EEE 431: Telecommunications 1

Quiz 3

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Instructor: Sinan Gezici

Name: ____________________

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Section: ____________________

Prob. 1: ________ / 30

Prob. 2: ________ / 30

Prob. 3: ________ / 40

Total: ________ / 100

Some trigonometric identities: $\sin(2x) = 2\sin(x)\cos(x)$
$\cos(2x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
$\sin(x)\cos(y) = 0.5\sin(x+y) + 0.5\sin(x-y)$
$\cos(x)\cos(y) = 0.5\cos(x+y) + 0.5\cos(x-y)$
$\sin(x)\sin(y) = 0.5\cos(x-y) - 0.5\cos(x+y)$.

Problem 1 This question has two independent parts:

(a) Consider a signal $s(t)$ that is defined over a duration of $[0,T]$ seconds. Suppose that $\psi_1(t)$ and $\psi_2(t)$ are two orthonormal basis functions for $s(t)$; i.e., $s(t)$ resides in the signal space generated by $\psi_1(t)$ and $\psi_2(t)$. Let the vector representation of $s(t)$ in this signal space be denoted by $\boldsymbol{s}$. Prove or disprove the following statement: "*The energy of $s(t)$ is always equal to the inner product of $\boldsymbol{s}$ with itself; i.e., $\boldsymbol{s}^T\boldsymbol{s}$.*"

(b) Write down expressions for three time domain signals $s_1(t)$, $s_2(t)$, and $s_3(t)$, defined over a duration of $[0,T]$ seconds, which satisfy all of the following conditions:

(i) The carrier frequencies of the signals are all equal to f_c Hz, where f_cT is an integer.

(ii) $s_2(t)$ is orthogonal to both $s_1(t)$ and $s_3(t)$.

(iii) $s_1(t)$ and $s_3(t)$ have the same energy.

(iv) The distance between $s_1(t)$ and $s_2(t)$ is equal to the distance between $s_1(t)$ and $s_3(t)$.

(*Your answer may not be unique.)*

Problem 2 This question has two independent parts:

(a) Consider a ternary communications system with the following signals:

$$\boldsymbol{s}_1 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{s}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Write down a mathematical expression for the maximum likelihood (ML) decision rule at the optimal receiver, and simplify it as much as possible. (Let r_i denote the i-th correlator output at the optimal receiver.)

(b) Consider a binary communications system with the following signals:

$$\boldsymbol{s}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{s}_2 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Considering the ML decision rule, calculate the conditional probability of error for message 1. In other words, find the probability of error when $\boldsymbol{s}_1$ is the transmitted signal. The noise at the correlator outputs is modeled as independent and identically distributed zero mean Gaussian random variables with variance $N_0/2$. (*You do not have to write down the ML decision rule.*) (*The probability that a zero-mean, unit variance Gaussian random variable is larger than x is defined as $Q(x)$, i.e., $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt$.*)

Problem 3 Consider two real-valued, independent, and wide-sense stationary (WSS) random processes $W(t)$ and $N(t)$, where $W(t)$ is a zero-mean white Gaussian process with a power spectral density level of η for all frequencies, and $N(t)$ is a Gaussian process with mean μ and autocorrelation function $R_N(\tau) = \sigma\,\mathrm{sinc}(\tau)$. Define a new random process as $X(t) = \alpha W(t) + \beta N(t)$.

(a) Find the autocorrelation function of $X(t)$.

(b) Find the power spectral density of $X(t)$, and plot it (mark all the values on the plot).

(c) Calculate the average power of $X(t)$.

(d) Suppose that $X(t)$ passes through an ideal low-pass filter with a cut-off frequency of 10 Hz, and let $Y(t)$ denote the output of this filter. Specify the probability density function of $Y(5)$.

Hint: The Fourier transform $S(f)$ of $s(t)$ is defined as $S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt$ and the inverse Fourier transform is given by $s(t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi ft}df$.

