I. PROBABILITY AND RANDOM VARIABLES

- **Conditional probability:**

$$P(A|B) = P(A \cap B)/P(B) = P(B|A)P(A)/P(B)$$

- **Random variable:** A mapping from sample space to real numbers.
- **Cumulative distribution function (CDF):**

$$F_X(x) = P(X \leq x)$$

- **Probability density function (PDF):** $f_X(x) \geq 0 \; \forall x$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$.
- **Scalar Gaussian PDF:** $X \sim \mathcal{N}(\mu, \sigma^2)$, for $x \in \mathrm{R}$ $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$
- **Jointly Gaussian PDF:** $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, for $\mathbf{x} \in \mathrm{R}^n$ $f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{0.5}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$
- **Q-function:** $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt$ (i.e., probability that $\mathcal{N}(0,1)$ is larger than x)
- $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx$
- **Variance:** $Var(X) = E[X^2] - (E[X])^2$
- **Covariance:** $Cov(X,Y) = E[XY] - E[X]E[Y]$
- X and Y are *uncorrelated* if $E[XY] = E[X]E[Y]$
- X and Y are *independent* if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$
- $f_{Y|X}(y|x) = f_{X,Y}(x,y)/f_X(x)$

II. SOURCE CODING

- **Discrete Memoryless Source (DMS):** Discrete-time and discrete amplitude source with i.i.d. outputs.
- **Entropy:** $H(X) = -\sum_{i=1}^{N} p_i \log_2(p_i)$, where X takes N values with probabilities $p_1, \ldots, p_N$.
- **Source Coding Theorem:** A DMS with entropy $H(X)$ can be encoded in a lossless manner by using $\bar{R} \geq H(X)$ bits per source output, where $\bar{R}$ is the average codeword length, $\bar{R} = \sum p(x)\, l(x)$.
- **Huffman Coding:** Sort outputs in decreasing order of probabilities and start combining from the least probable two outputs...
- **Lempel-Ziv Coding:** Does not need output probabilities, variable to fixed length coding...

III. ANALOG TO DIGITAL CONVERSION

- **Fourier Transform (F.T.):**

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt, \; x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

- **Sampling Theorem:** Suppose $x(t)$ is bandlimited to W Hz (i.e., $X(f) = 0$ for $|f| \geq W$). Then, $x(t)$ can perfectly be reconstructed from its samples $\{x(nT_s)\}_{n \in \mathbb{Z}}$ if $T_s \leq 1/(2W)$, where $2W$ is the Nyquist rate.
- **Quantization:** A scalar $N-$level ($\log_2 N-$bit) quantizer maps each input into one of N possible outputs, $\hat{x}_1, \ldots, \hat{x}_N$ (called reconstruction/quantization levels). An N-level quantizer has $N-1$ decision boundaries (thresholds), $a_1, \ldots, a_{N-1}$.
- **Mean Squared Error (MSE) Distortion:** $D = E[(X - Q(X))^2]$, where X is quantizer input and $Q(X)$ is quantizer output.
- **SQNR:** $SQNR = E[X^2]/E[(X - Q(X))^2]$
- $D = E[(X - Q(X))^2] = \int_{-\infty}^{\infty} (x - Q(x))^2 f_X(x)dx$ $= \sum_{i=0}^{N-1} \int_{a_i}^{a_{i+1}} (x - \hat{x}_{i+1})^2 f_X(x)dx$ where $a_0 \triangleq -\infty$ and $a_N \triangleq \infty$.
- **Lloyd-Max quantizer:** Iteratively optimize reconstruction levels and decision boundaries (equating partial derivatives of D to zero).

IV. PULSE CODE MODULATION (PCM)

Sample, quantize and encode an analog message signal.

