# EEE 431 – Homework 4: Solutions

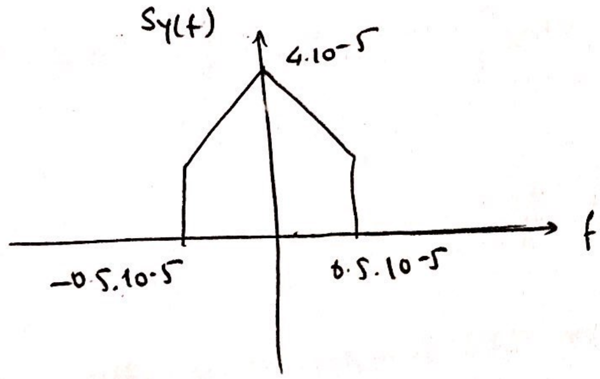
## 1) Problem 5.49

1. $$P_X = \int_{-\infty}^{\infty} S_X(f)\,df = \frac{1}{2}\cdot 2\cdot 10^5 \cdot 4\cdot 10^{-5} = \boxed{4\,W.}$$

2. The range of frequencies are $[-10^5, 10^5] \Rightarrow$ hence $\boxed{\text{bandwith} = 10^5\,Hz = 100\,kHz.}$

3. The transfer function of the ideal lowpass filter is $H(f) = \Pi\left(\frac{f}{10^5}\right)$, hence

$$S_Y(f) = S_X(f)\,|H(f)|^2 = 4\cdot 10^{-5}\,\Lambda\left(\frac{f}{10^5}\right)\Pi\left(\frac{f}{10^5}\right) = \begin{cases} 4\cdot 10^{-5}\,\Lambda\left(\frac{f}{10^5}\right), & |f| < 0.5\cdot 10^{-5} \\ 0, & \text{o.w.} \end{cases}$$




$$P_Y = \int_{-0.5\times 10^{-5}}^{0.5\times 10^{-5}} S_Y(f)\,df = 10^5 \times 2\times 10^{-5} + \frac{1}{2}\times 10^5 \times 2\times 10^{-5} = \boxed{3\,W}.$$

4. Since $X(t)$ is Gaussian, $X(0) \sim N(0, 4)$

$$\Rightarrow f_{X(0)}(x) = \frac{1}{\sqrt{8\pi}}\, e^{-\frac{x^2}{8}}.$$

5. Since for Gaussian random variables independence means uncorrelated, we need to find smallest $t_0$ s.t. $R_X(t_0) = 0$.

$$R_X(\tau) = \mathcal{F}^{-1}\{S_X(f)\} = 4\,\text{sinc}^2(10^5\tau), \quad 10^5 t_0 = 1 \Rightarrow \boxed{t_0 = 10^{-5}}$$

## 2) Problem 5.5 1

1. The impulse response $h(t)$ is given by

$$h(t) = \frac{1}{2}\,\delta'(t-1) + \delta(t-1)$$

$$\Rightarrow H(f) = \left(1+\frac{1}{2}j2\pi f\right)e^{-j2\pi f} = (1+j\pi f)e^{-j2\pi f}.$$

$$\boxed{E\{Y(t)\} = E\{X(t)\}\,H(0) = 2\cdot 1 = 2}$$

2. $$S_Y(f) = S_X(f)\,|H(f)|^2 = S_X(f)\cdot(1+\pi^2 f^2) = \begin{cases} 10^{-3}(1+\pi^2f^2), & |f| \le 200 \\ 0, & \text{o.w.} \end{cases}$$

3. $$P_Y = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)\,df = 2\int_0^{200} 10^{-3}(1+\pi^2f^2)\,df$$

$$= 0.4 + \frac{16\pi^2}{3}10^3$$

4. Since $X(t)$ is WSS, and the system is LTI $\Rightarrow$ $\boxed{Y(t) \text{ is WSS.}}$

5. Since $X(t)$ is Gaussian, and the system is LTI $\Rightarrow$ $\boxed{Y(t) \text{ is Gaussian.}}$

6. $$\left.\begin{array}{l} E[Y^2(1)] = R_Y(0) = 0.4 + \frac{16\pi^2}{3}.10^3 \\ E[Y(1)] = 2 \end{array}\right\}\; Y(1) \sim N\left(2,\, 0.4+\frac{16\pi^2}{3}.10^3\right)$$

