EEE 431: Telecommunications 1

Quiz 1

March 4, 2017, 10:40-11:40

Instructor: Sinan Gezici

Name: ______________________

Signature: ______________________

Bilkent ID: ______________________

Section: 1 (Thu. lectures) or 2 (Wed. lectures)

Prob. 1: ________ / 16

Prob. 2: ________ / 24

Prob. 3: ________ / 32

Prob. 4: ________ / 28

Total: ________ / 100

Hint: The probability density function (PDF) of a Gaussian random variable Z with mean μ and variance σ^2 is given by $f_Z(z) = (1/\sqrt{2\pi}\sigma)\, e^{-(z-\mu)^2/(2\sigma^2)}$, and the Q-function is defined as $Q(z) = (1/\sqrt{2\pi}) \int_z^\infty e^{-t^2/2}\, dt$.

Problem 1 Consider a signal $x(t)$ which is band limited to $100\,\text{kHz}$. That is, $X(f) = 0$ for $|f| \geq 100\,\text{kHz}$. Consider another signal $y(t)$ which is band limited to $200\,\text{kHz}$. That is, $Y(f) = 0$ for $|f| \geq 200\,\text{kHz}$. Then, find the minimum sampling frequencies for the following signals so that they can perfectly be reconstructed from their samples. Explain your reasoning for each case.

(a) $x(t) + y(t)$

(b) $x(t)y(t)$

Problem 2 Consider a discrete memoryless source (DMS) consisting of 6 symbols, $\{s_1, s_2, s_3, s_4, s_5, s_6\}$, in its alphabet with probabilities 0.13, 0.29, 0.23, 0.1, 0.17, and 0.08, respectively. Perform Huffman coding for this DMS and list the codewords. Calculate the average codeword length.

Problem 3 Consider two random variables X and Y, which are distributed according to the following joint probability density function (PDF):

$$f_{X,Y}(x,y) = \frac{1}{4\pi}\, e^{-(y^2+4x^2+1-2y)/8} \;, \quad (x,y) \in \mathbb{R}^2$$

(a) Find the marginal PDF of X, and the marginal PDF of Y.

(b) Are X and Y independent? Why/why not?

(c) Find the conditional expectation of $3X + 4Y$ given $Y = y$; that is, $E[3X + 4Y \,|\, Y = y]$.

(d) If possible, express the following probability in terms of the Q-function: $P(2X + Y < 3)$.

Problem 4 Let X denote a random variable with the following probability density function (PDF):

$$f_X(x) = \begin{cases} x/50, & \text{if } 0 \leq x \leq 10 \\ 0, & \text{otherwise} \end{cases} .$$

Suppose that X is input to a 4-level quantizer with decision boundaries a_1, a_2, a_3, and reconstruction (quantization) levels $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$, $\hat{x}_4$. (That is, if the input is lower than a_1, the reconstruction level is $\hat{x}_1$; if the input is between a_1 and a_2, the reconstruction level is $\hat{x}_2$; ... and so on.)

Design this quantizer in such a way that when the input X has the PDF specified above, the output of the quantizer is equal to $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$, or $\hat{x}_4$ with equal probabilities; i.e., with probability 0.25 each. In other words, specify a_1, a_2, a_3, $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$, and $\hat{x}_4$ for such a quantizer. (*The answer may not be unique.*)

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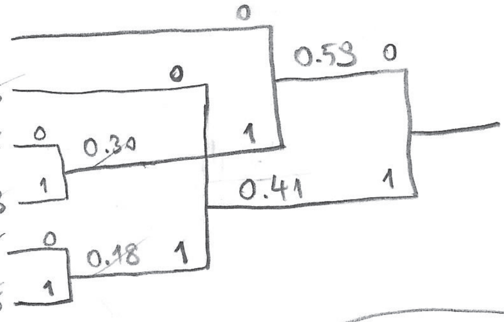
Quiz 1 Sol.

① ⓐ $x(t)+y(t) \rightarrow X(f)+Y(f) \rightsquigarrow$ max. freq. = 200 kHz $2W = \boxed{400\text{ kHz}}$

ⓑ $x(t)y(t) \rightarrow X(f)*Y(f) \rightsquigarrow$ max. freq. = 300 kHz $2W = \boxed{600\text{ kHz}}$

②

Codewords	
00	S_2 : 0.23
10	S_3 : 0.23
010	S_5 : 0.17
011	S_1 : 0.13
110	S_4 : 0.10
111	S_6 : 0.08



$\bar{R} = 2(0.23+0.23)+3(0.17+0.13+0.1+0.08)$

$= 1.04 + 1.44 = \boxed{2.48}$ bit/output

③ ⓐ $f_X(x) = \int_{-\infty}^{\infty} \frac{1}{4\pi} e^{-\frac{x^2}{2}} e^{-\frac{(y-1)^2}{8}} dy = \frac{1}{4\pi} e^{-\frac{x^2}{2}} \sqrt{2\pi}\, 2$ $\quad \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\, 2} e^{-\frac{(y-1)^2}{2\cdot 4}} dy = 1 \right)$

PDF of $N(1,4)$

$= \boxed{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}} \Rightarrow X \sim N(0,1)$

$f_Y(y) = \int_{-\infty}^{\infty} \frac{1}{4\pi} e^{-\frac{x^2}{2}} e^{-\frac{(y-1)^2}{8}} dx = \boxed{\frac{1}{\sqrt{2\pi}\, 2} e^{-\frac{(y-1)^2}{8}}} \Rightarrow Y \sim N(1,4)$

$\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1 \right)$

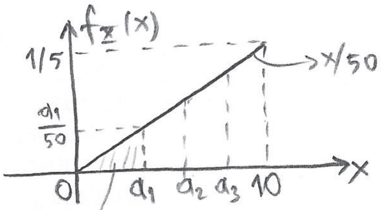
ⓑ Yes since $f_X(x) f_Y(y) = f_{X,Y}(x,y)$

ⓒ $E[3X+4Y \mid Y=y] = 3\underbrace{E[X \mid Y=y]}_{E[X]} + 4y = 3\cdot 0 + 4y = \boxed{4y}$

$E[X] \leftarrow$ since X & Y are independent.

ⓓ $2X+Y \sim N(1,8) \Rightarrow P(2X+Y<3) = 1 - Q\left(\frac{3-1}{\sqrt{8}}\right) = \boxed{1 - Q\left(\frac{2}{\sqrt{8}}\right)} = Q\left(-\frac{1}{\sqrt{2}}\right)$

④



Find a_1, a_2, a_3 s.t. $P(X<a_1) = P(X \in [a_1,a_2]) = P(X \in [a_2,a_3]) = P(X \geq a_3) = \frac{1}{4}$

$\frac{a_1^2}{100} = \frac{1}{4} \Rightarrow a_1 = 5$ $\frac{a_2^2}{100} = \frac{1}{2} \Rightarrow a_2 = 5\sqrt{2}$ $\frac{a_3^2}{100} = \frac{3}{4} \Rightarrow a_3 = 5\sqrt{3}$

$\hat{x}_1 \in (0,5)$ $\hat{x}_2 \in (5, 5\sqrt{2})$ $\hat{x}_3 \in (5\sqrt{2}, 5\sqrt{3})$ $\hat{x}_4 \in (5\sqrt{3}, 10)$