- **Bandwidth requirement for PCM:** $vf_s/2$ Hz, where v is the number of bits of the quantizer and f_s is the sampling rate of the sampler.
- **SQNR of Uniform PCM:** For large N (number of levels in quantizer), quantization error $\tilde{X} = X - Q(X)$ is approximately uniform r.v. over $[-\Delta/2, \Delta/2]$, where $\Delta = 2x_{\max}/N$ (signal is in $[-x_{\max}, x_{\max}]$).
  Then, $E[(X - Q(X))^2] = \Delta^2/12 = x_{\max}^2/(3N^2)$.
  $SQNR = 3E[X^2]4^v/x_{\max}^2$, where $N = 2^v$.
- **Non-uniform Quantizer:** A compressor and a uniform quantizer can be used to realize a non-uniform quantizer.
- **Differential PCM (DPCM):** Quantize the difference between consecutive samples for improved quantization performance.
- **Delta Modulation:** DPCM with two-level (one-bit) quantizer. Either increase or decrease signal level to get close to the message signal.
- **Adaptive Delta Modulation:** Delta modulation with adaptive delta parameter to reduce granular noise and slope overload distortion.

V. ANALOG MODULATION

Consider analog message signal $m(t)$ and insert that directly (without sampling and quantization) into a signal.

- **Full (conventional) amplitude modulation (AM):** $x(t) = (1 + k_a m(t))\, A_c \cos(2\pi f_c t)$
- **Double sideband suppressed carrier (DSB-SC) AM:** $x(t) = m(t) A_c \cos(2\pi f_c t)$
- **Single sideband (SSB) AM:** Filter a DSB-SC AM signal to keep only upper (or, lower) sidebands.
- **Phase modulation (PM):** $x(t) = A_c \cos(2\pi f_c t + k_p m(t))$
- **Frequency modulation (FM):** $x(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right)$

VI. Random Processes

An indexed family (ensemble) of random variables (equivalently, mapping from sample space to set of functions).

- **Mean (expectation) of a random process (r.p.):** $\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx$
- **Autocorrelation function of a r.p.:** $R_X(t_1, t_2) = E[X(t_1)X^*(t_2)]$ $= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2^* f_{X(t_1),X(t_1)}(x_1, x_2)dx_1 dx_2$
- **Autocovariance function of a r.p.:** $C_X(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))^*]$ $= R_X(t_1, t_2) - \mu_X(t_1)\mu_X^*(t_2)$
- **Strict Sense Stationary (SSS) r.p.:** $X(t)$ is SSS if $f_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1, \ldots, x_k) = f_{X(t_1),\ldots,X(t_k)}(x_1, \ldots, x_k)$ for all τ, k, $t_1, \ldots, t_k$.
- **Wide Sense Stationary (WSS) r.p.:** $X(t)$ is WSS if (i) $\mu_X(t) = \mu_X$ (i.e., constant) and (ii) $R_X(t_1, t_2) = R_X(t_1 - t_2)$.
- **Cyclostationary r.p.** if (i) $\mu_X(t) = \mu_X(t + T_0)$ and (ii) $R_X(t_1, t_2) = R_X(t_1 + T_0, t_2 + T_0)$. ($T_0$ is period.)
- For WSS $X(t)$, $R_X(\tau) = E[X(t + \tau)X^*(t)] = R_X^*(-\tau)$
- **Crosscorrelation function:** $R_{XY}(t_1, t_2) = E[X(t_1)Y^*(t_2)]$
- **Jointly WSS r.p.s:** $X(t)$ and $Y(t)$ are *jointly WSS* if (i) $X(t)$ is WSS, (ii) $Y(t)$ is WSS, and (iii) $R_{XY}(t_1, t_2) = R_{XY}(t_1 - t_2)$.
- $X(t)$ and $Y(t)$ are *independent* r.p.s if $(X(t_1), \ldots, X(t_k))$ and $(Y(u_1), \ldots, Y(u_l))$ are independent for all k, l, $(t_1, \ldots, t_k)$ and $(u_1, \ldots, u_l)$.
- $X(t)$ and $Y(t)$ are *uncorrelated* r.p.s if $X(t_1)$ and $Y(t_2)$ are uncorrelated r.v.s for all t_1 and t_2.
- A SSS r.p. is *ergodic* if time averages are equal to ensemble everages (expectations).
- **Filtering of a WSS r.p.:** If a WSS r.p. $X(t)$ passes through an LTI filter with impulse response $h(t)$, output $Y(t)$ is also WSS and $E[Y(t)] = H(0)E[X(t)]$, where $H(0) = \int_{-\infty}^{\infty} h(t)dt$.
- **Power Spectral Density (PSD):** Indicates distribution of average power among different frequencies. It is the Fourier transform (F.T.) of the autocorrelation function. $S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}d\tau$ $R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)e^{j2\pi f\tau}df$
- If a WSS r.p. $X(t)$ passes through an LTI filter with frequency response $H(f)$, output $Y(t)$ has the following PSD: $S_Y(f) = S_X(f)|H(f)|^2$
- $E[|X(t)|^2] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)df$
- **Cross-Spectral Density:** $S_{XY}(f)$ is F.T. of $R_{XY}(\tau)$.
- **Gaussian r.p.:** $X(t)$ is Gaussian r.p. if $\int_0^T g(t)X(t)dt$ is a Gaussian r.v. for all $g(\cdot)$.
- **Gaussian r.p.:** $X(t)$ is Gaussian r.p. if $X(t_1), \ldots, X(t_n)$ are jointly Gaussian for all n, $t_1, \ldots, t_n$.
- If a Gaussian r.p. is WSS, it is also SSS.
- Linear (stable) filtering of a Gaussian r.p. leads to another Gaussian r.p.
- **White Noise:** Zero-mean WSS r.p. with $S_W(f) = N_0/2$ for all f (i.e., $R_W(\tau) = 0.5N_0\delta(\tau)$).
- **Baseband Representation of *Deterministic* Bandpass Signals:** $x(t) = Re\{\tilde{x}(t)e^{j2\pi f_c t}\}$ $x(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)$
- One can also have baseband representation for bandpass random (noise) processes.

