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HOMEWORK 1

1)

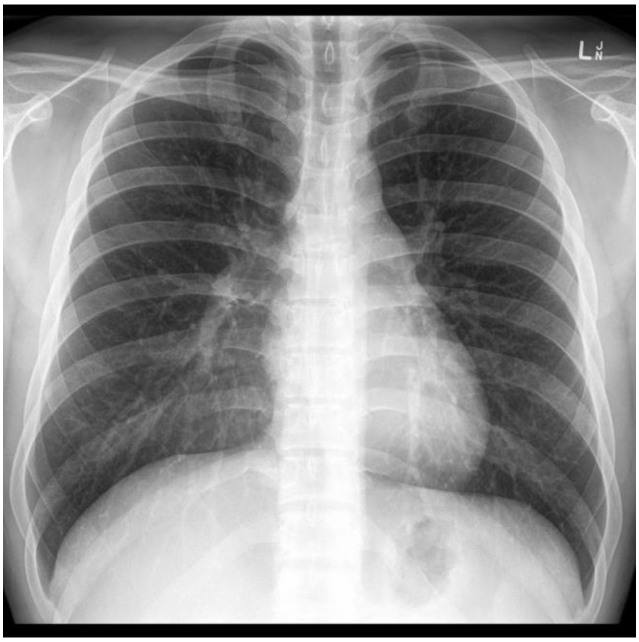

Figure 1.1 X-ray image. Diaphragm, heart, liver, lungs, rib cage, spine, and torso can be seen.
https://radiopaedia.org/cases/normal-frontal-chest-x-ray

Figure 1.2 CT image. Abdomen, liver, rib cage, and spine can be seen.
https://doi.org/10.1109/KCIC.2018.8628561

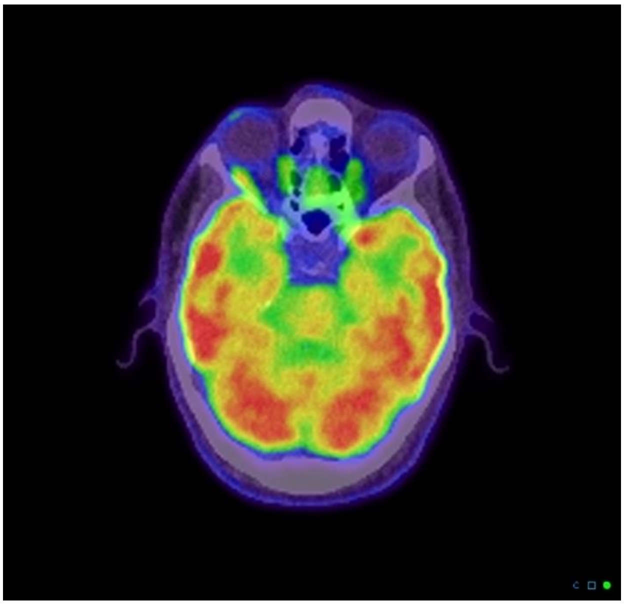

Figure 1.3 PET image. Brain, eyes, head, and nose can be seen.
https://radiopaedia.org/cases/normal-brain-pet?lang=us

Figure 1.4 Ultrasound image. Abdomen and pancreas can be seen.
https://radiopaedia.org/cases/normal-pancreas-ultrasound?lang=us

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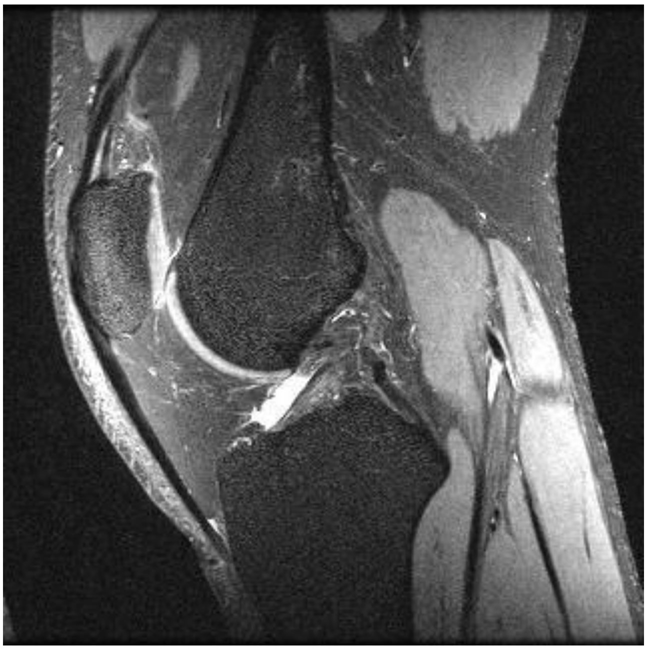

Figure 1.5 MRI image. Bone marrow, knee, and muscles can be seen.
http://mridata.org/list?project=Stanford%20Fullysampled%203D%20FSE%20Knees

Q2

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a) $g(x,y) = f(x,-1) + f(0,y)$

Let $f_1(x,y) = \sum_{k=1}^{K} w_k f_k(x,y)$ be the input.

