EEE 431-Homework 6 Solutions

1)

Problem 8.16

1) The optimum threshold is given by

$$\alpha^* = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{1-p}{p} = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$$

2) The average probability of error is ($\alpha^* = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$)

$$\begin{aligned}
P(e) &= p(a_m = -1) \int_{\alpha^*}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{\mathcal{E}_b})^2/N_0} dr \\
&\quad + p(a_m = 1) \int_{-\infty}^{\alpha^*} \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{\mathcal{E}_b})^2/N_0} dr \\
&= \frac{2}{3} Q\left[\frac{\alpha^* + \sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}}\right] + \frac{1}{3} Q\left[\frac{\sqrt{\mathcal{E}_b} - \alpha^*}{\sqrt{N_0/2}}\right] \\
&= \frac{2}{3} Q\left[\frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4} + \sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{3} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4}\right]
\end{aligned}$$

3) Here we have $P_e = \frac{2}{3} Q\left[\frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4} + \sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{3} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4}\right]$, substituting $\mathcal{E}_b = 1$ and $N_0 = 0.1$ we obtain

$$P_e = \frac{2}{3} Q\left[\frac{\sqrt{0.2} \times \ln 2}{4} + \sqrt{20}\right] + \frac{1}{3}\left[\sqrt{20} + \frac{\sqrt{0.2} \times \ln 2}{4}\right] = \frac{2}{3} Q(4.5496) - \frac{1}{3} Q(4.3946)$$

The result is $P_e = 3.64 \times 10^{-6}$.

2)

Problem 8.33:

1) Since $m_2(t) = -m_3(t)$ the dimensionality of the signal space is two.

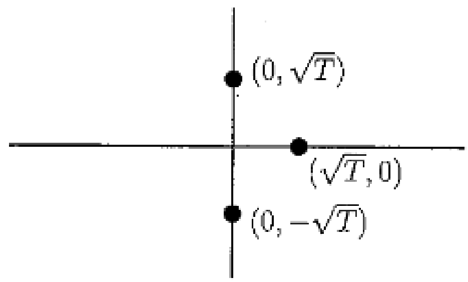
2) As a basis of the signal space we consider the functions

$$\psi_1(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \qquad \psi_2(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} < t \le T \\ 0 & \text{otherwise} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned}\mathbf{m}_1 &= [\sqrt{T},\ \ 0]\\ \mathbf{m}_2 &= [0,\ \ \sqrt{T}]\\ \mathbf{m}_3 &= [0,\ -\sqrt{T}]\end{aligned}$$

3) The signal constellation is depicted in the next figure



4) The three possible outputs of the matched filters, corresponding to the three possible transmitted signals are $(r_1, r_2) = (\sqrt{T} + n_1, n_2)$, $(n_1, \sqrt{T} + n_2)$ and $(n_1, -\sqrt{T} + n_2)$, where n_1, n_2 are zero-mean Gaussian random variables with variance $\frac{N_0}{2}$. If all the signals are equiprobable the optimum decision rule selects the signal that maximizes the metric

$$C(\mathbf{r} \cdot \mathbf{m}_i) = 2\mathbf{r} \cdot \mathbf{m}_i - |\mathbf{m}_i|^2$$

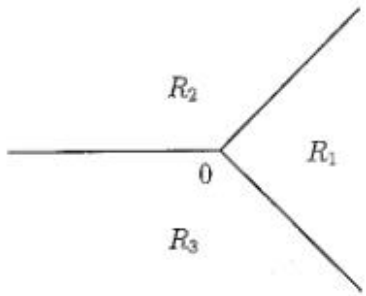
or since $|\mathbf{m}_i|^2$ is the same for all i,

$$C'(\mathbf{r} \cdot \mathbf{m}_i) = \mathbf{r} \cdot \mathbf{m}_i$$

Thus the optimal decision region R_1 for $\mathbf{m}_1$ is the set of points (r_1, r_2), such that $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_2$ and $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_3$. Since $(r_1, r_2) \cdot \mathbf{m}_1 = \sqrt{T} r_1$, $(r_1, r_2) \cdot \mathbf{m}_2 = \sqrt{T} r_2$ and $(r_1, r_2) \cdot \mathbf{m}_3 = -\sqrt{T} r_2$, the previous conditions are written as

$$r_1 > r_2 \quad \text{and} \quad r_1 > -r_2$$

Similarly we find that R_2 is the set of points (r_1, r_2) that satisfy $r_2 > 0$, $r_2 > r_1$ and R_3 is the region such that $r_2 < 0$ and $r_2 < -r_1$. The regions R_1, R_2 and R_3 are shown in the next figure.



