Name Lastname	
Student ID	
Signature	
Classroom #	EE-

Q1 (25 pts)	
Q2 (25 pts)	
Q3 (25 pts)	
Q4 (25 pts)	
TOTAL	

EEE 473/573 – Spring 2014-2015
MIDTERM EXAM #1
5 April 2015, 14:00-16:00

- Open book, open notes.
- Provide appropriate explanations in your solution and **show intermediate steps clearly. No credit will be given otherwise**.

1) [25 points] Answer the following questions. Simplify your answers as much as possible.

a) [5 points] Calculate the 2D convolution $\mathrm{rect}\left(\frac{x}{4}, y\right) * \delta(x-1, y-2)$.

b) [5 points] Calculate the 2D convolution $\mathrm{rect}\left(\frac{x}{4}, y\right) * \delta(y-2)$.

c) [5 points] Calculate the 2D convolution $\cos(2\pi x + 4\pi y) * \mathrm{sinc}(2x, 3y)$.

d) [5 points] Calculate the 2D convolution $\cos(2\pi u_0 x + 2\pi v_0 y) * \exp(-x^2 - y^2)$.

e) [5 points] What is the 2D Fourier transform of the following function?

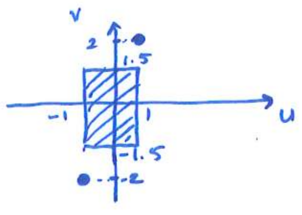
$f(x,y) = f(r) = \mathrm{rect}\left(\frac{r-a}{b}\right)$, where $a > b$.

a) $\mathrm{rect}\left(\frac{x}{4}, y\right) * \delta(x-1, y-2) = \mathrm{rect}\left(\frac{x-1}{4}, y-2\right)$

b) $\mathrm{rect}\left(\frac{x}{4}, y\right) * \delta(y-2) = \left(\mathrm{rect}\left(\frac{x}{4}\right) * 1\right) \cdot \left(\mathrm{rect}(y) * \delta(y-2)\right)$

$= 4 \cdot \mathrm{rect}(y-2)$ area under is 4

c) Take 2DFT : $\frac{1}{2}\left[\delta(u-1, v-2) + \delta(u+1, v+2)\right] \cdot \frac{1}{6} \cdot \mathrm{rect}\left(\frac{u}{2}, \frac{v}{3}\right) = 0$



impulses are outside the rect. multiplication is zero.

$\mathcal{F}_{2D}^{-1}\{0\} = 0$

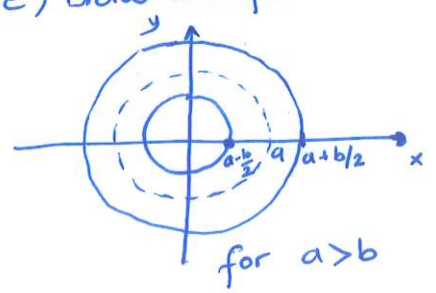
so, $\cos(2\pi x + 4\pi y) * \mathrm{sinc}(2x, 3y) = 0$

d) Take 2DFT : $\frac{1}{2}\left[\delta(u-u_0, v-v_0) + \delta(u+u_0, v+v_0)\right] \cdot \pi \cdot e^{-\pi^2(u^2+v^2)}$

$= \frac{\pi}{2}\left[\delta(u-u_0, v-v_0) + \delta(u+u_0, v+v_0)\right] \cdot e^{-\pi^2(u_0^2+v_0^2)}$ sifting property of delta

Take inverse 2DFT : $\pi \cos(2\pi u_0 x + 2\pi v_0 y)\, e^{-\pi^2(u_0^2+v_0^2)}$

e) Draw the function:



for a>b

$f(r) = \mathrm{rect}\left(\frac{r-a}{b}\right) = \mathrm{rect}\left(\frac{r}{2(a+b/2)}\right) - \mathrm{rect}\left(\frac{r}{2(a-\frac{b}{2})}\right)$

