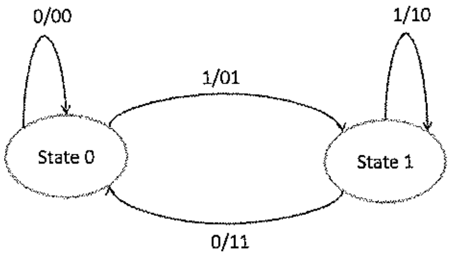
$$\underline{H} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

e) [4 Points] What is the minimum distance of the code?

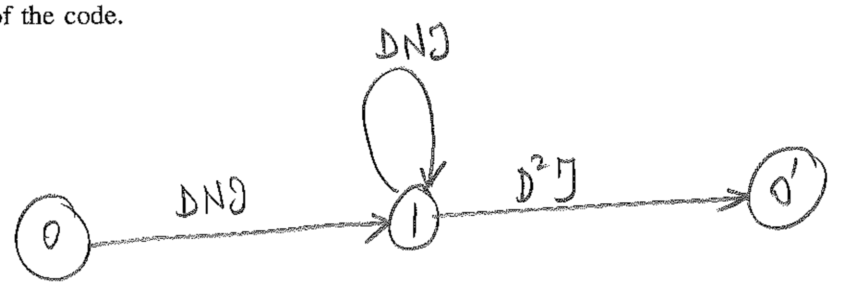
Min. # of columns of $\underline{H}$ adding to zero is 4 (e.g. 1^{st}, 4^{th}, 5^{th} & 7^{th}); hence

$$\boxed{d_{min} = 4}$$

Prb. 4 Consider a two-state convolutional code whose state-diagram is given below.



a) [6 Points] Find its input-output weight enumerating function $T(D, N, J)$, and determine the free distance of the code.



Node eqns:

$$X_1 = X_1\, DNJ + X_0\, DNJ \quad \Rightarrow \quad X_1 = \frac{DNJ}{1-DNJ} \cdot X_0$$

& $$X_{0'} = X_1 \cdot D^2 J \quad \Rightarrow \quad X_{0'} = \frac{D^3 N J^2}{1-DNJ} X_0$$

hence: $$\boxed{T(D,N,J) = \frac{D^3 N J^2}{1-DNJ}}$$

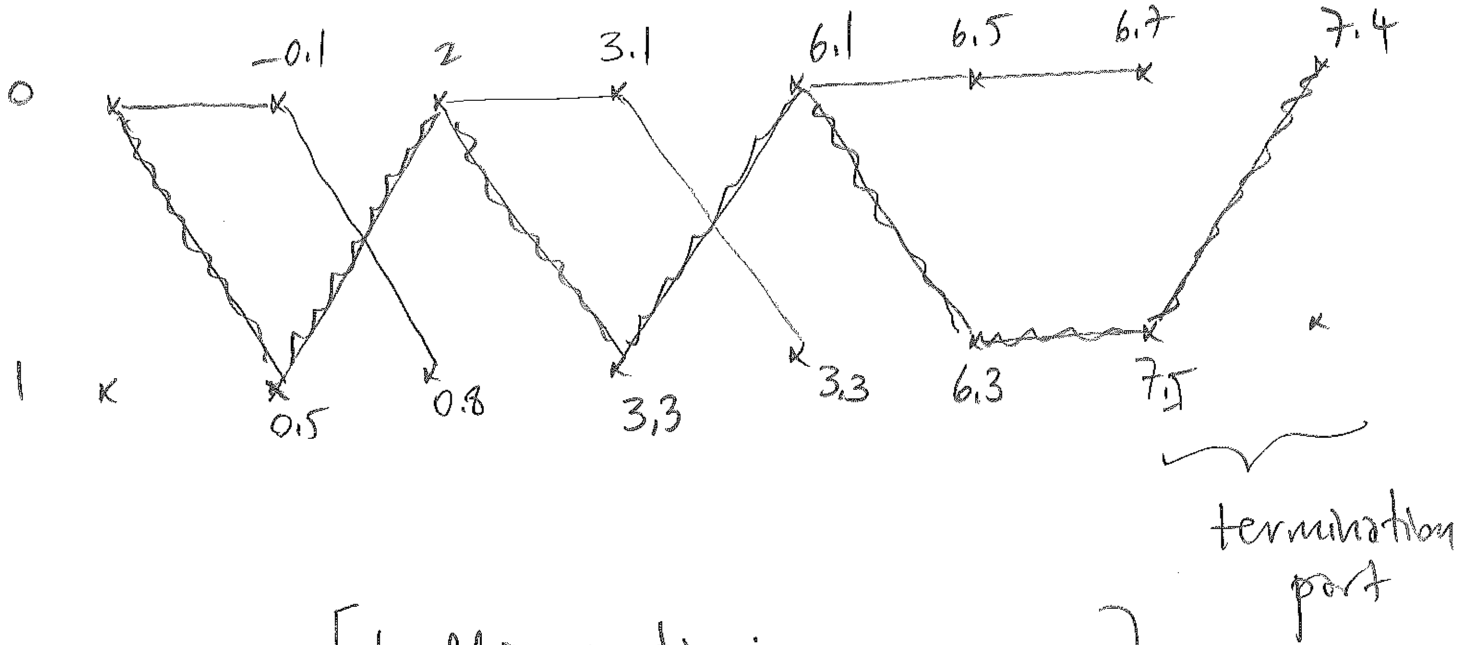
Since: $$T(D,N,J) = D^3 N J^2 + \cdots$$

