Ultrasound wave travels at the speed of sound

$$c = \sqrt{\frac{1}{K\rho}}$$

K : compressibility $(m.s^2/kg)$ } properties of material
ρ : density (kg/m^3)

$c = 330$ m/s in air

$c = 1480$ m/s in water

$c = 4080$ m/s in bone

$c = 1540$ m/s in tissue (average value)

* $\lambda = \frac{c}{f}$: wavelength

Plane waves

an acoustic wave that varies only in one spatial direction and time.

$$p(x,y,z,t) = p(z,t)$$

p ↑ acoustic pressure

$p(z,t)$ ↖ plane wave moving in $+z$ or $-z$ direction

Transducer

This wave has to satisfy the wave equation:

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \quad \text{(in 3D)} \qquad \left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

Laplacian

(reduce to 1D for $p(z,t)$)

$$\frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

General solution is:

$$p(z,t) = \underbrace{\phi_f(t - z/c)}_{\text{forward-traveling wave}} + \phi_b(t + z/c)$$

$\phi_b(t+z/c)$ ↖ backword-traveling wave

we are interested in forward traveling wave

e.g., $p(z,t) = \cos\left(k(z - ct)\right)$

↳ for constant z, pressure varies sinusoidally with $\omega = kc$

k ← called "wave number"

$$f = \frac{\omega}{2\pi} = \frac{kc}{2\pi}, \quad \lambda = \frac{c}{f} = \frac{2\pi}{k}$$

Example: $f = 2$ MHz sinusoidal wave. What is λ?

average $c = 1540$ m/s in tissue

$$\lambda = \frac{c}{f} = \frac{1540 \text{ m/s}}{2 \cdot 10^6 \text{ 1/s}} = 0.77 \text{ mm}$$

Example: An ultrasound transducer is pointing down the $+z$ axis. At time $t=0$, it starts to generate an acoustic pulse with form

$$\phi(t) = (1 - e^{-t/\tau_1})\, e^{-t/\tau_2}$$

* what is the forward-traveling wave? At what time does the leading edge of the impulse hit an interface 10 cm away from the transducer?

$$\phi_f(z,t) = \left(1 - e^{-(t - z/c)/\tau_1}\right) e^{-(t - z/c)/\tau_2}$$

it will take $\Delta t = \dfrac{10 \text{ cm}}{1540 \text{ m/s}} = 64.9\ \mu s$

Acoustic intensity: $I = \dfrac{p^2}{Z}$ ← pressure

intensity (or acoustic energy flux)

Z ← characteristic impedance of material ($Z = \rho . c$, ρ: density)

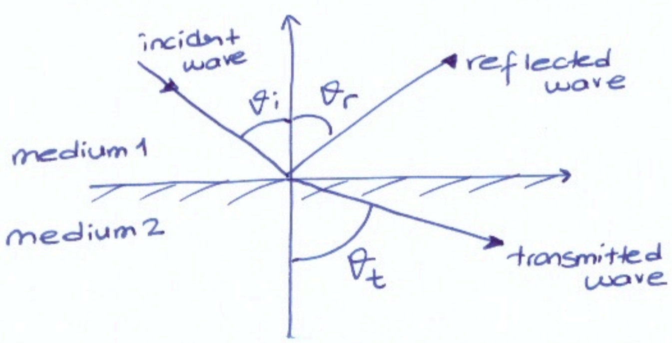
Analogy to electrical circuits:

pressure ⟷ voltage

Z ⟷ resistance

I ⟷ power

Reflection and Refraction at Plane Interfaces



$\theta_i = \theta_r$ (reflection)

Snell's Law:

$$\frac{\sin \theta_i}{\sin \theta_t} = \frac{c_1}{c_2}$$

Example: medium 1: fat , $\theta_i = 45°$, $\theta_r = ?$, $\theta_t = ?$
medium 2: liver

* $\theta_r = \theta_i = 45°$

$c_1 = 920$ m/s in fat
$c_2 = 1060$ m/s in liver
(Table 10.1)

$$\sin\theta_t = \sin\theta_i \cdot \frac{c_2}{c_1} = \frac{1}{\sqrt{2}} \cdot \frac{1060}{920} = 0.8147$$

$$\theta_t = 54.6°$$

Reflectivity:

$$R_I = \frac{I_r}{I_i} = \left(\frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}\right)^2 \quad \text{, intensity reflectivity}$$

$$T_I = \frac{I_t}{I_i} = \frac{4Z_1 Z_2 \cos^2\theta_i}{(Z_2\cos\theta_i + Z_1\cos\theta_t)^2} \quad \text{, intensity transmittivity}$$

Example: what fraction of acoustic intensity is reflected back from a fat/liver interface at normal incidence?

