DIGITAL MODULATION METHODS IN AN ADDITIVE WHITE GAUSSIAN NOISE (AWGN) CHANNEL

- Advantages of digital comm. over analog comm.
  - Resistance to channel impairments
  - Higher spectral (bandwidth) efficiency → more data per Hz. of bandwidth
  - Powerful error correction techniques
  - More efficient multiple access strategies
  - Better security and privacy
  - Cheaper implementation

  ⋮

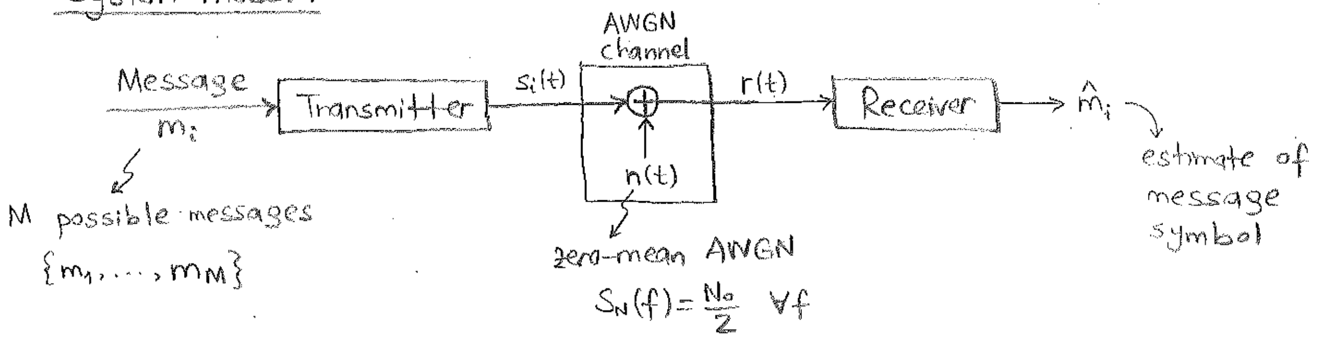
Introduction:

- Insert digital message (information) into waveform → Digital modulation:

$$m_i \longrightarrow s_i(t) \qquad i \in \{1, \dots, M\}$$

Digital demodulation: Extract message from noise corrupted waveform.

- System model:



- Commonly, each message represents a sequence of bits:

$$m = b_1 \cdots b_K \quad \text{where} \quad K = \log_2 M \quad (\text{or, } M = 2^K)$$

- Prior (a priori) probability of message m_i:

$$P_i = P(m_i \text{ sent}) \;,\; i = 1, \dots, M \quad \rightarrow \quad \sum_{i=1}^{M} P_i = 1$$

For equally likely messages (symbols):

$$P_i = \frac{1}{M} \;,\; i = 1, \dots, M \quad \rightarrow \text{equal priors.}$$

- Transmitted signal: $s_1(t), \dots, s_M(t)$ for $t \in [0, T)$ T: symbol interval

- Received signal: $\boxed{r(t) = \underbrace{s_i(t)}_{\text{real-valued energy signal}} + \underbrace{n(t)}_{\text{zero-mean AWGN}}}$ $\quad t \in [0, T]$

$$S_N(f) = \frac{N_0}{2} \quad \forall f$$

Geometric Representation of Signals:

$\rightsquigarrow$ Represent signals in a compact form and provide geometric intuition.

- Consider M energy signals $s_1(t), \ldots, s_M(t)$ for $t \in [0,T)$

  Express them as linear combinations of N orthonormal basis functions, $\psi_1(t), \ldots, \psi_N(t)$, where $N \leq M$.

$$\left.\begin{array}{lll} - \int_0^T \psi_i(t)\psi_j(t)\,dt = 0 & \forall i \neq j & \longrightarrow \text{Orthogonal} \\ - \int_0^T \psi_i^2(t)\,dt = 1 & \forall i & \longrightarrow \text{Normalized} \end{array}\right\} \text{Orthonormal basis fn.s}$$

Basis function representation of $s_1(t), \ldots, s_M(t)$:

$$(*) \quad \left.\begin{array}{ll} \boxed{s_i(t) = \sum_{j=1}^{N} s_{ij}\psi_j(t)} \quad t \in [0,T), & i = 1, \ldots, M \\ \boxed{s_{ij} = \int_0^T s_i(t)\psi_j(t)\,dt} \triangleq \langle s_i(t), \psi_j(t) \rangle, & i = 1, \ldots, M \\ & j = 1, \ldots, N \end{array}\right\} \quad s_i(t) = \sum_{j=1}^{N} \langle s_i(t), \psi_j(t) \rangle \psi_j(t)$$

- When $s_1(t), \ldots, s_M(t)$ can be represented as in $(*)$, we say that the basis functions $\psi_1(t), \ldots, \psi_N(t)$ span the set $\mathcal{S} = \{s_1(t), \ldots, s_M(t)\}$; or, we say that $s_i(t)$ resides in the signal space specified by $\psi_1(t), \ldots, \psi_N(t)$, $\forall i$.

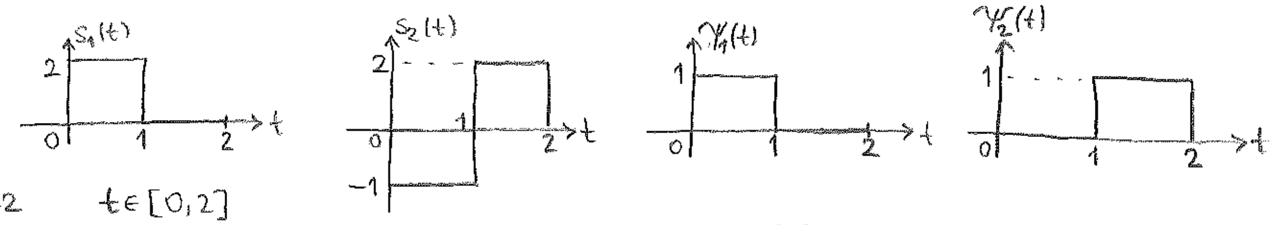
- Represent $s_i(t)$ as an N-dimensional vector:

$$\underline{s}_i = \begin{bmatrix} s_{i1} \\ \vdots \\ s_{iN} \end{bmatrix} \rightsquigarrow \text{Signal constellation point} \rightsquigarrow \text{Has one-to-one relationship with } s_i(t).$$

- Signal space: N-dimensional vector space with N perpendicular axes labeled as $\psi_1, \ldots, \psi_N$.

  $\underline{s}_1, \ldots, \underline{s}_M$ define M points in this signal space.

- Ex.

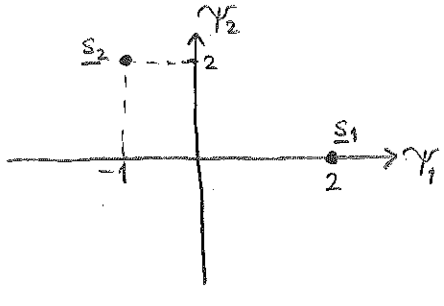


$M = N = 2 \qquad t \in [0,2]$

$$s_1(t) = 2\psi_1(t) + 0\,\psi_2(t)$$

$$s_2(t) = -1\,\psi_1(t) + 2\psi_2(t)$$

$$\underline{s}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \underline{s}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



Signal space representation

- Relations between function and vector operations:

$$\boxed{\langle f(t), g(t)\rangle \triangleq \int_{-\infty}^{\infty} f(t)g(t)\,dt} \rightsquigarrow \text{Inner product of two functions}$$

$$\underline{x}^T\underline{y} = \sum_i x_i y_i \rightsquigarrow \text{Inner product of two vectors.}$$

- $\boxed{E_i = \int_0^T s_i^2(t)\,dt = \|\underline{s}_i\|^2}$ $\qquad (\|\underline{s}_i\|^2 = \underline{s}_i^T\underline{s}_i)$

Proof: $$E_i = \int_0^T \Big(\sum_{j=1}^N s_{ij}\psi_j(t)\Big)^2 dt = \sum_{j=1}^N \sum_{k=1}^N s_{ij}s_{ik} \underbrace{\int_0^T \psi_j(t)\psi_k(t)\,dt}_{=\begin{cases}1, & j=k\\ 0, & j\neq k\end{cases}} = \sum_{j=1}^N s_{ij}^2 = \|\underline{s}_i\|^2 \checkmark$$

- $\boxed{\langle s_i(t), s_j(t)\rangle = \underline{s}_i^T\underline{s}_j}$

Proof: $$\langle s_i(t) s_j(t)\rangle = \int_0^T s_i(t)s_j(t)\,dt = \int_0^T \sum_{k=1}^N s_{ik}\psi_k(t) \sum_{l=1}^N s_{jl}\psi_l(t)\,dt$$

$$= \sum_{k=1}^N \sum_{l=1}^N s_{ik}s_{jl} \int_0^T \psi_k(t)\psi_l(t)\,dt = \sum_{k=1}^N s_{ik}s_{jk} = \underline{s}_i^T\underline{s}_j \checkmark$$

- $\boxed{\int_0^T (s_i(t)-s_j(t))^2\,dt = \|\underline{s}_i - \underline{s}_j\|^2} = \sum_{k=1}^N (s_{ik}-s_{jk})^2$

- Angle between two signals:

$$\underline{s}_i^T\underline{s}_j = \|\underline{s}_i\|\|\underline{s}_j\|\cos\theta_{ij} \Rightarrow \langle s_i(t), s_j(t)\rangle = \sqrt{E_i}\sqrt{E_j}\cos\theta_{ij}$$

$$\boxed{\cos\theta_{ij} = \frac{\langle s_i(t), s_j(t)\rangle}{\sqrt{E_iE_j}}}$$

- Gram-Schmidt Orthonormalization:

A technique for constructing an orthonormal basis for a given set of energy signals, $s_1(t), \ldots, s_M(t)$, for $t \in [0,T]$

Step-1: $\boxed{\psi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}}$, $\boxed{E_1 = \int_0^T s_1^2(t)\,dt}$

Step-2: $s_{21} = \int_0^T s_2(t)\psi_1(t)\,dt$

$d_2(t) = s_2(t) - s_{21}\psi_1(t)$

$$\boxed{\psi_2(t) = \frac{d_2(t)}{\sqrt{\int_0^T d_2^2(t)\,dt}}}$$

⋮

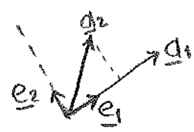
Step-i: $s_{ij} = \int_0^T s_i(t)\psi_j(t)\,dt, \quad j=1,\ldots,i-1$

$d_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\psi_j(t)$

$$\boxed{\psi_i(t) = \frac{d_i(t)}{\sqrt{\int_0^T d_i^2(t)\,dt}}}$$

Notation in textbook $s_{ij} \longleftrightarrow c_{ij}$

Similar to vector case:



$$\underline{e}_1 = \frac{\underline{a}_1}{\|\underline{a}_1\|}, \quad \underline{e}_2 = \frac{\underline{a}_2 - (\underline{a}_2^T\underline{e}_1)\underline{e}_1}{\|\underline{a}_2 - (\underline{a}_2^T\underline{e}_1)\underline{e}_1\|}, \ldots$$

$\leftarrow$ Continue until $i=M$, and consider only non-zero basis functions at the end.