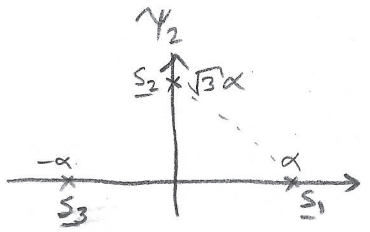
① ⓐ $\underline{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$ $s_1 = \int_0^T s(t)\psi_1(t)\,dt$ $s_2 = \int_0^T s(t)\psi_2(t)\,dt$ $\boxed{s(t) = s_1\psi_1(t) + s_2\psi_2(t)}$

$$E_s = \int_0^T s^2(t)\,dt = \int_0^T (s_1\psi_1(t) + s_2\psi_2(t))^2\,dt = s_1^2 \underbrace{\int_0^T \psi_1^2(t)\,dt}_{1} + 2s_1s_2 \underbrace{\int_0^T \psi_1(t)\psi_2(t)\,dt}_{0} + s_2^2 \underbrace{\int_0^T \psi_2^2(t)\,dt}_{1}$$

$$= s_1^2 + s_2^2 = \underline{s}^T\underline{s} \ \checkmark$$

ⓑ $s_1(t) = A\cos(2\pi f_c t)$
$s_2(t) = \sqrt{3}A\sin(2\pi f_c t)$ $t \in [0, T]$
$s_3(t) = -A\cos(2\pi f_c t)$

$\underbrace{\|\underline{s}_1 - \underline{s}_2\|}_{2\alpha} = \underbrace{\|\underline{s}_1 - \underline{s}_3\|}_{2\alpha}$

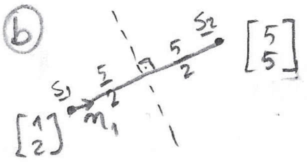


② ⓐ ML rule: $\hat{m} = \underset{i\in\{1,2,3\}}{\arg\min} \|\underline{r} - \underline{s}_i\|^2 = \underset{i\in\{1,2,3\}}{\arg\max} \underbrace{\left(\underline{r}^T\underline{s}_i - \frac{\|\underline{s}_i\|^2}{2}\right)}$

$\sqrt{2}r_1 - 1,\ i=1$
$r_2 - \frac{1}{2},\ i=2$
$r_3 - 2,\ i=3$

$$\boxed{\hat{m} = \begin{cases} 1, & \sqrt{2}r_1 - 1 \geq r_2 - \frac{1}{2} \ \&\ \sqrt{2}r_1 - 1 \geq r_3 - 2 \\ 2, & r_2 - \frac{1}{2} > \sqrt{2}r_1 - 1 \ \&\ r_2 - \frac{1}{2} \geq r_3 - 2 \\ 3, & r_3 - 2 > \sqrt{2}r_1 - 1 \ \&\ r_3 - 2 > r_2 - \frac{1}{2} \end{cases}}$$

ⓑ



$P_{e,1} = P_{e,2} = P_e = P\left(n_1 \geq \frac{5}{2}\right) = \boxed{Q\left(\frac{5/2}{\sqrt{N_0/2}}\right)}$

③ $X(t) = \alpha W(t) + \beta N(t)$

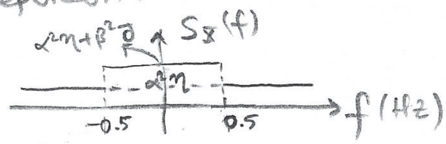
ⓐ $R_X(\tau) = E[(\alpha W(t+\tau) + \beta N(t+\tau))(\alpha W(t) + \beta N(t))]$

$= \alpha^2 R_W(\tau) + \alpha\beta\, \underbrace{E[W(t+\tau)N(t)]} + \alpha\beta\, \underbrace{E[N(t+\tau)W(t)]} + \beta^2 R_N(\tau)$

$= \boxed{\alpha^2\eta\,\delta(\tau) + \beta^2\sigma\,\text{sinc}(\tau)}$

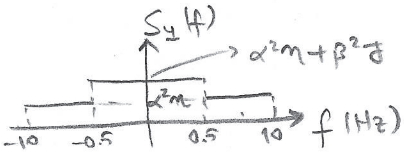
↳ zero since W(t) is zero mean & W(t) & N(t) are independent.

ⓑ $S_X(f) = \mathcal{F}\{R_X(\tau)\} = \boxed{\alpha^2\eta + \beta^2\sigma\,\Pi(f)}$



ⓒ $P_X = \int_{-\infty}^{\infty} S_X(f)\,df = \boxed{\infty}$

ⓓ $S_Y(f) = |H(f)|^2 S_X(f)$



$E[Y(t)] = H(0)\,E[X(t)] = 1\,(\alpha \underbrace{E[W(t)]}_{0} + \beta \underbrace{E[N(t)]}_{\mu}) = \boxed{\beta\mu}$

$E[Y^2(t)] = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)\,df = (\alpha^2\eta + \beta^2\sigma) + 19\alpha^2\eta = \boxed{20\alpha^2\eta + \beta^2\sigma}$

$$\boxed{Y(5) \sim \mathcal{N}\left(\beta\mu,\ 20\alpha^2\eta + \beta^2\sigma - \beta^2\mu^2\right)}$$

Gaussian