5) If the signals are equiprobable then,

$$P(e|\mathbf{m}_1) = P(|\mathbf{r}-\mathbf{m}_1|^2 > |\mathbf{r}-\mathbf{m}_2|^2|\mathbf{m}_1) + P(|\mathbf{r}-\mathbf{m}_1|^2 > |\mathbf{r}-\mathbf{m}_3|^2|\mathbf{m}_1)$$

When $\mathbf{m}_1$ is transmitted then $\mathbf{r} = [\sqrt{T}+n_1, n_2]$ and therefore, $P(e|\mathbf{m}_1)$ is written as

$$P(e|\mathbf{m}_1) = P(n_2 - n_1 > \sqrt{T}) + P(n_1 + n_2 < -\sqrt{T})$$

Since, n_1, n_2 are zero-mean statistically independent Gaussian random variables, each with variance $\frac{N_0}{2}$, the random variables $x = n_1 - n_2$ and $y = n_1 + n_2$ are zero-mean Gaussian with variance N_0. Hence,

$$\begin{aligned} P(e|\mathbf{m}_1) &= \frac{1}{\sqrt{2\pi N_0}}\int_{\sqrt{T}}^{\infty} e^{-\frac{x^2}{2N_0}}dx + \frac{1}{\sqrt{2\pi N_0}}\int_{-\infty}^{-\sqrt{T}} e^{-\frac{y^2}{2N_0}}dy \\ &= Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{T}{N_0}}\right] - 2Q\left[\sqrt{\frac{T}{N_0}}\right] \end{aligned}$$

When $\mathbf{m}_2$ is transmitted then $\mathbf{r} = [n_1, n_2 + \sqrt{T}]$ and therefore,

$$\begin{aligned} P(e|\mathbf{m}_2) &= P(n_1 - n_2 > \sqrt{T}) + P(n_2 < -\sqrt{T}) \\ &= Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{2T}{N_0}}\right] \end{aligned}$$

Similarly from the symmetry of the problem, we obtain

$$P(e|\mathbf{m}_2) = P(e|\mathbf{m}_3) = Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{2T}{N_0}}\right]$$

Since $Q[\cdot]$ is momononically decreasing, we obtain

$$Q\left[\sqrt{\frac{2T}{N_0}}\right] < Q\left[\sqrt{\frac{T}{N_0}}\right]$$

and therefore, the probability of error $P(e|\mathbf{m}_1)$ is larger than $P(e|\mathbf{m}_2)$ and $P(e|\mathbf{m}_3)$. Hence, the message $\mathbf{m}_1$ is more vulnerable to errors.

3)

a) We know that when an LTI filter is applied to a Gaussian process, the resulting output is also Gaussian. Hence $Y(t)$ is a Gaussian random process.

b) Since $Y(t) = X(t) - \frac{1}{2}X(t-1) - \frac{1}{2}X(t+1)$;

$$\mathbb{E}\{Y(t)\} = \mathbb{E}\{X(t)\} - \frac{1}{2}\mathbb{E}\{X(t-1)\} - \frac{1}{2}\mathbb{E}\{X(t+1)\} = 0,$$

and

$$\mathbb{E}\{Y(t)Y(t+\tau)\} = \mathbb{E}\left\{\left(X(t) - \tfrac{1}{2}X(t-1) - \tfrac{1}{2}X(t+1)\right)\left(X(t+\tau) - \tfrac{1}{2}X(t+\tau-1) - \tfrac{1}{2}X(t+\tau+1)\right)\right\}$$

$$= R_X(\tau) - \frac{1}{2}R_X(\tau-1) - \frac{1}{2}R_X(\tau+1)$$

$$- \frac{1}{2}R_X(\tau+1) + \frac{1}{4}R_X(\tau) + \frac{1}{4}R_X(\tau+2)$$

$$- \frac{1}{2}R_X(\tau-1) + \frac{1}{4}R_X(\tau-2) + \frac{1}{4}R_X(\tau)$$

$$= \underbrace{\frac{1}{4}R_X(\tau-2) - R_X(\tau-1) + \frac{3}{2}R_X(\tau) - R_X(\tau+1) + \frac{1}{4}R_X(\tau+2)}_{\text{depends on only } \tau \;\Rightarrow\; \text{hence WSS.}}$$

We know the fact that if a Gaussian process is WSS, it is also SSS.

Hence $Y(t)$ is SSS, also.

c) $Y(t_0) = X(t_0) - \frac{1}{2} X(t_0-1) - \frac{1}{2} X(t_0+1)$

$\Rightarrow \mathbb{E}\{Y(t_0)\} = 0$.

$\mathbb{E}\{Y(t_0)^2\} = R_Y(0) = \frac{1}{4} R_X(-2) - R_X(-1) + \frac{3}{2} R_X(0) - R_X(1) + \frac{1}{4} R_X(2)$

$= \frac{3}{2} R_X(0)$

Hence $Y(t_0) \sim N\left(0, \frac{3N_0}{4}\right)$.