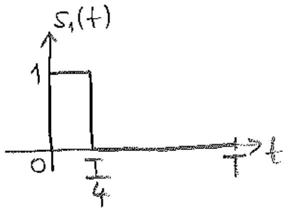
$\psi_1(t), \ldots, \psi_N(t), \quad N \leq M.$ $\quad$ ($N=M$ iff $s_1(t), \ldots, s_M(t)$ are linearly independent)

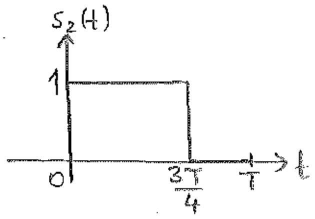
N : # non-zero $\{d_i(t)\}$.

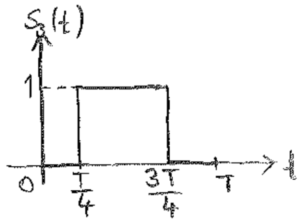
★ Note that the choice of orthonormal basis $\{\psi_i(t)\}_{i=1}^{N}$ is not unique.
But that choice does not affect the dimension of the signal space, and the norms and inner products of the vectors.

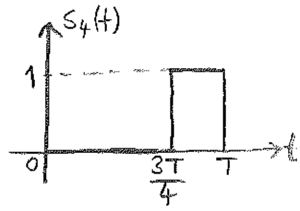
Example:

Find a set of orthonormal basis functions for the following signals.



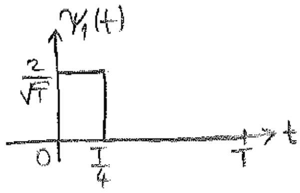






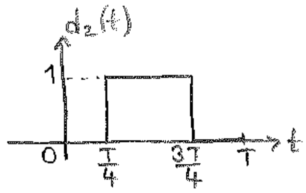
- $E_1 = \int_0^{T/4} (1)^2\, dt = \frac{T}{4}$

$$\psi_1(t) = \frac{s_1(t)}{\sqrt{\frac{T}{4}}} = \boxed{\frac{2}{\sqrt{T}} s_1(t)}$$

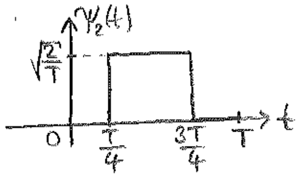


- $s_{21} = \int_0^T s_2(t)\psi_1(t)\, dt = \int_0^{T/4} 1 \cdot \frac{2}{\sqrt{T}}\, dt = \frac{\sqrt{T}}{2}$

$$d_2(t) = s_2(t) - s_{21}\psi_1(t) = s_2(t) - \frac{\sqrt{T}}{2}\psi_1(t) = s_2(t) - s_1(t) \rightarrow$$



$$\psi_2(t) = \frac{d_2(t)}{\sqrt{\int_0^T d_2^2(t)\, dt}} = \boxed{\frac{s_2(t) - s_1(t)}{\sqrt{\frac{T}{2}}}} \rightarrow$$



- $s_{31} = \int_0^T s_3(t)\psi_1(t)\, dt = 0$

$s_{32} = \int_0^T s_3(t)\psi_2(t)\, dt = \sqrt{\frac{T}{2}}$

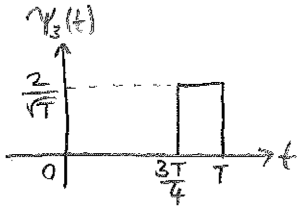
$d_3(t) = s_3(t) - \cancel{s_{31}\psi_1(t)}^{\,0} - s_{32}\psi_2(t) = s_3(t) - \sqrt{\frac{T}{2}}\psi_2(t) = \boxed{0}$ → No new basis function at this step!

- $s_{41} = \int_0^T s_4(t)\phi_1(t)\, dt = 0$

$s_{42} = \int_0^T s_4(t)\phi_2(t)\, dt = 0$

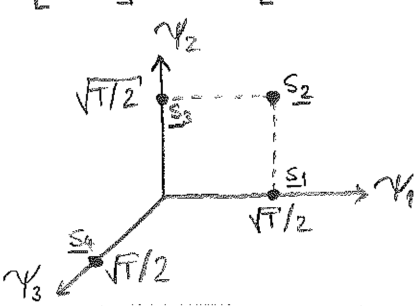
$d_4(t) = s_4(t) - s_{41}\psi_1(t) - s_{42}\psi_2(t) = s_4(t)$

So, $\psi_3(t) = \frac{s_4(t)}{\sqrt{\int_0^T s_4^2(t)\, dt}} = \boxed{\frac{2}{\sqrt{T}} s_4(t)}$



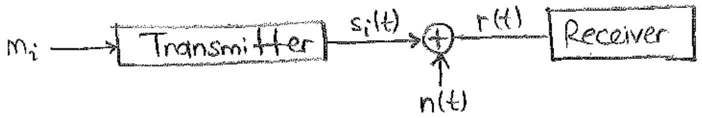
Overall,

$$\underline{s_1} = \begin{bmatrix} \sqrt{T}/2 \\ 0 \\ 0 \end{bmatrix} \quad \underline{s_2} = \begin{bmatrix} \sqrt{T}/2 \\ \sqrt{T}/2 \\ 0 \end{bmatrix} \quad \underline{s_3} = \begin{bmatrix} 0 \\ \sqrt{T}/2 \\ 0 \end{bmatrix} \quad \underline{s_4} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{T}/2 \end{bmatrix}$$



Optimal Receiver Structure for AWGN Channels:

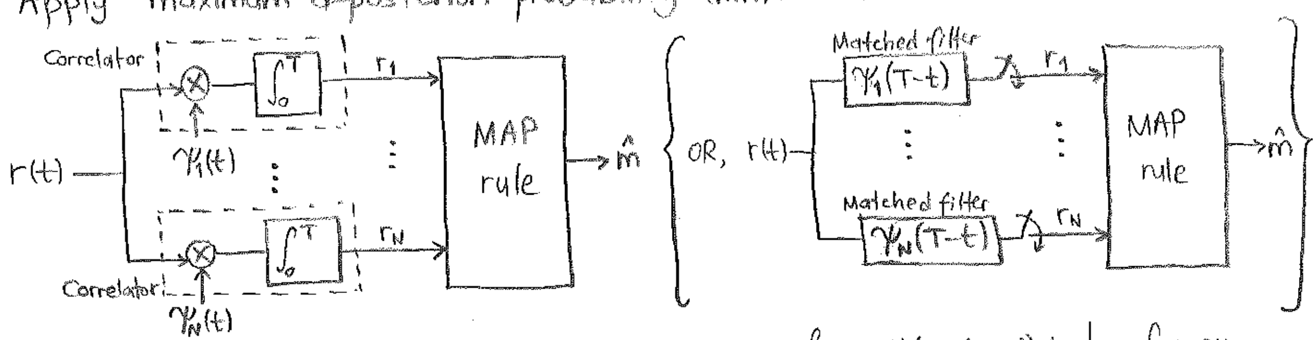
- Consider a communications system in which one of M possible messages is transmitted via signals $s_1(t), \dots, s_M(t)$, and the channel is modeled as AWGN channel with $S_N(f) = N_0/2 \quad \forall f$.



$i \in \{1, \dots, M\}$

- For the optimal receiver design:
  - Obtain a set of orthonormal basis functions $\psi_1(t), \dots, \psi_N(t)$ for signals $s_1(t), \dots, s_M(t)$.
  - Obtain projection of $r(t)$ onto the signal space specified by $\psi_1(t), \dots, \psi_N(t)$.
  - Apply maximum a-posteriori probability (MAP) decision rule to estimate the message

*



$$r_j = \int r(\tau)\psi_j(T-(t-\tau))\,d\tau \Big|_{t=T} = \int r(\tau)\psi_j(\tau)\,d\tau$$

- $\boxed{r(t) = s_i(t) + n(t)}, \quad t \in [0, T)$

$$r_j = \int_0^T r(t)\psi_j(t)\,dt = \int_0^T (s_i(t)+n(t))\psi_j(t)\,dt = \underbrace{\int_0^T s_i(t)\psi_j(t)\,dt}_{s_{ij}} + \underbrace{\int_0^T n(t)\psi_j(t)\,dt}_{n_j}$$

$$\boxed{r_j = s_{ij} + n_j}$$

- Statistics of $\underline{r} = [r_1 \cdots r_N]^T$:

$$\begin{bmatrix} r_1 \\ \vdots \\ r_N \end{bmatrix} = \begin{bmatrix} s_{i1} \\ \vdots \\ s_{iN} \end{bmatrix} + \begin{bmatrix} n_1 \\ \vdots \\ n_N \end{bmatrix} \longrightarrow \boxed{\underline{r} = \underline{s}_i + \underline{n}}$$

$\underline{r}$: Observation vector

$$E\{n_j\} = E\left\{\int_0^T n(t)\psi_j(t)\,dt\right\} = \int_0^T E\{n(t)\}\psi_j(t)\,dt = \boxed{0} \quad \forall j$$

$$E\{n_j n_k\} = E\left\{\int_0^T n(t_1)\psi_j(t_1)\,dt_1 \int_0^T n(t_2)\psi_k(t_2)\,dt_2\right\}$$

$$= \int_0^T\int_0^T \underbrace{E\{n(t_1)n(t_2)\}}_{\frac{N_0}{2}\delta(t_1-t_2)}\psi_j(t_1)\psi_k(t_2)\,dt_1\,dt_2$$

$$= \frac{N_0}{2}\int_0^T \psi_j(t)\psi_k(t)\,dt = \begin{cases} N_0/2, & \text{if } j=k \\ 0, & \text{if } j \neq k \end{cases}$$

$$\left\{ \begin{array}{l} n(t) \to \text{zero-mean AWGN} \\ S_N(f) = \frac{N_0}{2}, \ \forall f \\ R_N(\tau) = \frac{N_0}{2}\delta(\tau) \\ \int_0^T n(t)\phi_j(t)\,dt \to \text{Gaussian r.v.} \end{array} \right\}$$

So, $\boxed{\underline{n} \sim N\left(\underline{0}, \frac{N_0}{2} I\right)} \rightarrow$ ($n_1, \dots, n_N$ are i.i.d. Gaussian with $N(0, N_0/2)$.)

