EEE 431: Telecommunications 1

Quiz 2

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Name: ____________________

Section: ____________________

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Prob. 1: ______ / 25

Prob. 2: ______ / 25

Prob. 3: ______ / 25

Prob. 4: ______ / 25

Total: ______ / 100

Problem 1 Consider a DSB-SC AM signal given by

$$x(t) = A_c\, m(t)\, \cos(2\pi f_c t + \theta)$$

where $m(t)$ is the message signal and θ is the phase offset. If $m(t)$ is equal to 1 for $t \in [-2, 2]$ and equal to zero otherwise, calculate the Fourier transform of $x(t)$.

Reminder: The Fourier transform of a signal $s(t)$ is defined as $S(f) = \int_{-\infty}^{\infty} s(t)\, \mathrm{e}^{-j2\pi f t} dt$.

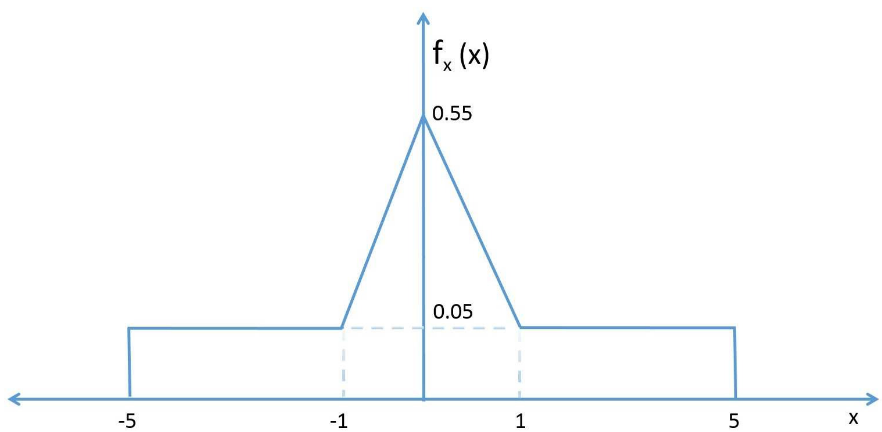
Can we transmit $x(t)$ over a channel with bandwidth $2f_c$ without any distortion? Justify your answer.

Problem 2 Consider the following random process: $X(t) = Y + \sin(2\pi f t)$, where Y is a random variable with zero mean and unit variance (i.e., variance of 1), and f is a discrete random variable which is equal to $10\,\mathrm{Hz}$ or $20\,\mathrm{Hz}$ with probability 0.5 each. It is assumed that Y and f are independent. Find the mean and the autocorrelation function of $X(t)$, and prove whether $X(t)$ is a WSS random process.

Hint: $\sin(a)\sin(b) = 0.5\cos(a - b) - 0.5\cos(a + b)$.

Problem 3 A signal $X(t)$ is modeled by a lowpass stationary process and has a probability density function (PDF) at any time t_0 as shown below. The bandwidth of this process is $10\,\mathrm{kHz}$, and we desire to transmit it using a PCM system. If sampling is performed at twice the Nyquist rate and a uniform quantizer with 16 levels is employed, what is the resulting SQNR? (You do not need to calculate the SQNR value exactly, just express the SQNR without any integrals and specify all the parameters.) What is the resulting bit rate?

Hint: $\mathrm{SQNR} = 3\,(4^v)\,\mathrm{E}[X^2(t)]/x_{\max}^2$, where v is the number of bits for the quantizer and $x_{\max}$ is the maximum value for the absolute value of the signal.



Problem 4 A zero-mean WSS Gaussian random process $X(t)$ has the following power spectral density (PSD):

$$S_X(f) = \begin{cases} (3 - |f|)^2\,, & |f| \le 3 \\ 0\,, & |f| > 3 \end{cases}$$

(a) Calculate the average power of $X(t)$.

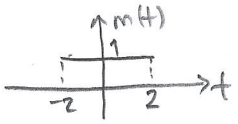
(b) What is the bandwidth of $X(t)$?

(c) Specify the probability distribution of $X(4)$. (No need for a probability density function (PDF) expression. Specify the distribution name and all the related parameter values.)

