EEE 431: Telecommunications 1

MIDTERM

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Section: ______________________________

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Problem 1. Consider the following signal: $s(t) = \cos(2000\pi t) + \sin(1000\pi t)$. Suppose $s(t)$ is sampled at every 0.25 millisecond for $-\infty < t < \infty$. Let X denote a random variable corresponding to these samples.

(a) Can we reconstruct $s(t)$ from these samples? Why or why not?

(b) Find the probability mass function (PMF) of X.

(c) Calculate the entropy of X.

(d) Perform Huffman coding of X. List all the codewords and calculate the average codeword length.

Problem 2. A source has a probability density function (PDF) as expressed below:

$$f_X(x) = \begin{cases} k\,e^{(x-1)^3}\,, & \text{if } 0 \le x \le 2 \\ k\,e\,x/2\,, & \text{if } 2 < x \le 4 \\ 0\,, & \text{otherwise} \end{cases}$$

where k is a suitable constant (no need to find it). This source is quantized by using the following 2-level quantizer:

$$Q(x) = \begin{cases} 1\,, & \text{if } x \in [0,2] \\ 3\,, & \text{if } x \in (2,4] \end{cases}$$

(a) Obtain the mean squared error distortion $E\{(X - Q(X))^2\}$ in its simplest form.

(b) Calculate the following ratio of probabilities: $\mathrm{P}(Q(X) = 1)/\mathrm{P}(Q(X) = 3)$. Simplify as much as possible.

(c) Suppose that the source produces 20000 samples (outputs) per second, and we would like to transmit the source outputs by using uniform PCM with 1 bit per sample, as specified by the quantizer above. What is the minimum bandwidth required to transmit this PCM signal?

(d) Propose a compander to improve the SQNR of this system. Namely, plot or define a function $g(x)$ corresponding to the "compressor" at the transmitter. Specify the domain and range of function $g(x)$. Why do you think the proposed compander will increase SQNR? (You do not need to calculate the SQNR. Your compander does not need to be optimal.)

Problem 3. Consider a wide sense stationary (WSS) Gaussian random process $X(t)$ with zero mean and autocorrelation function $R_X(\tau) = 2 - |\tau|$ if $|\tau| < 2$ and $R_X(\tau) = 0$ otherwise. This process is input to a system which generates the output process $Y(t)$ given by $Y(t) = X(t) - 3X(t-2)$.

(a) Determine the power spectral density of the input process $X(t)$.

(b) Determine the power spectral density of the output process $Y(t)$.

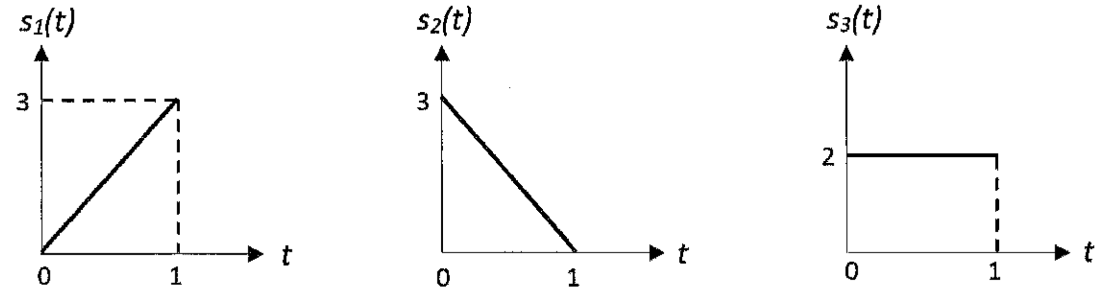
(c) Determine the autocorrelation function of $Y(t)$.

(d) Calculate the average powers of $X(t)$ and $Y(t)$.

(e) Specify the joint probability distribution of $Y(0)$ and $Y(1)$ (i.e., name the distribution and calculate all the necessary parameters).

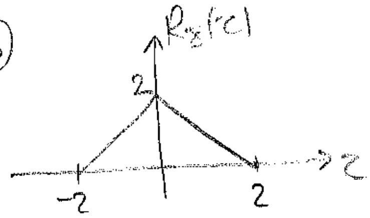
(f) Calculate the following expectation: $E[(2Y(0) - 3Y(1))^2]$.

Problem 4. (a) For the following signals, find a set of orthonormal basis functions (both write down their mathematical expressions and plot them), and represent each signal as a vector in the corresponding signal space.



(b) Find a signal $x(t)$ such that it has the same energy as $s_3(t)$ above, and the angle between $x(t)$ and $s_3(t)$ is equal to $\pi/3$ (i.e., 60 degrees). Both write down the mathematical expression of $x(t)$ and plot it.