Hence, $\boxed{\underline{r} \mid \underline{s}_i \sim N\left(\underline{s}_i, \frac{N_0}{2} I\right)}$

(or, $\underline{r} \mid m_i$)

{i.i.d. : Independent & identically distributed}

{$I \to N \times N$ identity matrix}

- Although $r(t)$ cannot be reconstructed perfectly from $\underline{r}$, $\underline{r}$ carries all the information related to the transmitted message.

$\underline{r} \longrightarrow$ Sufficient statistics

$$r(t) = s_i(t) + n(t) = \underbrace{\sum_{j=1}^{N} s_{ij}\psi_j(t)}_{s_i(t)} + \overbrace{\underbrace{\sum_{j=1}^{N} n_j \psi_j(t)}_{\text{Projection of noise on the signal space}} + \underbrace{n'(t)}_{\text{Remainder noise}}}^{n(t)}$$

Remainder noise $\rightsquigarrow$ Orthogonal to the signal space.

$$= \sum_{j=1}^{N} (s_{ij} + n_j)\psi_j(t) + n'(t)$$

$$(+) \quad \boxed{r(t) = \sum_{j=1}^{N} r_j \psi_j(t) + n'(t)} \qquad \underline{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_N \end{bmatrix}$$

$\left\{ \sum_{j=1}^{N} r_j \psi_j(t) \rightarrow \text{Projection of } r(t) \text{ onto the signal space} \right\}$ *

- Why is $\underline{r}$ sufficient statistics?

Because $n'(t)$ is independent of $r_1, \dots, r_N$; hence, carries no information related to which message is transmitted (see (+)).

Proof:

$$E\{n'(t) r_j\} = E\left\{\left(n(t) - \sum_{\ell=1}^{N} n_\ell \psi_\ell(t)\right)(s_{ij} + n_j)\right\}$$

$$= s_{ij}\, E\left\{n(t) - \sum_{\ell=1}^{N} n_\ell \psi_\ell(t)\right\} + E\left\{\underbrace{n(t) n_j} - \sum_{\ell=1}^{N} n_\ell n_j \psi_\ell(t)\right\}$$

$$= s_{ij}\left(E\{n(t)\} - \sum_{\ell=1}^{N} E\{n_\ell\}\psi_\ell(t)\right) + \int_0^T \underbrace{E\{n(t)n(\tau)\}}_{\frac{N_0}{2}\delta(t-\tau)}\psi_j(\tau)\,d\tau - \sum_{\ell=1}^{N} \underbrace{E\{n_\ell n_j\}}_{\begin{cases} N_0/2, & \ell = j \\ 0, & \ell \neq j \end{cases}}\psi_\ell(t)$$

$$= s_{ij}(0 - 0) + \frac{N_0}{2}\psi_j(t) - \frac{N_0}{2}\psi_j(t)$$

$$= \underline{0} \qquad \forall t,\ j \in \{1, .., N\}$$

Since, $E\{n'(t)\} = E\left\{n(t) - \sum_{j=1}^{N} n_j \psi_j(t)\right\} = 0$, we obtain $\boxed{E\{n'(t) r_j\} = E\{n'(t)\} E\{r_j\} = 0}$

Therefore $n'(t)$ and r_j are uncorrelated for all t and j.

Since they are also Gaussian, $n'(t)$ are r_j are independent $\forall\, t, j$.

- Equivalent vector channel model:

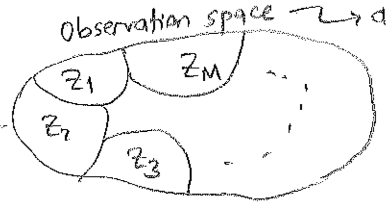
$\boxed{\underline{r} = \underline{s}_i + \underline{n}}$, $\quad \underline{n} \sim N\left(\underline{0}, \frac{N_0}{2} I\right)$, $\quad i \in \{1, \dots, M\}$

Given $\underline{r}$, find a decision rule that maps $\underline{r}$ into $\hat{m}$ in such a way that the average probability of error is minimized.

$\boxed{\underbrace{\hat{m} = g(\underline{r})}_{\text{decision rule (detector)}}}$ $\qquad g: \mathbb{R}^N \rightarrow \{1, .., M\}$

Observation space $\rightsquigarrow$ all possible $\underline{r}$ values.

$Z_1, \dots, Z_M \rightarrow$ Decision regions



$\boxed{\text{Decide } \hat{m} = m_i \text{ if } \underline{r} \in Z_i}$

- MAP Decision Rule:

$$\boxed{\text{Choose } \hat{m}=m_i \text{ if } P(m_i \text{ sent}\,|\,\underline{r}) \geq P(m_j \text{ sent}\,|\,\underline{r}) \text{ for all } j\neq i}$$

equivalent to

A-posteriori probability: (posterior)
$$P(m_i \text{ sent}\,|\,\underline{r}) = \frac{p(\underline{r}|m_i)\overbrace{P(m_i)}^{P_i}}{p(\underline{r})}$$

$P_i = P(m_i) \rightarrow$ Prior probability

$$\boxed{\text{Choose } \hat{m}=m_i \text{ if } P_i\, p(\underline{r}|m_i) \geq P_j\, p(\underline{r}|m_j) \text{ for all } j\neq i}$$

In other words, choose $\hat{m}=m_i$ if $\underline{r}\in Z_i$, where

$$Z_i = \{\underline{r} : P_i\, p(\underline{r}|m_i) \geq P_j\, p(\underline{r}|m_j) \quad \forall j \neq i\}, \quad i=1,\ldots,M.$$

- Maximum Likelihood (ML) Decision Rule:

Special case of MAP rule for equal priors ($P_i = \frac{1}{M}$, $i=1,\ldots,M$)

$$\boxed{\text{Choose } \hat{m}=m_i \text{ if } p(\underline{r}|m_i) \geq p(\underline{r}|m_j) \text{ for all } j\neq i}$$

equivalent to

Likelihood function: $\boxed{p(\underline{r}|m_i) \triangleq L(m_i)}$ $\rightarrow$ likelihood of m_i (conditional probability of $\underline{r}$ given m_i)

Log-likelihood function: $\boxed{\ell(m_i) \triangleq \log p(\underline{r}|m_i)}$ $\rightarrow$ easier to deal with.

$$\boxed{\text{Choose } \hat{m}=m_i \text{ if } \ell(m_i) \geq \ell(m_j) \text{ for all } j\neq i}$$

- ML based receiver for AWGN channel:

$$\underline{r} = \underline{s_i} + \underline{n}, \quad \underline{n} \sim N\left(0, \frac{N_0}{2}\underline{I}\right)$$

$$P_{\underline{N}}(\underline{n}) = \prod_{l=1}^{N} \frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0}{2}}} e^{-\frac{n_l^2}{2\cdot\frac{N_0}{2}}} = (\pi N_0)^{-\frac{N}{2}} \exp\left\{-\frac{1}{N_0}\sum_{j=1}^{N} n_j^2\right\} = (\pi N_0)^{-\frac{N}{2}} e^{-\frac{\|\underline{n}\|^2}{N_0}}$$

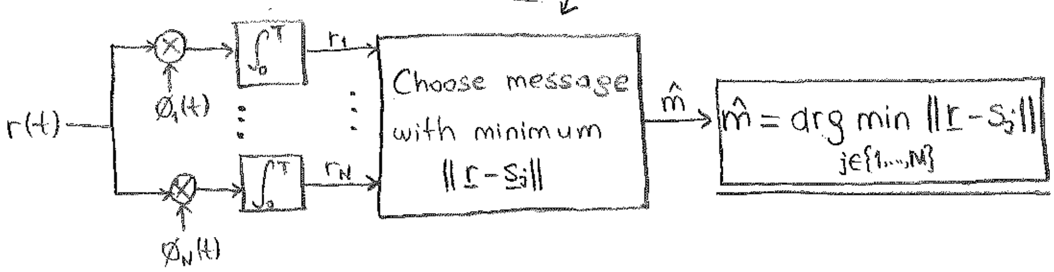
$$p(\underline{r}|m_i) = P_{\underline{N}}(\underline{r}-\underline{s_i}) = \boxed{(\pi N_0)^{-\frac{N_0}{2}} e^{-\frac{\|\underline{r}-\underline{s_i}\|^2}{N_0}}}$$

$$\ell(m_i) = \log p(\underline{r}|m_i) = -\frac{N_0}{2}\log(\pi N_0) - \frac{1}{N_0}\|\underline{r}-\underline{s_i}\|^2$$

ML rule: Choose m_i if $\ell(m_i)$ is the max of $\ell(m_1),\ldots,\ell(m_M)$.

★ Equivalently, choose m_i if $\|\underline{r}-\underline{s_i}\|$ is the minimum of $\|\underline{r}-\underline{s_1}\|,\ldots,\|\underline{r}-\underline{s_M}\|$.

$\Rightarrow$ Minimum distance decision rule.



- **Theorem:** MAP rule minimizes the probability of error.