Hence, $$f_{Y(1)}(\tau) = \frac{1}{\sqrt{2\pi\left(0.4+\frac{16\pi^2}{3}.10^3\right)}}\, e^{-\frac{(\tau-2)^2}{2\left(0.4+\frac{16\pi^2}{3}.10^3\right)}}$$

[redacted]

## 3) Problem 5.52

1. The transfer function from $X(t)$ to $Y(t)$, denoted as $h(t)$, given by

$$h(t) = \delta(t) + 2\,\delta'(t) \Rightarrow H(f) = 1 + j4\pi f$$

$$\boxed{E\{Y(t)\} = E\{X(t)\}\, H(0) = E\{X(t)\} = 0.}$$

$$\boxed{S_Y(f) = S_X(f)\, |H(f)|^2 = (1 + 16\pi^2 f^2)\,\frac{N_0}{2}}$$

2. $S_Z(f) = S_Y(f)\, \Pi\left(\frac{f}{2W}\right)$, hence

$$S_Z(f) = \begin{cases} (1+16\pi^2 f^2)\,\frac{N_0}{2}, & |f| \leq W \\ 0, & \text{o.w.} \end{cases}$$

3. Since $X(t)$ is WSS and the system is LTI $\Rightarrow$ $Z(t)$ is WSS, too.

4. $E\{Z(t)\} = 0$, $\quad E\{Z^2(t)\} = R_Z(0) = \int_{-\infty}^{\infty} S_Z(f)\,df$

$$\Rightarrow \sigma_Z^2 = 2\int_0^4 (1+16\pi^2 f^2)\,\frac{N_0}{2}\,df \approx 3372.8\,N_0.$$

5. Since the integral of $S_Y(f)$ from $-\infty$ to $\infty$, is infinite, the power of $Y(t)$ is infinite.

4)

a) $R_X(\tau) = 2-|\tau|$, for $|\tau|<2$.

$S_X(f) = \mathcal{F}\{R_X(\tau)\} = \mathcal{F}\{2\Lambda(\tau/2)\}$.

$\Lambda(\tau) \xleftrightarrow{\mathcal{F}} \text{sinc}^2(f)$ andusing the time-scaling property

$$\boxed{\mathcal{F}\{2\Lambda(\tau/2)\} = 4\,\text{sinc}^2(2f)}$$

b) $Y(t)$ is the output of a linear system, $X(t)$ is a Gaussian process

$\Rightarrow$ $\boxed{Y(t) \text{ is a Gaussian process.}}$

$E[Y(t_1)Y(t_2)] = E[(t_1X(t_1)+X(t_1-2))(t_2X(t_2)+X(t_2-2))]$

$= t_1t_2\,R_X(t_1-t_2) + t_1 R_X(t_1-t_2+2) + t_2 R_X(t_1-t_2-2) + R_X(t_1-t_2)$.

So $E[Y(t_1)Y(t_2)]$ is NOT a function of $t_1-t_2$, only.

Hence $Y(t)$ is NOT WSS.

c) $Z = Y(1)-3Y(2)$, $Y(1), Y(2)$ are jointly Gaussian $\Rightarrow Z$ is Gaussian.

$E[Z] = E[Y(1)-3Y(2)] = E[X(1)+X(-1)-3X(2)-X(0)] = 0$.

$E[Z^2] = E[(Y(1)-3Y(2))^2] = R_Y(1,1)+9R_Y(2,2)-6R_Y(1,2)$

$R_Y(1,1) = R_X(0)+R_X(2)+R_X(-2)+R_X(0) = 4$.

$R_Y(2,2) = 4R_X(0)+2R_X(2)+2R_X(-2)+R_X(0) = 10$

$R_Y(1,2) = 2R_X(-1)+1R_X(1)+2R_X(-3)+R_X(-1) = 2+1+0+1=4$.

$\Rightarrow E[Z^2] = 4+9\cdot 10-6\cdot 4 = 70 \Rightarrow Z\sim\mathcal{N}(0,70) \Rightarrow P(Z>5) = Q\left(\frac{5}{\sqrt{70}}\right)$.

