EEE 431: Telecommunications 1

Quiz 1

Oct. 23, 2021, 17:30-18:30

Instructor: Sinan Gezici

Name: ____________________

Signature: ____________________

Section: ____________________

Bilkent ID: ____________________

Prob. 1: ________ / 20

Prob. 2: ________ / 28

Prob. 3: ________ / 24

Prob. 4: ________ / 28

Total: ________ / 100

Hint: $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$, $\sin(\pi/6) = \cos(\pi/3) = 1/2$, $\sin(\pi/3) = \cos(\pi/6) = \sqrt{3}/2$.

Problem 1: Suppose that X_1 and X_2 are jointly Gaussian and the joint probability density function (PDF) of $\mathbf{X} = [X_1 \;\; X_2]^T$ is given by $\frac{1}{2\pi\sqrt{det(\mathbf{C})}} \exp\{-0.5(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}$, where $\boldsymbol{\mu} = [-1 \;\; 2]^T$ and $\mathbf{C} = \begin{bmatrix} 4 & -0.5 \\ -0.5 & 2 \end{bmatrix}$.

(a) Calculate the following probability in terms of the Q-function(s): $P\big(2(X_1)^2 - 5 < 1\big)$.

(b) Calculate the following expectation: $E[(X_1 + 3X_2)^2]$.

Hint: The Q-function is defined as $Q(b) = (1/\sqrt{2\pi}) \int_b^\infty e^{-t^2/2} dt$ (i.e., as the probability that a standard Gaussian random variable is larger than b).

Problem 2: Consider a source with alphabet $\{X, Y, Z, W\}$. Use Lempel-Ziv coding to encode the following sequence: $XXXXXXXXXXYYYYYYYYYZZZZZZZZZ$. (Assume that the initial dictionary is given by $X \to 1$, $Y \to 2$, $Z \to 3$, and $W \to 4$.) Compare the number of bits required to encode the original (uncoded) sequence and the compressed (coded) version.

Problem 3: A source output Y is modeled as a random variable with the probability density function (PDF) given by $f_Y(y) = 3y^2/16$ if $y \in [-2, 2]$ and $f_Y(y) = 0$ otherwise. This source output is being quantized using a 4-level quantizer $Q(\cdot)$ with the quantization regions $[-2, -1]$, $(-1, 0]$, $(0, 1]$, and $(1, 2]$. Determine the optimal reconstruction levels for the four quantization regions that minimize the mean squared error distortion, $D = E[(Y - Q(Y))^2]$.

Problem 4: Consider the following signal: $s(t) = 4\cos(1000\pi t/3)$ for $t \in [0, 1)$ second. Suppose $s(t)$ is sampled at every 1 millisecond over the duration of $t \in [0, 1)$ second. Let X denote a random variable corresponding to these samples.

(a) Can we recover $s(t)$ from these samples? Why or why not?

(b) Find the probability mass function (PMF) of X (i.e., the samples).

(c) The samples are processed by a two-level quantizer $Q(\cdot)$ that maps positive samples to A and negative samples to $-A$, where A is positive real number. First, write an expression, in terms of A, for the mean-squared error distortion, $D = E[(X - Q(X))^2]$. Then, find the optimal value of A that minimizes D.

(d) How much bandwidth (in Hertz) is needed to transmit the outputs of the quantizer in Part (c) assuming binary signaling?