Probability of error (P_e): $\boxed{P_e = \sum_{i=1}^{M} p_i \, P(\hat{m} \neq m_i \mid m_i \text{ sent})} \triangleq \sum_{i=1}^{M} p_i P_{e,i}$

Probability of correct decision $\rightarrow$ $P_c = \sum_{i=1}^{M} p_i \, P(\hat{m} = m_i \mid m_i \text{ sent}) \triangleq \sum_{i=1}^{M} p_i P_{c,i}$ (of course, $P_{c,i} = 1 - P_{e,i}$, $P_c = 1 - P_e$)

Proof: Show that MAP rule maximizes P_c.

$$P_c = \sum_{i=1}^{M} p_i \int_{Z_i} p(\underline{r} \mid m_i)\, d\underline{r} = \sum_{i=1}^{M} \int_{Z_i} p_i \, p(\underline{r} \mid m_i)\, d\underline{r}$$

For each $\underline{r}$, assign $\underline{r}$ to Z_i if $p_i \, p(\underline{r} \mid m_i) \geq p_j \, p(\underline{r} \mid m_j)$ $\forall j \neq i$ to maximize P_c.

$\Rightarrow$ MAP rule maximizes P_c, and minimizes P_e. ✓

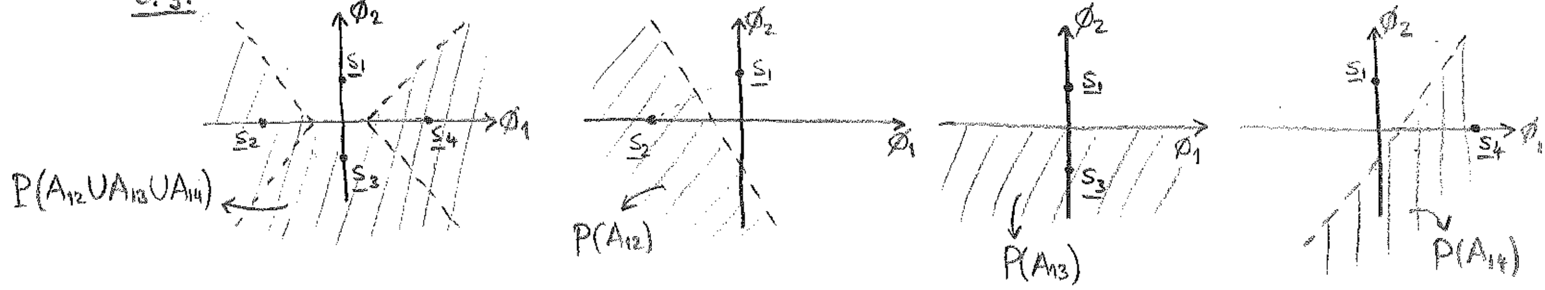
Union Bound on the Probability of Error:

- When it is difficult to calculate an exact expression for P_e, the union bound can be used to obtain an upper bound.
- Define event A_{ik} as follows:

A_{ik} : Event that observation vector $\underline{r}$ is closer to $\underline{s}_k$ than $\underline{s}_i$ when $\underline{s}_i$ (m_i) is sent.

$$P_{e,i} = P\Big(\bigcup_{\substack{k=1 \\ k \neq i}}^{M} A_{ik}\Big) \leq \sum_{\substack{k=1 \\ k \neq i}}^{M} \underbrace{P(A_{ik})}_{\text{pairwise error prob.}}, \quad i = 1, \dots, M \qquad \leftarrow \text{Union bound}$$

e.g.



$$P(A_{12} \cup A_{13} \cup A_{14}) \leq P(A_{12}) + P(A_{13}) + P(A_{14})$$

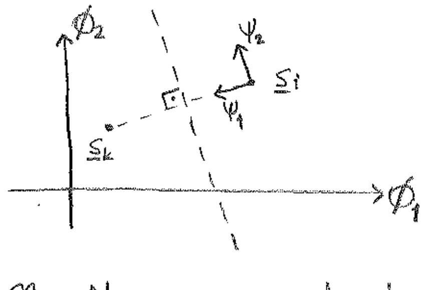
- Pairwise error probabilities are easy to obtain.

$$P(A_{ik}) = P(\|\underline{r} - \underline{s}_k\| < \|\underline{r} - \underline{s}_i\| \mid m_i)$$

$\eta \sim N(0, \frac{N_0}{2})$ $\quad = P\left(\eta_1 > \frac{\|\underline{s}_i - \underline{s}_k\|}{2}\right)$

$N(0,1) \leftarrow$ $\quad = P\left(\frac{\eta_1}{\sqrt{N_0/2}} > \frac{\|\underline{s}_i - \underline{s}_k\|}{2\sqrt{N_0/2}}\right)$

$$\boxed{P(A_{ik}) = Q\left(\frac{\|\underline{s}_i - \underline{s}_k\|}{\sqrt{2N_0}}\right)}$$



η_1 : Noise component along ψ_1.

So, $\boxed{P_{e,i} \leq \sum_{\substack{k=1 \\ k \neq i}}^{M} Q\left(\frac{\|\underline{s}_i - \underline{s}_k\|}{\sqrt{2N_0}}\right)}$ ★ Union bound for conditional probability of error when m_i is sent.

Hence, $P_e = \sum_{i=1}^{M} P_i P_{e,i} \Rightarrow \boxed{P_e \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k\neq i}}^{M} Q\left(\frac{\|\underline{s}_i - \underline{s}_k\|}{\sqrt{2N_0}}\right)}$ * Union bound for the probability of error.

- A looser version of union bound:

$$d_{min} = \min_{i,k} \|\underline{s}_i - \underline{s}_k\|$$

$$Q\left(\frac{\|\underline{s}_i - \underline{s}_k\|}{\sqrt{2N_0}}\right) \leq Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right) \quad \forall i,k \quad \text{since } Q(\cdot) \text{ is a monotone decreasing fn.}$$

Then, $\boxed{P_e \leq (M-1)\, Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)}$.

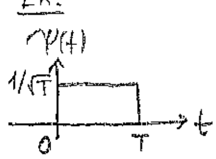
M-ARY PULSE AMPLITUDE MODULATION (PAM)

- Information is carried by the signal amplitude

$\boxed{s_i(t) = A_i\, p(t) = s_i \psi(t)}$, $\quad i=1,\dots,M, \quad t \in [0,T]$

$\boxed{\psi(t) = \frac{p(t)}{\sqrt{E_p}}}$ where $E_p = \int_{-\infty}^{\infty} |p(t)|^2 dt$ ⇝ Energy of $p(t)$

Ex.



Hence, $\boxed{s_i = A_i \sqrt{E_p}}$

- 1-dimensional signal space: $\underline{s}_i = [A_i \sqrt{E_p}]$; $i=1,\dots,M$
- Commonly, $\boxed{A_i = (2i-1-M)A}$, $i=1,\dots,M \Rightarrow s_i = (2i-1-M)\underbrace{A\sqrt{E_p}}_{\triangleq d} = (2i-1-M)\,d$

$$\underset{-(M-1)d}{\overset{m_1}{\bullet}} \quad \underset{-(M-3)d}{\overset{m_2}{\bullet}} \quad \cdots \quad \underset{-d}{\bullet} \quad \underset{d}{\bullet} \quad \cdots \quad \underset{(M-3)d}{\overset{m_{M-1}}{\bullet}} \quad \underset{(M-1)d}{\overset{m_M}{\bullet}} \to \psi$$

Assume equal priors → ML rule: $\hat{m} = \begin{cases} 1, & \text{if } r_1 < -(M-2)d \\ i, & \text{if } (2i-2-M)d \leq r_1 < (2i-M)d \\ M, & \text{if } r_1 > (M-2)d \end{cases}$ $\qquad r_1 = \int_0^T r(t)\psi(t)\,dt$

Ex. $M=4$ (2 bits)

$$\underset{00}{\overset{-3d}{\bullet}} \quad \underset{01}{\overset{-d}{\bullet}} \quad \underset{11}{\overset{d}{\bullet}} \quad \underset{10}{\overset{3d}{\bullet}} \to \psi$$

⇝ Gray encoding

- Carrier-Modulated PAM for Bandpass Channels (M-ary ASK): → Amplitude Shift Keying

$\boxed{u_i(t) = s_i(t)\cos(2\pi f_c t)}$, $\quad i=1,\dots,M$

$$s_i(t) \to \otimes \to u_i(t)$$
carrier $\cos(2\pi f_c t)$ ↑

$$E_{u_i} = \int_{-\infty}^{\infty} s_i^2(t)\cos^2(2\pi f_c t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} s_i^2(t)\,dt + \underbrace{\frac{1}{2}\int_{-\infty}^{\infty} s_i^2(t)\cos(4\pi f_c t)\,dt}_{\approx 0 \text{ for } f_c \gg \frac{1}{T}} \approx \frac{1}{2}\int_{-\infty}^{\infty} s_i^2(t)\,dt$$

$\boxed{u_i(t) = A_i\, p(t)\cos(2\pi f_c t)}$, $\quad i=1,\dots,M$ $\qquad \boxed{\psi(t) = \sqrt{\frac{2}{E_p}}\, p(t)\cos(2\pi f_c t)}$ ⇝ 1-dimensional signal space

Then, $\quad \underline{u}_i = \left[A_i \sqrt{\frac{E_p}{2}}\right]$, $i=1,\dots,M$

- Probability of Error:

Consider $s_i(t) = A_i p(t) = s_i \psi(t) \rightarrow \underline{s_i} = [A_i\sqrt{E_p}] = [(2i-1-M)d]$, $i=1,\dots,M$.