(error in book) in results

* $\theta_i = 0 \Rightarrow R_I = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2$

if $Z_1 = Z_2$, $R_I = 0$ no reflection!
if $Z_1 >> Z_2$, $R_I \approx 1$ all reflected
→ same for $Z_2 >> Z_1$

Using Table 10.1, $Z_{liver} = 1.66 \times 10^{-6}$ kg/m²/s
$Z_{fat} = 1.35 \times 10^{-6}$ kg/m²/s

$$R_I = \left(\frac{1.66 - 1.35}{1.66 + 1.35}\right)^2 = 0.0106$$

→ ~1 % reflected
99% transmitted through

* How about muscle/bone interface?

$Z_{bone} \sim 6 \times 10^{-6}$ kg/m²/s
$Z_{muscle} \sim 1.7 \times 10^{-6}$ kg/m²/s

$$R_I = \left(\frac{6 - 1.7}{6 + 1.7}\right)^2 = 0.32$$

→ 32% reflected
→ more difficult to see behind bone

Attenuation: amplitude of the wave decreases as wave propagates

due to absorption (conversion to thermal energy), scattering, mode conversion

$$p(z,t) = \underbrace{A_0 \cdot e^{-\mu_a z}}_{A_z} \cdot f(t - z/c)$$

μ_a: amplitude attenuation coefficient (cm⁻¹)

$$\frac{A_z}{A_0} = e^{-\mu_a z} \rightarrow 20\log_{10}\left(\frac{A_z}{A_0}\right) = -\mu_a z \cdot 20\log_{10} e$$

$$\text{(attenuation in dB)} = -z \cdot \underbrace{20(\log_{10} e)\mu_a}_{\alpha}$$

↓
phenomenological
(agrees with practice, but not easily supported by theory)

$\alpha \simeq 8.7 \mu a$

↳ attenuation coefficient in dB/cm.
(or absorption coefficient)

← very large for air & bones !!

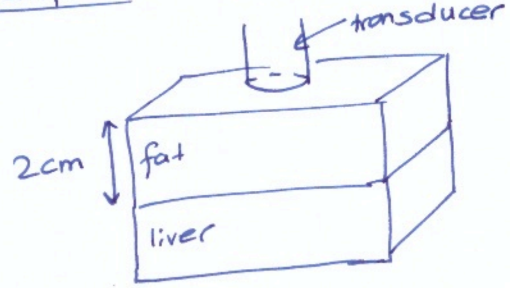
* α depends on frequency

$\alpha \approx a.f$ (approximately)

↳ dB/cm/MHz

* more attenuation at higher frequencies !!

Example : 5 MHz acoustic pulse



When does the "echo" reflected from the fat/liver interface arrive back at the transducer?

$c_{fat} = 1450$ m/s

$$t = \frac{2 \times 2\text{ cm}}{1450\text{ m/s}} = 27.6\ \mu s$$

(round-trip)

* What is the amplitude loss of the returning wave?

from before, $R_I = 0.103$ for fat/liver interface

↗ intensity reflectivity. remember $I = \frac{p^2}{z}$

So, amplitude reflectivity $= \sqrt{0.0106} = 0.103$

↳ convert to dB : $20 \log_{10}(0.103) = -19.7$ dB

(can also do $10.\log_{10}(0.0106) = -19.7$ dB)

Also, attenuation :

$a_{fat} = 0.63$ dB/cm/MHz

$\alpha = 0.63 \times 5 = 3.15$ dB/cm

* So, for 4 cm round-trip : $4 \times 3.15 = 12.6$ dB attenuation

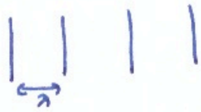
* In total

dB loss $= 20 \log_{10} \frac{A_z}{A_0} = -19.7 - 12.6 = -32.3$ dB

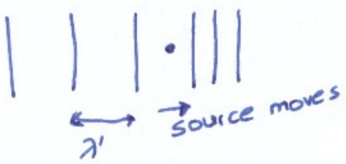
Doppler Effect

Change in frequency of sound due to relative motion of the source and receiver
(ambulance siren : approaching vs. moving away)
higher pitch ↑, lower pitch ↑

* Distance between wave peaks : $\lambda = \frac{c}{f_0}$



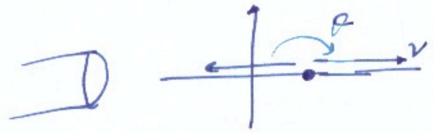
* if source moves : $\lambda' = \frac{c}{f_0} + \underbrace{\frac{v}{f_0}}_{\text{speed of source}} = \frac{c}{f'} \Rightarrow f' = \frac{c}{c+v} f_0$



more generally : $f' = \frac{c}{c - v\cos\theta} f_0$

$$f_D = f' - f_0 = \left(\frac{v\cos\theta}{c - v\cos\theta}\right) f_0$$

(Doppler freq)



