FALL 2021 – EEE 473/573 Medical Imaging
HW2 Solutions

1.

$$F(q) = 2\pi \int_0^{\infty} rect\left(\frac{r}{a}\right) J_0(2\pi qr) rdr$$
$$= 2\pi \int_0^{a/2} J_0(2\pi qr) rdr$$

Change of variables such that:

$$s = 2\pi qr$$
$$r = {}^{s}/_{2\pi q}$$
$$dr = {}^{ds}/_{2\pi q}$$
$$\frac{s}{2\pi q} = \frac{a}{2} \Rightarrow s = \pi qa$$

Then,

$$F(q) = 2\pi \int_0^{\pi qa} \frac{J_0(s)s}{2\pi q} \frac{ds}{2\pi q}$$
$$= \frac{1}{2\pi q^2} \int_0^{\pi qa} J_0(s) s ds$$

Here, we use the hint,

$$F(q) = \frac{1}{2\pi q^2} \pi qa J_1(\pi qa) = \frac{a}{2q} J_1(\pi qa)$$

Using, $\frac{J_1(\pi x)}{2x} = jinc(x)$,

$$F(q) = \frac{a}{a}\frac{a}{2q} J_1(\pi qa) = a^2 \frac{J_1(\pi qa)}{2qa} = a^2 jinc(qa)$$

2.

a. For the first system:

$$H_1(u,v) = 6e^{-\pi(4u^2+9v^2)}$$
$$MTF_1(u,v) = \frac{|H_1(u,v)|}{H_1(0,0)} = e^{-\pi(4u^2+9v^2)}$$

For the second system:

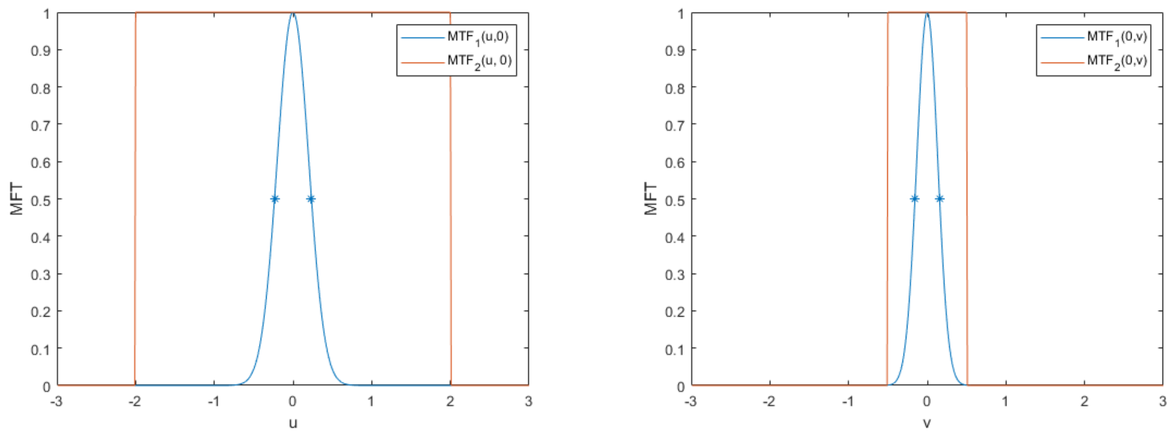
$$H_2(u,v) = \frac{1}{4} rect\left(\frac{u}{4}, v\right)$$
$$MTF_2(u,v) = \frac{|H_2(u,v)|}{H_2(0,0)} = rect\left(\frac{u}{4}, v\right)$$

b.

$$MTF_1(u,0) = \frac{1}{2} \Rightarrow u = \pm\sqrt{\frac{\ln(2)}{4\pi}}$$

$$MTF_1(0,v) = \frac{1}{2} \Rightarrow v = \pm\sqrt{\frac{\ln(2)}{9\pi}}$$

$MTF_2(u,v)$ does not have a specific ½ point since it's a rect function.



c. Modulation of the object is: $m_f = \frac{5-1}{5+1} = \frac{2}{3}$

Let $g(x,y) = f(x,y) * h(x,y) = F_{2D}^{-1}\{F(u,v)H(u,v)\}$

$$= F_{2D}^{-1}\left\{\left[3\delta(u,v) + \frac{2}{2j}\big(\delta(u-1,v-1) - \delta(u+1,v+1)\big)\right]H(u,v)\right\}$$

$$= F_{2D}^{-1}\left\{\left[3\delta(u,v)H(0,0) + \frac{2}{2j}\big(\delta(u-1,v-1)H(1,1) - \delta(u+1,v+1)H(-1,-1)\big)\right]\right\}$$