$\Rightarrow P(Y(t_0) > A) = P\left(\frac{Y(t_0)}{\sqrt{3N_0/4}} > \frac{A}{\sqrt{3N_0/4}}\right) = Q\left(\frac{A}{\sqrt{3N_0/4}}\right)$.

d) $E\{Y(t_1)Y(t_1+2)\} = R_Y(2)$

$= \frac{1}{4} R_X(0) =$

Hence $(Y(t_1), Y(t_1+2)) \sim N\left(\underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mu}, \underbrace{\begin{bmatrix} 3N_0/4 & N_0/8 \\ N_0/8 & 3N_0/4 \end{bmatrix}}_{\Sigma}\right)$

$\Rightarrow f_{Y(t_1), Y(t_1+2)}(y_1, y_2) = \frac{1}{2\pi\sqrt{\det\Sigma}} \cdot \exp\left\{\frac{-1}{2} [y_1 \; y_2] \Sigma^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right\}$

$\det\Sigma = \left(\frac{9}{16} - \frac{1}{64}\right) N_0^2 = \frac{35}{64} N_0^2$

$$\Sigma^{-1} = \frac{64}{35N_0^2}\begin{bmatrix} 3N_0/4 & -N_0/8 \\ -N_0/8 & 3N_0/4 \end{bmatrix} = \frac{64}{35N_0}\begin{bmatrix} 3/4 & -1/8 \\ -1/8 & 3/4 \end{bmatrix}$$

$$\Rightarrow f_{Y(t_1),Y(t_1+2)}(y_1,y_2) = \frac{1}{\frac{2\pi N_0}{8}\sqrt{35}} \exp\left\{\frac{-32}{35N_0}[y_1\ y_2]\begin{bmatrix} 3/4 & -1/8 \\ -1/8 & 3/4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right\}$$

$$= \frac{4}{\pi N_0\sqrt{35}} \exp\left\{\frac{-32}{35N_0}\left(\left(3/4\,y_1 - 1/8\,y_2\right)y_1 + \left(3/4\,y_2 - 1/8\,y_1\right)y_2\right)\right\}$$

$$= \frac{4}{\pi N_0\sqrt{35}} \exp\left\{\frac{-32}{35N_0}\left(\frac{3}{4}y_1^2 + \frac{3}{4}y_2^2 - \frac{1}{4}y_1y_2\right)\right\}$$

$$= \frac{4}{\pi N_0\sqrt{35}} \exp\left\{\frac{-8}{35N_0}\left(3y_1^2 + 3y_2^2 - y_1y_2\right)\right\}.$$

$$Y(t_2) - Y(t_1) = Y(t_1+2) - Y(t_1)$$

$$= X(t_1+2) - \frac{1}{2}X(t_1+1) - \frac{1}{2}X(t_1+3) - X(t_1) + \frac{1}{2}X(t_1-1) + \frac{1}{2}X(t_1+1)$$

$$= \underbrace{X(t_1+2) - X(t_1) + \frac{1}{2}(X_1-1) - \frac{1}{2}X(t_1+3)}_{Z(t_1)}$$

$$E\{Z(t_1)\} = 0, \quad E\{Z^2(t_1)\} = \frac{N_0}{2} + \frac{N_0}{2} + \frac{N_0}{8} + \frac{N_0}{8} = \frac{5N_0}{4}$$

$$\Rightarrow Z(t_1) \sim N\left(0, \frac{5N_0}{4}\right) \Rightarrow P\{|Z(t_1)| > A\} = P\{Z(t_1) > A\} + P\{Z(t_1) < -A\}$$

$$= 2Q\left(\frac{A}{\sqrt{5N_0/4}}\right).$$

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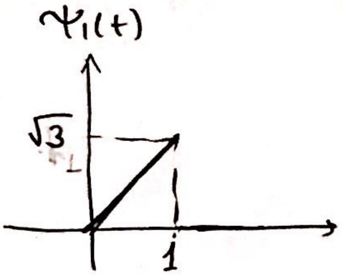
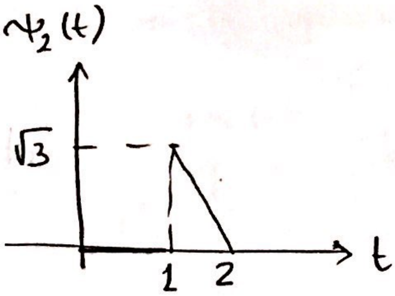
4)

a)





Let us prove $\{\psi_1(t), \psi_2(t)\}$ provides an orthonormal basis.