Then, $g_1(x,y) = f_1(x,-1) + f_1(0,y)$

$$= \sum_{k=1}^{K} w_k f_k(x,-1) + \sum_{k=1}^{K} w_k f_k(0,y)$$

$$= \sum_{k=1}^{K} w_k [f_k(x,-1) + f_k(0,y)] = \sum_{k=1}^{K} w_k g_k(x,y), \text{ where } f_k \xrightarrow{S} g_k$$

Hence, this system is $\boxed{\text{LINEAR}}$.

Let $f_2(x,y) = f(x-x_0, y-y_0)$ be the input.

Then, $g_2(x,y) = f_2(x,-1) + f_2(0,y) = f(x-x_0, -1-x_0) + f(-x_0, y-y_0) \neq g(x-x_0, y-y_0)$

because $g(x-x_0, y-y_0) = f(x-x_0, -1) + f(0, y-y_0)$. (Function only works on (x,y))

Hence, this system is $\boxed{\text{SHIFT-VARIANT}}$.

b) $g(x,y) = \max(f(x,y), 0)$

Let $f_1(x,y) = \sum_{k=1}^{K} w_k f_k(x,y)$ be the input.

Then, $g_1(x,y) = \max(f_1(x,y), 0)$

$$= \max\left(\sum_{k=1}^{K} w_k f_k(x,y), 0\right) \neq \sum_{k=1}^{K} w_k \max(f_k(x,y), 0)$$

For instance, for $K=2$, $w_k = \begin{cases} -5, & k=1 \\ 6, & k=2 \\ 0, & \text{o.w.} \end{cases}$, and $f_k(x,y) = \begin{cases} -1, & k=1 \\ 1, & k=2 \\ 0, & \text{o.w.} \end{cases}$, linearity requirement does not hold.

$\max\left(\sum_{k=1}^{K} w_k f_k(x,y), 0\right) = 11$, but $\sum_{k=1}^{K} w_k \max(f_k(x,y), 0) = 6$

Hence, this system is $\boxed{\text{NONLINEAR}}$.

Let $f_2(x,y) = f(x-x_0, y-y_0)$ be the input.

Then, $g_2(x,y) = \max(f_2(x,y), 0) = \max(f(x-x_0, y-y_0), 0) = g(x-x_0, y-y_0)$

because $g(x-x_0, y-y_0) = \max(f(x-x_0, y-y_0), 0)$ (Function only works on (x,y))

Hence, this system is $\boxed{\text{SHIFT-INVARIANT}}$.

$f(x,y) = e^{j2\pi(x+y)}$, $x_1 = 9$, $y_1 = 5$ Q3

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a) $f(x,y)\delta(x-x_1, y+y_1)$

$$f(x,y)\delta(x-9,y+5) \overset{\text{Sampling Property}}{=} f(9,-5)\delta(x-9,y+5)$$

$$= \boxed{e^{j2\pi 4}\delta(x-9,y+5)}$$

b) $f(x,y) * \delta(x-x_1, y+y_1)$

$$f(x,y) * \delta(x-9,y+5) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\varepsilon,\eta)\delta(x-9-\varepsilon, y+5-\eta)\, d\varepsilon\, d\eta$$

$$\overset{\text{Sampling}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-9,y+5)\delta(x-9-\varepsilon,y+5-\eta)\, d\varepsilon\, d\eta$$

This is also known as the sifting property of the Dirac delta function

$$= f(x-9,y+5)\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-9-\varepsilon,y+5-\eta)\, d\varepsilon\, d\eta}_{=1 \text{ (Definition of Dirac delta)}}$$

$$= f(x-9,y+5) = e^{j2\pi(x-9+y+5)} = \boxed{e^{-j2\pi 4} e^{j2\pi(x+y)}}$$

c) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-x_1, 3y+y_1) f(x,y)\,dx\,dy$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-9,3y+5)f(x,y)\,dx\,dy \overset{\text{Scaling Property}}{=} \frac{1}{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-9,y+\tfrac{5}{3})f(x,y)\,dx\,dy$$

$$\overset{\text{Sampling}}{=} \frac{1}{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-9,y+\tfrac{5}{3})f(9,-\tfrac{5}{3})\,dx\,dy$$

$$= \frac{1}{3}f(9,-\tfrac{5}{3})\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-9,y+\tfrac{5}{3})\,dx\,dy}_{1}$$

