CONTINUOUS-WAVE MODULATION:

- **Modulation:** Vary some parameters of a carrier-wave according to the message signal.
  ↳ analog signal in this case.

Examples: AM radio, FM radio, Analog TV broadcasting

Cont.-wave modulation

Carrier → Sinusoidal signal
- Amplitude mod. (AM)
- Angle mod.
  - FM
  - PM

- Message signal → Information-bearing, baseband signal
  → Modulating wave

Message signal (info.) → Modulator → Modulated wave
↑ Sinusoidal carrier wave (into Modulator)

Channel output → Demodulator → Estimate of message signal

AMPLITUDE MODULATION:

- Carrier wave: $\boxed{c(t) = A_c \cos(2\pi f_c t)}$

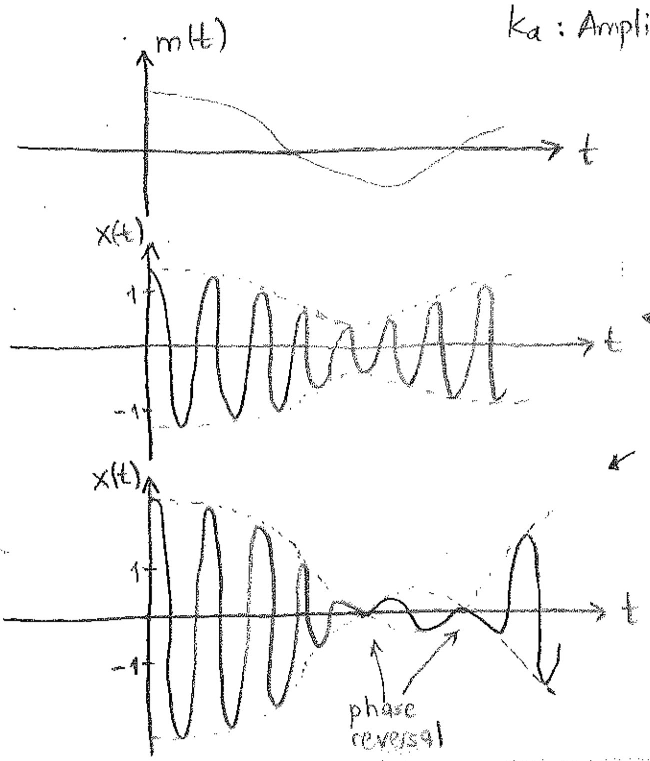
  A_c: Carrier amplitude
  f_c: Carrier frequency

  Message: $m(t)$ → Baseband signal w/ bandwidth W.

  Modulated signal: $\boxed{x(t) = A_c[1 + k_a m(t)] \cos(2\pi f_c t)}$ Full AM (Conventional AM)

  k_a: Amplitude sensitivity (1/volt)

Percentage modulation:
$100 \cdot \max |k_a m(t)|$



← $1 + k_a m(t) > 0 \quad \forall t$

← $1 + k_a m(t) < 0$ for some t.

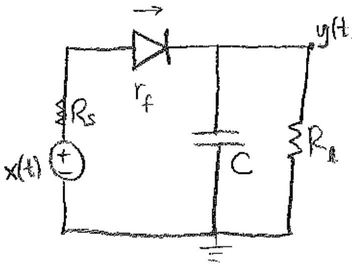
overmodulation

If $|k_a m(t)| < 1 \ \forall t$ & $f_c >> W$, envelope of $x(t)$ is essentially the same as $m(t)$.

(W: message bandwidth)

$\Downarrow$

Use of envelope detector becomes possible (simple implementation)



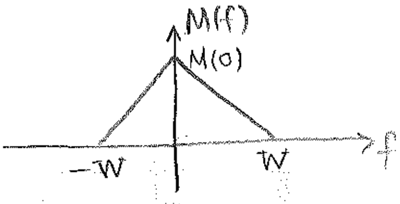
$\boxed{(R_s + r_f)C << \frac{1}{f_c}}$ (quickly charging the capacitor)

$\frac{1}{W} >> R_L C >> \frac{1}{f_c}$

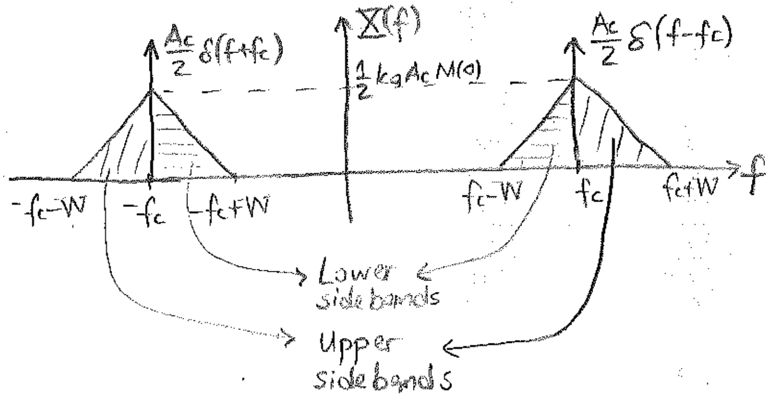
- Spectrum of AM signal:

$$\boxed{X(f) = \frac{A_c}{2}[\delta(f-f_c)+\delta(f+f_c)] + \frac{k_a A_c}{2}[M(f-f_c)+M(f+f_c)]}$$

Fourier transform of $x(t)$.



$\rightarrow$



Transmission bandwidth:

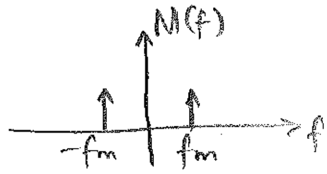
$\boxed{B_T = 2W}$

- Advantages of AM → easily realizable (inexpensive)

Disadvantages of AM → waste of power due to carrier-wave
waste of bandwidth since lower & upper sidebands are related (same info. content).

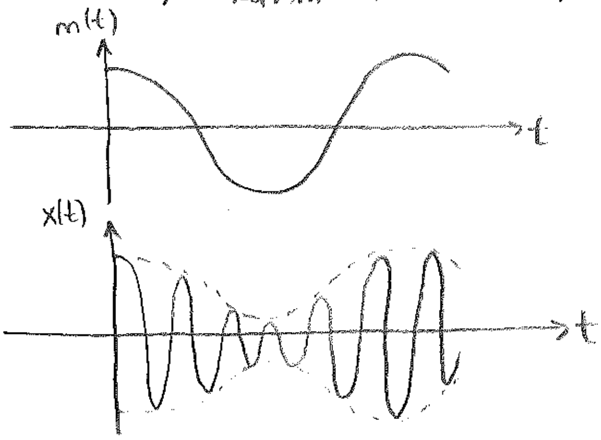
Ex. Single-tone modulation

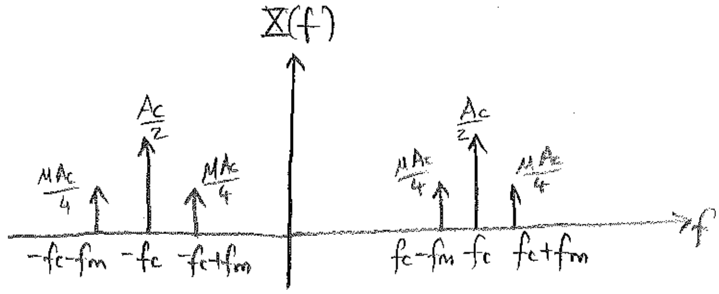
$m(t) = A_m \cos(2\pi f_m t)$, $f_c > f_m$



$$\boxed{x(t) = A_c[1+\mu \cos(2\pi f_m t)]\cos(2\pi f_c t)}$$

$\mu \triangleq k_a A_m < 1$ → modulation factor





$$X(f) = \frac{1}{2}A_c[\delta(f-f_c)+\delta(f+f_c)] + \frac{1}{4}\mu A_c[\delta(f-f_c-f_m)+\delta(f+f_c+f_m) + \delta(f-f_c+f_m)+\delta(f+f_c-f_m)]$$

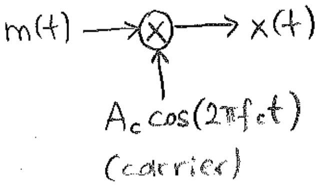
$$\frac{\text{Total sideband power}}{\text{Total power}} = \frac{\frac{1}{4}\mu^2 A_c^2}{\frac{1}{4}\mu^2 A_c^2 + \frac{1}{2}A_c^2} = \boxed{\frac{\mu^2}{\mu^2+2}}$$

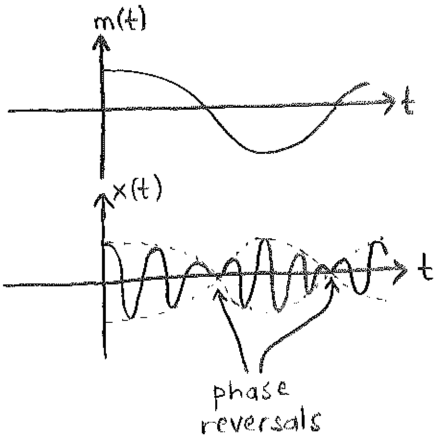
- In order to reduce power and bandwidth inefficiency of full AM:
  - → Double sideband-suppressed carrier (DSB-SC) modulation
  - → Single sideband (SSB) modulation

- $\boxed{\text{DSB-SC AM:}}$

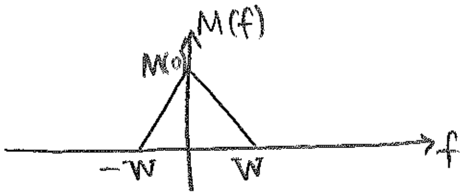
  Saves power by <u>not</u> transmitting a separate carrier.

