Q2

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a) $g(x,y) = f(x,-1) + f(0,y)$

Let $f_1(x,y) = \sum_{k=1}^{K} w_k f_k(x,y)$ be the input.

Then, $g_1(x,y) = f_1(x,-1) + f_1(0,y)$

$$= \sum_{k=1}^{K} w_k f_k(x,-1) + \sum_{k=1}^{K} w_k f_k(0,y)$$

$$= \sum_{k=1}^{K} w_k [f_k(x,-1) + f_k(0,y)] = \sum_{k=1}^{K} w_k g_k(x,y), \text{ where } f_k \xrightarrow{S} g_k$$

Hence, this system is $\boxed{\text{LINEAR}}$.

Let $f_2(x,y) = f(x-x_0, y-y_0)$ be the input.

Then, $g_2(x,y) = f_2(x,-1) + f_2(0,y) = f(x-x_0,-1-x_0) + f(-x_0, y-y_0) \neq g(x-x_0, y-y_0)$

because $g(x-x_0, y-y_0) = f(x-x_0,-1) + f(0, y-y_0)$. (Function only works on (x,y))

Hence, this system is $\boxed{\text{SHIFT-VARIANT}}$.

b) $g(x,y) = \max(f(x,y), 0)$

Let $f_1(x,y) = \sum_{k=1}^{K} w_k f_k(x,y)$ be the input.

Then, $g_1(x,y) = \max(f_1(x,y), 0)$

$$= \max\left(\sum_{k=1}^{K} w_k f_k(x,y), 0\right) \neq \sum_{k=1}^{K} w_k \max(f_k(x,y), 0)$$

For instance, for $K=2$, $w_k = \begin{cases} -5, & k=1 \\ 6, & k=2 \\ 0, & \text{o.w.} \end{cases}$, and $f_k(x,y) = \begin{cases} -1, & k=1 \\ 1, & k=2 \\ 0, & \text{o.w.} \end{cases}$, linearity requirement does not hold.

$\max\left(\sum_{k=1}^{K} w_k f_k(x,y), 0\right) = 11$, but $\sum_{k=1}^{K} w_k \max(f_k(x,y), 0) = 6$

Hence, this system is $\boxed{\text{NONLINEAR}}$.

Let $f_2(x,y) = f(x-x_0, y-y_0)$ be the input.

Then, $g_2(x,y) = \max(f_2(x,y), 0) = \max(f(x-x_0, y-y_0), 0) = g(x-x_0, y-y_0)$

because $g(x-x_0, y-y_0) = \max(f(x-x_0, y-y_0), 0)$ (Function only works on (x,y))

Hence, this system is $\boxed{\text{SHIFT-INVARIANT}}$.

$f(x,y) = e^{j2\pi(x+y)}$, $x_1 = 9$, $y_1 = 5$ Q3

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a) $f(x,y)\delta(x-x_1, y+y_1)$

$$f(x,y)\delta(x-9,y+5) \overset{\text{Sampling Property}}{=} f(9,-5)\delta(x-9,y+5)$$

$$= \boxed{e^{j2\pi 4}\delta(x-9,y+5)}$$

b) $f(x,y) * \delta(x-x_1, y+y_1)$

$$f(x,y) * \delta(x-9,y+5) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\varepsilon,\eta)\delta(x-9-\varepsilon, y+5-\eta)\,d\varepsilon\,d\eta$$

$$\overset{\text{Sampling}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-9,y+5)\delta(x-9-\varepsilon,y+5-\eta)\,d\varepsilon\,d\eta$$

