EEE 431: Telecommunications 1

Quiz 3

April 21, 2018, 9:00

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Name: ______________________________

Signature: ______________________________

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Prob. 1: ________ / 40

Prob. 2: ________ / 35

Prob. 3: ________ / 25

Total: ________ / 100

Some trigonometric identities: $\sin(2x) = 2\sin(x)\cos(x)$
$\cos(2x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
$\sin(x)\cos(y) = 0.5\sin(x+y) + 0.5\sin(x-y)$
$\cos(x)\cos(y) = 0.5\cos(x+y) + 0.5\cos(x-y)$
$\sin(x)\sin(y) = 0.5\cos(x-y) - 0.5\cos(x+y)$
$\sin(\pi/6) = \cos(\pi/3) = 1/2,\ \sin(\pi/3) = \cos(\pi/6) = \sqrt{3}/2.$

Problem 1 Consider the following random process: $X(t) = A\cos(2000\pi\,\alpha\,t + \theta)$, where $A > 0$ is a known fixed number, θ is a uniform random variable in the interval of $[0, 2\pi)$, and α is a discrete random variable, which is equal to 1 or 2 with equal probabilities.

(a) Find the autocorrelation function of $X(t)$.

(b) Find the power spectral density of $X(t)$.

(c) Calculate the average power of $X(t)$.

(d) Suppose that $X(t)$ passes through a linear time-invariant filter with the following frequency response:

$$H(f) = \begin{cases} (3000 - |f|)/1000\,, & \text{if } |f| \leq 3000 \\ 0\,, & \text{otherwise} \end{cases}$$

Find the autocorrelation function of the output of the filter.

<u>Hint:</u> The Fourier transform $S(f)$ of $s(t)$ is defined as $S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt$ and the inverse Fourier transform is given by $s(t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi ft}df$.

Problem 2 Consider a binary communication system operating over an additive white Gaussian noise channel with spectral density level of $N_0/2$. The transmitted signals are given by

$$s_1(t) = \sqrt{\frac{8}{T}}\cos\left(2\pi f_c t - \frac{\pi}{6}\right)\,, \quad s_2(t) = \sqrt{\frac{2}{T}}\sin\left(2\pi f_c t + \frac{4\pi}{3}\right)$$

for $t \in [0, T]$, and they are zero otherwise. Suppose that $f_c T$ is an integer.

(a) Calculate the energy of $s_1(t)$ and the energy of $s_2(t)$.

(b) Find orthonormal basis function(s) to represent these signals, and illustrate them as constellation points in the corresponding signal space.

(c) Show the structure of the optimal receiver.

(d) Calculate the probability of error of the optimal receiver.

Problem 3 Consider a binary communication system, where the vector representations of the transmitted signals are denoted by $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. For simplicity, you can assume that each vector is 2-dimensional. The system operates over an additive white Gaussian noise channel with spectral density level $N_0/2$. That is, the observation vector at the receiver is $\boldsymbol{r} = \boldsymbol{s}_i + \boldsymbol{n}$ for $i \in \{1, 2\}$, where the components of $\boldsymbol{n}$ are independent and identically distributed zero mean Gaussian random variables each with variance $N_0/2$. At the receiver, the maximum likelihood (ML) decision rule processes $\boldsymbol{r}$ and makes a decision about the transmitted signal. Let P_e denote the probability of error for this receiver. Now suppose that instead of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, we use the following signals:

$$\tilde{\boldsymbol{s}}_1 = k\boldsymbol{s}_1 + \boldsymbol{a}\,, \quad \tilde{\boldsymbol{s}}_2 = k\boldsymbol{s}_2 + \boldsymbol{a}$$

where $k > 0$ is a known scalar, and $\boldsymbol{a}$ is a known 2-dimensional vector. Considering the use of the ML decision rule based on $\tilde{\boldsymbol{r}} = \tilde{\boldsymbol{s}}_i + \boldsymbol{n}$, express the probability of error at the receiver in terms of P_e and any other parameter(s). (*The probability that a zero-mean, unit variance Gaussian random variable is larger than x is defined as $Q(x)$, i.e., $Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}}\,e^{-t^2/2}\,dt$. The $Q-$function is monotone decreasing.*)

(1) $X(t)=A\cos(2000\pi t\alpha+\theta)$ $\quad \theta\sim\mathcal{U}[0,2\pi)$ $\quad p(\alpha)=\begin{cases}0.5, & \text{if } \alpha=1\\ 0.5, & \text{if } \alpha=2\end{cases}$