$$= \frac{1}{3}f(9,-\tfrac{5}{3}) = \boxed{\frac{1}{3}e^{j2\pi(\frac{22}{3})}}$$

d) $f(x+1,-y_1) * \delta(x-x_1, y+1)$

$$f(x+1,-5) * \delta(x-9,y+1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\varepsilon+1,-5)\delta(x-9-\varepsilon,y+1-\eta)\,d\varepsilon\,d\eta$$

$$\overset{\text{Sampling}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-8,-5)\,\delta(x-9-\varepsilon,y+1-\eta)\,d\varepsilon\,d\eta$$

$$= f(x-8,-5)\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-9-\varepsilon,y+1-\eta)\,d\varepsilon\,d\eta}_{1}$$

$$= f(x-8,-5) = e^{j2\pi(x-8-5)} = \boxed{e^{-j2\pi 13}e^{j2\pi x}}$$

$x_1 = 9,\ y_1 = 5$

Q4

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a) $g(x,y) = \delta\left(\frac{x}{x_1}, y_1 y - 1\right)$

$g(x,y) = \delta\left(\frac{x}{9}, 5y-1\right) \xleftrightarrow[\text{Known pair}]{\text{2D FT}} \frac{1}{\frac{1}{9}\cdot 5} e^{-j2\pi(9u\cdot 0 + \frac{1}{5}v\cdot 1)}$

$= \boxed{\frac{9}{5} e^{-j2\pi\frac{v}{5}}}$ $(G(u,v))$

Scaling and time-shifting properties of 2DFT ~~is~~ are used

b) $g(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{j2\pi(nx+my)}$

$g(x,y)$ is separable because there are functions $g_1(x)$ and $g_2(y)$, s.t.,

$g(x,y) = g_1(x) g_2(y)$, where $g_1(x) = \sum_{n=-\infty}^{\infty} e^{j2\pi nx}$, $g_2(y) = \sum_{m=-\infty}^{\infty} e^{j2\pi my}$

$g(x,y)$ is separable $\Longleftrightarrow G(u,v) = G_1(u) G_2(v)$

$g_1(x) \xleftrightarrow{\text{1D FT}} G_1(u) = \sum_{n=-\infty}^{\infty} \delta(u-n)$

$g_2(y) \xleftrightarrow{\text{1D FT}} G_2(v) = \sum_{m=-\infty}^{\infty} \delta(v-m)$

This result can also be seen from another perspective: Fourier series representation of an impulse train with a period of 1 is $g_1(x)$. So, its 1D Fourier transform is also an impulse train with a period of 1 in the spectral domain.

$\Rightarrow G(u,v) = G_1(u) G_2(v) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(u-n)\delta(v-m)$

Hence, $\boxed{G(u,v) = \text{comb}(u,v)}$

c) $g(x,y) = \text{sinc}(3x + x_1, y_1 y - 1)$

$g(x,y) = \text{sinc}(3x+9, 5y-1) \xleftrightarrow[\text{Known pair}]{\text{2D FT}} \frac{1}{3.5} e^{j2\pi 3u} \cdot e^{-j2\pi\frac{v}{5}} \text{rect}\left(\frac{u}{3}, \frac{v}{5}\right)$

$3(x+3) \quad 5(y-\frac{1}{5})$

$= \boxed{\frac{1}{15} e^{j2\pi(3u - \frac{v}{5})} \text{rect}\left(\frac{u}{3}, \frac{v}{5}\right)}$ $(G(u,v))$

Scaling and time-shifting properties of 2DFT are used.

d) $g(x,y) = \text{rect}\left(x_1 x, \frac{y}{y_1}\right) e^{j2\pi(u_0 x + 4v_0 y)}$

$g(x,y) = \text{rect}\left(9x, \frac{y}{5}\right) e^{j2\pi(u_0 x + 4v_0 y)} \xleftrightarrow{\text{2D FT}} \frac{1}{9\cdot\frac{1}{5}} \text{sinc}\left(\frac{u-u_0}{9}, 5(v-4v_0)\right)$

$9(x) \quad \frac{1}{5}(y)$

$= \boxed{\frac{5}{9} \text{sinc}\left(\frac{u-u_0}{9}, 5(v-v_0)\right)}$ $(G(u,v))$

Scaling and frequency-shifting properties of 2DFT are used.

e) $g(x,y) = e^{-2\pi(4x^2+y^2)} * \cos(2\pi x + \pi y)$

$$e^{-2\pi(4x^2+y^2)} = e^{-\pi((2\sqrt{2}x)^2 + (\sqrt{2}y)^2)}$$

Known pair: $f(x,y) = e^{-\pi(x^2+y^2)} \xleftrightarrow{2DFT} F(u,v) = e^{-\pi(u^2+v^2)}$

$$\Rightarrow f(2\sqrt{2}x, \sqrt{2}y) = e^{-2\pi(4x^2+y^2)} \xleftrightarrow{2DFT} \frac{1}{2\sqrt{2}\cdot\sqrt{2}} F\left(\frac{u}{2\sqrt{2}}, \frac{v}{\sqrt{2}}\right)$$