$$\int_0^2 \psi_1^2(t)\,dt = \int_0^1 (\sqrt{3}\,t)^2\,dt = 3\int_0^1 t^2\,dt = \frac{3t^3}{3}\Big|_0^1 = \boxed{1}$$

$$\int_0^2 \psi_2^2(t)\,dt = \int_1^2 \left(\sqrt{3}\,(2-t)\right)^2\,dt = 3\int_1^2 (2-t)^2\,dt$$

$$= 3\int_0^1 u^2\,du = 3\,\frac{u^3}{3}\Big|_0^1 = \boxed{1}.$$

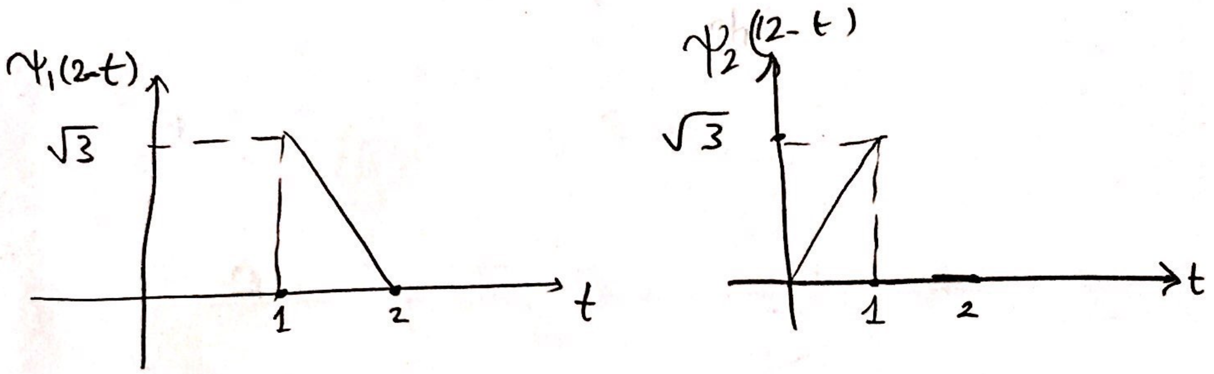
$$\int_0^2 \psi_1(t)\psi_2(t)\,dt = 0.$$

and $s_1(t) = \frac{1}{\sqrt{3}}\left(\psi_1(t) + \psi_2(t)\right)$ and

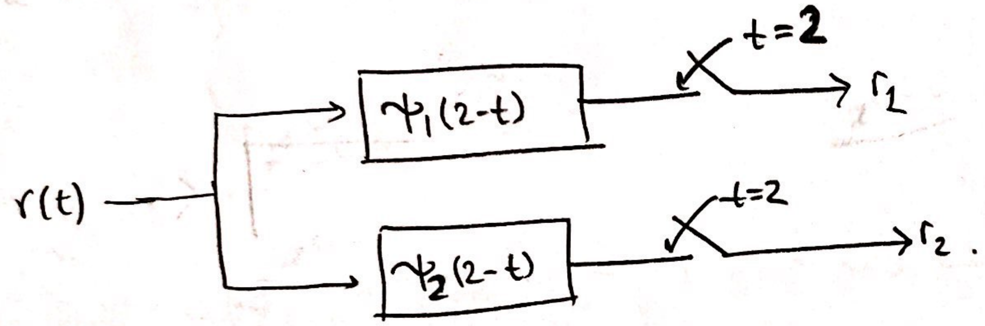
$s_2(t) = \frac{1}{\sqrt{3}}\psi_1(t)$, i.e., $s_1(t), s_2(t) \in \mathrm{span}\{\psi_1(t), \psi_2(t)\}$.

Hence $\{\psi_1(t), \psi_2(t)\}$ is an orthonormal basis for the signal space.

b) $\psi_1(2-t)$ and $\psi_2(2-t)$ can be plotted as



Then, optimal receiver structure is simply;



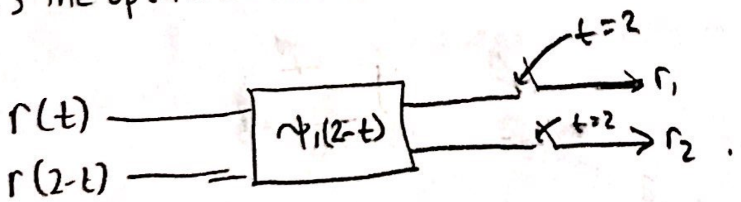
Note that $\psi_1(2-t) = \psi_2(t)$

Let us show that $r(t) * \psi_2(2-t)\big|_{t=2} = r(2-t) * \psi_1(2-t)\big|_{t=2}$.

$$r(t) * \psi_2(2-t)\big|_{t=2} = \int_0^2 \psi_2(2-\tau)\, r(2-\tau)\, d\tau = \int_0^2 \psi_2(u) r(u)\, du$$

$$r(2-t) * \underbrace{\psi_1(2-t)}_{\psi_2(t)}\big|_{t=2} = \int_0^2 r(2-\tau)\psi_2(2-\tau)\, d\tau = \int_0^2 \psi_2(u) r(u)\, du$$

