EEE 431: Telecommunications 1

Quiz 1

Oct. 20, 2018, 9:30-10:30

Instructor: Sinan Gezici

Name: ______________________________

Signature: ______________________________

Bilkent ID: ______________________________

Please put your cell phones/tablets/computers on teacher's desk.

Prob. 1: ________ / 18

Prob. 2: ________ / 20

Prob. 3: ________ / 12

Prob. 4: ________ / 15

Prob. 5: ________ / 35

Total: ________ / 100

Problem 1 Consider a binary communication system in which the transmitter sends either bit 0 or bit 1, and the receiver gets a scalar observation denoted by Y. If bit 0 is sent, Y is distributed as a Gaussian random variable with mean 1 and variance 4. If bit 1 is sent, Y is distributed as a Gaussian random variable with mean 4 and variance 4. The receiver decides that bit 1 is sent if $Y \geq 2$, and that bit 0 is sent if $Y < 2$.

(a) Calculate the probability that the receiver decides for bit 0 even though bit 1 is sent.

(b) Calculate the probability that the receiver decides for bit 1 even though bit 0 is sent.

(c) Suppose that bit 0 is sent with probability 0.2 and bit 1 is sent with probability 0.8. Calculate the probability that the receiver makes an erroneous decision.

Hint: The probability density function (PDF) of a Gaussian random variable with mean μ and variance σ^2 is given by $f_Y(y) = (1/\sqrt{2\pi}\sigma)e^{-(y-\mu)^2/(2\sigma^2)}$, and the Q-function is defined as $Q(x) = (1/\sqrt{2\pi})\int_x^\infty e^{-t^2/2}dt$.

Problem 2 Consider a discrete memoryless source (DMS) with 4 different outputs. Suppose a certain coding operation is performed and a set of codewords are generated. For each case below, explain if the employed coding operation can or cannot correspond to Huffman coding. If you say that it can be Huffman coding, then list a related property of those codewords. If you say that it cannot be Huffman coding, then list one property of the codewords that eliminates the Huffman coding possibility. (*No points without justification.*)

(a) Codewords: 0, 10, 110, 111

(b) Codewords: 1, 01, 10, 000

(c) Codewords: 00, 10, 01, 11

(d) Codewords: 1, 0, 11, 00

Problem 3 Find the maximum value of $g(\alpha)$ over the closed interval $[0, 1]$, where

$$g(\alpha) = 4\alpha^2(\log_2 \alpha)^2 + 4(1-\alpha)^2 (\log_2(1-\alpha))^2 + 8(\alpha - \alpha^2)(\log_2 \alpha)(\log_2(1-\alpha)) \ .$$

Justify your answer. (You can take $0 \log_2 0 = 0$.)

Problem 4 Suppose that X_1 and X_2 are jointly Gaussian and the joint probability density function (PDF) of $\mathbf{X} = [X_1 \ X_2]^T$ is given by $\frac{1}{2\pi\sqrt{det(\mathbf{C})}} \exp\{-0.5(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}$, where $\boldsymbol{\mu} = [-2 \ 3]^T$ and $\mathbf{C} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$. Calculate $E[2X_1 - (X_2)^2]$ and $P(X_1 > 0 \,|\, X_2 > 0)$.

Problem 5 Signal $s(t)$ is defined as

$$s(t) = \sum_{n=-\infty}^{\infty} p(t-4n)\,, \quad \text{where} \quad p(t) = \begin{cases} 4t - t^2\,, & \text{if } 0 \leq t \leq 4 \\ 0\,, & \text{otherwise} \end{cases} .$$

Suppose $s(t)$ is sampled at every 1 second, and the samples are quantized via the following quantizer:

$$Q(x) = \begin{cases} 3\,, & \text{if } x \geq 2 \\ 1\,, & \text{otherwise} \end{cases}$$

where x denotes the input to the quantizer.

(a) Plot $s(t)$ and mark all the crucial points on the figure.

(b) Let the samples of $s(t)$ be denoted by random variable X. Obtain the probability mass function (PMF) of X.