Scaling Property

$$= \boxed{\frac{1}{2} e^{-\pi\left(\frac{u^2}{8} + \frac{v^2}{2}\right)}}$$

$$\cos(2\pi x + \pi y) \xleftrightarrow{2DFT} \frac{1}{2}\left[\delta\left(u-1, v-\frac{1}{2}\right) + \delta\left(u+1, v+\frac{1}{2}\right)\right]$$

Hence, $G(u,v) = \frac{1}{4} e^{-\pi\left(\frac{u^2}{8} + \frac{v^2}{2}\right)}\left[\delta\left(u-1, v-\frac{1}{2}\right) + \delta\left(u+1, v+\frac{1}{2}\right)\right]$

$$= \frac{1}{4} e^{-\pi\left(\frac{1}{8}+\frac{1}{8}\right)} \delta\left(u-1, v-\frac{1}{2}\right) + \frac{1}{4} e^{-\pi\left(\frac{1}{8}+\frac{1}{8}\right)} \delta\left(u+1, v+\frac{1}{2}\right)$$

$$= \boxed{\frac{e^{-\frac{\pi}{4}}}{4}\left[\delta\left(u-1, v-\frac{1}{2}\right) + \delta\left(u+1, v+\frac{1}{2}\right)\right]}$$

f) $F(u,v) = \mathcal{F}_{2D}(f(x,y))$ and $f(x)$ is a real-valued function.

i) $f(x,y) = f(-x,-y) \overset{?}{\Rightarrow} F^*(u,v) = F(u,v)$

$f(x,y)$ is real $\Leftrightarrow$ $f^*(x,y) = f(x,y)$

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\,dy \Rightarrow F^*(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{j2\pi(ux+vy)} dx\,dy$$

$\downarrow \begin{matrix} x=-x \\ y=-y \end{matrix}$

Since $f(x,y) = f(-x,-y)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(-x,-y) e^{-j2\pi(ux+vy)} dx\,dy$$

$\downarrow f(x,y) = f(-x,-y)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\,dy$$

$$= F(u,v) \Rightarrow \boxed{F^*(u,v) = F(u,v)}$$

ii) $f(x,y)$ is real $\Leftrightarrow$ $f^*(x,y) = f(x,y)$

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\,dy \Rightarrow F^*(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{j2\pi(ux+vy)} dx\,dy$$

$\begin{matrix} x=-x \\ y=-y \end{matrix} \downarrow$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(-x,-y) e^{-j2\pi(ux+vy)} dx\,dy$$

$f(x,y) = -f(-x,-y)$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dx\,dy = -F(u,v)$$

$$\Rightarrow \boxed{F^*(u,v) = -F(u,v)}$$

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5)

a)

Figure 5.1.1 Phantom in spatial domain

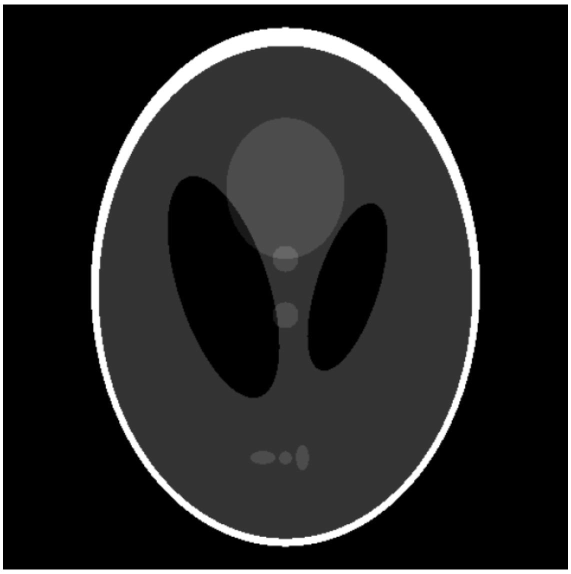
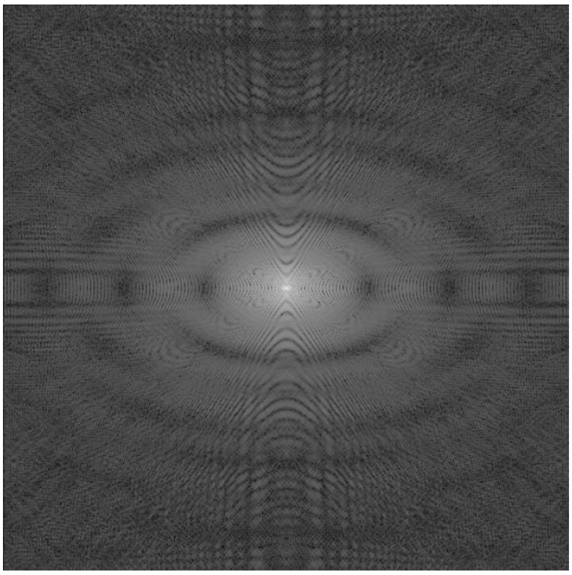