VII. Digital Modulation and Demodulation

- **M-ary communication system:** We have M different messages, $m_1, \ldots, m_M$. Transmitter modulates them as $s_1(t), \ldots, s_M(t)$ and receiver observes $r(t) = s_i(t) + n(t)$ for $t \in [0, T)$, where T is symbol duration and $n(t)$ is zero-mean additive white Gaussian noise (AWGN) with PSD of $S_n(f) = N_0/2$ for all f.
- **Basis function representation** of $s_1(t), \ldots, s_M(t)$: $s_i(t) = \sum_{j=1}^{N} s_{ij}\psi_j(t)$ with $s_{ij} = \int_0^T s_i(t)\psi_j(t)dt$, where $\psi_1(t), \ldots, \psi_N(t)$ are orthonormal basis functions that span the signals.
  Then, $\boldsymbol{s}_i = [s_{i1} \cdots s_{iN}]^T$ is vector representation of $s_i(t)$.
- **Properties:** $\int_0^T s_i^2(t)dt = \|\boldsymbol{s}_i\|^2$, $\int_0^T s_i(t)s_j(t)dt = \boldsymbol{s}_i^T\boldsymbol{s}_j$.
- Angle between two signals: $\cos(\theta_{ij}) = \int_0^T s_i(t)s_j(t)dt / \sqrt{\int_0^T s_i^2(t)dt \int_0^T s_j^2(t)dt}$
- Two ways to find orthonormal basis functions: (1) Gram-Schmidt, (2) Intuition, trial and error.

VIII. Miscellaneous Formulas

- $\text{sinc}(x) = \sin(\pi x)/(\pi x)$
- F.T. of $\text{sinc}(t)$ is a rectangular pulse of amplitude 1 between -0.5 and 0.5.
- F.T. of $\text{sinc}^2(t)$ is a triangular pulse between -1 and 1 with maximum amplitude of 1 at zero.
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(2x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$
- $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
- $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- $\sin(x)\cos(y) = 0.5\sin(x + y) + 0.5\sin(x - y)$
- $\cos(x)\cos(y) = 0.5\cos(x + y) + 0.5\cos(x - y)$
- $\sin(x)\sin(y) = 0.5\cos(x - y) - 0.5\cos(x + y)$
- $\cos(\pi/3) = \sin(\pi/6) = 1/2$
- $\cos(\pi/6) = \sin(\pi/3) = \sqrt{3}/2$
- $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$