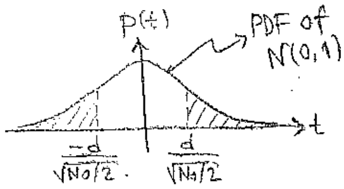
(The following result is valid for M-ary ASK as well if
$u_i(t) = A_i p(t)\cos(2\pi f_c t) \rightarrow \underline{u_i} = [A_i\sqrt{E_p/2}] \doteq [(2i-1-M)d]$, $i=1,\dots,M$)

$$P_e = \frac{1}{M}\sum_{i=1}^{M} P_{e,i}$$

For $s_i = (2i-1-M)d$, $i=1,\dots,M$, we have $P_{e,1} = P_{e,M}$ and $P_{e,2} = \dots = P_{e,M-1}$ due to symmetry.

$$\begin{aligned}
P_{e,1} = P(\hat{m} \neq m_1 \mid m_1 \text{ sent}) &= P(r_1 \geqslant -(M-2)d \mid m_1 \text{ sent}) \\
&= P(-(M-1)d + n_1 \geqslant -(M-2)d) \\
&= P(n_1 \geqslant d) = P\left(\frac{n_1}{\sqrt{N_0/2}} \geqslant \frac{d}{\sqrt{N_0/2}}\right) \qquad n_1 \sim N\left(0, \frac{N_0}{2}\right) \\
&= Q\left(\frac{d}{\sqrt{N_0/2}}\right) = \boxed{Q\left(\sqrt{\frac{2d^2}{N_0}}\right)}
\end{aligned}$$

$$\begin{aligned}
P_{e,2} = P(\hat{m} \neq m_2 \mid m_2 \text{ sent}) &= P(r_1 < -(M-2)d \text{ or } r_1 \geqslant -(M-4)d \mid m_2 \text{ sent}) \\
&= P(-(M-3)d + n_1 < -(M-2)d \text{ or } -(M-3)d + n_1 \geqslant -(M-4)d) \\
&= P(n_1 < -d \text{ or } n_1 \geqslant d) \\
&= P\left(\frac{n_1}{\sqrt{N_0/2}} < -\frac{d}{\sqrt{N_0/2}} \text{ or } \frac{n_1}{\sqrt{N_0/2}} \geqslant \frac{d}{\sqrt{N_0/2}}\right) \\
&= \boxed{2Q\left(\sqrt{\frac{2d^2}{N_0}}\right)}
\end{aligned}$$



So, $P_e = \frac{2}{M}P_{e,1} + \frac{(M-2)}{M}P_{e,2} = \boxed{\frac{2M-2}{M} Q\left(\sqrt{\frac{2d^2}{N_0}}\right)}$

$\left\{ 1^2 + 3^2 + \dots + (2j-1)^2 = \frac{j(2j-1)(2j+1)}{3} \right.$

Average energy per symbol: $E_s = \frac{1}{M}\sum_{i=1}^{M} A_i^2 = \frac{1}{M}\sum_{i=1}^{M}(2i-1-M)^2 d^2 = \boxed{\frac{(M^2-1)d^2}{3}}$

So, $P_e = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2}{N_0}\cdot\frac{3E_s}{(M^2-1)}}\right) = \boxed{\frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\bar{\gamma}_s}{M^2-1}}\right)}$ where $\boxed{\bar{\gamma}_s = \frac{E_s}{N_0}}$ average SNR per symbol

M-ARY PHASE-SHIFT KEYING (PSK):

- $s_i(t) = A\, p(t) \cos(2\pi f_c t + \phi_i)$, $i = 1, \dots, M$, $t \in [0, T_s)$

  information is carried in the phase

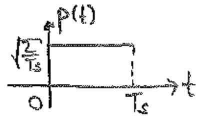
  Commonly, $\phi_i = \frac{2\pi(i-1)}{M}$, $i = 1, \dots, M$. Then,

  $$\boxed{s_i(t) = A \cdot p(t) \cos\left(2\pi f_c t + \frac{2\pi(i-1)}{M}\right)}, \quad i = 1, \dots, M.$$

  $$= A \cos\left(\frac{2\pi(i-1)}{M}\right) p(t) \cos(2\pi f_c t) - A \sin\left(\frac{2\pi(i-1)}{M}\right) p(t) \sin(2\pi f_c t)$$

- If $p(t)$ is a rectangular pulse with amplitude $\sqrt{\frac{2}{T_s}}$ over $[0, T_s]$ and $f_c T_s$ is an integer,

  $\boxed{\psi_1(t) = p(t) \cos(2\pi f_c t)}$ $\boxed{\psi_2(t) = -p(t) \sin(2\pi f_c t)}$



<u>Exercise:</u> Show that $\int_0^{T_s} \psi_1^2(t)\,dt = \int_0^{T_s} \psi_2^2(t)\,dt = 1$ and $\int_0^{T_s} \psi_1(t)\psi_2(t)\,dt = 0$

(Even if $p(t)$ is not rectangular and $f_c T_s$ is non-integer, these results hold approximately for $f_c \gg \frac{1}{T_s}$.) i.e., any pulse with energy 2 and domain $[0, T_s]$.

- <u>Vector representation:</u>

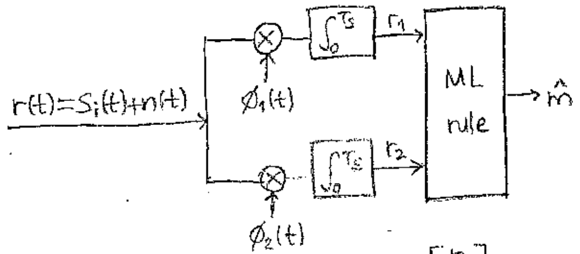
  $$\underline{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \end{bmatrix} = \begin{bmatrix} A \cos\left(\frac{2\pi(i-1)}{M}\right) \\ A \sin\left(\frac{2\pi(i-1)}{M}\right) \end{bmatrix}, \quad i = 1, \dots, M \quad \rightsquigarrow \text{Two-dimensional signal space}$$

- Symbol energy: $\|\underline{s}_i\|^2 = A^2 = E_s$

  we will denote it by E_b in the binary case $\rightsquigarrow$ bit energy

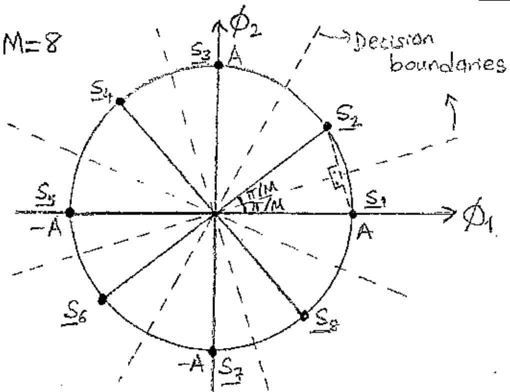
- ML rule:

$$\boxed{\hat{m} = i \quad \text{if} \quad (2i-3)\frac{\pi}{M} < \tan^{-1}\left(\frac{r_2}{r_1}\right) \leq (2i-1)\frac{\pi}{M}, \; i=1,\ldots,M}$$



$\underline{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

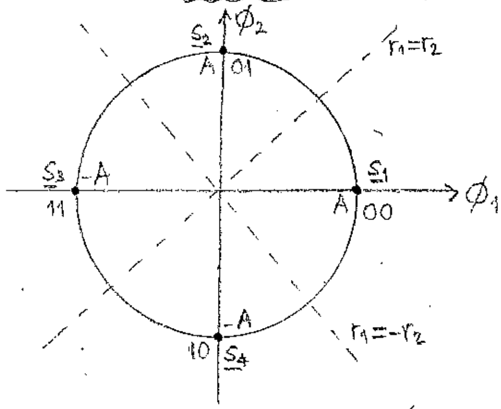
e.g. $M=8$



Decision regions:

$$\boxed{Z_i = \left\{ \underline{r} : (2i-3)\frac{\pi}{M} < \tan^{-1}\left(\frac{r_2}{r_1}\right) \leq (2i-1)\frac{\pi}{M} \right\}} \quad i=1,\ldots,M$$

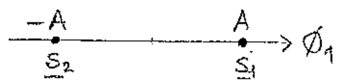
e.g. $M=4 \Rightarrow$ QPSK (quadri-phase shift keying)



$$\hat{m} = \begin{cases} 1, & \text{if } r_1 \geq r_2 \ \& \ r_1 \geq -r_2 \\ 2, & \text{if } r_1 < r_2 \ \& \ r_1 \geq -r_2 \\ 3, & \text{if } r_1 < r_2 \ \& \ r_1 < -r_2 \\ 4, & \text{if } r_1 \geq r_2 \ \& \ r_1 < -r_2 \end{cases}$$

$\underline{r} = \underline{s}_i + \underline{n}$, where $\underline{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$.

e.g. $M=2 \Rightarrow$ BPSK (binary phase shift keying) $\rightarrow$ can be regarded as BPAM as well ! (BASK)



$$\hat{m} = \begin{cases} 1, & \text{if } r_1 \geq 0 \\ 2, & \text{if } r_1 < 0 \end{cases} \qquad r_1 = \int_0^{T_s} r(t)\phi_1(t)\,dt = s_i + \underbrace{n_1}_{\mathcal{N}(0, \frac{N_0}{2})}$$

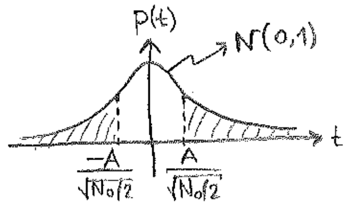
- Probability of Error:
  - BPSK (M=2):

$$P_e = \frac{1}{2}P_{e,1} + \frac{1}{2}P_{e,2} = \frac{1}{2}P(r_1<0 \mid m_1 \text{ sent}) + \frac{1}{2}P(r_1 \geq 0 \mid m_2 \text{ sent})$$

$$= \frac{1}{2}P(A+n_1<0) + \frac{1}{2}P(-A+n_1 \geq 0)$$

$$= \frac{1}{2}P\left(\frac{n_1}{\sqrt{N_0/2}} < \frac{-A}{\sqrt{N_0/2}}\right) + \frac{1}{2}P\left(\frac{n_1}{\sqrt{N_0/2}} \geq \frac{A}{\sqrt{N_0/2}}\right)$$

$$= \frac{1}{2}Q\left(\frac{A}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{A}{\sqrt{N_0/2}}\right)$$



$E_b = \|\underline{s}_i\|^2 = A^2 \rightarrow A = \sqrt{E_b}$ $\quad$ E_b: Bit energy.

$$\boxed{P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{2\gamma_b}\right)}$$ $\quad$ $\gamma_b = \frac{E_b}{N_0} \rightarrow$ SNR per bit

  - QPSK (M=4):

Due to symmetry, $P_{e,1} = P_{e,2} = P_{e,3} = P_{e,4} = P_e$

$$P_{e,1} = 1 - P_{c,1} = 1 - P(r_1 \geq r_2 \;\&\; r_1 \geq -r_2 \mid m_1 \text{ sent})$$

$$= 1 - P(A+n_1 \geq 0+n_2 \;\&\; A+n_1 \geq -(0+n_2))$$

$$= 1 - P(\underbrace{n_1-n_2}_{N(0,N_0)} \geq -A \;\&\; \underbrace{n_1+n_2}_{N(0,N_0)} \geq -A)$$

$$= 1 - P(n_1-n_2 \geq -A)\,P(n_1+n_2 \geq -A)$$

$$= 1 - Q\left(\frac{-A}{\sqrt{N_0}}\right)Q\left(\frac{-A}{\sqrt{N_0}}\right)$$

independent

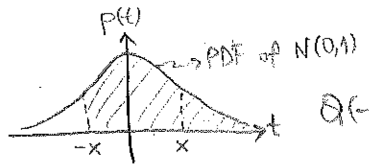
$n_1 \sim N\left(0,\frac{N_0}{2}\right)$, $n_2 \sim N\left(0,\frac{N_0}{2}\right)$

$E\{(n_1-n_2)(n_1+n_2)\}$
$= E\{n_1^2 - n_2^2\} = E\{n_1^2\} - E\{n_2^2\} = 0$

Since $E\{(n_1-n_2)(n_1+n_2)\} = E\{n_1-n_2\}E\{n_1+n_2\}$, n_1-n_2 & n_1+n_2 are uncorrelated.
Since n_1-n_2 and n_1+n_2 are Gaussian, they are also independent.