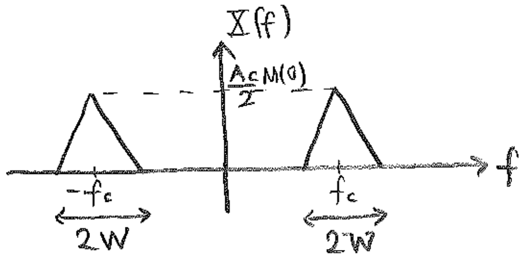
$$\boxed{x(t) = A_c\, m(t) \cos(2\pi f_c t)}$$



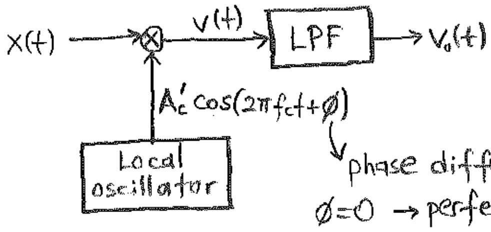


$$\boxed{X(f) = \frac{A_c}{2}\left(M(f-f_c) + M(f+f_c)\right)}$$





<u>Coherent Detection (Synchronous Demodulation):</u>



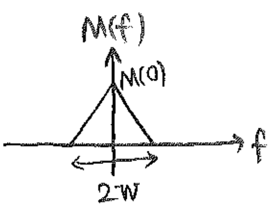
phase difference

$\phi = 0 \rightarrow$ perfect phase synchronization with $x(t)$.

$$x(t) = A_c\, m(t) \cos(2\pi f_c t)$$

$$v(t) = A_c A_c' \cos(2\pi f_c t) \cos(2\pi f_c t + \phi)\, m(t)$$
$$= \frac{A_c A_c'}{2} \cos(4\pi f_c t + \phi)\, m(t) + \frac{1}{2} A_c A_c' \cos\phi\, m(t)$$

After LPF : $\boxed{v_o(t) = \frac{A_c A_c'}{2} \cos\phi\, m(t)}$ $\rightsquigarrow$ $\phi = \mp\frac{\pi}{2} \Rightarrow$ quadrature null effect $(v_o(t) = 0)$



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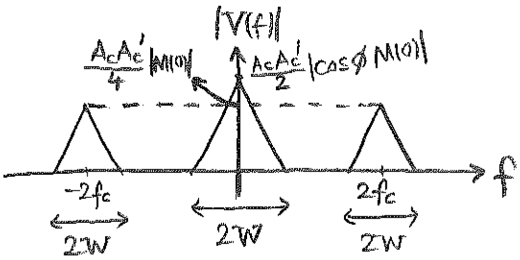
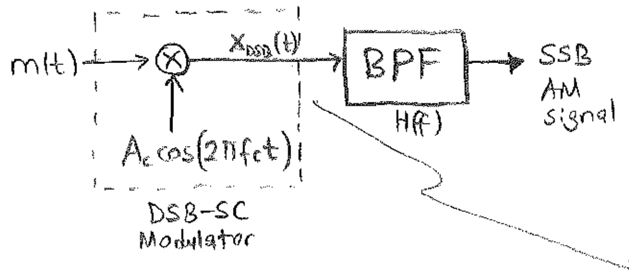
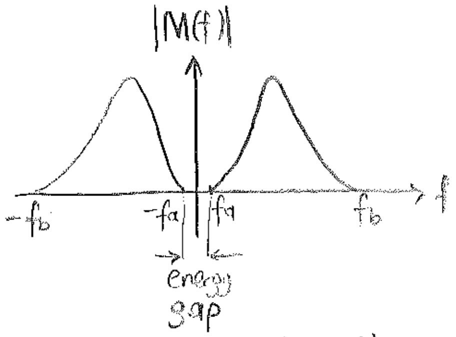


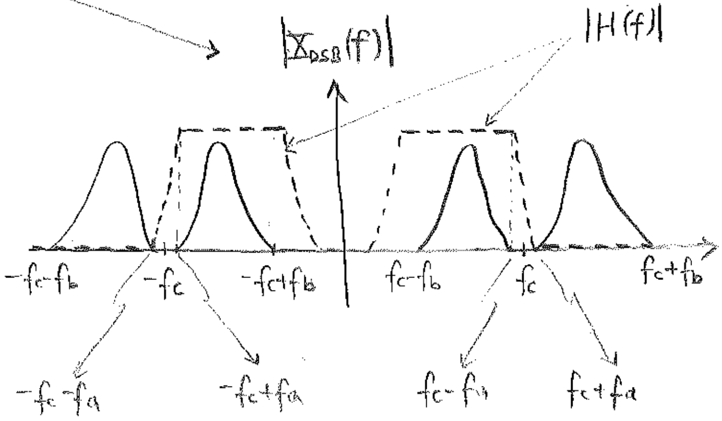
- SSB AM:

- Transmit only the upper (or lower) sidebands.
- One way to generate SSB AM signals:





(e.g. voice signals have it)

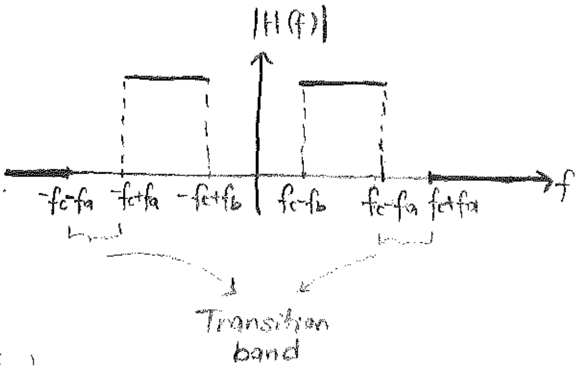


BPF requirements:

Desired sideband is in the passband

Unwanted sideband is in the stopband

Filter's transition band is no larger than $2f_a$.



- Another technique to generate SSB AM signals is based on Hilbert transform (Hartley modulator) → not discussed here.

- Vestigial sideband AM (VSB AM): → used for transmitting video in analog TV broadcasting

  Uses a non-ideal BPF (see H(f) above)

  ⇒ Simplified filter design at the cost of bandwidth increase.

ANGLE MODULATION:

- $\theta_i(t)$: Angle of modulated sinusoidal wave

$$\boxed{x(t) = A_c \cos[\theta_i(t)]}$$

Unmodulated carrier:
$\theta_i(t) = 2\pi f_c t + \phi$
$\downarrow$
No message (info.) in it.

Instantaneous frequency:

$$f_i(t) = \lim_{\Delta t \to 0} \frac{\theta_i(t+\Delta t) - \theta_i(t)}{2\pi \Delta t} = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}$$

- Phase Modulation (PM):

$$\boxed{\theta_i(t) = 2\pi f_c t + k_p m(t)}$$

$m(t)$: Message signal

k_p : Phase sensitivity (rad./volt)

$$\boxed{x(t) = A_c \cos[2\pi f_c t + k_p m(t)]} \rightarrow \text{PM signal}$$

- Frequency Modulation (FM):

$$\boxed{f_i(t) = f_c + k_f m(t)}$$

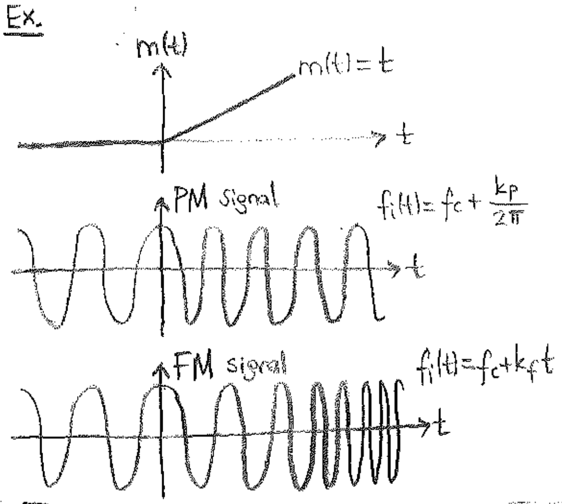
$m(t)$: Message signal

k_f : Frequency sensitivity (Hz./volt)

$$\theta_i(t) = 2\pi \int_0^t f_i(\tau)\,d\tau = 2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau$$

$$\boxed{x(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau\right]} \rightarrow \text{FM signal}$$

Ex.



