EEE 431: Telecommunications 1

Quiz 3

April 16, 2016, 10:30

Instructor: Sinan Gezici

Name: ______________________

Signature: ______________________

Bilkent ID: ______________________

Prob. 1: ______ / 33

Prob. 2: ______ / 32

Prob. 3: ______ / 35

Total: ______ / 100

Problem 1 Consider the following signals that are defined over $t \in [0,4]$.

$$s_1(t) = \begin{cases} 0\,, & t \in [0,2) \\ 1\,, & t \in [2,4] \end{cases} \qquad s_2(t) = \begin{cases} 0\,, & t \in [0,2) \\ t-2\,, & t \in [2,4] \end{cases} \qquad s_3(t) = \begin{cases} -1\,, & t \in [0,1) \\ 1\,, & t \in [1,2] \\ 0\,, & t \in (2,4] \end{cases}$$

Find a set of orthonormal basis functions for these signals, and express each of the signals as a vector in the corresponding signal space.

Problem 2 A random process $X(t)$ is expressed as $X(t) = s(t) + N(t)$, where $s(t)$ is a deterministic signal and $N(t)$ is a zero-mean white Gaussian process with a power spectral density level of σ^2. Let Y be obtained as

$$Y = \int_0^1 h(t)X(t)dt$$

where $h(t)$ is a deterministic signal. Suppose that $h(t)$ and $s(t)$ are given by

$$h(t) = \begin{cases} e^t - 1\,, & t \in [0,1] \\ 0\,, & \text{otherwise} \end{cases} \qquad s(t) = \begin{cases} e^{-t}\,, & t \in [0,1] \\ 0\,, & \text{otherwise} \end{cases}.$$

Calculate the mean and variance of Y and specify (name) the type of its probability distribution. (No need to write down the probability density function.)

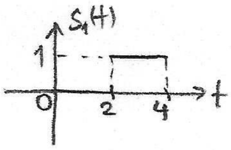
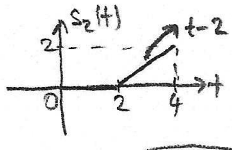
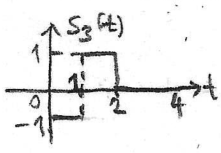
Problem 3 Consider a real-valued wide-sense stationary (WSS) random process $X(t)$, which has the mean function μ_X and the autocorrelation function $R_X(\tau)$. Define a new random process as $Y(t) = 2X'(t) - 4$, where $X'(t)$ denotes the derivative of $X(t)$.

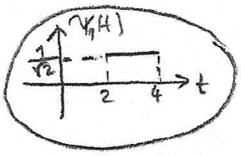
(a) Calculate the mean function for $Y(t)$ and simplify it as much as possible.

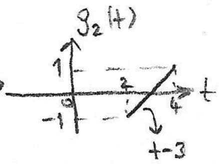
(b) Suppose that the power spectral density of $X(t)$ is given by $S_X(f) = 1/(1 + \pi^2 f^2)$ for $|f| \leq 4\,\text{kHz}$ and $S_X(f) = 0$ for $|f| > 4\,\text{kHz}$. Obtain a closed-form expression for the autocorrelation function of $Y(t)$.

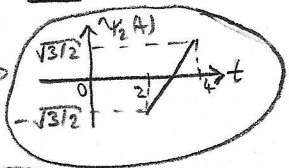
(c) If $X(t)$ is a Gaussian process, specify the probability distribution of $Y(4)$.

Hint: The Fourier transform $S(f)$ of $s(t)$ is defined as $S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt$ and the inverse Fourier transform is given by $s(t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi ft}df$.

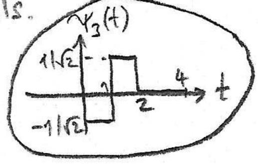
①   

- $\psi_1(t) = \dfrac{s_1(t)}{\sqrt{E_1}} = \dfrac{s_1(t)}{\sqrt{2}}$ ⤳ 

- $s_{21} = \int_0^4 s_2(t)\psi_1(t)\,dt = \int_2^4 \frac{1}{\sqrt{2}}(t-2)\,dt = \frac{1}{\sqrt{2}} \frac{(t-2)^2}{2}\Big|_2^4 = \boxed{\sqrt{2}} \Rightarrow g_2(t) = s_2(t) - \sqrt{2}\,\psi_1(t)$ ⤳ 

  $\psi_2(t) = \dfrac{g_2(t)}{\sqrt{\int g_2^2(t)\,dt}} = \dfrac{g_2(t)}{\sqrt{2/3}} = \sqrt{\dfrac{3}{2}}\, g_2(t)$ ⤳ 

- $s_3(t)$ is orthogonal to other signals.