* for $v << c$,

$$f_D \approx \frac{v\cos\theta}{c} \cdot f_0$$

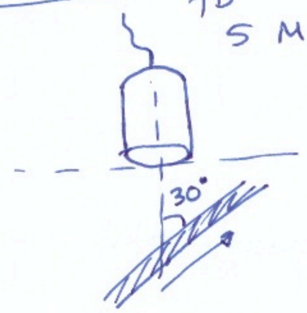
* In ultrasound this is double due to round-trip from source to object

$$f_D = \frac{2v \cdot \cos\theta}{c} f_0$$

if object moving away, $\theta = 180°$ → f_D is negative (lower pitch)

if " " closer, $\theta = 0$ → f_D positive (higher pitch)

Example : $f_D = +500$ Hz measured
5 MHz transducer, makes 30° angle with direction of blood motion in a vessel



$f_D > 0$ → moving toward transducer

$$V = \frac{c f_D}{2 . \cos\theta . f_0} = \frac{500\,Hz . 1540\,m/s}{2 . \sqrt{3}/2 \ . \ 5 . 10^6\ Hz}$$

$$= 0.0889\ m/s$$

Note : need to know angle. if we thought $\theta = 0°$, then

$$V' = \frac{c f_D}{2 f_0} = 0.077\ m/s$$

Received Signal amplitude goes down exponentially as time passes

Gain compensation: $g(t) = \frac{(ct)^2 e^{\mu_{ac} t}}{K}$

compensate for amplitude loss of signal

Problem: Depth of penetration

if wave travels too deep, the received signal level can go below the detection threshold of the system.

$d_p = \frac{L}{2af}$, where L: loss the system can hande (~ 80 dB)

depth of penetration (we cannot receive signal beyond this depth)

* assuming $a = 1\,dB.cm^{-1}.MHz^{-1}$, $L = 80\,dB$

freq (MHz)	Depth of Penetration (cm)
1	40
2	20
3	13
5	8
10	4
20	2

} high frequencies can only image superficially

Pulse Repetition Rate (TR)

Send a short pulse, wait until all echoes die out

Round trip distance $\sim 2d_p$

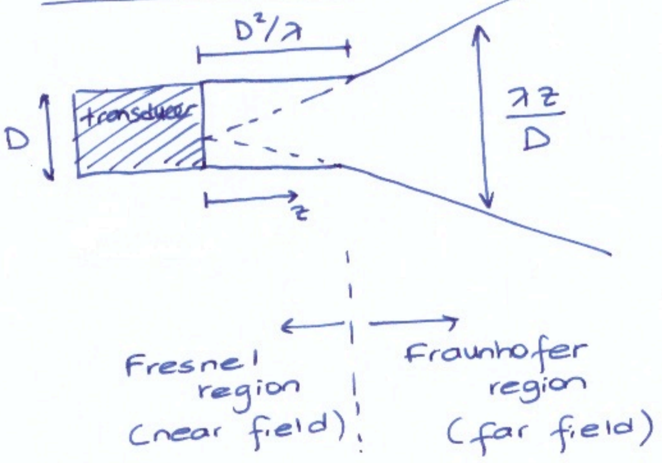
time it takes $\sim \frac{2d_p}{c}$

So, $TR \geq \frac{2d_p}{c}$

Example: $d_p = 2cm$ @ 20 MHz $\Rightarrow TR = \frac{2 \times 2 \times 10^{-2}}{1540} \approx 26\mu s$

$d_p = 40\,cm$ @ 1 MHz $\Rightarrow TR = \frac{2.40.10^{-2}}{1540} \approx 519\mu s$

Beam Pattern



beamwidth, $w(z) = \begin{cases} D, & z \le D^2/\lambda \\ \frac{\lambda z}{D}, & z > D^2/\lambda \end{cases}$ ← lateral resolution worsens deep in the body

Also; λ larger for lower freq $(\lambda = \frac{c}{f})$
→ worse resolution

"Optimum" transducer size

$\frac{D_{opt}^2}{\lambda} = z_{max}$ (remain in Fresnel region)

$\Rightarrow \quad D_{opt} = \sqrt{\lambda . z_{max}}$
↳ choose as small as possible, limited by attenuation.
(small λ → high f → more att.)

1) $z_{max} = 20$ cm (deep body imaging)

from table → $f = 2$ MHz, $\lambda = \frac{1540 \, m/s}{2.10^6 \, 1/s} = 0.77$ mm

$D_{opt} = \sqrt{0.77 \text{ mm} \times 20 \text{ cm}} \approx 1.2$ cm

2) $z_{max} = 4$ cm (thyroid imaging)

$f = 10$ MHz, $\lambda = \frac{1540}{10.10^6} = 0.15$ mm

$D_{opt} = \sqrt{0.15 \text{mm} \times 4 \text{ cm}} = 2.4$ mm

-END-