- FM can be thought of as a special case of PM and vice versa.

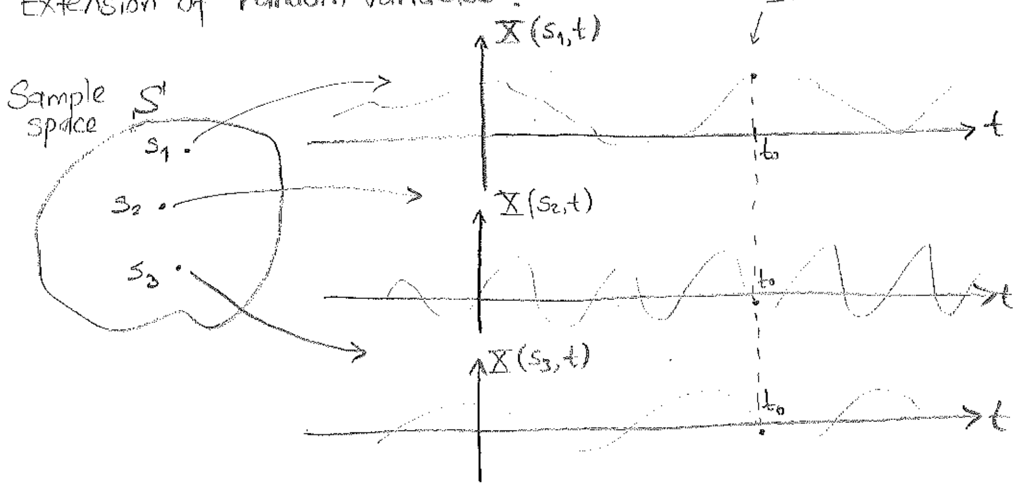
$$m(t) \rightarrow \boxed{PM} \rightarrow x(t) \quad \equiv \quad m(t) \rightarrow \boxed{\frac{d}{dt}} \rightarrow \boxed{FM} \rightarrow x(t)$$

$$m(t) \rightarrow \boxed{FM} \rightarrow x(t) \quad \equiv \quad m(t) \rightarrow \boxed{\int} \rightarrow \boxed{PM} \rightarrow x(t)$$

(Assume appropriate scalings.)

RANDOM PROCESSES:

- Extension of random variables!



Random process → Indexed family (ensemble) of random variables.

Fix s_0 → $X(s_0,t)$: Deterministic time signal

Fix t_0 → $X(s,t_0)$: Random variable (in short, $X(t_0)$)

$\underset{\substack{\uparrow \\ \text{index}}}{X(t)}$: random process → discrete-time, → continuous-time

e.g. $X(t) = A\sin(2\pi f_c t + \Theta)$, $\Theta \sim \mathcal{N}(\mu, \sigma^2)$

e.g. $X(t) = \cos(100\pi f t)$, $f \sim \mathcal{U}[10,20]$

- Complete statistical description:

For any positive integer n & $\forall (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$

$f_{X(t_1),\dots,X(t_n)}(x_1,\dots,x_n)$ is given.

M^{th} order statistics: $\forall n:\ 1 \le n \le M$ & $\forall (t_1,\dots,t_n) \in \mathbb{R}^n$

$f_{X(t_1),\dots,X(t_n)}(x_1,\dots,x_n)$ is given.

- Mean (expectation) of a random process:

$$\boxed{\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)\, dx}$$

- Autocorrelation function:

$$\boxed{R_X(t_1,t_2) = E[X(t_1)X^*(t_2)]}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2^* f_{X(t_1),X(t_2)}(x_1,x_2)\, dx_1\, dx_2$$

- Autocovariance function:

$$C_X(t_1,t_2) = E\left[(X(t_1)-\mu_X(t_1))(X(t_2)-\mu_X(t_2))^*\right]$$

$$= \boxed{R_X(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2)}$$

Ex. $X(t) = A\cos(2\pi f_c t + \Theta)$ $\quad \Theta \sim \mathcal{U}[0, 2\pi)$

$$\mu_X(t) = E[A\cos(2\pi f_c t + \Theta)]$$
$$= \int_0^{2\pi} \frac{1}{2\pi} A\cos(2\pi f_c t + \theta)\, d\theta = \frac{A}{2\pi} \sin(2\pi f_c t + \theta) \Big|_0^{2\pi}$$
$$= \boxed{0}$$

$$R_X(t_1, t_2) = E[A^2 \cos(2\pi f_c t_1 + \theta)\cos(2\pi f_c t_2 + \theta)]$$
$$= \frac{A^2}{2} E\left[\cos(2\pi f_c(t_1 - t_2)) + \cos(2\pi f_c(t_1 + t_2) + 2\theta)\right]$$
$$= \boxed{\frac{A^2}{2}\cos(2\pi f_c(t_1 - t_2))}$$

- **Stationary Processes:**

Random processes with statistical properties that do not change with time. (otherwise, nonstationary).

Strictly stationary process (stationary in the strict sense, SSS):

$$\boxed{F_{X(t_1+\tau),\dots,X(t_k+\tau)}(x_1,\dots,x_k) = F_{X(t_1),\dots,X(t_k)}(x_1,\dots,x_k)} \quad \forall \tau, k, (t_1,\dots,t_k) \in \mathbb{R}^k$$

(m^{th} order stationary if this holds for $\forall k \leq M$)

SSS $\rightarrow$ very strong condition!

Wide-sense stationary (WSS): (also called weakly stationary, 2^{nd} order stationary)

Weaker condition!

$$\left.\begin{aligned} &- \mu_X(t) = \mu_X \quad \text{Constant mean} \\ &- R_X(t_1,t_2) = R_X(t_1 - t_2) = R_X(\tau) \quad \tau = t_1 - t_2 \end{aligned}\right\} \Rightarrow \text{WSS}$$

SSS $\Rightarrow$ WSS (w/ finite 2^{nd} order moments) but WSS $\not\Rightarrow$ SSS

Proof of SSS $\Rightarrow$ WSS:

$X(t)$ is SSS.

- k=1: $F_{X(t)}(x) = F_{X(t+\tau)}(x) = \underbrace{F_X(x)}_{\text{indep. of time.}} \quad \forall t, \tau$

So, $\mu_X(t) = \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx \rightarrow$ independent of time. ✓

- k=2, $\tau = -t_1$: $F_{X(t_1), X(t_2)}(x_1, x_2) = F_{X(0), X(t_2 - t_1)}(x_1, x_2) \quad \forall t_1, t_2$ $\rightarrow$ depends only on the difference

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1,x_2)\, dx_1 dx_2$$ ✓

Ex. $X(t) = A\cos(2\pi f_c t + \Theta)$, $\Theta \sim \mathcal{U}[0, 2\pi)$

$$\left.\begin{array}{l} \mu_X(t) = 0 \\ R_X(t_1,t_2) = \frac{A^2}{2}\cos(2\pi f_c(t_1-t_2)) \end{array}\right\} X(t) \text{ is WSS.}$$

- Cyclostationary random processes: (wide-sense)

$\boxed{\mu_X(t) = \mu_X(t+T_0)}$ T_0 : period

$\boxed{R_X(t_1+T_0, t_2+T_0) = R_X(t_1,t_2)}$

Average autocorrelation function:

$$\bar{R}_X(\tau) = \frac{1}{T_0}\int_0^{T_0} R_X(t+\tau, t)\,dt$$

- Properties of $R_X(\tau)$:

$\boxed{R_X(\tau) = E[X(t+\tau)X^*(t)]}$ $\forall t$

1) $\boxed{R_X(\tau) = R_X^*(-\tau)}$

$$R_X^*(-\tau) = E[X(t)X^*(t-\tau)]$$
$$= E[X(t+\tau)X^*(t)] = R_X(\tau) \checkmark$$

2) $R_X(0)$ is real valued and $\boxed{R_X(0) \geq 0}$

$\boxed{R_X(0) = E[|X(t)|^2]}$

3) $\boxed{|R_X(\tau)| \leq R_X(0)} \leftarrow$ max value

Proof for real $X(t)$:

$$E[|X(t) \pm X(t-\tau)|^2] \geq 0$$

$$E[(X(t) \pm X(t-\tau))(X(t) \pm X(t-\tau))] \geq 0$$

$$E[|X(t)|^2] + E[|X(t-\tau)|^2] \pm E[X(t)X(t-\tau)] \pm E[X(t-\tau)X(t)] \geq 0$$

$$2R_X(0) \pm 2R_X(\tau) \geq 0$$

$$\boxed{-R_X(0) \leq R_X(\tau) \leq R_X(0)} \checkmark$$

- We usually say "stationary" to mean WSS.