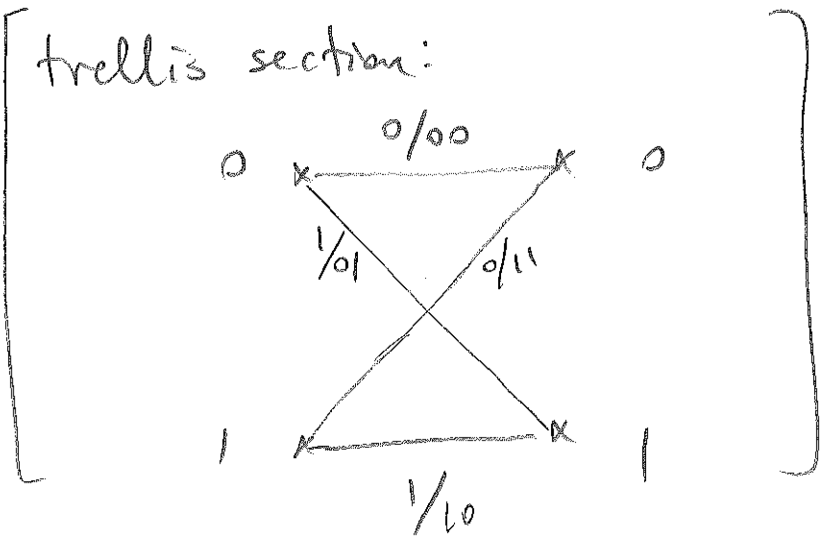
$$\boxed{d_{free} = 3}$$

b) [6 Points] Assume that this code is used over an AWGN channel with BPSK modulation ($0 \rightarrow -1$, $1 \rightarrow +1$). The received signal sequence is

$$(-0.2, 0.3), (0.3, 1.2), (-1.2, 0.1), (1.3, 1.5), (-0.3, -0.1), (0.5, -0.7), (0.2, -0.3).$$

Use ML decoding (with the correlation metric) to find the most likely sequence of message bits. (Assume that the initial state is 0 and the last bit is used for trellis termination).





decoded bits: 1 0 1 0 1 1 0

Prb. 5 Define a random process as $X(t) = A+2\cos(2000\pi t+\Theta_1)+\sin(4000\pi t+\Theta_2)$ where Θ_1 and Θ_2 are uniform random variables on $[-\pi,\pi)$, and A is a uniform random variable on $[1,2]$. Assume that A, Θ_1 and Θ_2 are independent.

$X(t)$ is input to an LTI system with frequency response

$$H(f) = \begin{cases} \frac{-|f-3000|}{1000} & \text{if } |f| \le 3000, \\ 0 & \text{else.} \end{cases}$$

The output is denoted by $Y(t)$.

a) [4 Points] Compute the mean and autocorrelation of the process $X(t)$. Is the process WSS?

$$E[X(t)] = E[A] + 0 + 0 = 3/2.$$

$$E[X(t+\tau)X(t)] = E\Big[\big(A + 2\cos(2000\pi(t+\tau)+\Theta_1) + \sin(4000\pi(t+\tau)+\Theta_2)\big)\big(A+2\cos(2000\pi t+\Theta_1)+\sin(4000\pi t+\Theta_2)\big)\Big]$$

$$= E[A^2] + 2\cos(2000\pi\tau) + \frac{1}{2}\cos(4000\pi\tau)$$

+ other terms which evaluate to zero.

