EEE 431: Telecommunications 1

Quiz 1

February 18, 2016, 18:30-19:40.

Instructor: Sinan Gezici

Name: ______________________

Signature: ______________________

Bilkent ID: ______________________

Prob. 1: ______ / 13

Prob. 2: ______ / 12

Prob. 3: ______ / 24

Prob. 4: ______ / 23

Prob. 5: ______ / 28

Total: ______ / 100

Problem 1 Consider a signal $x(t)$ which is band limited to W Hz. That is, $X(f) = 0$ for $|f| \geq W$. Then, define $y(t) = x(t)\cos(\pi W t)$. What is the minimum sampling frequency for signal $y(t)$ so that it can perfectly be reconstructed from its samples?

Problem 2 Consider a discrete memoryless source (DMS) consisting of 4 symbols, $\{s_1, s_2, s_3, s_4\}$, in its alphabet with probabilities p_1, p_2, p_3, and p_4, respectively.

(a) For what values of p_1, p_2, p_3, and p_4 is the minimum entropy achieved? What is the resulting entropy? (proof is not required)

(b) For what values of p_1, p_2, p_3, and p_4 is the maximum entropy achieved? What is the resulting entropy? (proof is not required)

Hint: The entropy of a DMS is given by $H = -\sum_{i=1}^{N} p_i \log_2(p_i)$.

Problem 3 Consider a discrete memoryless source (DMS) consisting of 6 symbols, $\{s_1, s_2, s_3, s_4, s_5, s_6\}$, in its alphabet with probabilities 0.12, 0.33, 0.08, 0.25, 0.14, and 0.08, respectively. Perform Huffman coding for this DMS and list the codewords. Calculate the average codeword length.

Problem 4 Let X denote a Gaussian random variable with mean 2 and variance 9, and Y be defined as $Y = X^2 + 2X$. Obtain an expression (in terms of the Q-function(s)) for the probability that Y is greater than or equal to 3, and simplify it as much as possible.

Hint: The probability density function (PDF) of a Gaussian random variable with mean μ and variance σ^2 is given by $f_X(x) = (1/\sqrt{2\pi}\sigma)\, e^{-(x-\mu)^2/(2\sigma^2)}$, and the Q-function is defined as $Q(x) = (1/\sqrt{2\pi}) \int_x^{\infty} e^{-t^2/2}\, dt$.

Problem 5 Consider two random variables X and Y, which are distributed according to the following joint probability density function (PDF):

$$f_{X,Y}(x,y) = \begin{cases} (1/\pi)\, e^{-(x^2+y^2)/2}\ , & \text{if } x \geq 0 \text{ and } y \geq 0 \\ (1/\pi)\, e^{-(x^2+y^2)/2}\ , & \text{if } x < 0 \text{ and } y < 0 \\ 0\ , & \text{otherwise} \end{cases}$$

(a) Find the marginal PDF of X, and the marginal PDF of Y.

(b) Are X and Y independent? Why/why not?

(c) Are X and Y jointly Gaussian? Why/why not?

(d) Find the conditional PDF of X given Y.

Hint: For a Gaussian random variable Z, its marginal PDF is given by $f_Z(z) = (1/\sqrt{2\pi}\sigma)\, e^{-(z-\mu)^2/(2\sigma^2)}$, where μ is the mean and σ^2 is the variance.

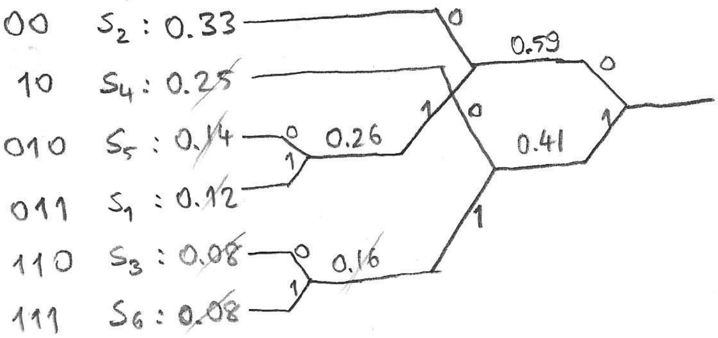
1) $y(t) = x(t)\cos\left(2\pi \frac{W}{2} t\right) \Rightarrow Y(f) = \frac{1}{2} X\left(f - \frac{W}{2}\right) + \frac{1}{2} X\left(f + \frac{W}{2}\right)$