This is also known as the sifting property of the Dirac delta function

$$= f(x-9,y+5)\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-9-\varepsilon,y+5-\eta)\,d\varepsilon\,d\eta}_{=1 \text{ (Definition of Dirac delta)}}$$

$$= f(x-9,y+5) = e^{j2\pi(x-9+y+5)} = \boxed{e^{-j2\pi 4}e^{j2\pi(x+y)}}$$

c) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-x_1, 3y+y_1)f(x,y)\,dx\,dy$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-9,3y+5)f(x,y)\,dx\,dy \overset{\text{Scaling Property}}{=} \frac{1}{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-9,y+\tfrac{5}{3})f(x,y)\,dx\,dy$$

$$\overset{\text{Sampling}}{=} \frac{1}{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-9,y+\tfrac{5}{3})f(9,-\tfrac{5}{3})\,dx\,dy$$

$$= \frac{1}{3}f(9,-\tfrac{5}{3})\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-9,y+\tfrac{5}{3})\,dx\,dy}_{1}$$

$$= \frac{1}{3}f(9,-\tfrac{5}{3}) = \boxed{\frac{1}{3}e^{j2\pi(\frac{22}{3})}}$$

d) $f(x+1,-y_1) * \delta(x-x_1, y+1)$

$$f(x+1,-5) * \delta(x-9,y+1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\varepsilon+1,-5)\delta(x-9-\varepsilon,y+1-\eta)\,d\varepsilon\,d\eta$$

$$\overset{\text{Sampling}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-8,-5)\,\delta(x-9-\varepsilon,y+1-\eta)\,d\varepsilon\,d\eta$$

$$= f(x-8,-5)\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-9-\varepsilon,y+1-\eta)\,d\varepsilon\,d\eta}_{1}$$

$$= f(x-8,-5) = e^{j2\pi(x-8-5)} = \boxed{e^{-j2\pi 13}e^{j2\pi x}}$$

$x_1 = 9,\ y_1 = 5$ [Q4]

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a) $g(x,y) = \delta\left(\frac{x}{x_1}, y_1 y - 1\right)$

$g(x,y) = \delta\left(\frac{x}{9}, 5y-1\right) \xleftrightarrow[\text{Known pair}]{\text{2D FT}} \frac{1}{\frac{1}{9}\cdot 5} e^{-j2\pi(9u\cdot 0 + \frac{1}{5}v\cdot 1)}$

$= \boxed{\frac{9}{5} e^{-j2\pi\frac{v}{5}}}$ $(G(u,v))$

Scaling and time-shifting properties of 2DFT are used

b) $g(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{j2\pi(nx+my)}$

$g(x,y)$ is separable because there are functions $g_1(x)$ and $g_2(y)$, s.t.,

$g(x,y) = g_1(x) g_2(y)$, where $g_1(x) = \sum_{n=-\infty}^{\infty} e^{j2\pi nx}$, $g_2(y) = \sum_{m=-\infty}^{\infty} e^{j2\pi my}$

$g(x,y)$ is separable $\Leftrightarrow G(u,v) = G_1(u) G_2(v)$

$g_1(x) \xleftrightarrow{\text{1D FT}} G_1(u) = \sum_{n=-\infty}^{\infty} \delta(u-n)$

$g_2(y) \xleftrightarrow{\text{1D FT}} G_2(v) = \sum_{m=-\infty}^{\infty} \delta(v-m)$

This result can also be seen from another perspective: Fourier series representation of an impulse train with a period of 1 is $g_1(x)$. So, its 1D Fourier transform is also an impulse train with a period of 1 in the spectral domain.

$\Rightarrow G(u,v) = G_1(u) G_2(v) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \delta(u-n)\delta(v-m)$

Hence, $\boxed{G(u,v) = \text{comb}(u,v)}$

c) $g(x,y) = \text{sinc}(3x + x_1, y_1 y - 1)$

$g(x,y) = \text{sinc}(\underbrace{3x+9}_{3(x+3)}, \underbrace{5y-1}_{5(y-\frac{1}{5})}) \xleftrightarrow[\text{Known pair}]{\text{2D FT}} \frac{1}{3.5} e^{j2\pi 3u} \cdot e^{-j2\pi\frac{v}{5}}\, \text{rect}\left(\frac{u}{3}, \frac{v}{5}\right)$

$= \boxed{\frac{1}{15} e^{j2\pi(3u - \frac{v}{5})} \text{rect}\left(\frac{u}{3}, \frac{v}{5}\right)}$ $(G(u,v))$

Scaling and time-shifting properties of 2DFT are used.

d) $g(x,y) = \text{rect}\left(x_1 x, \frac{y}{y_1}\right) e^{j2\pi(u_0 x + 4v_0 y)}$