5)
$$X(t) = A\cos(2\pi f_0 t + \Theta)$$

$$E\{X(t)\} \underset{\uparrow}{=} E\{A\}\,E\{\cos(2\pi f_0 t + \Theta)\}$$

(A & $\Theta$ independent)

$$E\{A\} = \int_1^2 x\,dx = \frac{x^2}{2}\Big|_1^2 = \frac{3}{2}.$$

$$E\{\cos(2\pi f_0 t + \Theta)\} = \frac{1}{3}\cos(2\pi f_0 t) + \frac{1}{3}\cos(2\pi f_0 t + \frac{\pi}{2}) + \frac{1}{3}\cos(2\pi f_0 t + \pi)$$

Since $\cos(\pi + 2\pi f_0 t) = -\cos(2\pi f_0 t)$

$$\Rightarrow E\{\cos(2\pi f_0 t + \Theta)\} = \frac{1}{3}\cos(2\pi f_0 t + \frac{\pi}{2}) = -\frac{1}{3}\sin(2\pi f_0 t)$$

$$\Rightarrow \boxed{E\{X(t)\} = -\frac{1}{2}\sin(2\pi f_0 t)}$$

(A & $\Theta$ ind.)

$$E\{X(t_1)\,X(t_2)\} \overset{\downarrow}{=} E\{A^2\}\,E\{\cos(2\pi f_0 t_1 + \Theta)\cos(2\pi f_0 t_2 + \Theta)\}$$

$$= \left(\int_1^2 x^2 dx\right)\frac{1}{2}\Big[\cos(2\pi f_0(t_1 - t_2)) + E\{\cos(2\pi f_0(t_1 + t_2) + 2\Theta)\}\Big]$$

$$= \frac{7}{6}\Big[\cos(2\pi f_0(t_1 - t_2)) + E\{\cos(2\pi f_0(t_1 + t_2) + 2\Theta)\}\Big]$$

$$E\{\cos(2\pi f_0(t_1 + t_2) + 2\Theta)\} = \frac{1}{3}\cos(2\pi f_0(t_1 + t_2)) + \frac{1}{3}\cos(2\pi f_0(t_1 + t_2) + \pi)$$

$$+ \frac{1}{3}\cos(2\pi f_0(t_1 + t_2) + 2\pi) = \frac{1}{3}\cos(2\pi f_0(t_1 + t_2)).$$

$$\Rightarrow \boxed{R_X(t_1,t_2) = E\{X(t_1)X(t_2)\} = \frac{7}{6}\cos(2\pi f_0(t_1-t_2)) + \frac{7}{18}\cos(2\pi f_0(t_1+t_2))}$$

Note that $E\{X(t)\} = E\{X(t+\frac{1}{f_0})\}$ and

$$R_X(t_1,t_2) = R_X(t_1+\frac{1}{2f_0}, t_2+\frac{1}{2f_0})$$

$\Rightarrow$ Hence the process is cyclo-stationary with period $\frac{1}{f_0}$.

Average auto-correlation $\overline{R_X}(\tau)$ is given by

$$\overline{R_X}(\tau) = \frac{1}{T_0}\int_0^{T_0} \frac{7}{6}\cos(2\pi f_0\tau) + \underbrace{\frac{7}{18}\cos(2\pi f_0(2t+\tau))}_{\text{integrates to } 0}\, dt, \text{ where } T_0=\frac{1}{f_0}$$

$$= \boxed{\frac{7}{6}\cos(2\pi f_0\tau)}$$

6)

a) $E\{X(t)\} = E\{A_1\}E\{\cos(2\pi 20kt + \Theta)\} + E\{A_2\}E\{\cos(2\pi 30kt + \Theta)\}$

($A_1$ & $\Theta$) are independent
($A_2$ & $\Theta$) are independent

$E\{A_1\} = 1$, $E\{A_2\} = 2$, $E\{\cos(2\pi 20kt + \Theta)\} = E\{\cos(2\pi 30kt + \Theta)\} = 0$

$$\Rightarrow \boxed{E\{X(t)\} = 0}$$

$$R_X(t_1,t_2) = E\{X(t_1)X(t_2)\} = E\Big\{\big(A_1\cos(2\pi 20k t_1 + \Theta) + A_2\cos(2\pi 30k t_1 + \Theta)\big)\cdot$$
$$\big(A_1\cos(2\pi 20k t_2 + \Theta) + A_2\cos(2\pi 30k t_2 + \Theta)\big)\Big\}$$

$$= \frac{E\{A_1^2\}}{2}\cdot\Big[\cos(2\pi 20k(t_1-t_2)) + \cancelto{0}{E[\cos(2\pi 20k(t_1+t_2)+2\Theta)]}\Big]$$

$$+ \frac{E\{A_2^2\}}{2}\Big[\cos(2\pi 30k(t_1-t_2)) + \cancelto{0}{E[\cos(2\pi 30k(t_1+t_2)+2\Theta)]}\Big]$$

$$+ \frac{E[A_1A_2]}{2}\Big[\cos\big(2\pi(20k t_1 - 30k t_2)\big) + \cancelto{0}{E[\cos(2\pi(20k t_1 + 30k t_2)+2\Theta)]}\Big]$$

$$+ \frac{E(A_1A_2)}{2}\Big[\cos\big(2\pi(30k t_1 - 20k t_2)\big) + \cancelto{0}{E[\cos(2\pi(30k t_1 + 20k t_2)+2\Theta)]}\Big]$$