Example:

$X(t) = A\cos(2\pi f_0 t)$ $\qquad A \sim \mathcal{U}[0,1]$

$$\mu_X(t) = E[X(t)] = \int_0^1 1.\,A\cos(2\pi f_0 t)\,dA = \cos(2\pi f_0 t)\,\frac{A^2}{2}\Big|_0^1 = \boxed{\frac{1}{2}\cos(2\pi f_0 t)}$$

periodic with period $\frac{1}{f_0}$

$\left(\mu_X(t) = \mu_X(t + \frac{k}{f_0}),\ k \to \text{integer}\right)$

$$R_X(t+\tau, t) = E[X(t+\tau)X^*(t)]$$

$$= E[A^2\cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t)]$$

$$= \cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t)\,E[A^2] \rightsquigarrow \int_0^1 1.\,A^2\,dA = \frac{1}{3}$$

$$= \boxed{\frac{1}{6}\cos(2\pi f_0\tau) + \frac{1}{6}\cos(2\pi f_0(2t+\tau))}$$

$\hookrightarrow$ periodic with period $\frac{1}{2f_0}$

So, $X(t)$ is cyclostationary with $T_0 = \frac{1}{f_0}$. $\quad \left(R_X(t+\tau+\frac{k'}{2f_0}, t+\frac{k'}{2f_0}) = R_X(t+\tau, t)\right)$

$k' \to$ integer

Average autocorrelation function:

$$\bar{R}_X(\tau) = \frac{1}{T_0}\int_0^{T_0} R_X(t+\tau, t)\,dt = f_0\int_0^{1/f_0}\left(\frac{1}{6}\cos(2\pi f_0\tau) + \frac{1}{6}\cos(2\pi f_0(2t+\tau))\right)dt$$

$$= \frac{1}{6}\cos(2\pi f_0\tau) + \frac{f_0}{6}\underbrace{\int_0^{1/f_0}\cos(2\pi f_0(2t+\tau))\,dt}_{0} = \boxed{\frac{1}{6}\cos(2\pi f_0\tau)}$$

Multiple Random Processes:

- $X(t)$ and $Y(t)$ are <u>independent</u> random processes if $(X(t_1), \ldots, X(t_n))$ and $(Y(\tau_1), \ldots, Y(\tau_m))$ are independent for all positive integers m, n, and for all $t_1, \ldots, t_n$ and $\tau_1, \ldots, \tau_m$.
- <u>Cross-correlation fn:</u>

$X(t)$, $Y(t)$

$\downarrow \qquad \downarrow$

$R_X(t,u) \quad R_Y(t,u)$

$$R(t,u) = \begin{bmatrix} R_X(t,u) & R_{XY}(t,u) \\ R_{YX}(t,u) & R_Y(t,u) \end{bmatrix} \leftarrow \text{Correlation matrix}$$

$$R_{XY}(t,u) = E[X(t)Y^*(u)]$$

$$R_{YX}(t,u) = E[Y(t)X^*(u)]$$

For stationary & <u>jointly</u> stationary processes:

$$R(\tau) = \begin{bmatrix} R_X(\tau) & R_{XY}(\tau) \\ R_{YX}(\tau) & R_Y(\tau) \end{bmatrix} \qquad R_{XY}(\tau) = R^*_{YX}(-\tau)$$

$X(t)$ & $Y(t)$ are <u>jointly WSS</u> if
- $X(t)$ & $Y(t)$ are individually WSS
- $R_{XY}(t_1, t_2)$ is only a function of $t_1 - t_2$.

⊛ ($X(t)$ & $Y(t)$ uncorrelated if $X(t_1)$ & $Y(t_2)$ uncorrelated $\forall t_1, t_2$)

Ex. $X_1(t) = X(t)\cos(2\pi f_c t + \Theta)$

$X_2(t) = X(t)\sin(2\pi f_c t + \Theta)$

$\Theta \sim \mathcal{U}[0, 2\pi]$ and indep. of $X(t)$. $\quad X(t) \to$ WSS

$$R_{12}(\tau) = E[X_1(t) X_2(t-\tau)]$$

independence of Θ & $X(t)$

$$= E[X(t)X(t-\tau)]\,E[\cos(2\pi f_c t + \Theta)\sin(2\pi f_c t - 2\pi f_c \tau + \Theta)]$$

$$= \frac{1}{2} R_X(\tau)\,E[\sin(4\pi f_c t - 2\pi f_c \tau + 2\Theta) - \sin(2\pi f_c \tau)]$$

$$= \boxed{-\frac{1}{2} R_X(\tau)\sin(2\pi f_c \tau)}$$

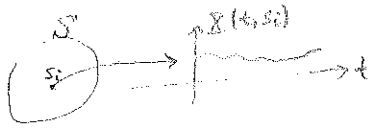
$R_{12}(0) = E[X_1(t)X_2(t)] = 0 \quad \leftarrow X_1(t)$ & $X_2(t)$ are orthogonal

- **Ergodic Processes:**
  - Expectation/ensemble average $\to$ Average across the process
    e.g. avg. of all possible values of sample functions at $t = t_k$; $E[X(t_k)]$.
  - $=$ ← Ergodic
  - Time/long-term (sample) average $\to$ Average along the process

$X(t) \to$ SSS r.p. $\qquad g(x) \to$ a fn.

Ensemble avg. of $g(X(t))$:

$$E[g(X(t))] = \int_{-\infty}^{\infty} g(x) f_{X(t)}(x)\,dx \underset{\text{stationary}}{=} \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$



Time avg. of $g(X(t))$:

Given a sample fn. $X(t, s_i)$

$$\langle g(X(t,s_i))\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g(X(t,s_i))\,dt$$

independent of t but depends on s_i

$\Downarrow$

a random variable!

★ Ergodic if $\langle g(X)\rangle = E[g(X(t))]$

i.e., Ergodic process if $\forall g(\cdot)$, $\forall s \in S$

$$\boxed{\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g(X(t,s))\,dt = \underbrace{\int_{-\infty}^{\infty} g(x) f_X(x)\,dx}_{E[g(X(t,s))]}}$$

• Transmission of a Random Process Thru an LTI Filter:

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

$h(t) \rightarrow$ impulse response

$X(t) \rightarrow$ stationary

$$Y(t)=\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\,d\tau_1$$

$$\mu_Y(t)=E[Y(t)]=E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\,d\tau_1\right]$$

Assuming $E[X(t)]$ is finite $\forall t$ & system is stable

$$=\int_{-\infty}^{\infty} h(\tau_1)\underbrace{E[X(t-\tau_1)]}_{\mu_X(t-\tau_1)}\,d\tau_1$$

$X(t) \rightarrow$ stationary

$$=\mu_X\int_{-\infty}^{\infty} h(\tau_1)\,d\tau_1$$

① So, $\boxed{\mu_Y=\mu_X.\underbrace{H(0)}_{\text{zero-freq. (DC) response of the system.}}}$ $\left(H(f)=\int_{-\infty}^{\infty} h(t)\,e^{-j2\pi ft}\,dt\right)$

$$R_Y(t,u)=E[Y(t)Y(u)]$$

$$=E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\,d\tau_1\int_{-\infty}^{\infty} h(\tau_2)X(u-\tau_2)\,d\tau_2\right]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\underbrace{E[X(t-\tau_1)X(u-\tau_2)]}_{R_X(\underbrace{t-u}_{\tau}-\tau_1+\tau_2)}\,d\tau_1\,d\tau_2$$

$\rightarrow$ depends only on the difference $t-u$.

So, $\tau \triangleq t-u$

② $$\boxed{R_Y(\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau-\tau_1+\tau_2)\,d\tau_1\,d\tau_2}$$ $\rightarrow$ $\boxed{=R_X(\tau)*h(\tau)*h(-\tau)}$

Exercise

By ① & ②, $Y(t) \rightarrow$ stationary.

$$\boxed{E[Y^2(t)]=R_Y(0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_2-\tau_1)\,d\tau_1\,d\tau_2}$$

- $\boxed{\text{Power Spectral Density:}}$ (PSD)

$h(t) \to$ impulse response

$H(f) \to$ frequency response

$$\underset{\text{stationary}}{X(t)} \longrightarrow \boxed{h(t)} \longrightarrow \underset{\text{stationary}}{Y(t)}$$

$$h(t) = \int_{-\infty}^{\infty} H(f)\, e^{j2\pi f t}\, df$$

$$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau_2 - \tau_1)\, d\tau_1\, d\tau_2$$

$$\tau \triangleq \tau_2 - \tau_1$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{h(\tau_2 - \tau)}\, h(\tau_2) R_X(\tau)\, d\tau\, d\tau_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(f)\, e^{j2\pi f \tau_2}\, e^{-j2\pi f \tau}\, h(\tau_2) R_X(\tau)\, df\, d\tau\, d\tau_2$$

$$= \int_{-\infty}^{\infty} H(f) \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f \tau} \underbrace{\int_{-\infty}^{\infty} h(\tau_2)\, e^{j2\pi f \tau_2}\, d\tau_2}_{H^*(f)}\, d\tau\, df$$

$$= \int_{-\infty}^{\infty} |H(f)|^2 \underbrace{\int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f \tau}\, d\tau}_{S_X(f)}\, df$$