Since $E_s = A^2$, $P_{e,1} = 1 - \left(Q\left(-\sqrt{\frac{E_s}{N_0}}\right)\right)^2$.

So, $$\boxed{P_e = 1 - \left[1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2}$$ ←



$Q(-x) = 1 - Q(x)$

$$= 1 - \left[1 - Q(\sqrt{\gamma_s})\right]^2 \qquad \gamma_s = \frac{E_s}{N_0} \rightarrow \text{SNR per symbol}$$

$$= 1 - \left[1 - Q(\sqrt{2\gamma_b})\right]^2 \qquad \text{Since } E_b = \frac{E_s}{2},\; \gamma_s = 2\gamma_b$$

Alternatively, $$\boxed{P_e = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - Q^2\left(\sqrt{\frac{E_s}{N_0}}\right)}.$$

At high SNRs,

$$P_e \approx 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

With gray encoding, assuming errors only between adjacent symbols at high SNRs → $P_b = \frac{1}{2}P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$ → same as BPSK!

P_b: probability of bit error (bit error rate)

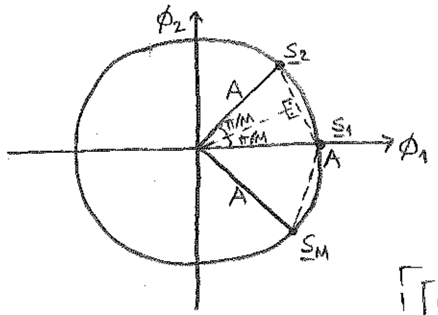
But QPSK can send twice the bit rate of BPSK for the same bandwidth!

- MPSK (M>4):

Since exact expression is difficult, we use the union bound.

$$P_{e,1} \leq \sum_{k=2}^{M} Q\left(\frac{\|\underline{s}_1-\underline{s}_k\|}{\sqrt{2N_0}}\right)$$

$$\|\underline{s}_1-\underline{s}_2\| = \|\underline{s}_1-\underline{s}_M\| = 2A\sin\left(\frac{\pi}{M}\right)$$



Looser bound (see p. 5) $\rightsquigarrow \leq (M-1)\, Q\left(\frac{2A\sin(\pi/M)}{\sqrt{2N_0}}\right)$

Nearest neighbor approximation:

$$\boxed{P_e \approx \underbrace{M_{d_{min}}}_{\substack{\text{number of neighbors at}\\ \text{min. distance } d_{min}}} Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)} \rightsquigarrow P_{e,1} \approx 2\,Q\left(\frac{2A\sin(\pi/M)}{\sqrt{2N_0}}\right)$$

[Craig, 1991]

Exact P_e for MPSK:

$$P_e = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \exp\left\{-\frac{\sin^2(\pi/M)\,\gamma_s}{\sin^2\phi}\right\} d\phi$$

Due to symmetry, $P_{e,1} = \cdots = P_{e,M} = P_e$.

$$P_e \approx 2\,Q\left(\frac{\sqrt{2}\,A\sin(\pi/M)}{\sqrt{N_0}}\right) = \boxed{2\,Q\left(\sqrt{2\gamma_s}\,\sin(\pi/M)\right)} \qquad \gamma_s = \frac{E_s}{N_0} = \frac{A^2}{N_0}$$

M-ARY QUADRATURE AMPLITUDE MODULATION (QAM):

- $\boxed{s_i(t) = A_i\cos\theta_i\, g(t)\cos(2\pi f_c t) - A_i\sin\theta_i\, g(t)\sin(2\pi f_c t)}$, $t\in[0,T_s)$, $i=1,\ldots,M$.
- Orthonormal basis functions: $\boxed{\psi_1(t) = g(t)\cos(2\pi f_c t)}$, $\boxed{\psi_2(t) = -g(t)\sin(2\pi f_c t)}$.

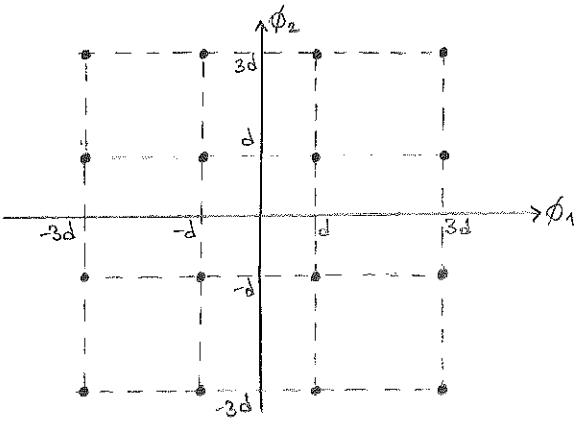
  consider similar arguments as in page 10.

  $$\boxed{\underline{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2}\end{bmatrix} = \begin{bmatrix} A_i\cos\theta_i \\ A_i\sin\theta_i\end{bmatrix}}, \quad i=1,\ldots,M$$

- Energy of $s_i(t)$: $E_{s_i} = \|\underline{s}_i\|^2 = A_i^2$
- Square constellation: Even number of bits ($\log_2 M \to$ even)

  Cross constellation: Odd number of bits ($\log_2 M \to$ odd)

- Ex. $M=16$ (4 bits/symbol)



For square constellations:

$$MQAM \equiv LPAM \times LPAM$$

$$M = L^2$$

• Probability of Error:

Consider square constellation with $M = L^2$.

$$MQAM \equiv LPAM \times LPAM \rightsquigarrow \boxed{A_i = (2i-1-L)d}, \quad i = 1, \ldots, L$$ ← for each branch (in-phase & quadrature).

Prob. of correct decision for LPAM:

$$1 - \frac{2(L-1)}{L} Q\left(\sqrt{\frac{2d^2}{N_0}}\right)$$ ← see page 9.

Prob. of correct decision for MQAM:

$$P_c = \left(1 - \frac{2(L-1)}{L} Q\left(\sqrt{\frac{2d^2}{N_0}}\right)\right)^2$$

Prob. of error for MQAM:

$$P_e = 1 - P_c$$

$$= \boxed{\frac{4(\sqrt{M}-1)}{\sqrt{M}} Q\left(\sqrt{\frac{2d^2}{N_0}}\right) - \frac{4(\sqrt{M}-1)^2}{M} Q^2\left(\sqrt{\frac{2d^2}{N_0}}\right)}$$

At high SNRs → $$P_e \approx 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{2d^2}{N_0}}\right)$$

$$\boxed{P_e \approx 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\bar{\gamma}_s}{M-1}}\right)} \qquad \bar{\gamma}_s = \frac{\bar{E}_s}{N_0}$$

• Average energy of MQAM signals:

$$\underline{s}_i = \begin{bmatrix} \mp(2j-1)d \\ \mp(2\ell-1)d \end{bmatrix}, \quad j = 1, \ldots, L/2, \quad \ell = 1, \ldots, L/2$$

$$\bar{E}_s = \frac{1}{M} \sum_{i=1}^{M} \|\underline{s}_i\|^2$$

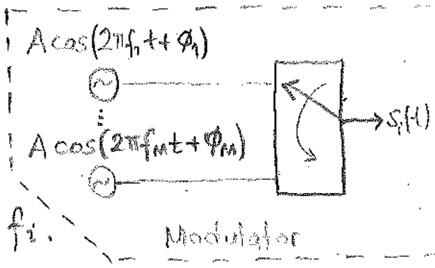
$$= \frac{4}{M} \sum_{j=1}^{L/2} \sum_{\ell=1}^{L/2} \left[(2j-1)^2 + (2\ell-1)^2\right] d^2$$

Exercise

$$= \boxed{\frac{2d^2(M-1)}{3}} \qquad (L^2 = M)$$

FREQUENCY-SHIFT KEYING (FSK):

• $\boxed{s_i(t) = A\cos(2\pi f_i t + \phi_i)}$, $t \in [0, T_s)$, $i = 1, 2, \ldots, M$.



Each symbol corresponds to a different frequency f_i.

ϕ_i : Phase of the i^{th} carrier.

First, assume that the value of ϕ_i is known at the receiver (via phase estimation).

⇒ Coherent receiver: A receiver that uses the phase info.

• Assume $f_i \neq f_j \; \forall i \neq j$ and $f_i T_s \to$ integer $\forall i$.

