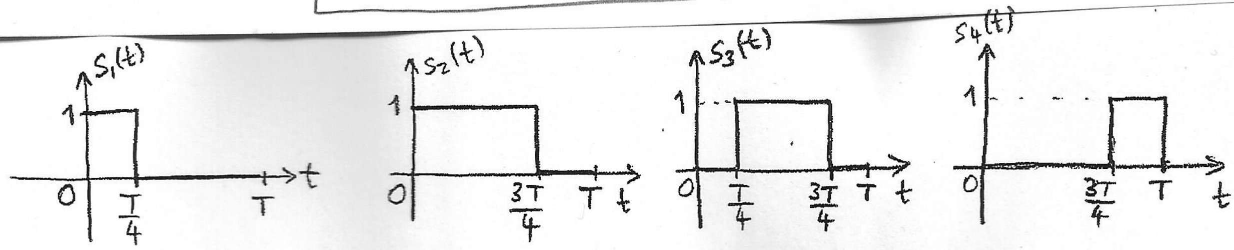
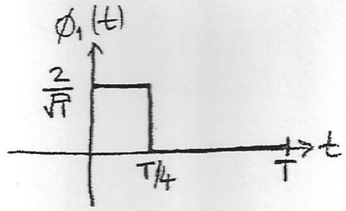
EEE-431 HW#5 Solutions

1



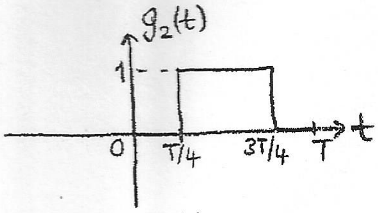
- $E_1 = \int_0^{T/4} (1)^2\, dt = \frac{T}{4}$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\frac{T}{4}}} = \boxed{\frac{2}{\sqrt{T}} s_1(t)}$$

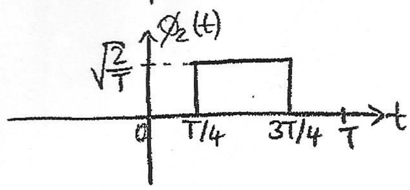


- $s_{21} = \int_0^T s_2(t)\phi_1(t)\,dt = \int_0^{T/4} 1 \cdot \frac{2}{\sqrt{T}}\, dt = \frac{\sqrt{T}}{2}$

$$g_2(t) = s_2(t) - s_{21}\phi_1(t) = s_2(t) - \frac{\sqrt{T}}{2}\phi_1(t)$$
$$= s_2(t) - s_1(t)$$



$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)\,dt}} = \boxed{\frac{s_2(t) - s_1(t)}{\sqrt{\frac{T}{2}}}}$$



- $s_{31} = \int_0^T s_3(t)\phi_1(t)\,dt = 0$

$$s_{32} = \int_0^T s_3(t)\phi_2(t)\,dt = \sqrt{\frac{T}{2}}$$

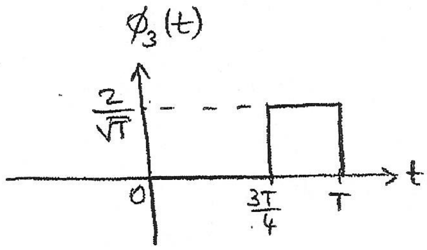
$$g_3(t) = s_3(t) - \cancel{s_{31}\phi_1(t)} - s_{32}\phi_2(t).$$
$$= s_3(t) - \sqrt{\frac{T}{2}}\phi_2(t) = 0 \rightarrow \text{No new basis fn. at this step!}$$

- $s_{41} = \int_0^T s_4(t)\phi_1(t)\,dt = 0$

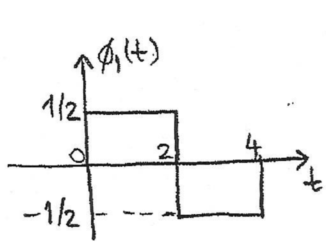
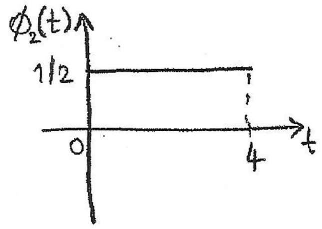
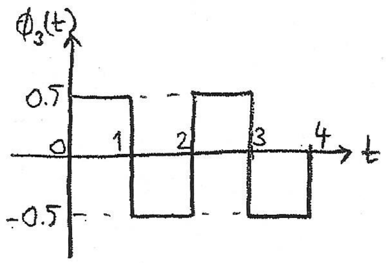
$$s_{42} = \int_0^T s_4(t)\phi_2(t)\,dt = 0$$

$$g_4(t) = s_4(t) - s_{41}\phi_1(t) - s_{42}\phi_2(t) = s_4(t)$$

So, $\phi_3(t) = \frac{s_4(t)}{\sqrt{\int_0^T s_4^2(t)\,dt}} = \boxed{\frac{2}{\sqrt{T}} s_4(t)}$



