EEE 431: Telecommunications 1

Quiz 1

Feb. 24, 2018, 9:00-10:00

Instructor: Sinan Gezici

Name: ____________________

Signature: ____________________

Bilkent ID: ____________________

Prob. 1: ______ / 16

Prob. 2: ______ / 24

Prob. 3: ______ / 32

Prob. 4: ______ / 28

Total: ______ / 100

Problem 1 Consider a time-domain signal $x(t)$. The Fourier transform of $x(t)$ is denoted by $X(f)$, that is, $X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$.

(a) Derive the Fourier transform of $x(3t)$ in terms of $X(f)$ by starting from the definition of the Fourier transform.

(b) Suppose that $x(t)$ is bandlimited to $100\,\text{kHz}$, that is, $X(f) = 0$ for all $|f| \geq 100\,\text{kHz}$. Define $y(t)$ as $y(t) = (x(3t))^2$. Then, find the minimum sampling frequency for $y(t)$ so that it can perfectly be reconstructed from its samples. Explain your reasoning.

Problem 2 Consider a discrete memoryless source (DMS) consisting of two symbols, a and b, in its alphabet with probabilities $1/3$ and $2/3$, respectively. Perform Huffman coding for this DMS by considering blocks of three symbols. List the codewords. Calculate the average codeword length per symbol.

Problem 3 Consider two discrete random variables X and Y, which are distributed according to the following joint probability mass function (PMF):

$$P(X = i, Y = j) = \begin{cases} 1/6\,, & \text{if } i = 1,\ j = 2 \\ 1/4\,, & \text{if } i = 1,\ j = 1 \\ 1/6\,, & \text{if } i = 0,\ j = 1 \\ 1/3\,, & \text{if } i = 0,\ j = 2 \\ 1/24\,, & \text{if } i = -1,\ j = 1 \\ 1/24\,, & \text{if } i = -1,\ j = 2 \end{cases}$$

(a) Are X and Y independent? Why/why not?

(b) Find the conditional PMF of Y given that $X^2 > 0.5$.

(c) Calculate $E[(2X^2 - 4Y^3)\,|\,Y = 2]$.

Problem 4 Let X denote a random variable with the following probability density function (PDF):

$$f_X(x) = \begin{cases} 2(x+2)/9, & \text{if } -2 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}.$$

Suppose that X is input to a 2-level (1-bit) quantizer with the decision boundary at zero and the reconstruction (quantization) levels of $-a$ and a, where $a > 0$. (That is, if the input is negative, the reconstruction level is $-a$, and if the input is non-negative, the reconstruction level is a.)

Find the optimal value of a that minimizes the mean-squared error distortion D, where $D = E[(X - Q(X))^2]$, with $Q(X)$ denoting the quantizer output.

③ ⓐ $P(X=i)=\begin{cases}1/12, & i=-1,\\ 1/2, & i=0,\\ 5/12, & i=1,\end{cases}$ $P(Y=j)=\begin{cases}11/24, & j=1\\ 13/24, & j=2\end{cases}$

$P(X=i\ \&\ Y=j)\neq P(X=i)P(Y=j) \leftarrow \exists\, i,j$ e.g. take $i=1$, $j=1$ $\left(\frac{5}{12}\frac{11}{24}\neq\frac{1}{4}\right)$

Not independent.

ⓑ $P(Y=j\,|\,X^2>\frac{1}{2})=P(Y=j\,|\,X=-1 \text{ or } 1)=\dfrac{P(Y=j\ \&\ (X=-1 \text{ or } 1))}{\underbrace{P(X=-1 \text{ or } 1)}_{1/2}}=\begin{cases}\dfrac{\frac{1}{4}+\frac{1}{24}}{1/2}=\frac{7}{12}, & j=1\\ \dfrac{\frac{1}{6}+\frac{1}{24}}{1/2}=\frac{5}{12}, & j=2\end{cases}$