Then $\boxed{\psi_i(t) = \sqrt{\frac{2}{T_s}} \cos(2\pi f_i t + \phi_i)}$, $t \in [0, T_s]$, $i = 1, \ldots, M$

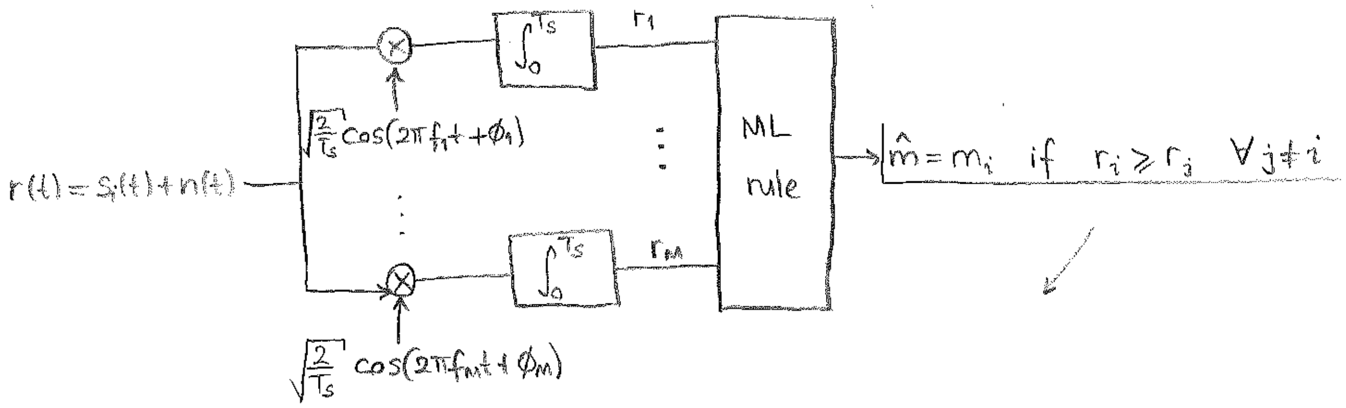
Orthonormal basis functions: $\psi_1(t), \ldots, \psi_M(t)$.

$$\int_0^{T_s} \psi_i(t)\psi_j(t)\,dt = \frac{2}{T_s} \int_0^{T_s} \cos(2\pi f_i t + \phi_i)\cos(2\pi f_j t + \phi_j)\,dt$$

$$= \frac{1}{T_s} \int_0^{T_s} \left[\underbrace{\cancel{\cos(2\pi(f_i+f_j)t + \phi_i + \phi_j)}}_{0} + \cos(2\pi(f_i - f_j)t + \phi_i - \phi_j)\right] dt = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

• $s_1 = [A\sqrt{T_s/2} \; 0 \ldots 0]^T \quad s_2 = [0 \; A\sqrt{T_s/2} \; 0 \ldots 0]^T \quad \ldots \quad s_M = [0 \ldots 0 \; A\sqrt{T_s/2}]$

- Optimal coherent FSK receiver (Assume equal priors)



ML rule: Choose m_i if $\|\underline{r} - \underline{s}_i\| \leq \|\underline{r} - \underline{s}_j\|, \ \forall j \neq i$

$$\left(\underline{r}^T \underline{s}_i = \int r(t) s_i(t) dt = \underbrace{A\sqrt{\tfrac{T_s}{2}}}_{k} \underbrace{\int r(t)\phi_i(t)\, dt}_{r_i} \right) \Longrightarrow \quad -2\underbrace{\underline{r}^T \underline{s}_i}_{k \cdot r_i} + \cancel{\|\underline{s}_i\|^2} \leq -2\underbrace{\underline{r}^T \underline{s}_j}_{k \cdot r_j} + \cancel{\|\underline{s}_j\|^2}$$

← Equal-energy symbols

$$\boxed{r_i \geq r_j, \ \forall j \neq i}$$

- Let $\boxed{f_i = f_c + \alpha_i \Delta f_c}$, $i = 1, \dots, M$

  Then, $\boxed{s_i(t) = A\cos\left(2\pi f_c t + 2\pi \alpha_i \Delta f_c t + \phi_i\right)}$, $t \in [0, T_s]$, $i = 1, \dots, M$

  with $\boxed{\alpha_i = (2i - 1 - M)}$. → Frequency spacing of $2\Delta f_c$

- Minimum-Shift Keying (MSK):

  A special case of binary FSK where $\phi_1 = \phi_2$, and $|f_1 - f_2| = \frac{1}{2T_s}$.

  Let $\phi_1 = \phi_2 = 0$. $s_1(t) = A\cos(2\pi f_1 t)$, $s_2(t) = A\cos(2\pi f_2 t)$, $t \in [0, T_s)$

$$\langle s_1(t), s_2(t) \rangle = \int_0^{T_s} A^2 \frac{1}{2}\left[\cos(2\pi(f_1+f_2)t) + \underbrace{\cos(2\pi(f_1-f_2)t)}_{\text{Half osc. over } [0,T_s]}\right] dt = 0$$

* $\frac{1}{2T_s}$ is the minimum freq. separation in FSK to provide orthogonality.

- Non-coherent receiver for FSK:

  A non-coherent receiver doesn't use the phase information.

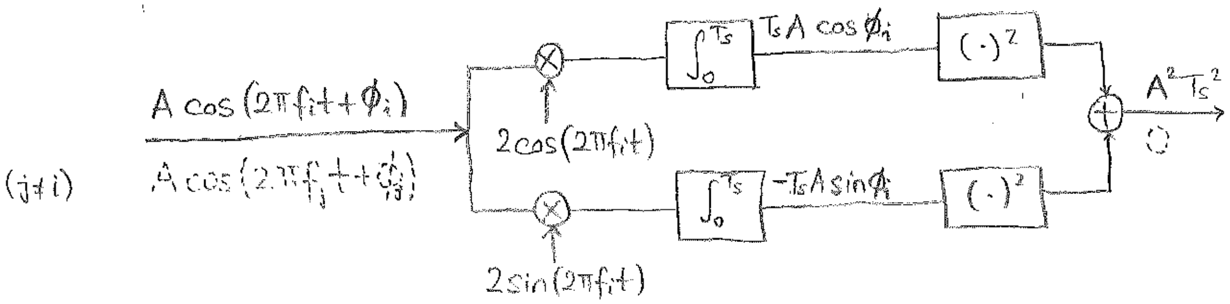
  Advantage: No need for phase estimation at the receiver → less complex / costly

  Disadvantage: Worse error performance than a coherent receiver.

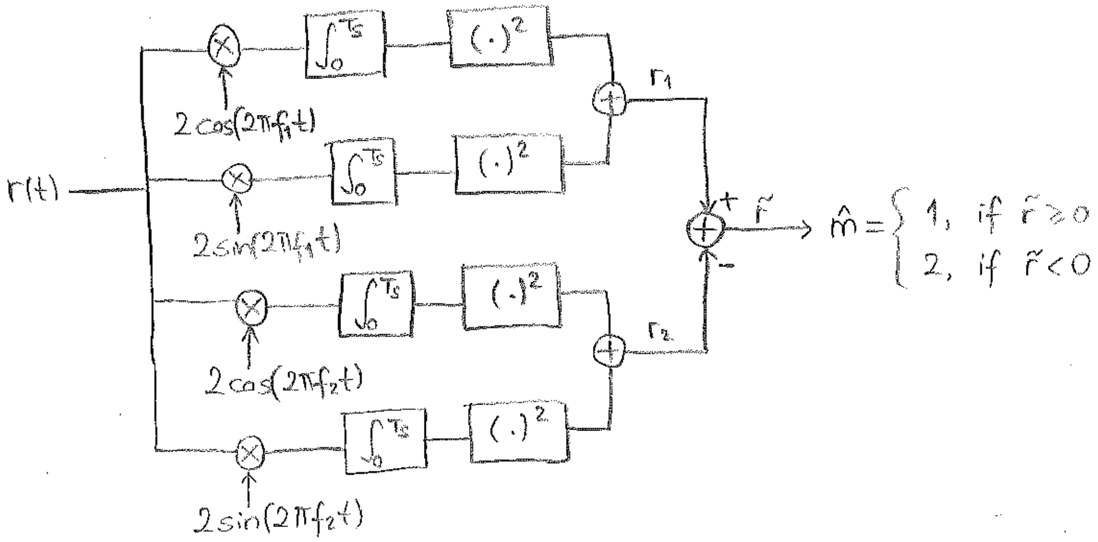
$$s_i(t) = A\cos(2\pi f_i t + \phi_i)$$
$$= A\cos\phi_i \cos(2\pi f_i t) - A\sin\phi_i \sin(2\pi f_i t)$$

Since ϕ_i is unknown for a non-coherent receiver, we should correlate with both $\cos(2\pi f_i t)$ and $\sin(2\pi f_i t)$ to collect all the energy of the signal.

To provide intuition, ignore noise and consider the following:

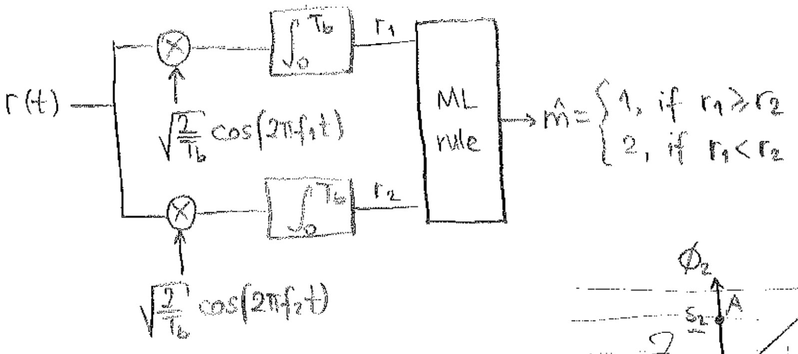


Non-coherent BFSK receiver:



• Error Probability for Coherent BFSK:

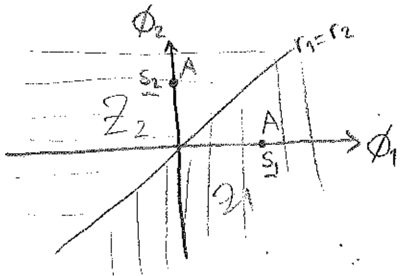
$$s_1(t) = A\sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t), \quad s_2(t) = A\sqrt{\frac{2}{T_b}}\cos(2\pi f_2 t), \quad t \in [0, T_b)$$



Basis fn.s:

$$\phi_i(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_i t), \quad i = 1,2$$

$$\underline{s}_1 = \begin{bmatrix} A \\ 0 \end{bmatrix}, \quad \underline{s}_2 = \begin{bmatrix} 0 \\ A \end{bmatrix}$$



$$P_{e,1} = P(r_1 < r_2 \mid m_1 \text{ sent})$$

$$= P(A + n_1 < 0 + n_2) = P(\underbrace{n_2 - n_1}_{\mathcal{N}(0, N_0)} > A) = P\left(\frac{n_2 - n_1}{\sqrt{N_0}} > \frac{A}{\sqrt{N_0}}\right)$$

$$= Q\left(\frac{A}{\sqrt{N_0}}\right)$$

$$P_{e,2} = P(r_1 \geq r_2 \mid m_2 \text{ sent}) = P(0 + n_1 \geq A + n_2) = P(n_1 - n_2 \geq A) = Q\left(\frac{A}{\sqrt{N_0}}\right)$$

$$\boxed{P_e = Q\left(\frac{A}{\sqrt{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q(\sqrt{\gamma_b})}$$

$E_b = A^2$ ← bit energy

$\gamma_b = \frac{E_b}{N_0}$ ← SNR per bit

(Compared to BPSK, BFSK needs 3dB more SNR to achieve the same error rate. BPSK → $P_e = Q(\sqrt{2\gamma_b})$)

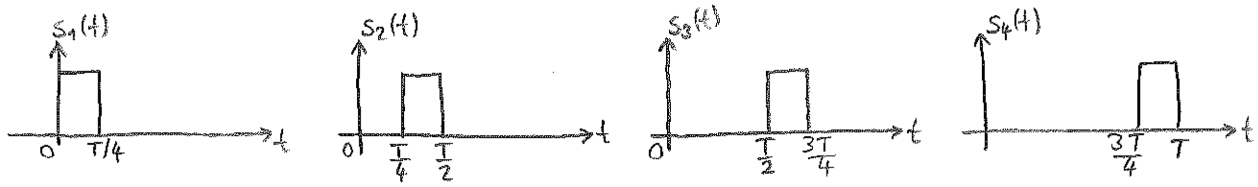
- FSK is an example of orthogonal signaling.