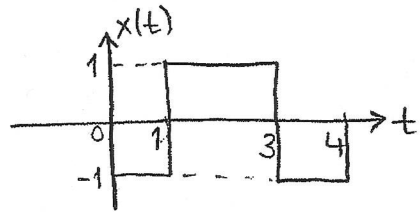
$$\underline{s}_1 = \begin{bmatrix} \sqrt{T}/2 \\ 0 \\ 0 \end{bmatrix} \quad \underline{s}_2 = \begin{bmatrix} \sqrt{T}/2 \\ \sqrt{T}/\sqrt{2} \\ 0 \end{bmatrix} \quad \underline{s}_3 = \begin{bmatrix} 0 \\ \sqrt{T/2} \\ 0 \end{bmatrix} \quad \underline{s}_4 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{T}/2 \end{bmatrix}$$

②   

a) • $\int_0^4 \phi_1^2(t)dt = \int_0^4 \phi_2^2(t)dt = \int_0^4 \phi_3^2(t)dt = 1$

• $\int_0^4 \phi_1(t)\phi_2(t)dt = \int_0^4 \phi_1(t)\phi_3(t)dt = \int_0^4 \phi_2(t)\phi_3(t)dt = 0$

$\Bigg\}$ $\phi_1(t), \phi_2(t), \phi_3(t)$ are orthonormal.

b) 

$$X_1 = \int_0^4 x(t)\phi_1(t)\,dt = -\frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} = 0$$

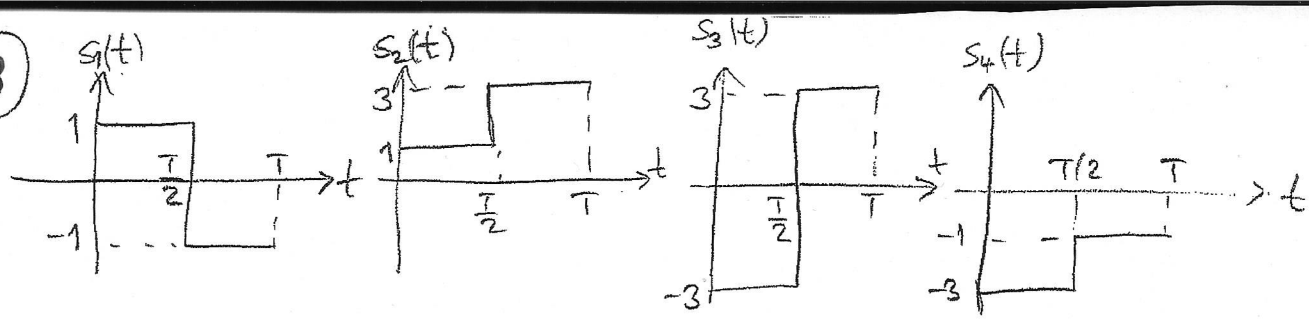
$$X_2 = \int_0^4 x(t)\,\phi_2(t)\,dt = -\frac{1}{2} + 1 - \frac{1}{2} = 0$$

$$X_3 = \int_0^4 x(t)\,\phi_3(t)\,dt = -\frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 0$$

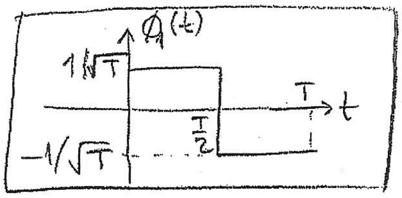
Since $\sum_{i=1}^{3} x_i \phi_i(t) \neq x(t)$, $x(t)$ does <u>not</u> reside in the 3-dimensional signal space specified by $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$.

In fact, $x(t)$ is orthonogonal to that space, since it does not have any components along $\phi_1(t)$, $\phi_2(t)$ or $\phi_3(t)$.