$S_X(f) \to$ power spectral density (W/Hz)
(power spectrum)
of stationary process $X(t)$

$$\boxed{E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\, df}$$

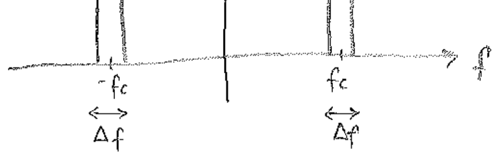
$$E[Y^2(t)] = \int_{-\infty}^{\infty} S_Y(f)\, df = R_Y(0)$$

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f)}$$

$$\boxed{\begin{aligned} S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f \tau}\, d\tau \\ R_X(\tau) &= \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f \tau}\, df \end{aligned}}$$

$$\underbrace{R_X(0) = E[|X(t)|^2]}_{\text{average power}} = \int_{-\infty}^{\infty} S_X(f)\, df$$

Ex.



$$E[Y^2(t)] \approx 2\Delta f\, S_X(f_c)$$

if $S_X(f) \approx S_X(f_c)$ for $|f \mp f_c| < \frac{\Delta f}{2}$

- Properties of PSD:

1) $\boxed{S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau}$

2) $\boxed{E[|X(t)|^2] = \int_{-\infty}^{\infty} S_X(f)\, df = R_X(0)}$ $\rightarrow$ Mean-square value (average power) of $X(t)$.

3) $\boxed{S_X(f) \geq 0 \quad \forall f}$

Previous example : $\underbrace{E[Y^2(t)]}_{\geq 0} \approx 2\Delta f \underbrace{S_X(f_c)}_{\geq 0}$

4) If $X(t)$ is real valued,

$\boxed{S_X(f) = S_X(-f)}$ $\leftarrow$ even fn.

Proof:

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{j2\pi f\tau}\, d\tau \underset{u=-\tau}{=} \int_{-\infty}^{\infty} \underbrace{R_X(-u)}_{=R_X(u)}\, e^{-j2\pi f u}\, du = S_X(f) \quad \checkmark$$

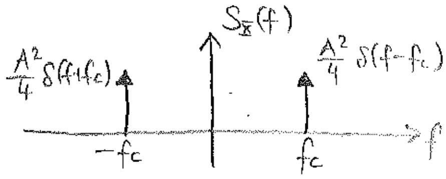
5) $\boxed{P_X(f) \triangleq \dfrac{S_X(f)}{\int_{-\infty}^{\infty} S_X(f)\, df}}$ $\quad \underbrace{P_X(f) \geq 0 \;\; \forall f \quad \& \quad \int P_X(f)\, df = 1}_{\text{PDF properties.}}$

Ex. $X(t) = A\cos(2\pi f_c t + \Theta)$, $\Theta \sim \mathcal{U}[0, 2\pi)$

$$R_X(\tau) = E[A^2 \cos(2\pi f_c(t+\tau) + \Theta)\cos(2\pi f_c t + \Theta)]$$

$$= \boxed{\frac{A^2}{2}\cos(2\pi f_c \tau)}$$

$$\boxed{S_X(f) = \frac{A^2}{4}[\delta(f-f_c) + \delta(f+f_c)]}$$



$$P_X = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df = \boxed{\frac{A^2}{2}}$$

Ex. $Y(t) = X(t)\cos(2\pi f_c t + \Theta)$ $\quad \Theta \sim \mathcal{U}[0, 2\pi)$ $\quad X(t)$ & Θ $\rightarrow$ independent

$$R_Y(\tau) = E[Y(t+\tau)Y(t)]$$

indep. of $X(t)$ & Θ

$$= E[X(t+\tau)X(t)]\, E[\cos(2\pi f_c t + 2\pi f_c \tau + \Theta)\cos(2\pi f_c t + \Theta)]$$

$$= \frac{1}{2} R_X(\tau)\cos(2\pi f_c \tau)$$

$$\boxed{S_Y(f) = \frac{1}{4}[S_X(f-f_c) + S_X(f+f_c)]}$$

- Cross-Spectral Densities:

Specifies frequency inter-relationship between two random processes.

$R_{XY}(\tau)$, $R_{YX}(\tau) \rightarrow$ Cross-correlation functions.

$$\boxed{\begin{aligned} S_{XY}(f) &= \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j2\pi f\tau}\, d\tau \\ S_{YX}(f) &= \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-j2\pi f\tau}\, d\tau \end{aligned}}$$

not necessarily real.

$$\boxed{\begin{aligned} R_{XY}(\tau) &= \int_{-\infty}^{\infty} S_{XY}(f)\, e^{j2\pi f\tau}\, df \\ R_{YX}(\tau) &= \int_{-\infty}^{\infty} S_{YX}(f)\, e^{j2\pi f\tau}\, df \end{aligned}}$$

Since $R_{XY}(\tau) = R_{YX}(-\tau)$, $\leftarrow$ for real processes

$$\boxed{S_{XY}(f) = S_{YX}(-f) = S_{YX}^{*}(f)}$$

Ex. $X(t)$, $Y(t) \rightarrow$ zero-mean, jointly stationary

$Z(t) = X(t) + Y(t) \rightarrow$ Find its PSD.

$$R_Z(t,u) = E[(X(t)+Y(t))(X(u)+Y(u))]$$

$$= E[X(t)X(u)] + E[X(t)Y(u)] + E[Y(t)X(u)] + E[Y(t)Y(u)]$$

jointly stat.

$$\Rightarrow \boxed{R_Z(\tau) = R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau)}$$

$$\boxed{\tau = t-u}$$

$$\boxed{S_Z(f) = S_X(f) + S_{XY}(f) + S_{YX}(f) + S_Y(f)}$$

If $X(t)$ & $Y(t)$ are uncorrelated:

$$R_{XY}(\tau) = E[X(t+\tau)Y(t)]$$

uncorrelated

$$= E[X(t+\tau)]\, E[Y(t)]$$

zero-mean

$$= 0$$

Then, $\boxed{S_Z(f) = S_X(f) + S_Y(f)}$

Ex.

$$\underline{X}(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

$\underline{X}(t)$ & $Y(t)$ $\rightarrow$ jointly stationary

$$R_{XY}(\tau) = E\Big[\underline{X}(t+\tau)\overbrace{\int_{-\infty}^{\infty} h^*(u)\,\underline{X}^*(t-u)\,du}^{Y^*(t)}\Big]$$

$$= \int_{-\infty}^{\infty} h^*(u)\,R_X(\tau+u)\,du$$

$$= \int_{-\infty}^{\infty} h^*(-u)\,R_X(\tau-u)\,du$$

$$= R_X(\tau) * h^*(-\tau)$$

$$\boxed{S_{XY}(f) = S_X(f)\,H^*(f)}$$

Exercise:

Show that $\boxed{S_{YX}(f) = S_X(f)\,H(f)}$

Ex.

$$X(t) \to \boxed{h_1(t)} \to V(t) \qquad Y(t) \to \boxed{h_2(t)} \to Z(t) \qquad S_{VZ}(f) = ?$$

$X(t)$ & $Y(t)$ → jointly stationary

$$R_{VZ}(t,u) = E[V(t)Z(u)]$$

$$= E\left[\int_{-\infty}^{\infty} h_1(\tau_1) X(t-\tau_1)\,d\tau_1 \int_{-\infty}^{\infty} h_2(\tau_2) Y(u-\tau_2)\,d\tau_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1) h_2(\tau_2) \underbrace{E[X(t-\tau_1)Y(u-\tau_2)]}_{R_{XY}(t-u-\tau_1+\tau_2)}\,d\tau_1\,d\tau_2 \quad \leftarrow X(t)\,\&\,Y(t) \text{ jointly stationary.}$$

$\tau = t-u$

$$R_{VZ}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1) h_2(\tau_2) R_{XY}(\tau-\tau_1+\tau_2)\,d\tau_1\,d\tau_2$$