In general, M-ary orthogonal signals satisfy

$$\int_0^T s_i(t)\, s_j(t)\, dt = 0 \quad \forall j \neq i \qquad s_1(t), \ldots, s_M(t) \rightarrow \text{M-ary signaling}$$

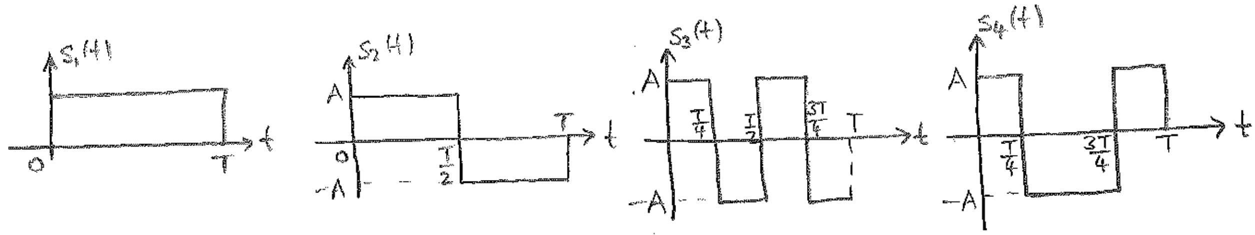
Basis functions: $\boxed{\psi_i(t) = \frac{s_i(t)}{\sqrt{E_{s_i}}}}$, $i = 1, \ldots, M$ → M-dimensional signal space.

Ex.



Pulse position modulation (PPM).

Ex.

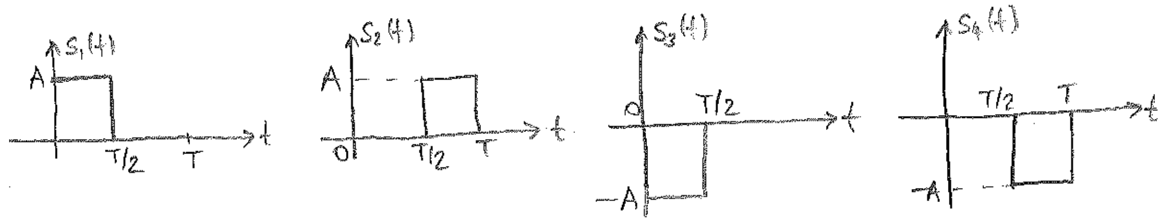


Biorthogonal Signals:

Constructed from M/2 orthogonal signals and their negatives.

$$\underbrace{s_1(t), \ldots, s_{M/2}(t)}_{\text{orthogonal}}, -s_1(t), \ldots, -s_{M/2}(t) \qquad t \in [0, T_s]$$

Ex.



Simplex Signals:

Constructed from M orthogonal signals by subtracting the average of the M orthogonal signals from each signal.

$$\boxed{\tilde{s}_i(t) = s_i(t) - \frac{1}{M} \sum_{k=1}^{M} s_k(t)} \qquad s_1(t), \ldots, s_M(t) \rightarrow \text{orthogonal, each with energy } E_s.$$

$i = 1, \ldots, M.$

$$\tilde{E}_s = \int_0^{T_s} \Big(s_i(t) - \frac{1}{M} \sum_{k=1}^{M} s_k(t) \Big)^2 dt = \int_0^{T_s} s_i^2(t)\, dt - \frac{2}{M} \sum_{k=1}^{M} \int_0^{T_s} s_i(t) s_k(t)\, dt + \frac{1}{M^2} \sum_{k=1}^{M} \sum_{l=1}^{M} \int_0^{T_s} s_k(t) s_l(t)\, dt$$

$$= E_s - \frac{2}{M} E_s + \frac{1}{M^2} M . E_s = \boxed{\Big(1 - \frac{1}{M}\Big) E_s}$$

→ each signal has smaller energy

$$\int_0^{T_s} \tilde{s}_i(t)\, \tilde{s}_j(t)\, dt = \boxed{-\frac{E_s}{M-1}}, \text{ if } i \neq j$$

COMPARISON OF MODULATION METHODS:

- Consider fixed bit rate R_b.
- Required channel bandwidth for transmission:

  - M-PAM: Pulse duration: $T_s \rightarrow$ Bandwidth $\approx \frac{1}{2T_s}$
    $\log_2 M$ bits per symbol $\rightarrow T_s = \frac{\log_2 M}{R_b}$
    $$\left.\right\} \boxed{W = \frac{R_b}{2\log_2 M}}$$
    Bandwidth
    ← multiply by 2 for double-sideband

  - QAM: $W \approx \frac{1}{T_s}$, $T_s = \frac{2\log_2 M_{PAM}}{R_b}$ $\Rightarrow W = \frac{R_b}{2\log_2 M_{PAM}} = \boxed{\frac{R_b}{\log_2 M_{QAM}}}$
    due to two quadrature carriers

  - M-PSK: (M>2) $W = \frac{1}{T_s}$, $T_s = \frac{\log_2 M}{R_b}$ $\Rightarrow \boxed{W = \frac{R_b}{\log_2 M}}$

  - M-FSK: $W = \frac{M}{2T_s} = \frac{M}{2\frac{\log_2 M}{R_b}} = \boxed{\frac{M R_b}{2\log_2 M}}$
    $\frac{1}{2T_s} \rightarrow$ minimum frequency separation for orthogonality.
    Valid for orthogonal signaling in general

  - Biorthogonal signaling: $\boxed{W = \frac{M R_b}{4\log_2 M}}$ ← since it uses half bandwidth of orthogonal signaling

- For compact and meaningful comparison, consider normalized data rate (spectral bit rate) versus SNR per bit (E_b/N_0).

$$r = \frac{R_b}{W} \text{ bits/sec/Hz.}$$

M-PAM: $r = 2\log_2 M$

M-PSK: $r = \log_2 M$ (M>2)

BPSK: $r = 2\log_2 M$

QAM: $r = \log_2 M$

Orthogonal: $r = \frac{2\log_2 M}{M}$

Biorthogonal: $r = \frac{4\log_2 M}{M}$

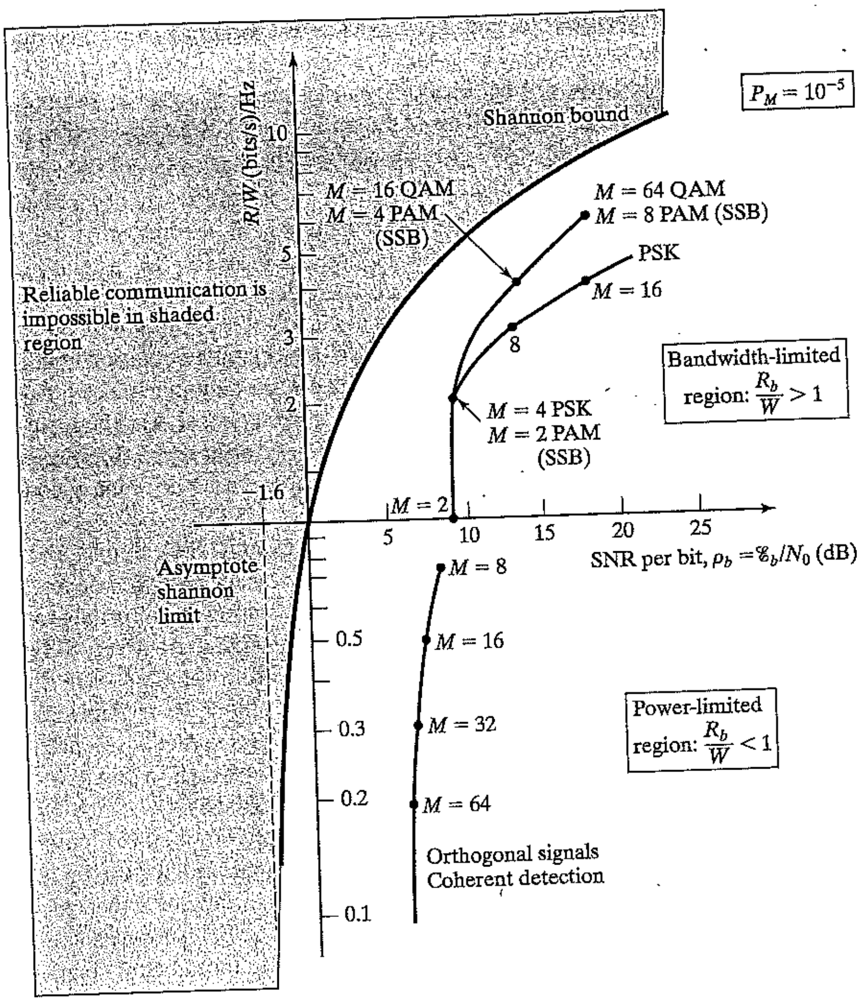


Figure 9.29 Comparison of several modulation methods at 10^{-5} symbol rate.

TABLE 9.2 QAM SIGNAL CONSTELLATIONS

Number of signal points M	Increase in average power (dB) relative to $M = 2$
4	3
8	6.7
16	10.0
32	13.2
64	16.2
128	19.2

End

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