$H_1(u,v)$, is a real and even function, i.e. $H_1(u,v) = H_1(-u,-v)$, then,

$$g_1(x,y) = F_{2D}^{-1}\left\{\left[3\delta(u,v)H_1(0,0) + \frac{2H_1(1,1)}{2j}\big(\delta(u-1,v-1) - \delta(u+1,v+1)\big)\right]\right\}$$
$$= 3H_1(0,0) + 2H_1(1,1)\sin\big(2\pi(x+y)\big)$$

then,

$$m_{g_1} = \frac{2}{3}\frac{H_1(1,1)}{H_1(0,0)} = \frac{2}{3}MTF_1(1,1) = \frac{2}{3}e^{-\pi 13}$$

$H_2(u,v)$ is a real and even function, i.e. $H_2(u,v) = H_2(-u,-v)$, then,

$$g_2(x,y) = F_{2D}^{-1}\left\{\left[3\delta(u,v)H_2(0,0) + \frac{2H_2(1,1)}{2j}\big(\delta(u-1,v-1) - \delta(u+1,v+1)\big)\right]\right\}$$
$$= 3H_2(0,0) + 2H_2(1,1)\sin\big(2\pi(x+y)\big)$$

then,

$$m_{g_2} = \frac{3}{2}\frac{H_2(1,1)}{H_2(0,0)} = \frac{2}{3}\,MTF_2(1,1) = 0$$

3.

a. FWHM of $h_1(x,y)$ along the x- and y- directions:

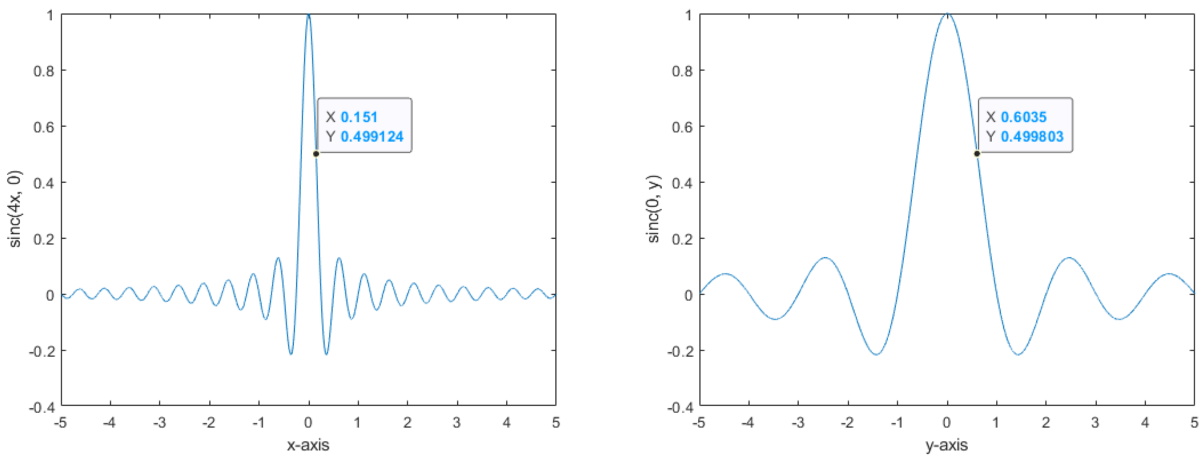
$$h_1(x,0) = e^{-\frac{\pi x^2}{4}} = \frac{1}{2}\ x = \pm\sqrt{\frac{4\ln(2)}{\pi}} \Longrightarrow FWHM_{h_1,x} = 2\sqrt{\frac{4\ln(2)}{\pi}} =\ 1.88$$

Similarly,

$$h_1(0,y) = e^{-\frac{\pi y^2}{9}} = \frac{1}{2}$$

$$x = \pm\sqrt{\frac{9\ln(2)}{\pi}} \Longrightarrow FWHM_{h_1,y} = 2\sqrt{\frac{9\ln(2)}{\pi}} = 2.82$$

$h_2(x,y) = sinc(4x,y)$ does not have an analytical solution, so we will check their plots from Matlab:



$$FWHM_{h_2,x} = 0.151 \times 2 \approx 0.30$$
$$FWHM_{h_2,y} = 0.6032 \times 2 \approx 1.206$$