32 (3)



(5) (a) $S_X(f) = \mathcal{F}\{R_X(\tau)\}$

$= \boxed{4\,\mathrm{sinc}^2(2f)}$

$$\begin{cases} \Lambda(\tau) \longleftrightarrow \mathrm{sinc}^2 f \\ 2\Lambda\left(\frac{\tau}{2}\right) \longleftrightarrow 4\,\mathrm{sinc}^2(2f) \end{cases}$$

(7) (b) $S_Y(f) = S_X(f)\,|H(f)|^2$ $\qquad Y(f) = X(f)\underbrace{\left(1 - 3e^{-j2\pi f 2}\right)}_{H(f)}$

$= 4\,\mathrm{sinc}^2(2f)\left((1-3\cos(4\pi f))^2 + 9\sin^2(4\pi f)\right)$

$= \boxed{4\,\mathrm{sinc}^2(2f)\left(10 - 6\cos(4\pi f)\right)}$

(6) (c) $R_Y(\tau) = E[Y(t+\tau)Y(t)] = E[(X(t+\tau) - 3X(t+\tau-2))(X(t) - 3X(t-2))]$

$= \boxed{10R_X(\tau) - 3R_X(\tau-2) - 3R_X(\tau+2)}$

(3) (d) $P_X = R_X(0) = \boxed{2}$ (1) $\quad P_Y = R_Y(0) = 10\underbrace{R_X(0)}_{2} - 3R_X(-2) - 3R_X(2) = \boxed{20}$ (2)

(7) (e) $\begin{bmatrix} Y(0) \\ Y(1) \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 & 7 \\ 7 & 20 \end{bmatrix}\right)$ (1) (1) (2) (3)

$E[Y(1)Y(0)] = R_Y(1)$

$= 10R_X(1) - 3R_X(-1) - \cancel{3R_X(3)}$ 0

$= \boxed{7}$

(4) (f) $4E[Y^2(0)] - 12E[Y(0)Y(1)] + 9E[Y^2(1)]$

$= 4\cdot 20 - 12\cdot 7 + 9\cdot 20 = \boxed{206}$

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21 (1) (a) (2) $2f_{max} = 2000$ Hz $\qquad f_{samp} = \dfrac{1}{0.25\,\text{ms}} = 4000$ Hz $\qquad f_{samp} > 2f_{max} \longrightarrow$ Yes ✓

10 (b)

t (ms)	0	0.25	0.5	0.75	1	1.25	1.5	1.75
$s(t)$	1	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$-\frac{1}{\sqrt{2}}$	-2	$-\frac{1}{\sqrt{2}}$

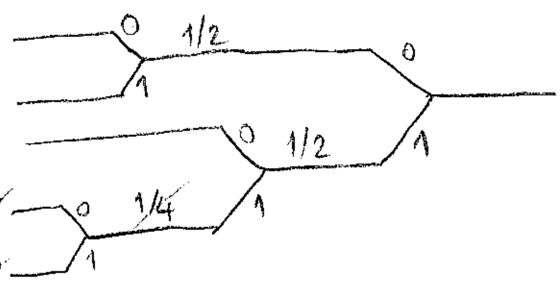
← 1 period

$X = \{-2, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1\}$ w/ probs $\{\frac{1}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}\}$, respectively.

(2) (c) $H(X) = 2\,\frac{1}{8}\log_2 8 + 3\,\frac{1}{4}\log_2 4 = \frac{3}{4} + \frac{6}{4} = \boxed{\frac{9}{4}\text{ bits/sample}}$

(7) (d)

Codeword		
00	1	: 1/4
01	$1/\sqrt{2}$	: 1/4
10	$-1/\sqrt{2}$	: 1/4
110	0	: 1/8
111	-2	: 1/8



Avg. codeword length $= 2\cdot\frac{1}{4}\,3 + 3\,\frac{1}{8}\cdot 2 = \boxed{\frac{9}{4}}$ bits/sample

21 (2) $f_X(x)=\begin{cases} k e^{(x-1)^3}, & 0\le x\le 2 \\ kex/2, & 2<x\le 4 \\ 0, & \text{o.w.}\end{cases}$ $\quad Q(x)=\begin{cases}1, & 0\le x\le 2\\ 3, & 2<x\le 4\end{cases}$

12 (a) $E[(X-Q(X))^2] \overset{(4)}{=} \int_0^2 (x-1)^2 k e^{(x-1)^3}dx + \int_2^4 (x-3)^2 \frac{kex}{2}dx \rightarrow \frac{ke}{2}\int_2^4 (x-3)^3+3(x-3)^2 dx$

$u=(x-1)^3 \rightarrow du=3(x-1)^2dx$

$\overset{(8)}{=} \frac{k}{3} e^u\Big|_{-1}^{1} + \frac{ke}{2}\left(\frac{(x-3)^4}{4}\Big|_2^4 + (x-3)^3\Big|_2^4\right)$

$= \frac{k}{3}(e-e^{-1}) + \frac{ke}{2}\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{4}}+1+1\right) = \boxed{\frac{k}{3}(e-e^{-1})+ke}$