Figure 5.1.2 Magnitude of the spectral domain



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b)

i)

Figure 5.2.1 PSF h_1

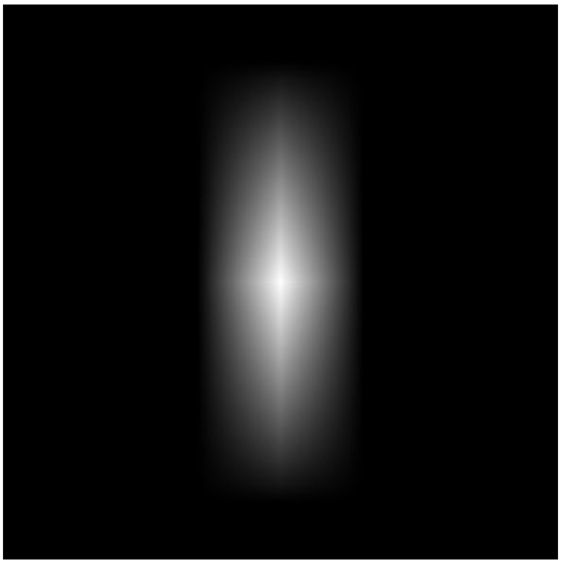
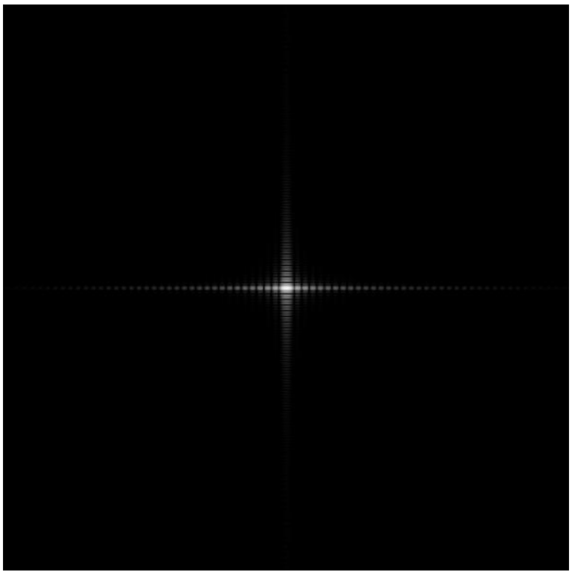


Figure 5.2.2 Transfer function $|H_1|$



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ii)

Figure 5.3.1 PSF h_2

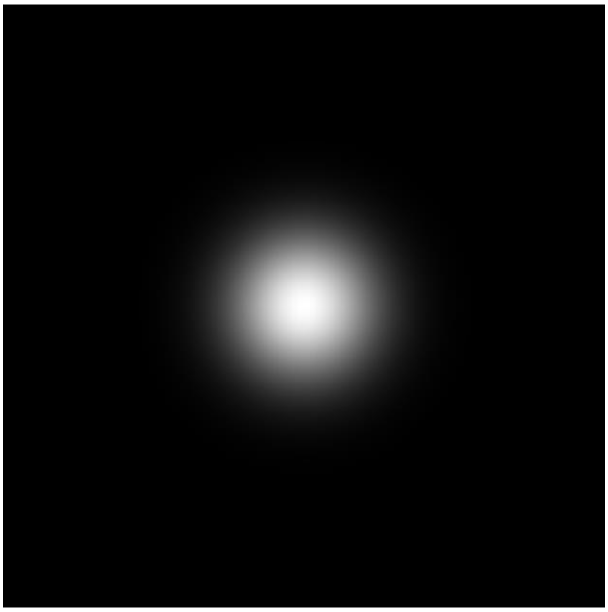
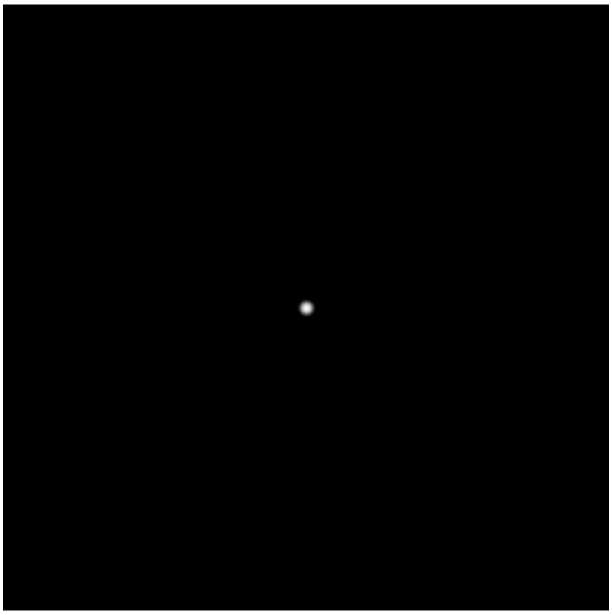