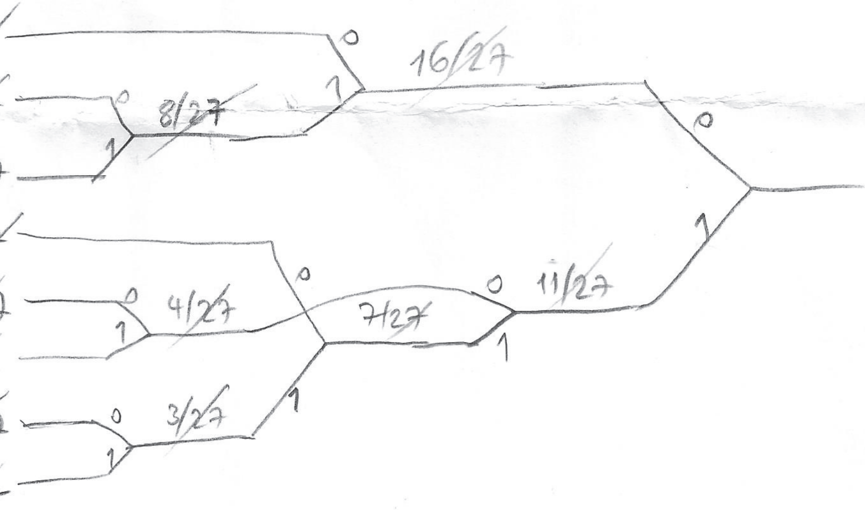
ⓒ $E[2X^2-4Y^3\,|\,Y=2]=2E[X^2\,|\,Y=2]-4(2)^3=2\left(\frac{1}{13}(-1)^2+\frac{8}{13}(0)^2+\frac{4}{13}(1)^2\right)-32$

$P(X=i\,|\,Y=2)=\begin{cases}1/24/13/24, & i=-1\\ 1/3/13/24, & i=0\\ 1/6/13/24, & i=1\end{cases}=\begin{cases}\frac{1}{13}, & i=-1\\ \frac{8}{13}, & i=0\\ \frac{4}{13}, & i=1\end{cases}$

$=\boxed{\frac{10}{13}-32}$

②

alternative sol.	Codeword		
00	00	$a_2a_2a_2$	8/27
10	010	$a_2a_2a_1$	4/27
010	011	$a_2a_1a_2$	4/27
110	110	$a_1a_2a_2$	4/27
0110	100	$a_2a_1a_1$	2/27
0111	101	$a_1a_2a_1$	2/27
1110	1110	$a_1a_1a_2$	2/27
1111	1111	$a_1a_1a_1$	1/27



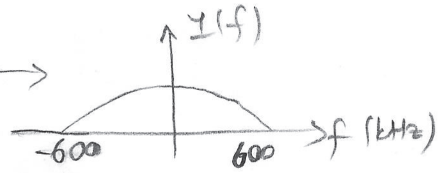
↑ alternative sol.

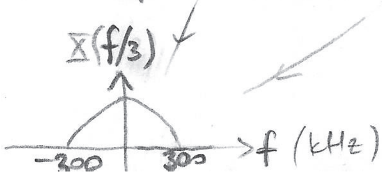
$R=\frac{1}{27}\left((2)8+(3)4.3+(3)2.2+(4)2+(4)1\right)=\frac{76}{27}$ bits/3 symbols

$=\boxed{\frac{76}{51}}$ bits/symbol

① ⓐ $F\{x(3t)\}=\int_{-\infty}^{\infty}x(3t)e^{-j2\pi ft}\,dt \xrightarrow{u=3t} \int_{-\infty}^{\infty}x(u)\,e^{-j2\pi\frac{f}{3}u}\frac{du}{3}=\boxed{\frac{1}{3}X\left(\frac{f}{3}\right)}$

ⓑ $y(t)=(x(3t))^2 \Rightarrow Y(f)=\left(\frac{1}{3}X\left(\frac{f}{3}\right)\right)*\left(\frac{1}{3}X\left(\frac{f}{3}\right)\right)$ →





Nyquist rate $=\boxed{1200\text{ kHz}}$.

(4) $D = E\left[(X - g(X))^2\right] = \int_{-2}^{0} (x+a)^2 \frac{2(x+2)}{9}\,dx + \int_{0}^{1} (x-a)^2 \frac{2(x+2)}{9}\,dx$

$$\frac{\partial D}{\partial A} = \frac{4}{9}\int_{-2}^{0} (x+a)(x+2)\,dx - \frac{4}{9}\int_{0}^{1} (x-a)(x+2)\,dx = 0$$

$$\Rightarrow \left(\frac{x^3}{3} + (a+2)\frac{x^2}{2} + 2ax\right)\Bigg|_{-2}^{0} - \left(\frac{x^3}{3} + (2-a)\frac{x^2}{2} - 2ax\right)\Bigg|_{0}^{1} = 0$$

$$\frac{8}{3} - 2(a+2) + 4a - \frac{1}{3} - (2-a)\frac{1}{2} + 2a = 0 \Rightarrow \boxed{a = \frac{16}{27}}$$