An alternative solution could be using the Taylor series expansion of sine, up to 2nd term yields similar results.

b. FWHM of the cascaded system is:

$$FWHM_x = \sqrt{FWHM_{h_1,x}^2 + FWHM_{h_2,x}^2} =\ 1.9$$

$$FWHM_y = \sqrt{FWHM_{h_1,y}^2 + FWHM_{h_2,y}^2} =\ 3.1$$

4.

a.

$$Prevelance = \frac{183 + 320}{183 + 72 + 7425 + 320} \cong 6.29\%$$

$$sensitivity = \frac{183}{183 + 320} \cong 36.4\%$$

$$specifity = \frac{7425}{7425 + 72} \cong 99\%$$

$$PPV\ = \frac{183}{183 + 72} \cong 71.8\%$$

$$NPV = \frac{7425}{7425 + 320} \cong 95.9\%$$

b.

$$Prevelance = \frac{143 + 360}{143 + 39 + 7458 + 360} \cong 6.29\%$$

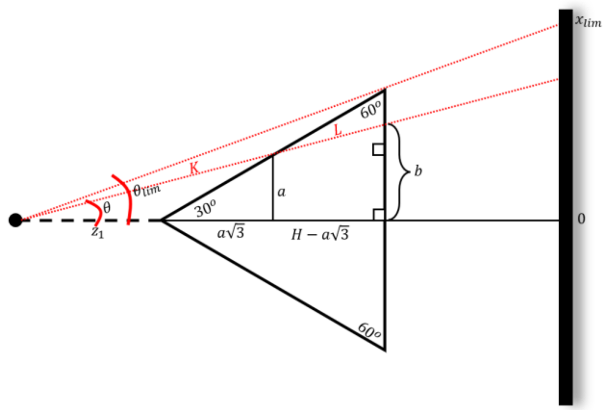
$$sensitivity = \frac{143}{143 + 360} \cong 28.4\%$$

$$specifity = \frac{7458}{7458 + 39} \cong 99.5\%$$

$$PPV = \frac{143}{143 + 39} \cong 78.6\%$$

$$NPV = \frac{7458}{7458 + 360} \cong 95.4\%$$

5.



θ is dependent on the position on the detector:

$$\tan(\theta) = \frac{x}{d} = \frac{a}{z_1 + a\sqrt{3}} \Rightarrow a = \frac{z_1 \tan(\theta)}{1 - \tan(\theta)\sqrt{3}}.$$

Using the triangle properties, the length b is:

$$b = \frac{a(z_1 + H)}{z_1 + a\sqrt{3}} = \tan(\theta)\,(z_1 + H),$$

and K is:

$$K = \frac{a}{\sin(\theta)},$$

Finally, L can be found as:

$$L = \frac{b}{\sin(\theta)} - \frac{a}{\sin(\theta)} = \left(z_1 + H - \frac{z_1}{1 - \tan(\theta)\sqrt{3}}\right)\frac{\tan(\theta)}{\sin(\theta)}$$

$$= \left(z_1 + H - \frac{z_1 d}{d - x\sqrt{3}}\right)\frac{1}{\cos(\theta)}$$

Then, the intensity on the detector is:

$$I_d(x,0) = \begin{cases} I_0 \cos^3(\theta)\, e^{-\mu_0 L}, & |x| \le x_{lim} \\ I_0 \cos^3(\theta) & , otherwise \end{cases}$$

with $x_{lim} = \frac{dH}{(z_1+H)\sqrt{3}}$.

a. Turning all variables to cm for $z_1 = 50cm$ yields:

$$I_d(x,0) = \begin{cases} I_0 \cos^3(\theta)\, e^{-0.1\left(60-\frac{5000}{100-x\sqrt{3}}\right)\frac{1}{\cos(\theta)}}, & |x| \le \frac{100}{6\sqrt{3}} \\ I_0 \cos^3(\theta) & , otherwise \end{cases}$$

with $\theta = \arctan\left(\frac{x}{100}\right)$.

b. Turning all variables to cm for $z_1 = 80cm$ yields:

$$I_d(x,0) = \begin{cases} I_0 \cos^3(\theta)\, e^{-0.1\left(90-\frac{8000}{100-x\sqrt{3}}\right)\frac{1}{\cos(\theta)}}, & |x| \le \frac{100}{9\sqrt{3}} \\ I_0 \cos^3(\theta) & , otherwise \end{cases}$$

with $\theta = \arctan\left(\frac{x}{100}\right)$.

6.