Note $E[A_1^2] = Var(A_1) + E(A_1)^2 = 1+4 = 5$, $E[A_2^2] = Var(A_2) + E(A_2)^2 = 8$

$E[A_1A_2] = C(1,2) + E(A_1)E(A_2) = 3.$

$$\Rightarrow \boxed{R_X(t_1,t_2) = \frac{5}{2}\cos(2\pi 20k(t_1-t_2)) + 4\cos(2\pi 30k(t_1-t_2)) + \frac{3}{2}\cos(2\pi 10k(2t_1-3t_2)) + \frac{3}{2}\cos(2\pi 10k(3t_1-2t_2))}$$

Mean: constant

$R_X(t_1,t_2) = R_X\left(t_1 + \frac{1}{10k}, t_2 + \frac{1}{10k}\right) \Rightarrow X(t)$ is cyclo-stationary.

$X(t)$ is <u>NOT</u> WSS, $R_X(t_1,t_2)$ is not dependent $t_1-t_2$ only.

b) $$\overline{R_x}(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} R_x(t+\tau, t)\,dt \qquad \left(T_0 = \frac{1}{10k}\right)$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \frac{5}{2}\cos(2\pi 20k\tau) + 4\cos(2\pi 30k\tau) + \frac{3}{2}\cos(2\pi 10k(2\tau - t)) + \frac{3}{2}\cos(2\pi 10k(3\tau + t))\,dt$$

$$= \boxed{\frac{5}{2}\cos(2\pi 20k\tau) + 4\cos(2\pi 30k\tau).}$$

$$\boxed{\begin{aligned} S_x(f) &= \mathcal{F}\{\overline{R_x}(\tau)\} \\ &= \frac{5}{4}\delta(f-20k) + \frac{5}{4}\delta(f+20k) + 2\delta(f-30k) + 2\delta(f+30k) \end{aligned}}$$

7)

a) $$E\{X(t)\} = A_1 \underbrace{E\{\cos(1000\pi t + \Theta_1)\}}_{0} - A_2 \underbrace{E\{\sin(1000\pi t + \Theta_2)\}}_{0}$$

$$= 0.$$

$$R_x(t_1, t_2) = E\{X(t_1)\,X(t_2)\}$$

$$\begin{aligned} &= A_1^2 E\{\cos(1000\pi t_1 + \Theta_1)\cos(1000\pi t_2 + \Theta_1)\} \\ &+ A_2^2 E\{\sin(1000\pi t_1 + \Theta_2)\sin(1000\pi t_2 + \Theta_2)\} \\ &- A_1 A_2 E\{\cos(1000\pi t_1 + \Theta_1)\sin(1000\pi t_2 + \Theta_2)\} \\ &- A_1 A_2 E\{\sin(1000\pi t_1 + \Theta_2)\cos(1000\pi t_2 + \Theta_1)\}. \end{aligned}$$

$$E\{\cos(1000\pi t_1 + \Theta_1)\cos(1000\pi t_2 + \Theta_1)\}$$

$$= \frac{1}{2}\left[\cos(1000\pi(t_1-t_2)) + \cancelto{0}{E\{\cos(1000\pi(t_1+t_2)+2\Theta_1)\}}\right]$$

$$= \frac{\cos(1000\pi(t_1-t_2))}{2}.$$

$$E\{\sin(1000\pi t_1 + \Theta_2)\sin(1000\pi t_2 + \Theta_2)\}$$

$$= \frac{1}{2}\left[\cos(1000\pi(t_1-t_2)) - \cancelto{0}{E\{\cos(1000\pi(t_1+t_2)+2\Theta_1)\}}\right]$$

$$= \frac{1}{2}\cos(1000\pi(t_1-t_2))$$

As $\Theta_1$ and $\Theta_2$ are independent;