$$\Rightarrow \boxed{R_X(\tau) = \frac{7}{3} + 2\cos(2000\pi\tau) + \frac{1}{2}\cos(4000\pi\tau)}$$

mean is constant & autocorrelation is a function of τ only; hence $X(t)$ is WSS.

b) [4 Points] Compute the power spectral density of $X(t)$.

$$S_X(f) = \mathcal{F}\{R_X(\tau)\}$$

$$= \frac{7}{3}\delta(f) + \delta(f-1000) + \delta(f+1000)$$

$$+ \frac{1}{4}\delta(f-2000) + \frac{1}{4}\delta(f+2000).$$

c) [4 Points] Compute the power spectral density of $Y(t)$.

$$S_Y(f) = S_X(f) \cdot |H(f)|^2$$

$$= 21\,\delta(f) + 4\,\delta(f-1000) + 4\,\delta(f+1000)$$

$$+ \frac{1}{4}\delta(f-2000) + \frac{1}{4}\delta(f+2000)$$

d) [4 Points] What is the total average power content of $Y(t)$? What is its average power content in the frequency band $(500, 1500)$ Hz?

$$P_Y = \int_{-\infty}^{\infty} S_Y(f)\,df = 21 + 4 + 4 + \frac{1}{4} + \frac{1}{4} = \boxed{\frac{59}{2}} \text{ units}$$

power content of $Y(t)$ in the band $(500, 1500)$ Hz

$$= \int_{-1.5k}^{-0.5} S_Y(f)\,df + \int_{0.5}^{1.5} S_Y(f)\,df$$

$$= 4 + 4 = \boxed{8} \text{ units}$$

e) [4 Points] What is the mean and autocorrelation of the output process $Y(t)$? Is it WSS?

$Y(t)$ is the output of an LTI system to a WSS input, hence it is also WSS.

$$E[Y(t)] = E[X(t)]\cdot H(0) = \frac{3}{2}\cdot(-3) = \boxed{-\frac{9}{2}}$$

$$R_{YY}(\tau) = F^{-1}\{S_Y(f)\}$$

$$\Rightarrow \boxed{R_{YY}(\tau) = 21 + 8\cos(2000\pi\tau) + \frac{1}{2}\cos(4000\pi\tau)}$$

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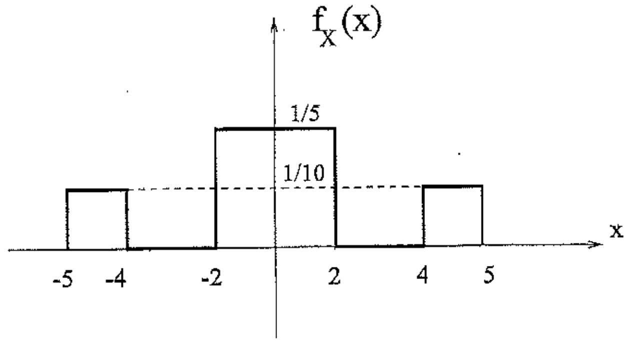
Prb.6 A source has a probability density function (PDF) shown in the following figure. The PDF clearly shows that the source outputs are limited to the range $[-5, 5]$.



Assume that it produces 20000 samples per second. We would like to transmit the source outputs using uniform PCM with 10 bits per sample.

a) [9 Points] What is the resulting SQNR (in dBs) of the system?

$$P_X = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{-5}^{-4} x^2 \frac{1}{10}\,dx + \int_{-2}^{2} x^2 \frac{1}{5}\,dx + \int_{4}^{5} x^2 \frac{1}{10}\,dx$$

$$\Rightarrow \boxed{P_X = \frac{77}{15}}$$

$$\mathrm{SQNR}\Big|_{dB} = 10 \log_{10}\left(\frac{P_X}{x_{max}^2}\right) + 6.02\nu + 4.8$$

$$= 10 \log_{10}\left(\frac{77}{15} \cdot \frac{1}{5^2}\right) + 6.02 \times 10 + 4.8$$

$$\boxed{\mathrm{SQNR}\Big|_{dB} = 58.1 \text{ dB.}}$$

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b) [5 Points] What is the minimum bandwidth required to transmit the PCM signal?