(3)

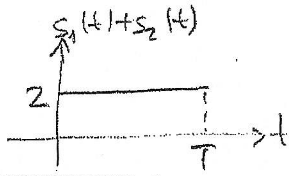


(a) • $\phi_1(t) = \dfrac{s_1(t)}{\sqrt{E_1}} = \boxed{\dfrac{s_1(t)}{\sqrt{T}}}$

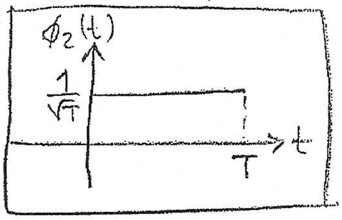


• $s_{21} = \displaystyle\int_0^T s_2(t)\,\phi_1(t)\,dt = \frac{T}{2}\cdot\frac{1}{\sqrt{T}} + \frac{T}{2}\,\frac{-3}{\sqrt{T}} = -\sqrt{T}$

$g_2(t) = s_2(t) - s_{21}\,\phi_1(t) = s_2(t) + \sqrt{T}\,\phi_1(t) = s_2(t) + s_1(t)$



$\phi_2(t) = \dfrac{s_1(t)+s_2(t)}{\sqrt{\int_0^T (s_1(t)+s_2(t))^2\,dt}} = \dfrac{s_1(t)+s_2(t)}{\sqrt{4T}}$



• $s_{31} = \displaystyle\int_0^T s_3(t)\,\phi_1(t)\,dt = \frac{T}{2}\,\frac{-3}{\sqrt{T}} + \frac{T}{2}\,\frac{-3}{\sqrt{T}} = -3\sqrt{T}$

$s_{32} = \displaystyle\int_0^T s_3(t)\,\phi_2(t)\,dt = \frac{-3}{\sqrt{T}}\,\frac{T}{2} + \frac{3}{\sqrt{T}}\,\frac{T}{2} = 0$

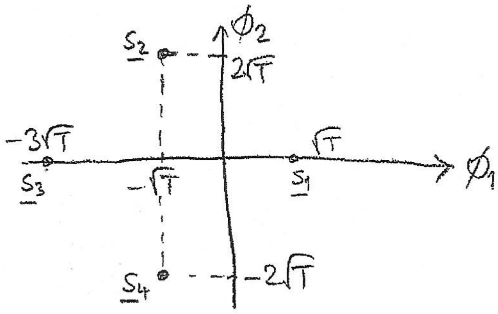
$g_3(t) = s_3(t) - s_{31}\,\phi_1(t) - s_{32}\,\phi_2(t) = s_3(t) + 3\sqrt{T}\,\phi_1(t) = \boxed{0}$ No new basis fn.

• $s_{41} = \displaystyle\int_0^T s_4(t)\,\phi_1(t)\,dt = -\sqrt{T}$

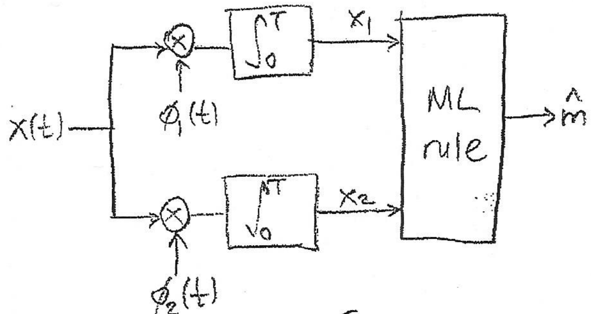
$s_{42} = \displaystyle\int_0^T s_4(t)\,\phi_2(t)\,dt = -2\sqrt{T}$

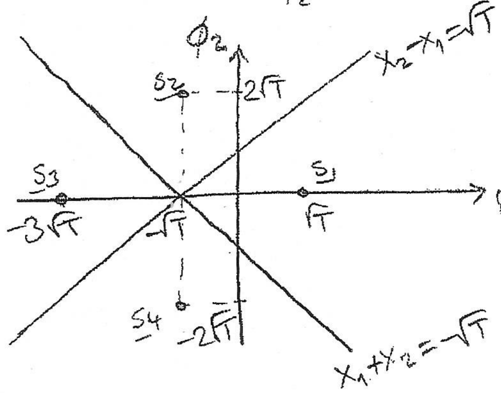
$g_4(t) = s_4(t) - s_{41}\,\phi_1(t) - s_{42}\,\phi_2(t) = s_4(t) + \sqrt{T}\,\phi_1(t) + 2\sqrt{T}\,\phi_2(t) = \boxed{0}$ No new basis fn.