Hence, the optimal receiver can be given as



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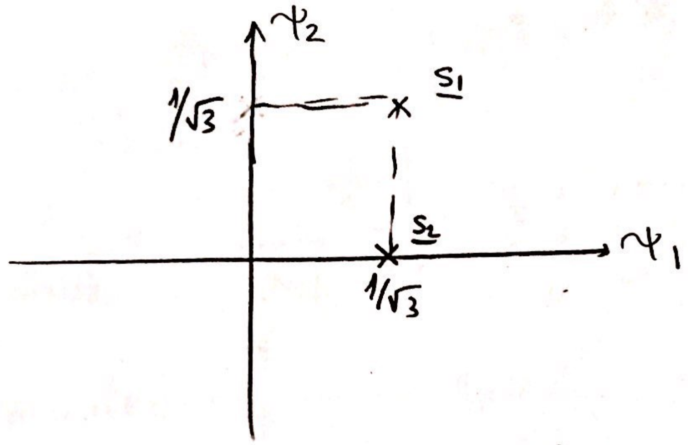
c) $\underline{s}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \qquad s_2 = \begin{bmatrix} 1/\sqrt{3} \\ 0 \end{bmatrix}$



$$\hat{m} = \begin{cases} 1, & \|\underline{r} - \underline{s}_1\| \leq \|\underline{r} - \underline{s}_2\| \\ 2, & \text{o.w.} \end{cases} \qquad \left(\underline{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \right)$$

$$\|\underline{r} - \underline{s}_1\| \leq \|\underline{r} - s_2\| \iff \left(r_2 - \frac{1}{\sqrt{3}}\right)^2 \leq r_2^2 \iff \frac{1}{3} \leq \frac{2r_2}{\sqrt{3}} \iff r_2 \geq \frac{1}{2\sqrt{3}}$$

$$P_{e,1} = P\left(r_2 < 1/2\sqrt{3} \mid \underline{s_1} \text{ is sent} \right)$$

$$= P\left(n_2 + \frac{1}{\sqrt{3}} < \frac{1}{2\sqrt{3}} \right) = P\left(n_2 < -\frac{1}{2\sqrt{3}}\right) = P\left(n_2 > \frac{1}{2\sqrt{3}} \right)$$

$$= P\left(\frac{n_2}{\sqrt{N_0/2}} > \frac{1}{2\sqrt{\frac{3N_0}{2}}} \right) = Q\left(\frac{1}{\sqrt{6N_0}} \right)$$

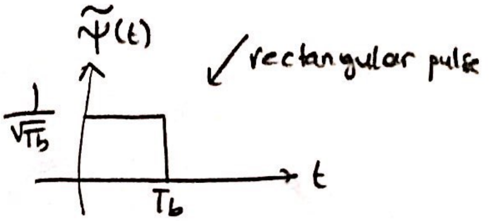
$$P_{e,2} = P\left(r_2 \geq \frac{1}{2\sqrt{3}} \mid \underline{s_2} \text{ is sent} \right) = P\left(n_2 \geq \frac{1}{2\sqrt{3}} \right) = Q\left(\frac{1}{\sqrt{6N_0}} \right)$$

$$P_e = Q\left(\frac{1}{\sqrt{6N_0}} \right), \quad \text{average energy} = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{3}\right) + \frac{1}{2}\cdot\left(\frac{1}{3}\right) = \frac{1}{2}$$

$$\gamma_b = \frac{1/2}{N_0} = \frac{1}{2N_0} \Rightarrow \boxed{P_e = Q\left(\sqrt{\frac{\gamma_b}{3}} \right)}$$

5)

a) $P_{e,1} = Pr\{ r < r_{th} \mid 1 \text{ is sent} \}$



$$= Pr\left\{ \int_0^{T_b} \left(\frac{p(t) + n(t)}{\sqrt{T_b}} \right) dt < r_{th} \right\}$$

$$\int_0^{T_b} p(t)\,dt = \int_0^{T_b} \left(\cos\left(\frac{\pi t}{T_b}\right) + \sin\left(\frac{\pi t}{T_b}\right) \right) dt = \sin\left(\frac{\pi t}{T_b}\right)\frac{T_b}{\pi} - \cos\left(\frac{\pi t}{T_b}\right)\frac{T_b}{\pi} \Big|_0^{T_b}$$

$$= \frac{2T_b}{\pi}.$$

$$\tilde{n} = \int_0^{T_b} \frac{n(t)}{\sqrt{T_b}} dt, \quad \mathbb{E}\{\tilde{n}\} = \int_0^{T_b} \frac{\mathbb{E}\{n(t)\}}{\sqrt{T_b}} dt = 0.$$

$$\mathbb{E}\{\tilde{n}^2\} = N_0/2,$$

Since $\tilde{n}$ is Gaussian, $\tilde{n} \sim N\left(0, \frac{N_0}{2}\right)$

$$\Rightarrow P_{e,1} = Pr\left\{ \tilde{n} < r_{th} - \frac{2\sqrt{T_b}}{\pi} \right\}$$

$$= Pr\left\{ \frac{\tilde{n}}{\sqrt{N_0/2}} < \frac{r_{th} - \frac{2\sqrt{T_b}}{\pi}}{\sqrt{N_0/2}} \right\}$$

$$= Q\left(\frac{\frac{2\sqrt{T_b}}{\pi} - r_{th}}{\sqrt{N_0/2}} \right).$$

$$P_{e,0} = \Pr\{ r \geq r_{th} \mid 0 \text{ is sent} \}$$

$$= \Pr\left\{ \tilde{n} - \frac{2\sqrt{T_b}}{\pi} \geq r_{th} \right\} = \Pr\left\{ \tilde{n} \geq r_{th} + \frac{2\sqrt{T_b}}{\pi} \right\}$$

$$= \Pr\left\{ \frac{\tilde{n}}{\sqrt{N_0/2}} \geq \frac{r_{th} + \frac{2\sqrt{T_b}}{\pi}}{\sqrt{N_0/2}} \right\} = Q\left(\frac{r_{th} + \frac{2\sqrt{T_b}}{\pi}}{\sqrt{N_0/2}} \right)$$