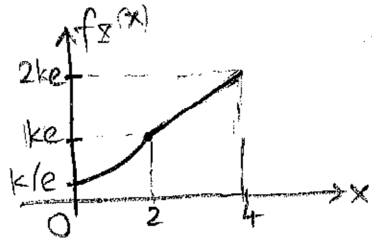
5 (b) $P(Q(X)=1)=1-P(Q(X)=3)$

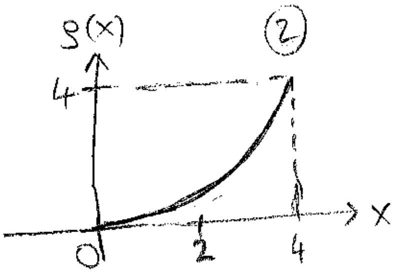
$P(Q(X)=3)=P(X\in(2,4])=\int_2^4 ke\frac{x}{2}dx=\frac{ke}{4}(16-4)=\boxed{3ke}$

so $\frac{P(Q(X)=1)}{P(Q(X)=3)} = \frac{1-3ke}{3ke}$

1 (c) 20000 bits/sec $\rightarrow$ 10000 Hz needed.

3 (d)

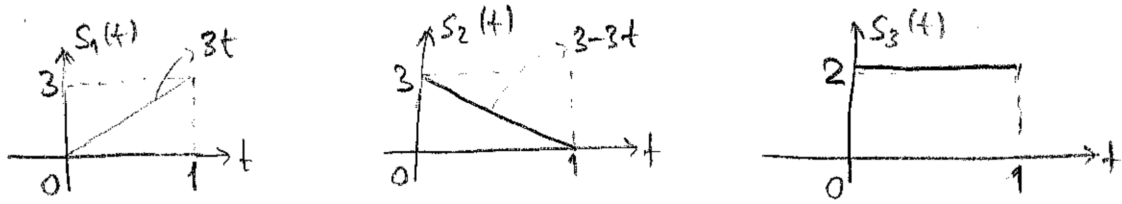




(2)

(1) More prob. values mapped to larger regions

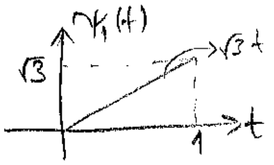
(4)



(13) (a)

$$\psi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \qquad E_1 = \int_0^1 (3t)^2\,dt = 9\,\frac{t^3}{3}\Big|_0^1 = \boxed{3}$$

(4) $$\psi_1(t) = \frac{s_1(t)}{\sqrt{3}} = \boxed{\sqrt{3}\,t}$$



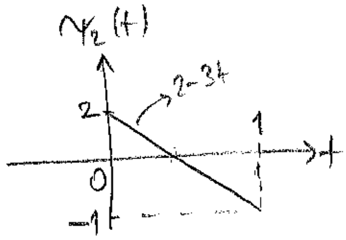
(9)

$$s_{21} = \int_0^1 s_2(t)\psi_1(t)\,dt = \int_0^1 (3-3t)\sqrt{3}\,t\,dt = \left(3\sqrt{3}\,\frac{t^2}{2} - 3\sqrt{3}\,\frac{t^3}{3}\right)\Big|_0^1 = \frac{\sqrt{3}}{2}$$

$$d_2(t) = s_2(t) - s_{21}\psi_1(t) = (3-3t) - \frac{\sqrt{3}}{2}\sqrt{3}\,t = 3 - \frac{9t}{2}$$

$$\int_0^1 \left(3 - \frac{9t}{2}\right)^2 dt = 9 - \frac{27}{2} + \frac{\cancel{81}^{27}}{4}\,\frac{1}{\cancel{3}} = \frac{9}{4}$$

$$\psi_2(t) = \frac{d_2(t)}{\sqrt{9/4}} = \frac{2}{3}\left(3 - \frac{9t}{2}\right) = \boxed{2-3t}$$



Since $s_3(t) = \frac{2}{3}s_1(t) + \frac{2}{3}s_2(t)$, no need for another basis fn.

(6) $$\underline{s}_1 = \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix} \qquad \underline{s}_2 = \begin{bmatrix} \sqrt{3}/2 \\ 3/2 \end{bmatrix} \qquad \underline{s}_3 = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

$$\left(s_{22} = \int_0^1 s_2(t)\psi_2(t)\,dt = \int_0^1 (3-3t)(2-3t)\,dt = \frac{3}{2} \right)$$

(7) (b) $x(t) \rightarrow$ energy 4, $\quad \cos\theta = \frac{1}{2}$

$$\frac{1}{2} = \frac{\langle x(t), s_3(t)\rangle}{\sqrt{4\cdot 4}} \Rightarrow \langle x(t), s_3(t)\rangle = 2 \Rightarrow x_1\sqrt{3} + x_2 1 = 2$$

any x_1, x_2 satisfying this.

Let $\boxed{x(t) = x_1\psi_1(t) + x_2\psi_2(t)}$

e.g. $x_1 = 0$, $x_2 = 2$

Then, $\boxed{x(t) = 2\psi_2(t)}$