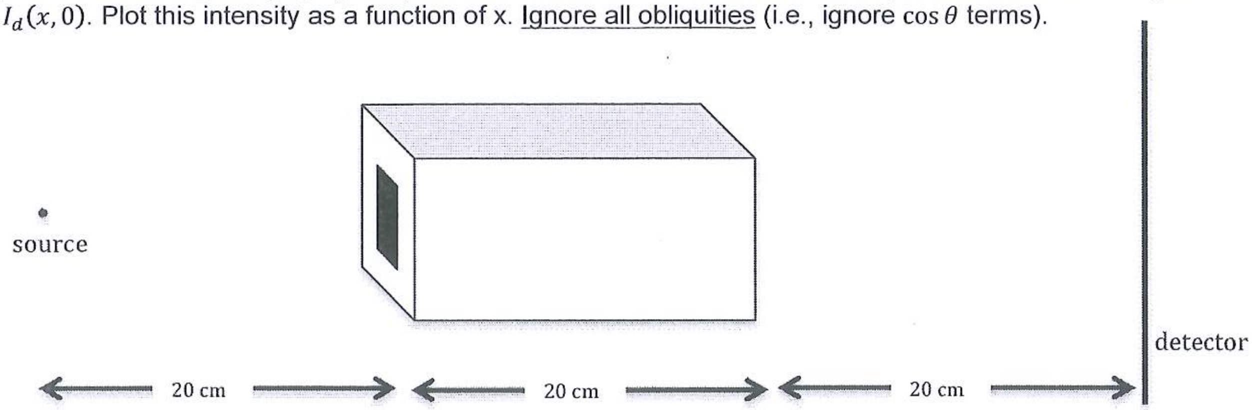
$= \mathrm{rect}\left(\frac{r}{2a+b}\right) - \mathrm{rect}\left(\frac{r}{2a-b}\right)$

$F(\rho) = (2a+b)^2 \cdot \mathrm{jinc}\left((2a+b)\rho\right) - (2a-b)^2 \cdot \mathrm{jinc}\left((2a-b)\rho\right)$

from Hankel transform and scaling.

where $\mathrm{jinc}(\rho) = \frac{J_1(\pi\rho)}{2\rho}$

2) [25 points] A <u>hollow</u> prism (i.e., with a hole at its center) has length L = 20 cm, outer width 8cm X 8cm, inner width 2cm X 2cm, and a constant linear attenuation coefficient of $\mu_0 = 0.05\ \text{cm}^{-1}$. This prism is imaged with a point source x-ray imaging system, as shown below. Formulate the intensity on the detector along the x-axis, $I_d(x,0)$. Plot this intensity as a function of x. <u>Ignore all obliquities</u> (i.e., ignore $\cos\theta$ terms).



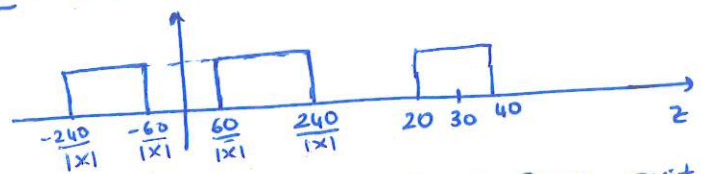
$$\mu(x,y,z) = \mu_0 \left[\text{rect}\left(\frac{x}{8}, \frac{y}{8}, \frac{z-30}{20}\right) - \text{rect}\left(\frac{x}{2}, \frac{y}{2}, \frac{z-30}{20}\right)\right]$$

Then,

$$I_d(x,0) = I_0 . \exp\left\{-\int_z \mu\left(\frac{x}{M}, \frac{0}{M}, z\right) dz\right\} \quad , \text{ where } \quad M = \frac{d}{z} = \frac{60}{z}$$

$$= I_0 . \exp\left\{-\mu_0 \int_{20}^{40} \left[\text{rect}\left(\frac{x . z}{8.60}\right) - \text{rect}\left(\frac{x . z}{2.60}\right)\right] . \text{rect}\left(\frac{z-30}{20}\right) dz\right\}$$

<u>Zone 1</u> : No overlap (rays outside the prism)