$\underline{s_1} = \begin{bmatrix} \sqrt{T} \\ 0 \end{bmatrix}$, $\underline{s_2} = \begin{bmatrix} -\sqrt{T} \\ 2\sqrt{T} \end{bmatrix}$, $\underline{s_3} = \begin{bmatrix} -3\sqrt{T} \\ 0 \end{bmatrix}$, $\underline{s_4} = \begin{bmatrix} -\sqrt{T} \\ -2\sqrt{T} \end{bmatrix}$



(b)

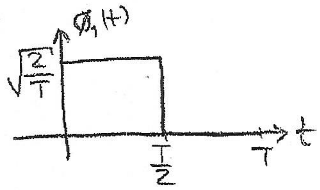


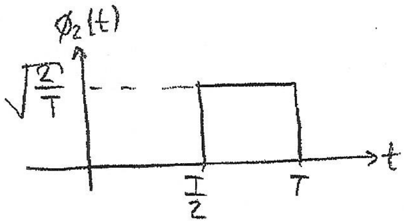


$$\hat{m} = \begin{cases} 1, & \text{if } x_2 - x_1 < \sqrt{T} \ \& \ x_1 + x_2 \geq -\sqrt{T} \\ 2, & \text{if } x_2 - x_1 \geq \sqrt{T} \ \& \ x_1 + x_2 \geq -\sqrt{T} \\ 3, & \text{if } x_2 - x_1 \geq \sqrt{T} \ \& \ x_1 + x_2 < -\sqrt{T} \\ 4, & \text{if } x_2 - x_1 < \sqrt{T} \ \& \ x_1 + x_2 < -\sqrt{T} \end{cases}$$

Shorter solution:

(a)



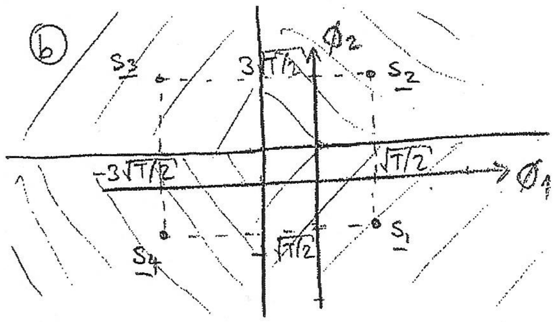


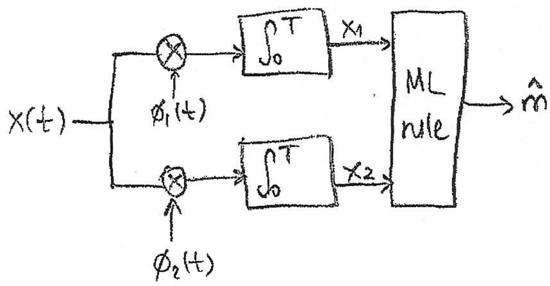
Obviously, $\int_0^T \phi_1^2(t)\,dt = \int_0^T \phi_2^2(t)\,dt = 1$

and $\int_0^T \phi_1(t)\,\phi_2(t)\,dt = 0$

$$\underline{s}_1 = \begin{bmatrix} \sqrt{T/2} \\ -\sqrt{T/2} \end{bmatrix} \quad \underline{s}_2 = \begin{bmatrix} \sqrt{T/2} \\ 3\sqrt{T/2} \end{bmatrix} \quad \underline{s}_3 = \begin{bmatrix} -3\sqrt{T/2} \\ 3\sqrt{T/2} \end{bmatrix} \quad \underline{s}_4 = \begin{bmatrix} -3\sqrt{T/2} \\ -\sqrt{T/2} \end{bmatrix}$$

(b)





$$\hat{m} = \begin{cases} 1, & \text{if } x_1 \geq -\sqrt{T/2} \ \& \ x_2 < \sqrt{T/2} \\ 2, & \text{if } x_1 \geq -\sqrt{T/2} \ \& \ x_2 \geq \sqrt{T/2} \\ 3, & \text{if } x_1 < -\sqrt{T/2} \ \& \ x_2 \geq \sqrt{T/2} \\ 4, & \text{if } x_1 < -\sqrt{T/2} \ \& \ x_2 < \sqrt{T/2} \end{cases}$$