EEE 431: Telecommunications 1

Quiz 2

Nov. 7, 2021, 13:30-14:40

Instructor: Sinan Gezici

Name: ______________________

Signature: ______________________

Section: ______________________

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Prob. 1: ________ / 26

Prob. 2: ________ / 30

Prob. 3: ________ / 14

Prob. 4: ________ / 30

Total: ________ / 100

Some trigonometric identities: $\sin(2x) = 2\sin(x)\cos(x)$
$\cos(2x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$
$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
$\sin(x)\cos(y) = 0.5\sin(x + y) + 0.5\sin(x - y)$
$\cos(x)\cos(y) = 0.5\cos(x + y) + 0.5\cos(x - y)$
$\sin(x)\sin(y) = 0.5\cos(x - y) - 0.5\cos(x + y)$.

Problem 1 Consider an analog message signal $m(t)$ given by $m(t) = 0.5\cos(2000\pi t) + \sin(1000\pi t)$. This message is transmitted via conventional AM, where the modulated signal is expressed as $x(t) = 5(1+m(t))\cos(200000\pi t)$.

(a) Write down the Fourier transforms of $m(t)$ and $x(t)$. Also, plot them.

(b) Determine the ratio of the average power in the sidebands to the overall average power in $x(t)$.

(c) Can we use an envelope detector to demodulate $x(t)$? Why or why not?

Problem 2 X is a random variable with the following PDF: $f_X(x) = 2x$ if $0 \leq x \leq 1$, and $f_X(x) = 0$ otherwise.

(a) Calculate $E[X^2]$.

(b) For this part, suppose that X is input to a 2-level (1-bit) uniform quantizer with the decision boundary at 0.5 and the reconstruction (quantization) levels of 0.25 and 0.75. Let $Q(X)$ denote the output of this uniform quantizer. Calculate $E\big[(X - Q(X))^2\big]$.

(c) For this part, suppose that we first transform X into Y as $Y = X^2$, and then quantize Y with the same uniform quantizer as in Part (b). Let $Q(Y)$ denote the output of the quantizer. Calculate $E\big[(Y - Q(Y))^2\big]$. Do you get a smaller or larger value than that in Part (b)? Why? (explain intuitively).

Hint: *First, express the CDF of Y in terms of the CDF of X, and then take derivative to obtain the PDF of Y in terms of the PDF of X.*

Problem 3 For a strict sense stationary (SSS) random process, the following holds for any k, τ, $t_1, \ldots, t_k$:

$$f_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1, \ldots, x_k) = f_{X(t_1+\tau),X(t_2+\tau),\ldots,X(t_k+\tau)}(x_1, \ldots, x_k) \tag{1}$$

Prove or disprove the following statement: "*For an SSS random process, the following expectation depends only on the time difference, i.e., $t_1 - t_2$: $E\big[(X(t_1))^2\cos(X(t_2))\big]$.*" (No points without theoretical justification.)

Hint: *Consider the SSS condition in equation (1) for $k = 2$.*

Problem 4 Consider the following random process: $Y(t) = A\cos(2\pi f\, t + \theta)$, where A is a Gaussian random variable with mean 3 and variance 2, f is a uniform (continuous) random variable in the closed interval of $[100, 1000]$, and θ is a constant (fixed). Assume that A and f are independent.

(a) Calculate the mean of $Y(t)$.

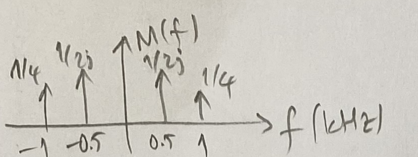
(b) Calculate the autocorrelation function of $Y(t)$.

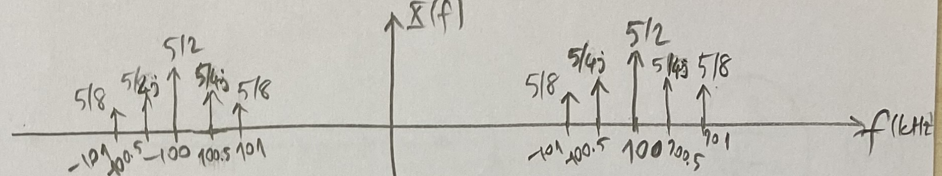
(c) Is $Y(t)$ wide-sense stationary (WSS)? Why or why not?