$\frac{240}{|x|} < 20$

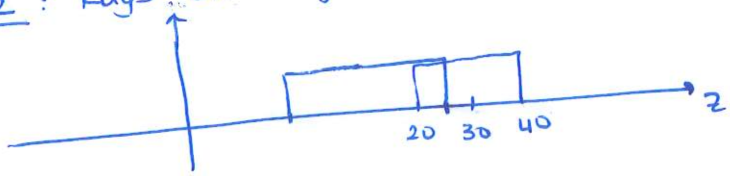
$|x| > 12$ cm , $I_d(x,0) = I_0$

<u>Zone 2</u> : Rays enter from front face, exit through the side

$20 < \frac{240}{|x|} < 40$ <u>AND</u> $\frac{60}{|x|} < 20$

$6 < |x| < 12$ <u>AND</u> $|x| > 3$

$\underbrace{\qquad\qquad\qquad\qquad}_{6 \text{ cm} < |x| < 12 \text{ cm}}$



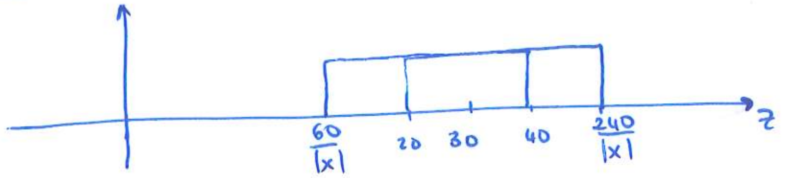
Then, $I_d(x,0) = I_0 . \exp\left\{-\mu_0 \left(\frac{240}{|x|} - 20\right)\right\}$

inserting $\mu_0 = 0.05 \text{cm}^{-1}$ $\Rightarrow$ $I_d(x,0) = I_0 . \exp\left\{-\left(\frac{12 - |x|}{|x|}\right)\right\}$

<u>Zone 3</u> : Full overlap (rays fully pass through prism)

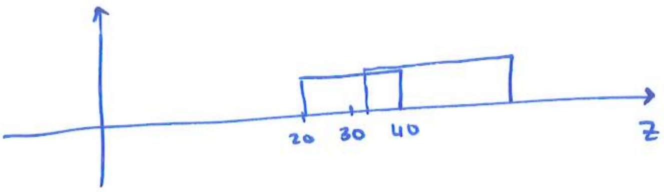
$\frac{240}{|x|} > 40$ <u>AND</u> $20 > \frac{60}{|x|}$

$3 \text{ cm} < |x| < 6 \text{ cm}$



Then, $I_d(x,0) = I_0 . \exp\left\{-\mu_0 . (40-20)\right\} = I_0 . e^{-1}$

Zone 4 : Rays enter from inner side, exit at the back

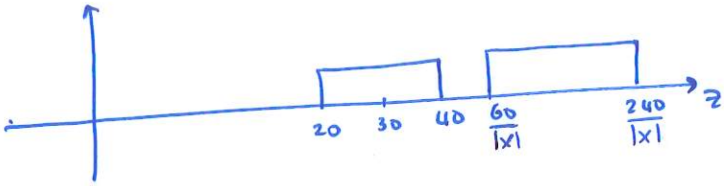


$$20 < \frac{60}{|x|} < 40 \quad \text{AND} \quad \frac{240}{|x|} > 40$$

$$\underbrace{1.5\text{cm} < |x| < 3\text{cm} \quad \text{AND} \quad |x| < 6\text{cm}}_{1.5\text{cm} < |x| < 3\text{cm}}$$

Then,
$$I_d(x,0) = I_0 \cdot \exp\left\{-\mu_0\left(40 - \frac{60}{|x|}\right)\right\}$$
$$= I_0 \cdot \exp\left\{-\left(\frac{2|x|-3}{|x|}\right)\right\}$$

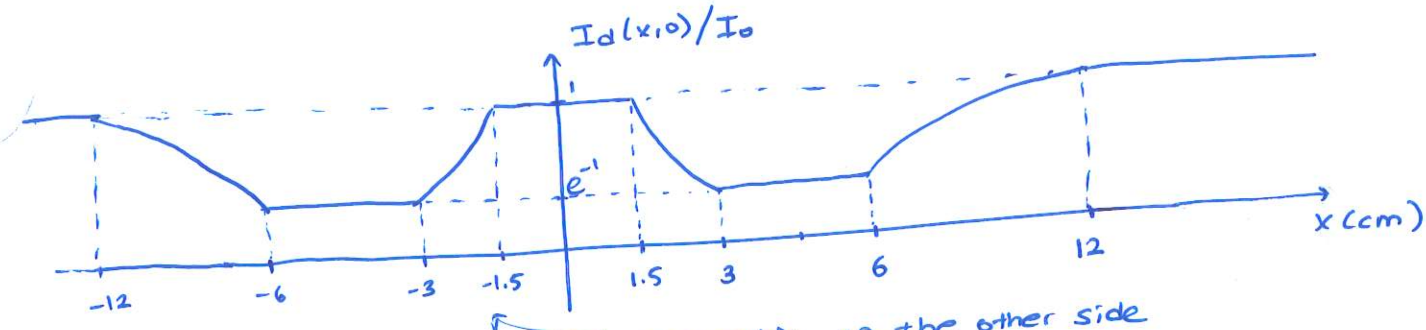
Zone 5 : Rays pass through the hollow center