Figure 5.3.2 Transfer function $|H_2|$



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iii)

Figure 5.4.1 PSF h_3

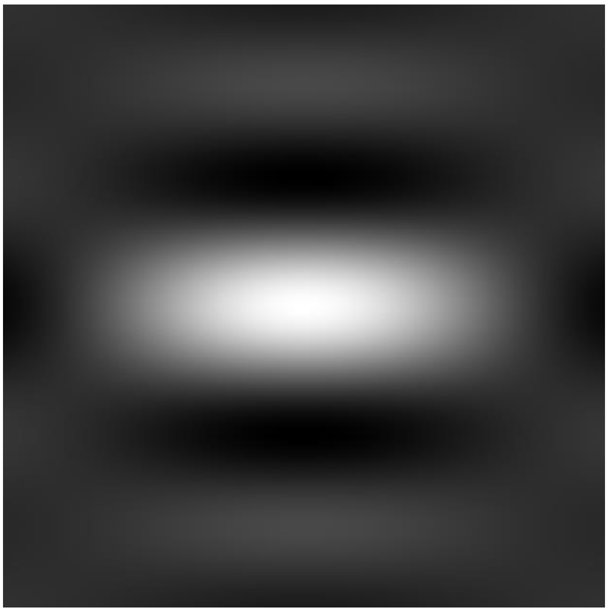
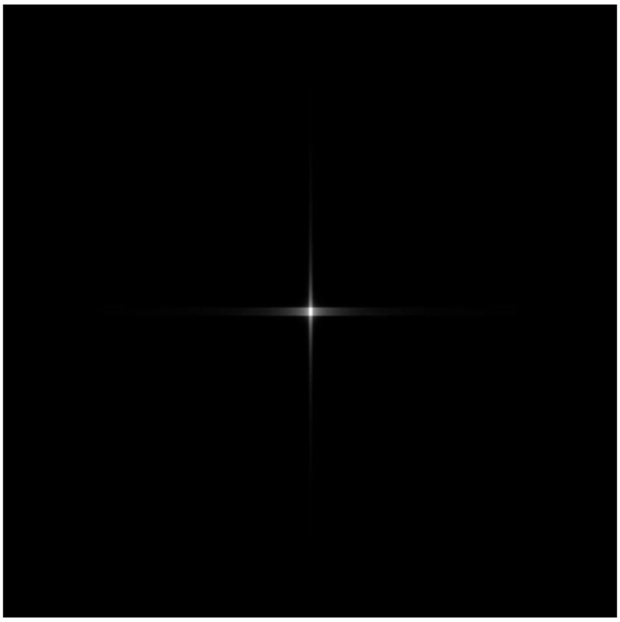


Figure 5.4.2 Transfer function $|H_3|$



c)

Figure 5.5.1 Output of system 1 in the spectral domain (magnitude)