$$E\{\cos(1000\pi t_1 + \Theta_1)\sin(1000\pi t_2 + \Theta_2)\}$$

$$= E\{\cos(1000\pi t_1 + \Theta_1)\}\, E\{\sin(1000\pi t_2 + \Theta_2)\} = 0.$$

and, $E\{\sin(1000\pi t_1 + \Theta_2)\cos(1000\pi t_2 + \Theta_1)\}$

$$= E\{\sin(1000\pi t_1 + \Theta_2)\}\, E\{\cos(1000\pi t_2 + \Theta_1)\} = 0.$$

Hence, $R_X(t_1,t_2) = \frac{A_1^2 + A_2^2}{2}\cos(1000\pi(t_1-t_2))$

$\Rightarrow$ Autocorrellation depends on $t_1 - t_2$ only, also mean is constent

$\Rightarrow$ Hence, $X(t)$ is WSS.

$$\boxed{R_X(\tau) = \frac{A_1^2 + A_2^2}{2}\cos(1000\pi\tau).}$$

b) $S_X(f) = \mathcal{F}\{R_X(\tau)\} = \frac{A_1^2+A_2^2}{4}\left(\delta(f-500)+\delta(f+500)\right)$

$H(f) = \frac{4000}{2000}\,\Lambda\left(\frac{f}{2000}\right) = 2\,\Lambda(f/2000)$

$S_Y(f) = S_X(f)\,|H(f)|^2$

$= (A_1^2+A_2^2)\left(\delta(f-500)\left(\Lambda(1/4)\right)^2 + \delta(f+500)\left(\Lambda(-1/4)\right)^2\right)$

$\Lambda(1/4) = \frac{3}{4}, \quad \Lambda(-1/4) = 3/4$

$\Rightarrow S_Y(f) = \frac{9}{16}(A_1^2+A_2^2)\left(\delta(f-500)+\delta(f+500)\right).$

c) $\boxed{P_X = R_X(0) = \frac{A_1^2+A_2^2}{2}}$

$P_Y = \int_{-\infty}^{\infty} S_Y(f)\,df = \frac{9}{8}(A_1^2+A_2^2)$

Power in the band $[900, 1100]$ Hz is <u>zero</u>.

[Since for both processes $\int_{-1100}^{-900} S(f)\,df + \int_{900}^{1100} S(f)\,df = 0$ ].

8) $X(t) = Y\cos(2\pi f_0 t) + Z\sin(2\pi f_0 t)$

$$E\{X(t_1)X(t_2)\} = E\{Y^2\}\cos(2\pi f_0 t_1)\cos(2\pi f_0 t_2)$$

$$+E\{Z^2\}\sin(2\pi f_0 t_1)\sin(2\pi f_0 t_2) + E\{YZ\}\left[\sin(2\pi f_0 t_1)\cos(2\pi f_0 t_2) + \sin(2\pi f_0 t_2)\cos(2\pi f_0 t_1)\right]$$

$$= \frac{1}{2}E\{Y^2\}\left[\cos(2\pi f_0(t_1-t_2)) + \cos(2\pi f_0(t_1+t_2))\right]$$

$$+\frac{1}{2}E\{Z^2\}\left[\cos(2\pi f_0(t_1-t_2)) - \cos(2\pi f_0(t_1+t_2))\right]$$

$$+E\{YZ\}\sin(2\pi f_0(t_1+t_2))$$

- Assume $X(t)$ is WSS.

Then $R_X(t_1,t_2)$ must be a function of $t_1-t_2$ only,

hence $E\{YZ\}=0$ and $E\{Y^2\}=E\{Z^2\}$ must be satisfied.

As $E\{Y\}=E\{Z\}=0 \Rightarrow E\{YZ\}=E\{Y\}E\{Z\} \Rightarrow Y$ & $Z$ are uncorrelated.

Moreover by combining $E\{Y^2\}=E\{Z^2\}$ and $E\{Y\}=E\{Z\}=0$

$$\Rightarrow Var(Y)=Var(Z)$$

- Assume $Var(Y)=Var(Z)$ and $E\{YZ\}=E\{Y\}E\{Z\}$.

Since $E\{Y\}=E\{Z\}=0 \Rightarrow E\{YZ\}=0$ and $E\{Y^2\}=E\{Z^2\}$.

Then, $R_X(t_1,t_2) = E\{Y^2\}\cos(2\pi f_0(t_1-t_2)) \Rightarrow X(t)$ is WSS.