Assuming transmitted bits are equally likely;

$$P_e = \frac{Q\left(\frac{\frac{2\sqrt{T_b}}{\pi} - r_{th}}{\sqrt{N_0/2}} \right) + Q\left(\frac{r_{th} + \frac{2\sqrt{T_b}}{\pi}}{\sqrt{N_0/2}} \right)}{2}$$

Since $\frac{\partial}{\partial x} Q(x) = \frac{-1}{\sqrt{2\pi}} e^{-x^2/2}$;

$$\frac{\partial P_e}{\partial r_{th}} = \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{N_0/2}} e^{-\left(\frac{\frac{2\sqrt{T_b}}{\pi} - r_{th}}{\sqrt{N_0/2}} \right)^2 / 2} - \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{N_0/2}} e^{-\left(\frac{r_{th} + \frac{2\sqrt{T_b}}{\pi}}{\sqrt{N_0/2}} \right)^2 / 2} \right) = 0$$

if $\exp\left\{ -\left(\frac{\frac{2\sqrt{T_b}}{\pi} - r_{th}}{\sqrt{N_0/2}} \right)^2 \right\} = \exp\left\{ -\left(\frac{\frac{2\sqrt{T_b}}{\pi} + r_{th}}{\sqrt{N_0/2}} \right)^2 \right\}$

if $\left(\frac{2\sqrt{T_b}}{\pi} - r_{th} \right)^2 = \left(\frac{2\sqrt{T_b}}{\pi} + r_{th} \right)^2 \Rightarrow \boxed{r_{th} = 0}$.

<u>Note</u>: This problem can be solved when transmitted bits have unequal prior, in a similar way.

b) When $r_{th}=0$, $P_e = Q\left(\sqrt{\frac{8T_b}{\pi^2 N_0}}\right)$

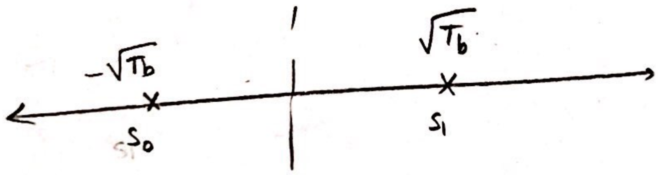
$$E_b = \int_0^{T_b} p^2(t)\,dt = \int_0^{T_b}\Big(\underbrace{\sin^2\left(\frac{\pi t}{T_b}\right)+\cos^2\left(\frac{\pi t}{T_b}\right)}_{1}+\sin\left(\frac{2\pi t}{T_b}\right)\Big)\,dt$$

$$= T_b + \int_0^{T_b}\sin\left(\frac{2\pi t}{T_b}\right)dt =$$

$$= T_b - \cos\left(\frac{2\pi t}{T_b}\right)\cdot\frac{T_b}{2\pi}\Bigg|_0^{T_b} = T_b. \Rightarrow P_e = Q\left(\sqrt{\frac{8}{\pi^2}\frac{E_b}{N_0}}\right)$$

c) Take $\psi(t) = \frac{p(t)}{\sqrt{T_b}}$, hence $\{\psi(t)\}$ forms an orthonormal basis for the signal space.

$$s_1 = \int_0^{T_b} p(t)\,\psi(t)\,dt = \sqrt{T_b}, \quad s_0 = \int_0^{T_b}(-p(t))\,\psi(t)\,dt = -\sqrt{T_b}.$$



Hence, optimal decision rule is

$$\hat{m} = \begin{cases} 1, & \text{if } r \geq 0 \\ 0, & \text{o.w.} \end{cases}, \text{ where } r = \int_0^{T_b} r(t)\,\psi(t)\,dt.$$

$$P_{e,1} = P(r<0 \mid 1 \text{ is sent}) = P\left(\sqrt{T_b}+n'<0\right) =$$

$$= P\left(n' < -\sqrt{T_b}\right)$$

$$= P\left(\frac{n'}{\sqrt{N_0/2}} > \sqrt{\frac{T_b}{N_0/2}}\right) = Q\left(\sqrt{\frac{T_b}{N_0/2}}\right).$$

$$\left(n' = \int_0^{T_b} n(t)\,\psi(t)\,dt\right)$$

$$P_{e,0} = P\left(r \geq 0 \mid 0 \text{ is sent}\right)$$

$$= P\left(n' + \sqrt{T_b} \geq 0\right) = Q\left(\sqrt{\frac{T_b}{N_0/2}}\right)$$