(d) Is $Y(t)$ cyclostationary? Why or why not?

1 (a) $M(f) = \frac{1}{4}\delta(f-1000) + \frac{1}{4}\delta(f+1000) + \frac{1}{2j}\delta(f-500) + \frac{1}{2j}\delta(f+500)$

$X(f) = \frac{5}{2}\delta(f-100000) + \frac{5}{2}\delta(f+100000) + \frac{5}{2}M(f-100000) + \frac{5}{2}M(f+100000)$





(b) $$\frac{4\left(\frac{5}{8}\right)^2 + 4\left(\frac{5}{4}\right)^2}{4\left(\frac{5}{8}\right)^2 + 4\left(\frac{5}{4}\right)^2 + 2\left(\frac{5}{2}\right)^2} = \frac{\frac{125}{16}}{\frac{325}{16}} = \frac{125}{325} = \boxed{\frac{5}{13}}$$

(c) No. Because $1+m(t)$ is <u>not always</u> positive.

3 $$E[X^2(t_1)\cos(X(t_2))] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2 \cos x_2 \underbrace{f_{X(t_1),X(t_2)}(x_1,x_2)} dx_1 dx_2 \quad (*)$$

SSS, $k=2 \Rightarrow f_{X(t_1),X(t_2)}(x_1,x_2) = f_{X(t_1+\tau),X(t_2+\tau)}(x_1,x_2) \quad \forall t_1,t_2,\tau$

set $\tau = -t_1 \Rightarrow \quad = f_{X(0),X(t_2-t_1)}(x_1,x_2) \longrightarrow$ So, $f_{X(t_1),X(t_2)}$ depends on t_1-t_2 only.

Hence, (*) depends on (t_1-t_2) only, as well.

2 (a) $E[X^2] = \int_0^1 x^2 \cdot 2x\,dx = \boxed{\frac{1}{2}}$

(b) $$E[(X-Q(X))^2] = \int_0^{0.5}\left(x-\frac{1}{4}\right)^2 2x\,dx + \int_{0.5}^{1}\left(x-\frac{3}{4}\right)^2 2x\,dx$$

$= 0.00521 + 0.22396 = \boxed{0.22917}$

(c) $Y = X^2 \rightarrow F_Y(y) = P(X^2 \le y) = P(X \le \sqrt{y}) \longrightarrow f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) = \boxed{1}, \; y \in [0,1]$

$$E[(Y-Q(Y))^2] = \int_0^{0.5}\left(y-\frac{1}{4}\right)^2 1\,dy + \int_{0.5}^{1}\left(y-\frac{3}{4}\right)^2 \cdot 1\,dy = 0.010417 + 0.010417$$

$= \boxed{0.02083}$

Lower since uniform quantizer is optimal for uniform input.

4 (a) $$E[A\cos(2\pi f t+\theta)] = E[A]\int_{100}^{1000}\frac{1}{900}\cos(2\pi f t+\theta)\,df = \boxed{\frac{1}{300}\frac{1}{2\pi t}\left(\sin(2000\pi t+\theta) - \sin(200\pi t+\theta)\right)}$$

(b) $$E[A^2\cos(2\pi f t_1+\theta)\cos(2\pi f t_2+\theta)] = E[A^2]\,E\left[\frac{1}{2}\cos(2\pi f(t_1+t_2)+2\theta) + \frac{1}{2}\cos(2\pi f(t_1-t_2))\right]$$

$$= \frac{11}{2}\left(\frac{1}{2\pi(t_1+t_2)}\left(\sin(2000\pi(t_1+t_2)+2\theta) - \sin(200\pi(t_1+t_2)+2\theta)\right) + \frac{1}{2\pi(t_1-t_2)}\left(\sin(2000\pi(t_1-t_2)) - \sin(200\pi(t_1-t_2))\right)\right)$$

(c) Not WSS, mean depends on t.

(d) Not cyclostationary, mean is not periodic with t.