$\tau - \tau_1 = \tilde{\tau}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau-\tilde{\tau})\, h_2(\tau_2)\, R_{XY}(\tilde{\tau}+\tau_2)\,d\tilde{\tau}\,d\tau_2 \quad = h_1(\tau) * \left(h_2(-\tau) * R_{XY}(\tau)\right)$$

$$h_1(t) = \int_{-\infty}^{\infty} H_1(f)\, e^{j2\pi f t}\,df$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underbrace{\int_{-\infty}^{\infty} H_1(f)\, e^{j2\pi f\tau} e^{-j2\pi f\tilde{\tau}}\,df}_{h_1(\tau-\tilde{\tau})}\; h_2(\tau_2) R_{XY}(\tilde{\tau}+\tau_2)\,d\tilde{\tau}\,d\tau_2$$

$$= \int_{-\infty}^{\infty} H_1(f)\, e^{j2\pi f\tau} \int_{-\infty}^{\infty} h_2(\tau_2) \underbrace{\int_{-\infty}^{\infty} R_{XY}(\tilde{\tau}+\tau_2)\, e^{-j2\pi f\tilde{\tau}}\,d\tilde{\tau}}_{S_{XY}(f)\, e^{+j2\pi f\tau_2}}\,d\tau_2\,df$$

$$= \int_{-\infty}^{\infty} H_1(f)\, e^{j2\pi f\tau} S_{XY}(f) \underbrace{\int_{-\infty}^{\infty} h_2(\tau_2)\, e^{+j2\pi f\tau_2}\,d\tau_2}_{H_2^*(f)}\,df$$

$$R_{VZ}(\tau) = \int_{-\infty}^{\infty} \underbrace{H_1(f) H_2^*(f) S_{XY}(f)}_{S_{VZ}(f)}\, e^{j2\pi f\tau}\,df$$

$$\boxed{S_{VZ}(f) = H_1(f)\, H_2^*(f)\, S_{XY}(f)}$$

Gaussian Processes:

- **Definition-1:** $X(t)$ is a Gaussian process if

$$\boxed{Y = \int_0^T g(t) X(t)\,dt}$$ is a Gaussian random variable $\forall g(t)$.

linear functional of $X(t)$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_Y} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

Definition-2: $X(t)$ is a Gaussian process if

$X(t_1), \dots, X(t_n)$ are jointly Gaussian $\forall n,\ \forall (t_1, \dots, t_n) \in \mathbb{R}^n$

- $\underline{X} = [X(t_1) \cdots X(t_n)]^T \rightarrow$ jointly Gaussian $\rightarrow$
  - Any subset is also jointly Gaussian
  - Marginal PDFs are Gaussian
  - Any linear combination is jointly Gaussian
  - Any subset conditioned on any other subset is jointly Gaussian.

$\underline{X} \sim N(\underline{\mu}, \Sigma)$

$\underline{\mu} = [E[X(t_1)] \cdots E[X(t_n)]]^T \longrightarrow$ mean vector

$\Sigma \rightarrow n \times n$ covariance matrix

$$\Sigma_{ij} = C_X(t_i, t_j) = E\Big[\big(X(t_i) - E[X(t_i)]\big)\big(X(t_j) - E[X(t_j)]\big)\Big]$$

$$\boxed{f_{X(t_1),\dots,X(t_n)}(\underbrace{x_1, \dots, x_n}_{\underline{x}}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(\Sigma)}} \exp\left\{-\frac{1}{2}(\underline{x} - \underline{\mu})^T \Sigma^{-1} (\underline{x} - \underline{\mu})\right\}}$$

$\underline{\mu}$ & Σ characterize the PDF completely.

- **Central Limit Theorem:**

$X_1, \dots, X_N \rightarrow$ independent & identically distributed w/ mean μ_X & variance σ_X^2

Then $$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{(X_i - \mu_X)}{\sigma_X} \longrightarrow N(0,1)$$

converges in distribution

- **Properties of Gaussian Processes:**

1) If a Gaussian process is stationary, it is also SSS.

WSS

Proof: WSS $\rightarrow$ $E[X(t)] = \mu_X$ (indep. of time) $\quad C_X(t_i, t_j) = C_X(t_i - t_j)$ (depends only on the difference)

Since $\underline{\mu}$ & Σ completely characterize the joint PDF (see above), all the joint PDFs are invariant to the shifts in the time instants.

2) $\mu_X(t)$ and $R_X(t_1, t_2)$ (or, $C_X(t_1, t_2)$) completely characterize a Gaussian random process.

3) $X(t) \longrightarrow \boxed{\text{Linear stable filter}} \longrightarrow Y(t)$ $Y(t)$ is also a Gaussian process.

$X(t)$: Gaussian process $[0,T]$

Proof: $$Y(t) = \int_0^T X(\tau)\, \underbrace{h_\tau(t)}_{\text{Filter response at time } t \text{ to } \delta(t-\tau)}\, d\tau \qquad 0 \le t < \infty$$

$$\int_0^{T'} g_y(t) Y(t)\, dt = \int_0^{T'} g_y(t) \int_0^T X(\tau) h_\tau(t)\, d\tau\, dt = \int_0^T X(\tau) \underbrace{\int_0^{T'} g_y(t) h_\tau(t)\, dt}_{g(\tau)}\, d\tau$$

$$= \int_0^T X(\tau) g(\tau)\, d\tau \rightarrow \text{Gaussian r.v. } \forall g(\tau) \text{ since } X(t) \text{ is Gaussian r.p.}$$

4) If $X(t_1), \ldots, X(t_n)$ are uncorrelated, they are also statistically independent.

Proof: $E[(X(t_i) - \mu_{X(t_i)})(X(t_j) - \mu_{X(t_j)})] = 0 \quad \forall\, i \neq j$

$\rho_{XY} = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$

Then, $$\Sigma = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} \qquad \sigma_i^2 = E[(X(t_i) - \mu_{X(t_i)})^2]$$

Can show that $$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^n f_{X_i}(x_i)$$

where $X_i = X(t_i)$ & $$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left\{-\frac{1}{2\sigma_i^2}(x_i - \mu_{X_i})^2\right\}$$

5) $X(t)$ & $Y(t)$ are jointly Gaussian processes if

$X(t_1), \ldots, X(t_n), Y(\tau_1), \ldots, Y(\tau_m)$ are jointly Gaussian

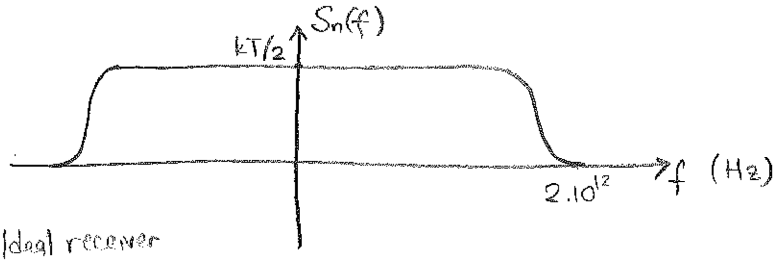
$\forall n, m, \quad \forall (t_1, \ldots, t_n) \in \mathbb{R}^n, \quad \forall (\tau_1, \ldots, \tau_m) \in \mathbb{R}^m$

Thermal Noise:

- Due to thermal agitation, random movement of electrons

$\Rightarrow$ Induced current is sum of currents due to movements of individual electrons

$\Rightarrow$ Gaussian current (Central Limit Theorem)

- 

Ideal receiver

$$\boxed{S_n(f) = \frac{hf}{2\left(e^{hf/(kT)} - 1\right)}}$$

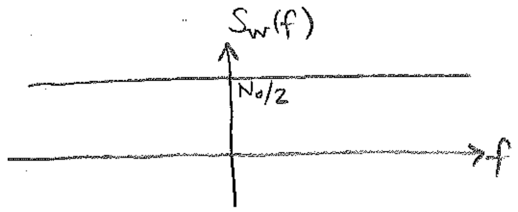
$h = 6.6 \times 10^{-34}$ Jsec (Planck's constant)

$k = 1.38 \times 10^{-23}$ J/K (Boltzman's constant)

T = Temperature in Kelvin

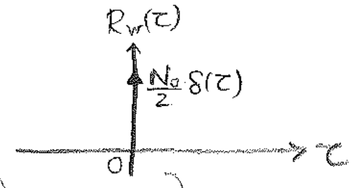
White Noise:

- An idealized stationary noise process. (zero-mean)



$$\boxed{S_w(f) = \frac{N_o}{2}}$$

$$\boxed{R_w(\tau) = \frac{N_o}{2}\delta(\tau)}$$



any two different samples are <u>uncorrelated</u>.

$\boxed{N_o = kT}$ $\rightarrow$ one-sided power spectral density

- Called "white" since it includes all frequencies similar to white light, which includes all frequencies within the visible band.

- $P_w = \int_{-\infty}^{\infty} S_w(f)\,df = \infty$ $\Rightarrow$ White process is <u>NOT</u> physically realizable.

As long as bandwidth of noise at system input >> bandwidth of the system, white noise model can be reasonable.

- If a white noise is <u>also</u> Gaussian, then any two samples at different time instants are independent.

$$E[W(t_1)W(t_2)] = R_w(t_1 - t_2) = 0 \quad \text{for } t_1 \neq t_2 \qquad (\text{Note that } E[W_1]E[W_2] = 0)$$

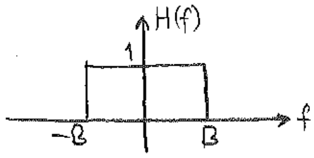
$W_1(t_1)$ & $W(t_2)$ are uncorrelated, hence <u>independent</u> (since Gaussian).