(d) If $X(t)$ passes through an ideal lowpass filter with a cutoff frequency of $1\,\mathrm{Hz}$ and a gain of 1, determine and plot the PSD of the output by marking all the critical values on the plot.

(1) $x(t) = A_c\, m(t)\cos(2\pi f_c t + \theta) = \frac{A_c}{2} m(t)\left(e^{j\theta} e^{j2\pi f_c t} + e^{-j\theta} e^{-j2\pi f_c t}\right)$

$$\boxed{X(f) = \frac{A_c}{2}\left(e^{j\theta} M(f-f_c) + e^{-j\theta} M(f+f_c)\right)}$$



$$M(f) = \int_{-2}^{2} 1\, e^{-j2\pi f t}\,dt = -\frac{1}{j2\pi f}\left(e^{-j2\pi f 2} - e^{j2\pi f 2}\right) = \frac{2\cancel{j}\sin(4\pi f)}{\cancel{j}2\pi f} = \boxed{\frac{\sin(4\pi f)}{\pi f}}$$

Has infinite bandwidth. ← No.

(2) $X(t) = Y + \sin(2\pi f t)$ $\quad E[Y]=0 \quad Var[Y] = E[Y^2] - (E[Y])^2 = E[Y^2] = 1$

$$E[X(t)] = E[Y] + E[\sin(2\pi f t)] = \boxed{0 + 0.5\sin(20\pi t) + 0.5\sin(40\pi t)}$$

$$R_X(t_1,t_2) = E[X(t_1)X^*(t_2)] = E[(Y + \sin(2\pi f t_1))(Y + \sin(2\pi f t_2))]$$

$$= E[Y^2] + \overset{0}{\cancel{E[Y]}} E[\sin(2\pi f t_2)] + \overset{0}{\cancel{E[Y]}} E[\sin(2\pi f t_1)] + E[\sin(2\pi f t_1)\sin(2\pi f t_2)]$$

$$= \boxed{1 + 0.5\sin(20\pi t_1)\sin(20\pi t_2) + 0.5\sin(40\pi t_1)\sin(40\pi t_2)} \rightsquigarrow \text{not WSS}$$

(3) Bit rate: $\nu f_s = 4(2 \times 20\text{ kHz}) = \boxed{160 \text{ kbps}}$

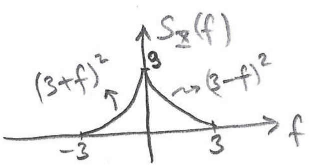
$$E[X^2(t)] = 2\int_0^5 x^2 \underbrace{f_{X(t)}(x)}_{f_X(x)\ \leftarrow \text{stationary}} dx = 2\left(\int_0^1 x^2(0.55 - 0.5x)\,dx + \int_1^5 x^2 (0.05)^2 dx\right)$$

$$= 2\left(0.55\frac{1^3}{3} - 0.5\frac{1^4}{4} + 0.0025\left(\frac{5^3}{3} - \frac{1^3}{3}\right)\right)$$

$$= 2\left(\frac{0.55}{3} - \frac{0.5}{4} + 0.0025\frac{124}{3}\right)$$

$$SQNR = \frac{3 \cdot 4^4\, E[X^2(t)]}{5^2}$$

(4)

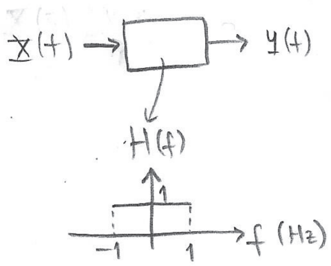


(a) $P_X = E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f)\,df = 2\int_0^3 (3-f)^2 df = 2\left.\frac{(f-3)^3}{3}\right|_0^3 = 2 \cdot \frac{3^3}{3} = \boxed{\frac{54}{3}}$

(b) BW = 3 Hz.

(c) $X(4) \sim \underset{\text{Gaussian}}{\mathcal{N}}\left(0, \frac{54}{3}\right)$ $\qquad Var(X(t)) = E[X^2(t)] - (\underset{0}{E[X(t)]})^2 = \frac{54}{3}$

(d)



$$S_Y(f) = |H(f)|^2 S_X(f) = \begin{cases} (3-|f|)^2, & |f| \le 1 \\ 0, & \text{o.w.} \end{cases}$$