  So, $\psi_3(t) = \dfrac{s_3(t)}{\sqrt{E_3}} = \dfrac{s_3(t)}{\sqrt{2}}$ ⤳ 

$\underline{s}_1 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix} \quad \underline{s}_2 = \begin{bmatrix} \sqrt{2} \\ \sqrt{2/3} \\ 0 \end{bmatrix} \quad \underline{s}_3 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \end{bmatrix}$

② $X(t) = s(t) + N(t)$

$$Y = \int_0^1 h(t)\big(s(t)+N(t)\big)\,dt = \int_0^1 h(t)s(t)\,dt + \underbrace{\int_0^1 h(t)N(t)\,dt}_{\tilde{N}} = \int_0^1 e^{-t}(e^t-1)\,dt + \tilde{N} \rightarrow N\left(0, \sigma^2\int_0^1 h^2(t)\,dt\right)$$

$$E[\tilde{N}] = \int_0^1 E[N(t)]h(t)\,dt = 0$$

$$E[\tilde{N}^2] = \int_0^1\int_0^1 \underbrace{E[N(t_1)N(t_2)]}_{\sigma^2\delta(t_1-t_2)} h(t_1)h(t_2)\,dt_1\,dt_2 = \sigma^2\int_0^1 h^2(t)\,dt$$

$$\int_0^1 h^2(t)\,dt = \int_0^1 (e^{2t} - 2e^t + 1)\,dt = \left(\frac{1}{2}e^{2t} - 2e^t + t\right)\Big|_0^1 = \boxed{\frac{1}{2}e^2 - 2e + \frac{5}{2}}$$

$$\int_0^1 e^{-t}(e^t-1)\,dt = \int_0^1 (1-e^{-t})\,dt = (t+e^{-t})\Big|_0^1 = \boxed{e^{-1}}$$

So, $\boxed{Y \sim N\left(e^{-1}, \sigma^2\left(\frac{1}{2}e^2 - 2e + \frac{5}{2}\right)\right)}$

③ $Y(t) = 2X'(t) - 4$ $\qquad X(t) \to$ WSS, μ_X, $R_X(\tau)$

ⓐ $E[Y(t)] = 2E[X'(t)] - 4 = 2(\underbrace{E[X(t)]}_{\mu_X})' - 4 = \boxed{-4}$

ⓑ $S_X(f) = \frac{1}{1+\pi^2 f^2}$, $|f| \leq 4\,kHz$

$R_Y(t_1,t_2) = E[Y(t_1)Y(t_2)] = E[(2X'(t_1)-4)(2X'(t_2)-4)]$

$= 4E[X'(t_1)X'(t_2)] - 8E[X'(t_1)] - 8E[X'(t_2)] + 16$

$= 4\frac{d}{dt_1}\left(\frac{d}{dt_2}\underbrace{E[X(t_1)X(t_2)]}_{R_X(t_1-t_2)}\right) - 8\underbrace{(\mu_X)'}_{0} - 8\underbrace{(\mu_X)'}_{0} + 16$

$R_X(\tau) = \mathcal{F}^{-1}\left\{\frac{1}{1+\pi^2 f^2}\Pi\left(\frac{f}{8000}\right)\right\} = \underbrace{\mathcal{F}^{-1}\left\{\frac{1}{1+\pi^2 f^2}\right\}}_{e^{-2|\tau|}} * (8000\,\mathrm{sinc}(8000\tau))$

$= 8000\int_{-\infty}^{\infty} e^{-2|x|}\,\mathrm{sinc}(8000(\tau - x))\,dx$ $\qquad \tau = t_1 - t_2$

$\left(\mathcal{F}^{-1}\left\{\frac{1}{1+\pi^2 f^2}\right\} = \int_{-\infty}^{\infty}\frac{1}{1+\pi^2 f^2}e^{j2\pi f\tau}\,df = e^{-2|\tau|}\right)$ using integration by parts.

$\boxed{R_Y(t_1,t_2) = 16 + 4\frac{d}{dt_1}\left(\frac{d}{dt_2}\,8000\int_{-\infty}^{\infty} e^{-2|x|}\,\mathrm{sinc}(8000(t_1-t_2-x))\,dx\right)}$ → no need for further simplifications

ⓒ $X(t) \to \boxed{LTI} \to 2X'(t)$ → Gaussian process → $2X'(t) \sim N(0, \ldots)$

$H(f) = j4\pi f$ $\qquad \mu_X H(0) \leftarrow 0$

$Y(4) = 2X'(4) - 4 \sim \boxed{N(-4, R_Y(4,4) - 16)}$