13 (a) $R_X(t_1,t_2)=E[X(t_1)X(t_2)]=\frac{A^2}{2}E\left[\cancel{\cos(2000\pi\alpha(t_1+t_2)+2\theta)}^{\to 0}+\cos(2000\pi\alpha(t_1-t_2))\right]$

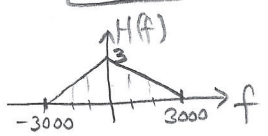
$$=\boxed{\frac{A^2}{2}\left(\frac{1}{2}\cos(2000\pi(t_1-t_2))+\frac{1}{2}\cos(4000\pi(t_1-t_2))\right)}$$

$\xrightarrow{\tau=t_1-t_2}$

(b) $S_X(f)=\mathcal{F}\{R_X(\tau)\}=\boxed{\frac{A^2}{8}\left(\delta(f-1000)+\delta(f+1000)+\delta(f-2000)+\delta(f+2000)\right)}$

(c) $P_X=R_X(0)=\int_{-\infty}^{\infty}S_X(f)\,df=\boxed{\frac{A^2}{2}}$

(d) $X(t)\rightarrow\boxed{H(f)}\rightarrow Y(t)$



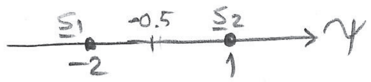
$$S_Y(f)=S_X(f)\,|H(f)|^2$$
$$=\frac{A^2}{2}\left(\delta(f-1000)+\delta(f+1000)\right)+\frac{A^2}{8}\left(\delta(f-2000)+\delta(f+2000)\right)$$
$$R_Y(\tau)=\boxed{A^2\cos(2000\pi\tau)+\frac{A^2}{4}\cos(4000\pi\tau)}$$

(2) (a) $E_1=\int_0^T\frac{8}{T}\cos^2\left(2\pi f_c t-\frac{\pi}{6}\right)dt=\frac{4}{T}\int_0^T\left(1+\cancel{\cos\left(4\pi f_c t-\frac{\pi}{3}\right)}^{\to 0}\right)dt=\boxed{4}$ $\quad$ Similarly, $E_2=\boxed{1}$.

(b) $s_1(t)=2\sqrt{\frac{2}{T}}\left(\frac{\sqrt{3}}{2}\cos(2\pi f_c t)+\frac{1}{2}\sin(2\pi f_c t)\right)$, $\quad s_2(t)=\sqrt{\frac{2}{T}}\left(-\frac{1}{2}\sin(2\pi f_c t)-\frac{\sqrt{3}}{2}\cos(2\pi f_c t)\right)$

So, $\boxed{s_1(t)=-2\,s_2(t)}$. Choose basis as $\boxed{\psi(t)=s_2(t)}$. → has unit energy.

Then, $\underline{s}_1=[-2]$, $\underline{s}_2=[1]$



← binary PAM

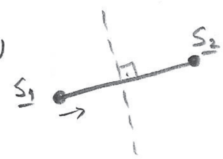
(c)

$r(t)\longrightarrow\otimes\longrightarrow\boxed{\int_0^T}\xrightarrow{r_1}\boxed{\text{ML rule}}\rightarrow\hat{m}=\begin{cases}1, & \text{if } r_1<-0.5\\ 2, & \text{if } r_1>-0.5\end{cases}$

$\uparrow$
$\psi(t)$

(d) $P_e=P_{e,1}=P_{e,2}=P(r_1>-0.5\mid s_1(t)\text{ sent})=P(-2+n_1>-0.5)=\boxed{Q\left(\frac{1.5}{\sqrt{N_0/2}}\right)}$

(3) $\underline{r}=\underline{s}_i+\underline{n}$ $\quad i\in\{1,2\}$ $\quad\longrightarrow\quad P_e=Q\left(\frac{\|\underline{s}_1-\underline{s}_2\|/2}{\sqrt{N_0/2}}\right)=Q\left(\frac{\|\underline{s}_1-\underline{s}_2\|}{\sqrt{2N_0}}\right)$ (*)



$\tilde{\underline{r}}=\tilde{\underline{s}}_i+\underline{n}$ $\quad i\in\{1,2\}$ $\quad\longrightarrow\quad\tilde{P}_e=Q\left(\frac{\|\tilde{\underline{s}}_1-\tilde{\underline{s}}_2\|}{\sqrt{2N_0}}\right)=Q\left(\frac{k\|\underline{s}_1-\underline{s}_2\|}{\sqrt{2N_0}}\right)$

From (*), $Q^{-1}(P_e)=\frac{\|\underline{s}_1-\underline{s}_2\|}{\sqrt{2N_0}}$. So, $\boxed{\tilde{P}_e=Q\left(k\,Q^{-1}(P_e)\right)}$.