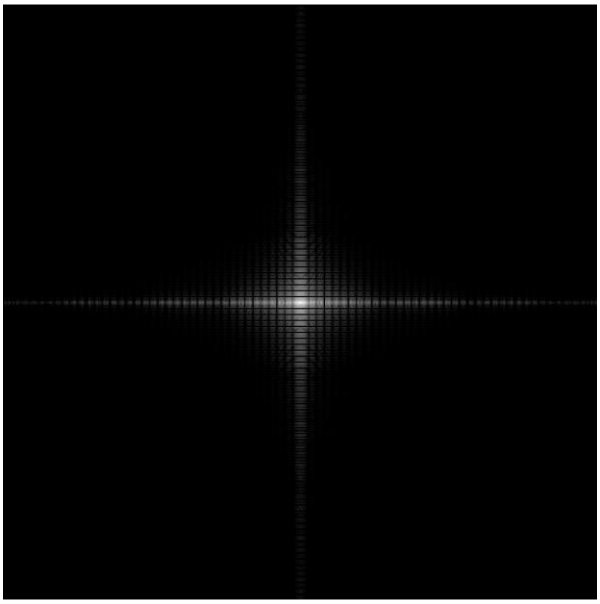


Figure 5.5.2 Output image of system 1

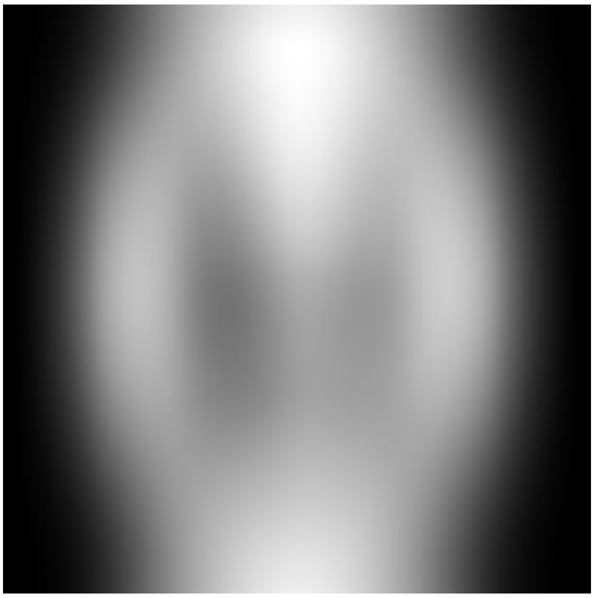


Figure 5.6.1 Output of system 2 in the spectral domain (magnitude)

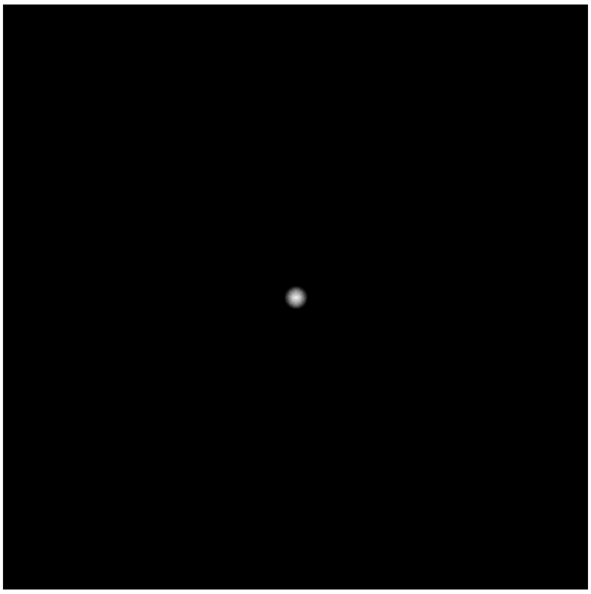


Figure 5.6.2 Output image of system 2

Figure 5.7.1 Output of system 3 in the spectral domain (magnitude)

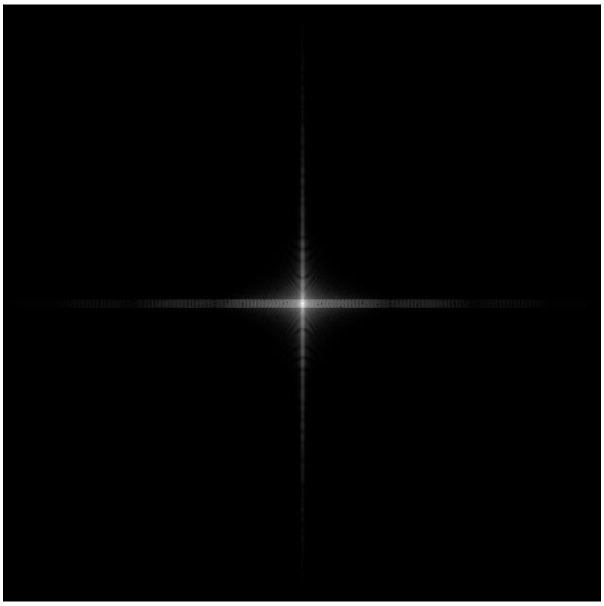
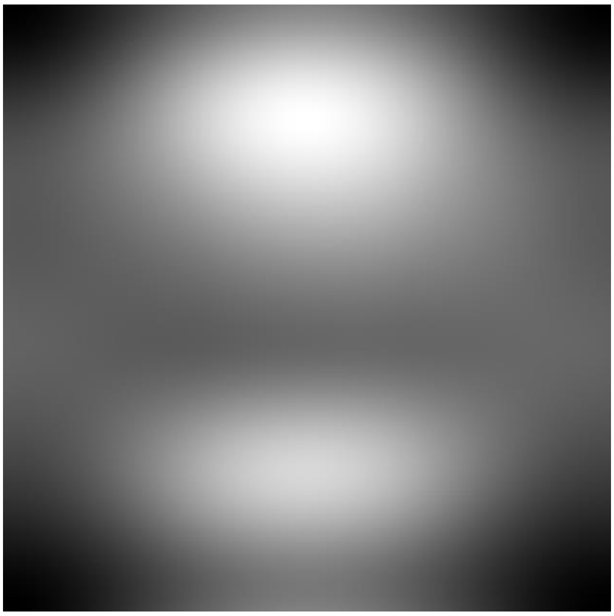