$$\frac{60}{|x|} > 40$$
$$|x| < 1.5\text{ cm}$$

Then,
$$I_d(x,0) = I_0.$$

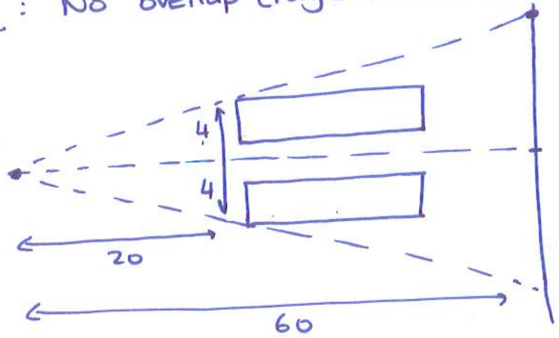
So,
$$I_d(x,0) = \begin{cases} I_0 & , \text{ if } |x| < 1.5\text{cm} \text{ OR } |x| > 12\text{cm} \\ I_0 \cdot \exp\left\{-\left(\frac{2|x|-3}{|x|}\right)\right\} & , \text{ if } 1.5\text{cm} < x < 3\text{cm} \\ I_0 \cdot e^{-1} & , \text{ if } 3\text{cm} < |x| < 6\text{cm} \\ I_0 \cdot \exp\left\{-\left(\frac{12-|x|}{|x|}\right)\right\} & , \text{ if } 6\text{ cm} < |x| < 12\text{ cm} \end{cases}$$



mirror symmetric on the other side

Alternative Solution for Question 2 (from Geometry)

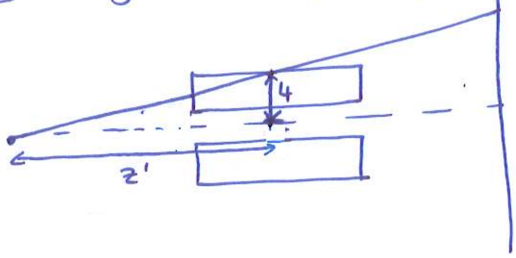
Zone 1: No overlap (rays miss the prism)



if $\frac{|x|}{60} > \frac{4}{20} \Rightarrow |x| > 12\,cm$

Then, $I_d(x,0) = I_0$

Zone 2: Rays enter from front face, exit through the side



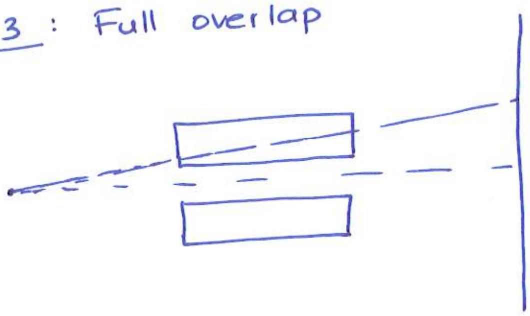
if $\underbrace{\frac{4}{20} > \frac{|x|}{60} \;\&\; \frac{|x|}{60} > \frac{4}{40}}_{6\,cm < |x| < 12\,cm}$

Then, $I_d(x,0) = I_0 . e^{-\mu(z'-20)}$

where $\frac{4}{z'} = \frac{|x|}{60} \Rightarrow z' = \frac{240}{|x|}$

so, $I_d(x,0) = I_0 . e^{-\mu\left(\frac{240}{|x|} - 20\right)} = I_0 . e^{-\left(\frac{12}{|x|} - 1\right)}$

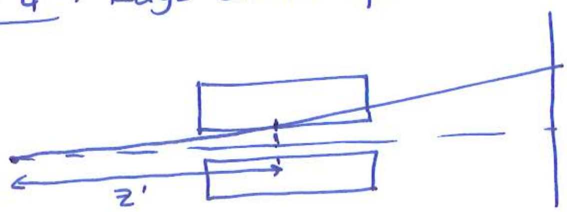
Zone 3: Full overlap



if $\underbrace{\frac{4}{40} > \frac{|x|}{60} \;\&\; \frac{|x|}{60} > \frac{1}{20}}_{3\,cm < |x| < 6\,cm}$