Hence $\boxed{P_e = Q\left(\sqrt{\frac{2T_b}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$

In part (a) we found that $P_e = Q\left(\sqrt{\frac{8}{\pi^2}\frac{E_b}{N_0}}\right)$

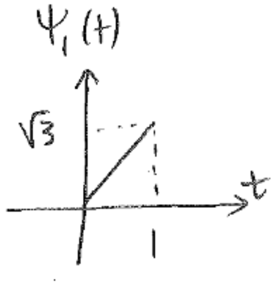
Note that $\frac{8}{\pi^2} \cong 0.81 \leq 2$

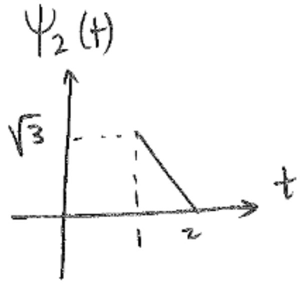
~~We~~ The loss in SNR is simply $10\log_{10}(2) - 10\log_{10}\left(\frac{8}{\pi^2}\right) \cong 3.92\,dB.$

CamScanner ile tarandı

6)

a-b)

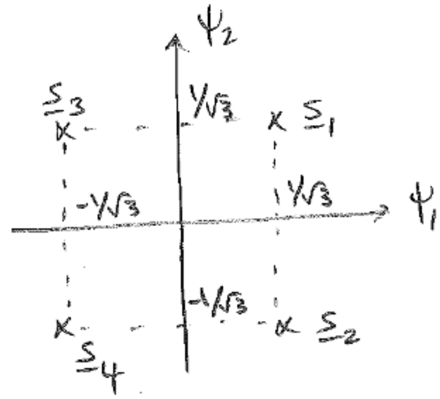




will work.

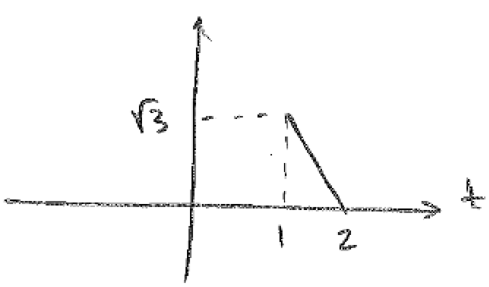
Then $\underline{s}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ $\quad \underline{s}_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

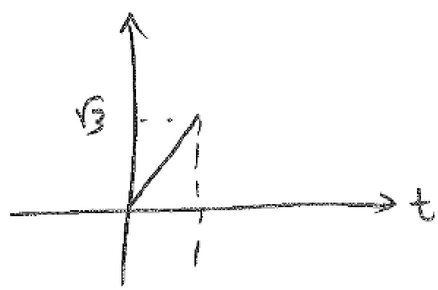
$\underline{s}_2 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$ $\quad \underline{s}_4 = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$



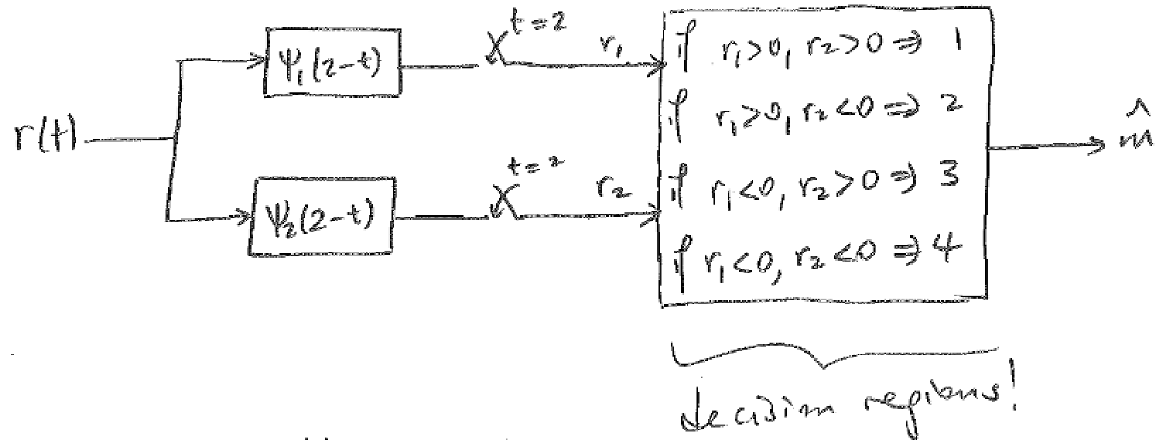
c)

