EEE 431: Telecommunications I
Homework 7

1) Rank the following three $4-$PAM signal constellations in terms of their power efficiency:

$$\{-3,-1,1,3\}, \ \{-5,-3,-1,1\}, \ \{-2,-1,1,3\}.$$

Also quantify their difference in terms of the signal to noise ratio (in dB) needed for the same level of error probability at high signal to noise ratios.

2) Consider a ternary ($M = 3$) communication system employing the three signals

$$s_1(t) = -\sqrt{6}\ t, \quad s_2(t) = 0, \quad s_3(t) = 2\sqrt{6}\ t, \quad \text{for } t \in [0, 2).$$

The symbol i is transmitted by the signal $s_i(t)$, $i = 1, 2, 3$, and the three symbols are equally likely. The symbol period is $T = 2$ units. Clearly, this is a one dimensional signal set, and a basis for it is $\psi(t) = \sqrt{\frac{3}{8}}\ t$ for $t \in [0, 2)$, and 0 else. Assume that transmission is over an additive white Gaussian noise (AWGN) channel where the noise $n(t)$ is zero mean, and it has a power spectral density $S_n(f) = \frac{N_0}{2}$.

At the receiver, we compute the projection of the received signal $r(t)$ for $t = [0, 2)$ with the basis function and obtain the decision variable $r = \int_0^2 r(t)\psi(t)dt$.

a) Determine the decision rule (based on r) that minimizes the probability of error, and compute the corresponding average symbol error probability. Express your answer in terms of the signal to noise ratio per symbol $\gamma_s = \frac{E_s}{N_0}$ (where E_s is the average energy of the constellation).

b) Assume that the mapping from bits to the three symbols is as follows: $00 \to 1$, $01 \to 2$, $11 \to 3$. Assuming that the decision rule in the previous part is used, determine the average bit error probability. Express your answer in terms of the signal to noise ratio per symbol $\gamma_s = \frac{E_s}{N_0}$ as in part a.

c) For this part, assume that the receiver correlates the received signal with the the function $\psi'(t) = \frac{1}{\sqrt{2}}$ for $t \in [0, 2)$, i.e., it computes $r' = \int_0^2 r(t)\psi'(t)dt$, and it applies a decision rule based on this (incorrect) correlator output r' only. What should be the decision rule so that the probability of error is minimized? What is the corresponding average symbol error probability? Express your answer in terms of γ_s and compare the result with the one in part a.

d) Assume that the receiver uses the correct basis function as in part a, and implements the correlation calculation using a matched filter (matched to the correct basis function). However, instead of taking the output sample at time $t = 2$, it takes it at $t = 1.9$. Assume also that the the transmission takes place in isolation, i.e., only a single symbol is transmitted with no neighboring symbols, i.e., there is no contribution from any previously transmitted or future symbols to the output sample of the matched filter. Except for the incorrect sampling instance, the receiver is the same as the one in part a. Determine the average symbol error probability (in terms of E_s/N_0 and compare your result with the one in part a).

3) Three symbols (m_1, m_2 and m_3) are being transmitted using pulse amplitude modulation over an additive white Gaussian noise (AWGN) channel with power spectral density $N_0/2$. Assume that m_i is transmitted using the signal $s_i(t) = A_i g(t)$ where $g(t)$ is a pulse defined on the interval $[0, T)$ (T being the symbol period), and $A_1 = -A$, $A_2 = 0$ and $A_3 = A$ for some $A > 0$. Also assume that the prior probabilities of m_1 and m_3 are p, while the prior probability of m_2 is $1-2p$ (where $0 \le p \le 1/2$), i.e., the symbols are not equally likely. Denote the energy of the pulse $g(t)$ by E_g, and the received signal by $r(t)$ (for $t \in [0, T)$).

a) Determine the optimal decision rule that minimizes the average symbol error probability.

b) Determine the range of values for p (in terms of the parameters A, E_g, N_0) for which the receiver never decides m_2. What is the decision rule in this case?

4) Consider transmission of four equally likely symbols $(1, 2, 3$ and $4)$ using digital modulation over an additive white Gaussian noise (AWGN) channel with noise power spectral density $S_n(f) = \frac{N_0}{2}$. Assume that the signal space is two dimensional and an orthonormal basis to it is given by $\psi_1(t)$ and $\psi_2(t)$. Also assume that to transmit the symbol i, the signal $s_i(t)$ with the vector representation $\mathbf{s}_i$ is used. Also, $\mathbf{s}_1 = [2 \;\; 1]^T$, $\mathbf{s}_2 = [-2 \;\; 2]^T$, $\mathbf{s}_3 = [2 \;\; -2]^T$, and $\mathbf{s}_4 = [-2 \;\; -1]^T$. The receiver projects the received signal onto the signal space and obtains its vector representation as $\mathbf{r} = [r_1 \;\; r_2]^T$ (with respect to the same basis $\{\psi_1(t), \psi_2(t)\}$).

a) Assume that the decision rule adopted at the receiver is as follows: if $r_1, r_2 > 0$, select "1"; if $r_1 < 0,\ r_2 > 0$, select "2"; if $r_1 > 0, r_2 < 0$, select "3"; if $r_1, r_2 < 0$, select "4". Note that this decision rule is not optimal.

   Determine the average symbol error probability. Express your answer in terms of average signal to noise ratio per symbol E_s/N_0.

b) Determine the decision region for the first symbol $\mathbf{s}_1$ for the optimal decision rule, i.e., the one that minimizes the average symbol error probability.

c) Determine the union bound on the error probability corresponding to the optimal decision rule (do not use the loose version of the bound), and compare your answer to the error probability computed in part a for high signal to noise ratios.

5) Four equally likely symbols are transmitted using QPSK over an AWGN channel. Assuming that the mapping of bits to symbols follows natural binary coding, i.e., 00, 01, 10, 11 are mapped to the phases $0, \pi/2, \pi, 3\pi/2$, respectively, determine the average bit error probability in terms of the signal to noise ratio per bit $\gamma_b = E_b/N_0$. Compare the result with the case of Gray mapping.

6) Compare the 16−PSK and 16−QAM (with a square constellation) in terms of their error rate performance by comparing their (normalized) minimum distances. Which one is more power efficient? How much is the difference between their performance at high signal to noise ratios?

7) Four equally likely signals are transmitted over an additive noise channel using the waveforms $s_1(t) = 0$, $s_2(t) = -2A\cos(2\pi f_0 t)$, $s_3(t) = 2A\sin(2\pi f_0 t)$ and $s_4(t) = 2A\cos(2\pi f_0 t)$ for $t \in [0, T_s]$; T_s being the symbol period. The noise is white Gaussian with power spectral density $N_0/2$, and $f_0 = N/T$ where $N \gg 1$ is an integer. Assume that the receiver employs coherent detection, and there is no synchronization error, i.e., the received signal is given by $r(t) = s_i(t) + n(t)$ where $s_i(t)$ is the transmitted signal and $n(t)$ is the noise process.

a) Find an orthonormal set of basis functions for this signal set and plot the signal constellation.

b) Describe the optimal receiver structure (give a complete block diagram, and be specific).

c) Determine the exact (conditional) probability of error given that $s_1(t)$ is transmitted. Express your answer in terms of $\bar{E}_b/N_0$ where $\bar{E}_b$ is the average energy per bit.