- Thermal noise is commonly modeled as zero-mean, stationary, ergodic, additive white Gaussian noise (AWGN) process.

Ex. $W(t)$: zero-mean, white noise process with $S_W(f) = N_0/2$

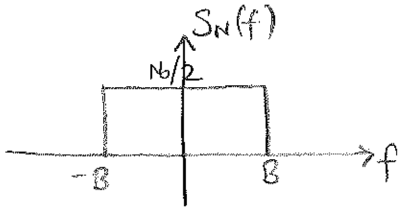
★

$$W(t) \longrightarrow \boxed{H(f)} \longrightarrow N(t)$$

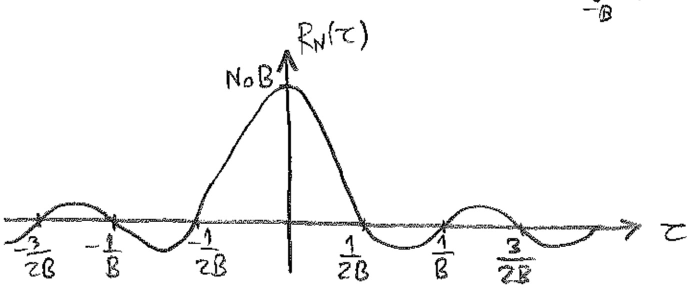


Find the autocorrelation function of $N(t)$.

$$S_N(f) = S_W(f)\,|H(f)|^2 = \begin{cases} N_0/2, & |f| \le B \\ 0, & \text{otherwise} \end{cases}$$



$$R_N(\tau) = \mathcal{F}^{-1}\{S_N(f)\} = \int_{-B}^{B} \frac{N_0}{2} e^{j2\pi f\tau}\, df = \boxed{N_0 B \,\mathrm{sinc}(2B\tau)}$$



$N(t)$ & $N(t + \frac{k}{2B})$ are <u>uncorrelated</u>

for $k = \ldots, -2, -1, 1, 2, \ldots$

{ Note that $E[N(t)] = 0$ }

If $W(t)$ is Gaussian, $N(t)$ is also <u>Gaussian</u>. Hence, its samples at rate $2B$ samples per second are <u>independent</u>.

Ex. $Y = \int_0^T W(t)\, h(t)\, dt$ $\qquad W(t) \rightarrow$ zero-mean white noise w/ $S_W(f) = N_0/2$.

$$E[Y] = \int_0^T h(t)\, E[W(t)]\, dt = \boxed{0}$$

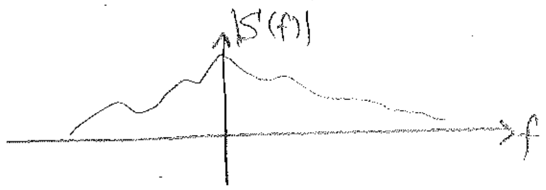
$$E[Y^2] = \int_0^T\!\!\int_0^T h(t_1) h(t_2) \underbrace{E[W(t_1) W(t_2)]}_{\frac{N_0}{2}\delta(t_1 - t_2)} dt_1\, dt_2 = \int_0^T h^2(t_1) \frac{N_0}{2}\, dt_1 = \boxed{\frac{N_0}{2} \int_0^T h^2(t)\, dt}$$

If $W(t)$ is also Gaussian, then

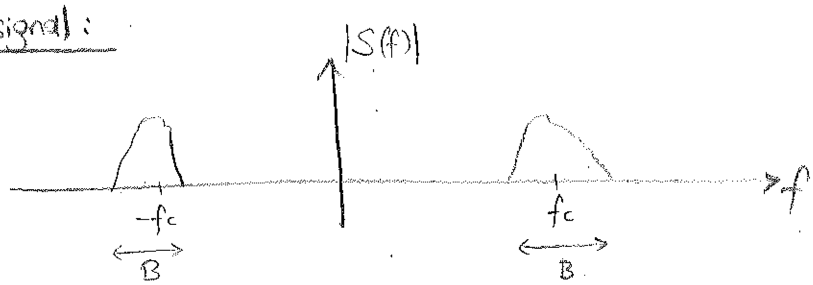
$$\boxed{Y \sim N\left(0, \frac{N_0}{2} \int_0^T h^2(t)\, dt\right)}.$$

Baseband Representation of Deterministic Bandpass Signals:

- Baseband signal (low-pass) : Includes frequencies around zero compared to its highest frequency.



Band-pass signal:

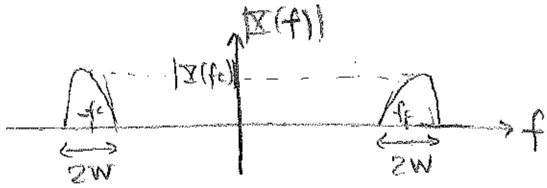


$f_c >> B \rightarrow$ narrowband signal.

- Canonical Representation of Band-pass Signals:

$x(t) \rightarrow$ a narrowband signal with Fourier transform $X(f)$

Real!



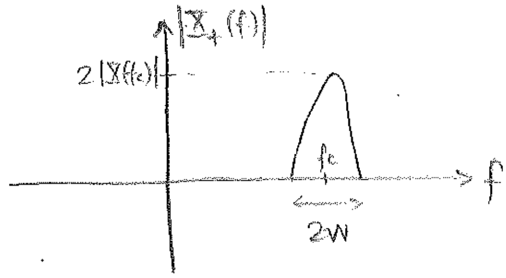
★ The represantation depends on the selection of f_c.

Pre-envelope (analytic signal) of $x(t)$: Signal with only positive frequencies in $x(t)$

$x_+(t)$

$$X_+(f) = 2\,\underbrace{U(f)}_{\text{step fn.}}\,X(f) \qquad \left\{ \delta(t) + \frac{j}{\pi t} \xleftrightarrow{FT} U(f) \right\}$$

$$= \begin{cases} 2X(f), & f>0 \\ X(0), & f=0 \\ 0, & f<0 \end{cases}$$

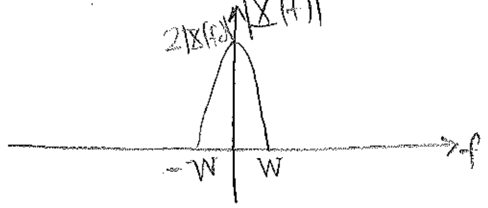


$\hat{x}(t) \Longleftrightarrow -j\,\mathrm{sgn}(f)\,X(f)$

$$= X(f) + \mathrm{sgn}(f) \cdot X(f)$$

$$\boxed{x_+(t) = x(t) + j\,\underbrace{\hat{x}(t)}_{\text{Hilbert transform}}}$$

1) $\boxed{\tilde{x}(t) = x_+(t)\, e^{-j2\pi f_c t}}$ $\longrightarrow$ <u>Complex envelope</u> of $x(t)$ *



$$\tilde{X}(f) = X_+(f + f_c)$$

$x(t) = \mathrm{Re}\{x_+(t)\}$ $\leftarrow$ Prev. page

① $\boxed{x(t) = \mathrm{Re}\{\tilde{x}(t)\, e^{j2\pi f_c t}\}}$

2) $\boxed{\tilde{x}(t) = x_I(t) + j\, x_Q(t)}$

in-phase — quadrature

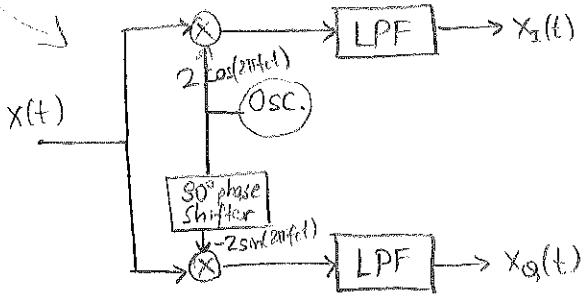
② $\boxed{x(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)}$ Canonical form (standard)

3) $\boxed{\tilde{x}(t) = a(t)\, e^{j\phi(t)}}$ where $a(t) = \sqrt{x_I^2(t) + x_Q^2(t)}$, $\phi(t) = \tan^{-1}\left(\dfrac{x_Q(t)}{x_I(t)}\right)$ $\left(\begin{array}{l} x_I(t) = a(t)\cos(\phi(t)) \\ x_Q(t) = a(t)\sin(\phi(t)) \end{array} \right)$