$f_s \geq 3W$

2) a) $P_1 = 1,\ P_2 = P_3 = P_4 = 0 \Rightarrow H = \boxed{0}$

b) $P_1 = P_2 = P_3 = P_4 = \frac{1}{4} \Rightarrow H = -\sum_{i=1}^{4} \frac{1}{4} \log_2 \frac{1}{4} = \boxed{\frac{1}{2} \text{ bit}}$

3)



$\bar{R} = 2(0.33 + 0.25) + 3(0.14 + 0.12 + 0.08 + 0.08)$

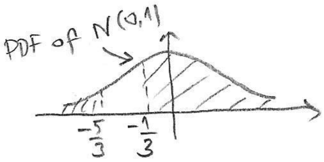
$= \boxed{2.42 \text{ bits/symbol}}$

4) $X \sim N(2,3)$ $\quad Y = X^2 + 2X$

$P(Y \geq 3) = P(X^2 + 2X - 3 \geq 0) = P((X+3)(X-1) \geq 0) = P(X \geq 1 \text{ or } X \leq -3)$

$= P\left(\frac{X-2}{3} \geq \frac{1-2}{3} \text{ or } \frac{X-2}{3} \leq \frac{-3-2}{3}\right)$ $\quad N(0,1)$ $\quad N(0,1)$

PDF of $N(0,1)$



$= \boxed{Q\left(-\frac{1}{3}\right) + Q\left(\frac{5}{3}\right)}$

5) a) $f_X(x) = \begin{cases} \frac{1}{\pi}\int_0^\infty e^{-\frac{(x^2+y^2)}{2}} dy, & \text{if } x \geq 0 \\ \frac{1}{\pi}\int_{-\infty}^0 e^{-\frac{(x^2+y^2)}{2}} dy, & \text{if } x < 0 \end{cases} = \begin{cases} \frac{e^{-x^2/2}}{\pi}\int_0^\infty e^{-y^2/2} dy, & x \geq 0 \\ \frac{e^{-x^2/2}}{\pi}\int_{-\infty}^0 e^{-y^2/2} dy, & x < 0 \end{cases} = \boxed{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$ $\quad x \in \mathbb{R}$

$\Downarrow$ Gaussian $N(0,1)$

$\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-y^2/2} dy = \frac{1}{2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 e^{-y^2/2} dy$

By a similar derivation:

$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}}$, $y \in \mathbb{R}$

Gaussian $\quad Y \sim N(0,1)$

b) No. $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$.

c) No. Although X & Y are marginally Gaussian, they are not jointly Gaussian since $f_{X,Y}(x,y)$ is not in the form of jointly Gaussian PDF, which is non-zero for all $(x,y) \in \mathbb{R}^2$ (but the given joint PDF is zero in some regions).

d) $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{\pi} e^{-(x^2+y^2)/2}}{\frac{1}{\sqrt{2\pi}} e^{-y^2/2}}$, for $x \geq 0$ if $y \geq 0$, for $x < 0$ if $y < 0$

$= \boxed{\sqrt{\frac{2}{\pi}} e^{-x^2/2}}$

If $y \geq 0$, $= \begin{cases} \sqrt{\frac{2}{\pi}} e^{-x^2/2}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$

If $y < 0$, $= \begin{cases} \sqrt{\frac{2}{\pi}} e^{-x^2/2}, & x < 0 \\ 0, & \text{o.w.} \end{cases}$