Bandwidth needed: $\frac{f_s\, v}{2} = \frac{20k \times 10}{2} = \boxed{100\,kHz}$

c) [16 Points] In order to improve the system performance, we decide to use a compander with the "compressor" at the transmitter given by

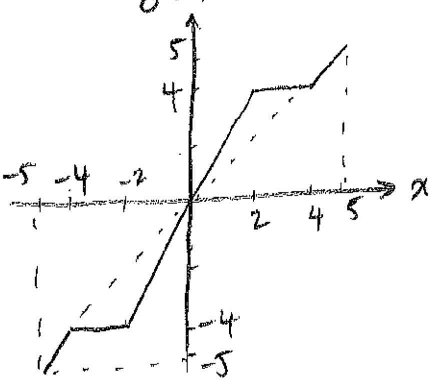
$$g(x) = \begin{cases} 2x & \text{if } |x| \leq 2 \\ 4 & \text{if } 2 < x < 4 \\ -4 & \text{if } -4 < x < -2 \\ x & \text{if } 4 \leq |x| \leq 5 \end{cases}$$

Determine the expander to be used at the receiver corresponding to $g(x)$, and explain why this compander may be a good choice to improve the SQNR.

Compute the resulting SQNR of this non-uniform PCM system. Compare this with the value you computed in part a.

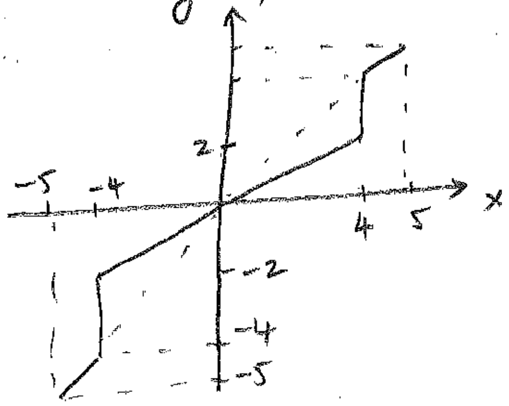
Given

$g(x)$



then, the expander is:

$g^{-1}(x)$



(symmetric around $y=x$ axis)

& $g^{-1}(x) = \begin{cases} x/2 & \text{if } |x| \leq 4 \\ x & \text{if } 4 < |x| \leq 5 \end{cases}$

This is a good choice since the more likely output range $x \in [-2, 2]$ is mapped to a larger number of quantization levels, reducing the quantization error.

SQNR analysis:

for $|x| \le 2$ we use $\frac{8N}{10} \cong 0.8N$ levels

for $|x| \in [4,5]$ we use $\simeq 0.2N$ levels.

} these are not exact since $N = 2^{\nu}$, but doesn't matter as N is large.

hence for $|x| \le 2$, the quantization error is uniform on $\left(-\frac{\Delta_1}{2}, \frac{\Delta_1}{2}\right)$ with $\Delta_1 = \frac{4}{0.8N} = \frac{5}{N}$

& for $|x| \in [4,5]$, it is uniform on $\left(-\frac{\Delta_2}{2}, \frac{\Delta_2}{2}\right)$ with

$$\Delta_2 = \frac{2}{0.2N} = \frac{N}{10}.$$

Then, denoting by $\tilde{X}$ the quantization error, we obtain:

$$E[\tilde{X}^2] = P(|X| \le 2) \cdot E[\tilde{X}^2 \mid |X| \le 2] + P(|X| \in [4,5]) \cdot E[\tilde{X}^2 \mid |X| \in [4,5]]$$

$$= 0.8 \times \int_{-\frac{\Delta_1}{2}}^{\Delta_1/2} x^2 \frac{1}{\Delta_1}\, dx + 0.2 \times \int_{-\Delta_2/2}^{\Delta_2/2} x^2 \frac{1}{\Delta_2}\, dx = \frac{10}{3N^2} = \frac{10}{3 \cdot 4^{\nu}}$$

&

$$\text{SQNR}\Big|_{dB} = 10 \log_{10}\left(\frac{E(X^2)}{E(\tilde{X}^2)}\right) = 10 \log_{10}\left(\frac{77/15}{10/3 \cdot 4^{\nu}}\right)$$

$$\boxed{\text{SQNR}\Big|_{dB} = 62.1 \text{ dB.}}$$

which is larger than the value computed in part a. (by about 4 dB).