$g(x,y) = \text{rect}\left(\underbrace{9x}_{9(x)}, \underbrace{\frac{y}{5}}_{\frac{1}{5}(y)}\right) e^{j2\pi(u_0 x + 4v_0 y)} \xleftrightarrow{\text{2D FT}} \frac{1}{9\cdot\frac{1}{5}} \text{sinc}\left(\frac{u-u_0}{9}, 5(v-4v_0)\right)$

$= \boxed{\frac{5}{9} \text{sinc}\left(\frac{u-u_0}{9}, 5(v-v_0)\right)}$ $(G(u,v))$

Scaling and frequency-shifting properties of 2DFT are used.

e) $g(x,y) = e^{-2\pi(4x^2+y^2)} * \cos(2\pi x + \pi y)$

$$e^{-2\pi(4x^2+y^2)} = e^{-\pi((2\sqrt{2}x)^2 + (\sqrt{2}y)^2)}$$

Known pair: $f(x,y) = e^{-\pi(x^2+y^2)} \xleftrightarrow{2DFT} F(u,v) = e^{-\pi(u^2+v^2)}$

$$\Rightarrow f(2\sqrt{2}x, \sqrt{2}y) = e^{-2\pi(4x^2+y^2)} \xleftrightarrow{2DFT} \frac{1}{2\sqrt{2}\cdot\sqrt{2}} F\left(\frac{u}{2\sqrt{2}}, \frac{v}{\sqrt{2}}\right)$$ Scaling Property

$$= \boxed{\frac{1}{2} e^{-\pi\left(\frac{u^2}{8} + \frac{v^2}{2}\right)}}$$

$$\cos(2\pi x + \pi y) \xleftrightarrow{2DFT} \frac{1}{2}\left[\delta\left(u-1, v-\frac{1}{2}\right) + \delta\left(u+1, v+\frac{1}{2}\right)\right]$$

Hence, $G(u,v) = \frac{1}{4} e^{-\pi\left(\frac{u^2}{8} + \frac{v^2}{2}\right)}\left[\delta\left(u-1, v-\frac{1}{2}\right) + \delta\left(u+1, v+\frac{1}{2}\right)\right]$

$$= \frac{1}{4} e^{-\pi\left(\frac{1}{8}+\frac{1}{8}\right)}\delta\left(u-1, v-\frac{1}{2}\right) + \frac{1}{4} e^{-\pi\left(\frac{1}{8}+\frac{1}{8}\right)}\delta\left(u+1, v+\frac{1}{2}\right)$$

$$= \boxed{\frac{e^{-\frac{\pi}{4}}}{4}\left[\delta\left(u-1, v-\frac{1}{2}\right) + \delta\left(u+1, v+\frac{1}{2}\right)\right]}$$

f) $F(u,v) = \mathcal{F}_{2D}(f(x,y))$ and $f(x)$ is a real-valued function.

i) $f(x,y) = f(-x,-y) \overset{?}{\Rightarrow} F^*(u,v) = F(u,v)$

$f(x,y)$ is real $\Leftrightarrow f^*(x,y) = f(x,y)$

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\,dy \Rightarrow F^*(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{j2\pi(ux+vy)} dx\,dy$$

$\downarrow \begin{matrix} x=-x \\ y=-y \end{matrix}$

Since $f(x,y) = f(-x,-y)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(-x,-y) e^{-j2\pi(ux+vy)} dx\,dy$$

$\downarrow f(x,y) = f(-x,-y)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\,dy$$

$$= F(u,v) \Rightarrow \boxed{F^*(u,v) = F(u,v)}$$

ii) $f(x,y)$ is real $\Leftrightarrow f^*(x,y) = f(x,y)$

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\,dy \Rightarrow F^*(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{j2\pi(ux+vy)} dx\,dy$$

$\begin{matrix} x=-x \\ y=-y \end{matrix} \downarrow$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(-x,-y) e^{-j2\pi(ux+vy)} dx\,dy$$

$f(x,y) = -f(-x,-y)$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\,dy = -F(u,v)$$

$$\Rightarrow \boxed{F^*(u,v) = -F(u,v)}$$