③ $\boxed{x(t) = a(t)\cos[2\pi f_c t + \phi(t)]}$

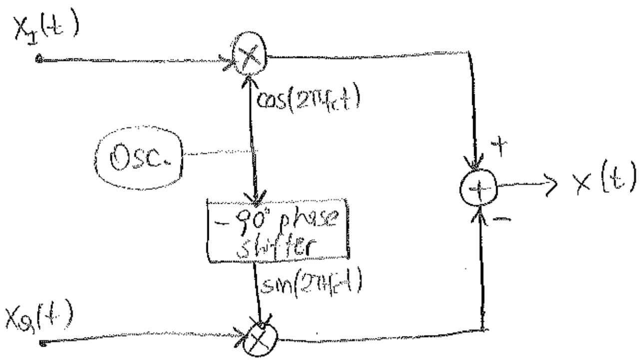
(natural) envelope — phase

$\underline{a(t) = |\tilde{x}(t)| = |x_+(t)|}$ (From 1) & 3))



OR

$$\left\{ \begin{array}{l} \tilde{x}(t) = x_+(t)\, e^{-j2\pi f_c t} \\ x_I(t) + j x_Q(t) = (x(t) + j\hat{x}(t))\, e^{-j2\pi f_c t} \\ \Downarrow \\ x_I(t) = x(t)\cos(2\pi f_c t) + \hat{x}(t)\sin(2\pi f_c t) \\ x_Q(t) = \hat{x}(t)\cos(2\pi f_c t) - x(t)\sin(2\pi f_c t) \end{array} \right\}$$



- $x(t) = Re\{\tilde{x}(t) e^{j2\pi f_c t}\}$

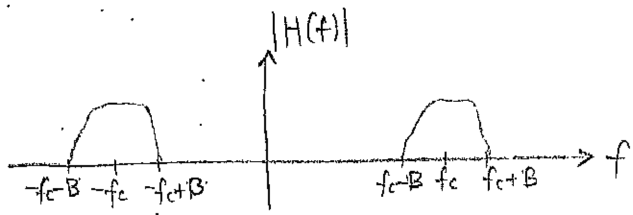
$$= \frac{1}{2}\tilde{x}(t) e^{j2\pi f_c t} + \frac{1}{2}\tilde{x}^*(t) e^{-j2\pi f_c t}$$

$$\boxed{X(f) = \frac{1}{2}\tilde{X}(f-f_c) + \frac{1}{2}\tilde{X}^*(-f-f_c)}$$

- Band-pass Systems:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$x(t)$: narrowband signal (2W)



$$\boxed{B \leq W << f_c}$$

$$h(t) = Re\{\tilde{h}(t) e^{j2\pi f_c t}\}$$

$$= \frac{1}{2}\tilde{h}(t) e^{j2\pi f_c t} + \frac{1}{2}\tilde{h}^*(t) e^{-j2\pi f_c t}$$

$$\boxed{2H(f) = \tilde{H}(f-f_c) + \tilde{H}^*(-f-f_c)}$$

$$Y(f) = X(f)H(f)$$

$$= \frac{1}{4}\left[\tilde{X}(f-f_c) + \tilde{X}^*(-f-f_c)\right]\left[\tilde{H}(f-f_c) + \tilde{H}^*(-f-f_c)\right]$$

$$= \frac{1}{4}\left[\tilde{X}(f-f_c)\tilde{H}(f-f_c) + \tilde{X}^*(-f-f_c)\tilde{H}^*(-f-f_c)\right] = \frac{1}{2}\left[\tilde{Y}(f-f_c) + \tilde{Y}^*(-f-f_c)\right]$$

$$\boxed{\tilde{Y}(f) = \frac{1}{2}\tilde{X}(f)\tilde{H}(f)}$$

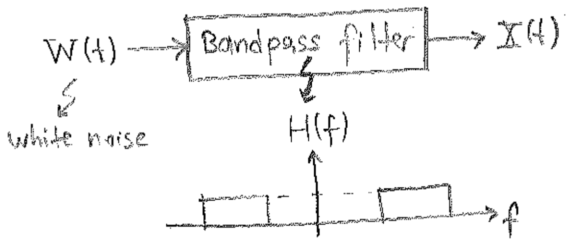
$$\boxed{2\tilde{y}(t) = \tilde{x}(t) * \tilde{h}(t)}$$

$$\tilde{x}(t) \longrightarrow \boxed{\tilde{h}(t)} \longrightarrow 2\tilde{y}(t)$$

*

Filtered Noise Processes:

- Receivers typically have front-end bandpass filters that pass the desired signal undistorted but limit the bandwidth of the white noise.
  $\Rightarrow$ Output of bandpass filter contains bandpass noise process.



$$S_X(f) = S_W(f)\,|H(f)|^2 = \boxed{\frac{N_0}{2}\,|H(f)|^2}$$

- $X(t) = \underbrace{X_c(t)}_{\text{in-phase component}} \cos(2\pi f_c t) - \underbrace{X_s(t)}_{\text{quadrature component}} \sin(2\pi f_c t)$

  $X_c(t), X_s(t), A(t), \theta(t)$ lowpass random processes (baseband)

  Alternatively, $X(t) = A(t)\cos(2\pi f_c t + \theta(t))$

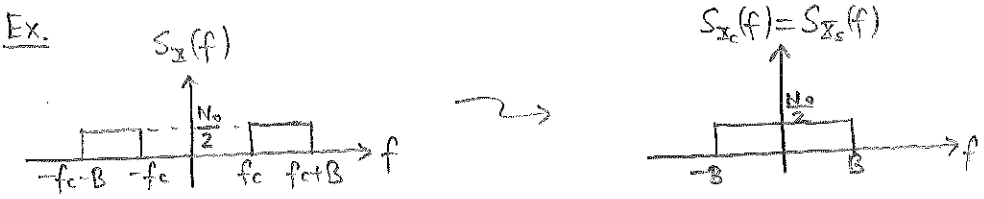
- Properties of In-phase and Quadrature Processes for Filtered White Gaussian Noise:

1) $X_c(t)$ and $X_s(t)$ are zero-mean, lowpass, jointly stationary, and jointly Gaussian random processes.
2) Power in process $X(t)$, $X_c(t)$, and $X_s(t)$ are the same.

$$P_X = P_{X_c} = P_{X_s} = \int_{-\infty}^{\infty} S_X(f)\,df$$

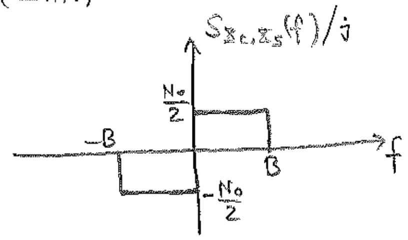
3) $S_{X_c}(f) = S_{X_s}(f) = \begin{cases} S_X(f-f_c) + S_X(f+f_c), & -B \le f \le B \\ 0, & \text{otherwise} \end{cases}$ $\quad B \to$ bandwidth

Ex.



4) $S_{X_c,X_s}(f) = -S_{X_s,X_c}(f) = \begin{cases} j\left(S_X(f+f_c) - S_X(f-f_c)\right), & -B \le f \le B \\ 0, & \text{otherwise} \end{cases}$ $\quad (*)$

Ex. (Cont.)



(For $S_X(f)$ in the previous example)

5) If $S_X(f)$ is even symmetric around f_c, then $X_c(t)$ and $X_s(t)$ are independent processes.

Proof: $S_{X_c,X_s}(f) = 0$ (see $(*)$) $\rightsquigarrow$ $R_{X_c,X_s}(\tau) = 0 = E[X_c(t+\tau)X_s(t)] \rightarrow$ independent since zero-mean Gaussian

- **Noise Equivalent Bandwidth:**

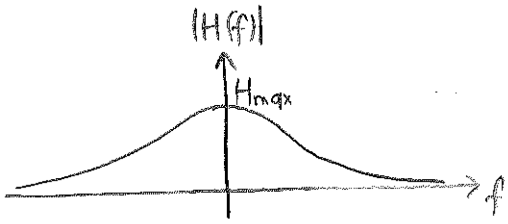
- Noise equivalent bandwidth of a filter with frequency response $H(f)$ is

$$\boxed{B_{neq} = \frac{\int_{-\infty}^{\infty} |H(f)|^2 df}{2 H_{max}^2}}$$

where $H_{max} \rightarrow$ Maximum of $|H(f)|$ in the passband of the filter.

- Ex. $H(f) = \dfrac{1}{1 + j 2\pi f \tau}$

$$|H(f)| = \frac{1}{\sqrt{1 + 4\pi^2 \tau^2 f^2}}$$



$H_{max} = 1$

$$\int_{-\infty}^{\infty} |H(f)|^2 df = \int_{-\infty}^{\infty} \frac{1}{1 + 4\pi^2 \tau^2 f^2} df$$

$u = 2\pi f \tau$

$$= 2 \int_{0}^{\infty} \frac{1}{1+u^2} \frac{du}{2\pi\tau}$$

$$= \frac{1}{\pi\tau} \arctan(u) \Big|_0^{\infty}$$

$$= \frac{1}{\pi\tau} \left(\frac{\pi}{2} - 0 \right)$$

$$= \frac{1}{2\tau}$$

$$B_{neq} = \frac{\frac{1}{2\tau}}{2(1)^2} = \boxed{\frac{1}{4\tau}}$$

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