Figure 5.7.2 Output image of system 3



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d)

<u>First Medical System:</u> The PSF of this system is also known as the first order hold method used in interpolation (triangle filter). However, this system's PSF is scaled distinctively in both axes. Since the y-axis has been upscaled more than the x-axis, the resulting image is blurred drastically in the y-axis compared to the x-axis. The reason behind this is that we lose more information about the higher frequency components as we upscale the spatial axes.

<u>Second Medical System:</u> The PSF of this system is a Gaussian curve, and the system is a Gaussian filter. The Fourier transform of a Gaussian curve is also another Gaussian curve because Gaussian curves are the eigenvectors of Fourier transform. The filter behaves like a low-pass filter, so important details such as edges are lost due to the disappearance of the high frequency components.

<u>Third Medical System:</u> The PSF of this system is a 2D sinc function. Ideally, we would expect to get an ideal low-pass filter as its transfer function, however we cannot generate a sinc function which occupies the whole spatial space. As we transform smaller portions of the PSF, the transfer function becomes less ideal. The "less" ideal term corresponds to the transfer function having nonzero values along the axes outside the rectangle. Also, since we are scaling the PSF unevenly, the blurring also occurs unevenly along the axes. The blurring is more apparent along the x-axis due to the distinct scalings.

<u>APPENDIX</u>

MATLAB code for Q5:

```
P = phantom("Modified Shepp-Logan",500);

%%
%part a

figure;
imshow(P);
title("Figure 5.1.1 Phantom in spatial domain");

spectrum = fft2c(P);

figure;
imshow(log(abs(spectrum)+1), []);
title("Figure 5.1.2 Magnitude of the spectral domain");

%%
%part b

%i
basis = linspace(-10,10,500);

x1 = (1-abs(basis/3)).*(abs(basis) <= 3);
y1 = (1-abs(basis/8)).*(abs(basis) <= 8);
```

```
[X1, Y1] = meshgrid(x1, y1);

%h1
h1 = X1.*Y1;
figure;
imshow(h1);
title("Figure 5.2.1 PSF h_1");

%H1
H1 = fft2c(h1);
figure;
imshow(log(abs(H1)+1), []);
title("Figure 5.2.2 Transfer function |H_1|");

%%
%ii
basis = linspace(-6,6,500);

x2 = (1/(2*pi)).*(exp(-(basis.^2)./2));
y2 = (1/(2*pi)).*(exp(-(basis.^2)./2));

[X2, Y2] = meshgrid(x2, y2);

%h2
h2 = X2.*Y2;
figure;
imshow(h2, []);
title("Figure 5.3.1 PSF h_2");

%H2
H2 = fft2c(h2);
figure;
imshow(log(abs(H2)+1), []);
title("Figure 5.3.2 Transfer function |H_2|");

%%
%iii
basis = linspace(-10,10,500);

x3 = sinc(basis/8);
y3 = sinc(basis/3);

[X3, Y3] = meshgrid(x3, y3);

%h3
h3 = X3.*Y3;
figure;
```

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```
imshow(h3, []);
title("Figure 5.4.1 PSF h_3");

%H3
H3 = fft2c(h3);
figure;
imshow(log(abs(H3)+1), []);
title("Figure 5.4.2 Transfer function |H_3|");

%%
%part c

%first imaging system
spectrum1 = spectrum.*H1;

figure;
imshow(log(abs(spectrum1)+1), []);
title("Figure 5.5.1 Output of system 1 in the spectral
domain (magnitude)");

output1 = fftshift(ifft2(ifftshift(spectrum1)));

figure;
imshow(output1, []);
title("Figure 5.5.2 Output image of system 1");

%%
%second imaging system
spectrum2 = spectrum.*H2;

figure;
imshow(log(abs(spectrum2)+1), []);
title("Figure 5.6.1 Output of system 2 in the spectral
domain (magnitude)");

output2 = fftshift(ifft2(ifftshift(spectrum2)));

figure;
imshow(output2, []);
title("Figure 5.6.2 Output image of system 2");

%%
%third imaging system
spectrum3 = spectrum.*H3;

figure;
imshow(log(abs(spectrum3)+1), []);
```

```
title("Figure 5.7.1 Output of system 3 in the spectral
domain (magnitude)");

output3 = fftshift(ifft2(ifftshift(spectrum3)));

figure;
imshow(output3, []);
title("Figure 5.7.2 Output image of system 3");
```