(c) Calculate the following conditional expectation: $E[(2X^2 - 3) \,|\, X \geq 1]$.

(d) Find the probability that the quantizer output is equal to 3, i.e., $P(Q(X) = 3)$.

(e) Calculate $D = E[(X - Q(X))^2]$.

(1) 7 a) $P(Y<2 \mid \text{bit 1 sent}) = 1 - Q\left(\frac{2-4}{2}\right) = \boxed{Q(1)}$ $\quad Y \mid \text{bit 1 sent} \sim N(4,4)$

7 b) $P(Y \geq 2 \mid \text{bit 0 sent}) = Q\left(\frac{2-1}{2}\right) = \boxed{Q(0.5)}$ $\quad Y \mid \text{bit 0 sent} \sim N(1,4)$

4 c) $P_e = \boxed{0.2\, Q(0.5) + 0.8\, Q(1)}$

(2) 5 a) 0, 10, 110, 111 → prefix-free, uniquely decodable → can be Huffman ✓

5 b) 1, 01, 10, 000 → not prefix-free → cannot be Huffman ✗

5 c) 00, 10, 01, 11 → prefix-free, uniquely decodable → can be Huffman ✓

5 d) 1, 0, 11, 00 → not prefix-free, not uniquely decodable → cannot be Huffman ✗

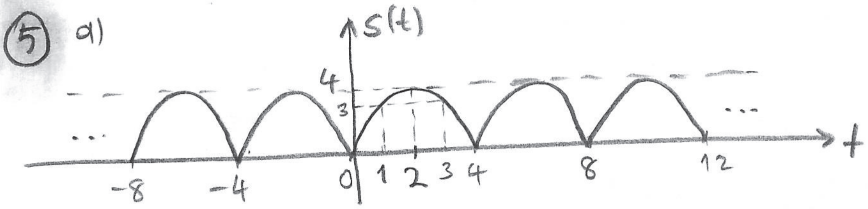
(3) $g(\alpha) = 4\Big(\underbrace{-\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha)}_{\text{binary entropy fn.}}\Big)^2$

4 binary entropy fn. → entropy of a binary DMS w/ prob.s α & $1-\alpha$

↓ max. for $\alpha = 1/2$

8 $g(\alpha) \leq 4\left(-\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2}\right)^2 = \boxed{4}$ ← achieved for $\alpha = 1/2$

(5) a)

5 

9 b)

t	... 0	1	2	3	4	5	6	7	8 ...
$s(t)$	... 0	3	4	3	0	3	4	3	0 ...

$P(X=k) = \begin{cases} 1/2, & \text{if } k=3 \\ 1/4, & \text{if } k=4 \\ 1/4, & \text{if } k=0 \end{cases}$

8 c) $P(X=k \mid X \geq 1) = \begin{cases} 2/3, & \text{if } k=3 \\ 1/3, & \text{if } k=4 \end{cases}$

$E[(2X^2-3) \mid X \geq 1] = (2(3)^2-3)\frac{2}{3} + (2(4)^2-3)\frac{1}{3} = \boxed{\frac{59}{3}}$

4 d) $P(Q(X)=3) = P(X \geq 2) = \boxed{\frac{3}{4}}$

9 e) $D = \frac{1}{2}(3-3)^2 + \frac{1}{4}(4-3)^2 + \frac{1}{4}(0-1)^2 = \boxed{\frac{1}{2}}$

(4) X_1 & X_2 are actually independent. $X_1 \sim N(-2,2)$, $X_2 \sim N(3,4)$

8 $E[2X_1 - (X_2)^2] = 2E[X_1] - (Var(X_2) + (E[X_2])^2) = 2(-2) - (4+3^2) = \boxed{-17}$

7 $P(X_1 > 0 \mid X_2 > 0) \underset{\text{indep.}}{=} P(X_1 > 0) = Q\left(\frac{2}{\sqrt{2}}\right) = \boxed{Q(\sqrt{2})}$