$\psi_1(T-t) = \psi_1(2-t)$ & $\psi_2(2-t)$



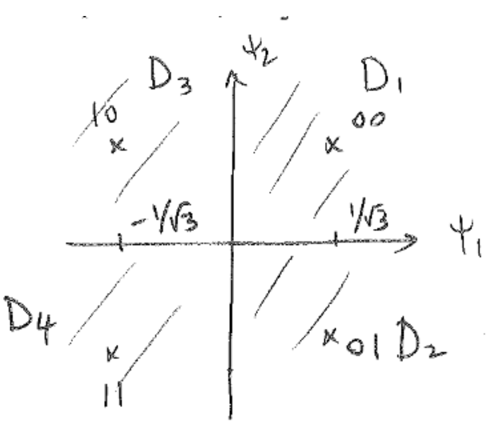


d)



ML receiver,

e)



P (error in the first bit)

$$= P\left(n_1 > 1/\sqrt{3}\right)$$

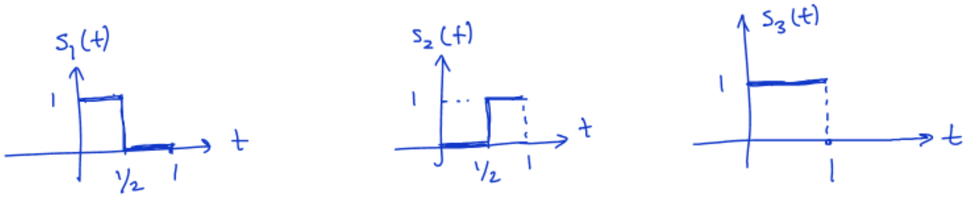
$$= Q\left(\sqrt{\frac{2}{3N_0}}\right)$$

$$P(\text{error in the second bit}) = P\left(n_2 > 1/\sqrt{3}\right) = Q\left(\sqrt{2/3N_0}\right)$$

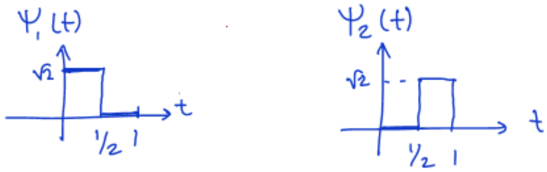
hence : $\boxed{P_b = Q\left(\sqrt{\frac{2}{3N_0}}\right)}$

7)

a)

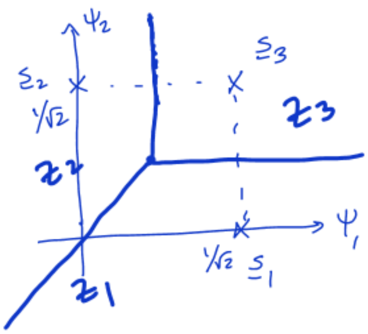


By inspection:



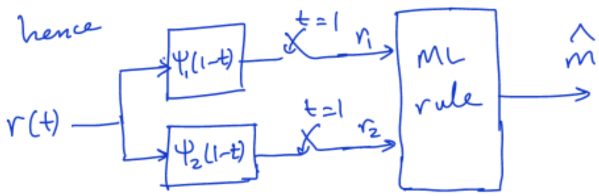
will work

Constellation:



Z_i's are the decision regions.

hence



$$\underline{s}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \end{bmatrix} \quad \underline{s}_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\underline{s}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \end{bmatrix}$$

where

$$\hat{m} = \begin{cases} 1 & \text{if } r_1 > r_2,\ r_2 < \frac{1}{2\sqrt{2}} \\ 2 & \text{if } r_1 < r_2,\ r_1 < \frac{1}{2\sqrt{2}} \\ 3 & \text{if } r_1, r_2 > \frac{1}{2\sqrt{2}} \end{cases}$$

b)

$$P_{e,3} = 1 - P\left(\underline{r} \in Z_3 \mid \underline{s}_3 \text{ is sent}\right)$$

$$= 1 - P\left(r_1 > \frac{1}{2\sqrt{2}},\ r_2 > \frac{1}{2\sqrt{2}} \;\middle|\; \underline{s}_3 \text{ is sent}\right)$$

$$= 1 - P\left(\frac{1}{\sqrt{2}} + n_1 > \frac{1}{2\sqrt{2}},\ \frac{1}{\sqrt{2}} + n_2 > \frac{1}{2\sqrt{2}}\right)$$

where n_1, n_2 are independent and both $\sim \mathcal{N}(0, N_0/2)$

$$\Rightarrow\ P_{e,3} = 1 - P\left(n_1 > -\frac{1}{2\sqrt{2}}\right) \cdot P\left(n_2 > -\frac{1}{2\sqrt{2}}\right)$$

$$= 1 - Q^2\left(\frac{-1/2\sqrt{2}}{\sqrt{N_0/2}}\right) = 1 - Q^2\left(-\frac{1}{\sqrt{4N_0}}\right)$$

$$\Rightarrow$$

$$\boxed{P_{e,3} = 2 \cdot Q\left(\frac{1}{\sqrt{4N_0}}\right) - Q^2\left(\frac{1}{\sqrt{4N_0}}\right)}$$