Then, $I_d(x,0) = I_0 . e^{-\mu . 20} = I_0 . e^{-1}$

Zone 4: Rays enter from inner side, exit at the back face

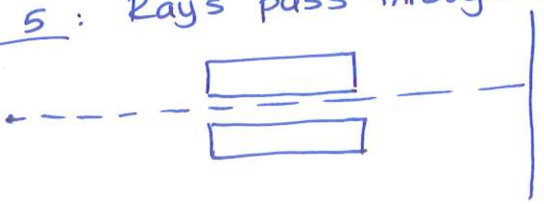


if $\underbrace{\frac{1}{20} > \frac{|x|}{60} \;\&\; \frac{|x|}{60} > \frac{1}{40}}_{1.5\,cm < |x| < 3\,cm}$

Then, $I_d(x,0) = I_0 . e^{-\mu(40 - z')}$

where $\frac{1}{z'} = \frac{|x|}{60} \Rightarrow z' = \frac{60}{|x|}$, So, $I_d(x,0) = I_0 . e^{-\mu\left(40 - \frac{60}{|x|}\right)}$

Zone 5: Rays pass through the hollow hole.



if $\frac{|x|}{60} < \frac{1}{4.0} \Rightarrow |x| < 1.5\,cm$

$I_d(x,0) = I_0$

3) [25 points] A 2D function $f(x,y)$ (or $f(r,\emptyset)$) produces 1D projections given by

$$g(l,\theta) = 4\,sinc(2l)\cos(4\pi l)$$

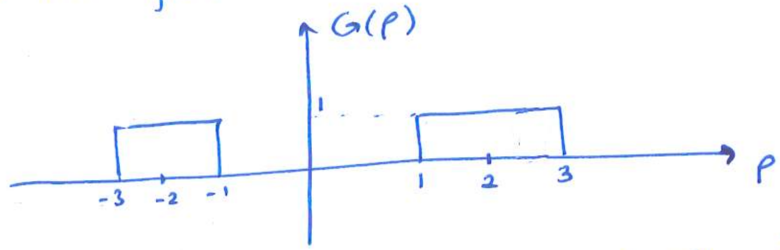
a) [13 points] Determine the 2D function $f(x,y)$ (or $f(r,\emptyset)$).

b) [12 points] If the CT image reconstruction is performed with a filtered backprojection system using a modified filter $|\rho|\mathrm{rect}\left(\frac{\rho}{2\rho_0}\right)$, determine the resultant reconstructed image as a function of ρ_0. Simplify your answer as much as possible.

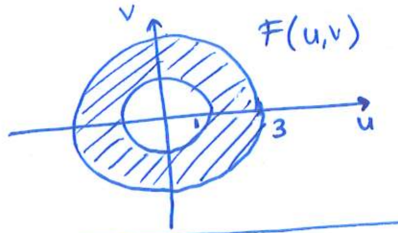
a) $G(\rho) = \mathcal{F}_{1D}\{g(l,\theta)\} = 4\cdot\frac{1}{2}\mathrm{rect}\left(\frac{\rho}{2}\right) * \frac{1}{2}\left[\delta(\rho-2) + \delta(\rho+2)\right]$

$= \mathrm{rect}\left(\frac{\rho-2}{2}\right) + \mathrm{rect}\left(\frac{\rho+2}{2}\right)$

Draw this function:



From projection-slice theorem : $G(\rho,\theta) = F(\rho\cos\theta, \rho\sin\theta)$



So,

$F(u,v) = F(\rho) = \mathrm{rect}\left(\frac{\rho}{6}\right) - \mathrm{rect}\left(\frac{\rho}{2}\right)$

Hence, $\boxed{f(r) = 36\cdot jinc(6r) - 4\cdot jinc(2r)}$, where $jinc(r) = \frac{J_1(\pi r)}{2r}$

b) When we use the "ideal" ramp filter, we get $f(x,y)$ as reconstructed image:

$f(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} |\rho|\,G(\rho,\theta)\,e^{j2\pi\rho(x\cos\theta + y\sin\theta)}\,d\rho\,d\theta$ (Eq. 6.22 of the book)

Now, we will use a modified filter

$\hat{f}(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} |\rho|\cdot\mathrm{rect}\left(\frac{\rho}{2\rho_0}\right)\cdot G(\rho,\theta)\cdot e^{j2\pi(x\cos\theta + y\sin\theta)}\,d\rho\,d\theta$

$= \int_0^{\pi}\int_{-\infty}^{\infty} |\rho|\cdot\underbrace{\left\{\mathrm{rect}\left(\frac{\rho}{2\rho_0}\right)\cdot G(\rho,\theta)\right\}}_{}\,e^{j2\pi(x\cos\theta + y\sin\theta)}\,d\rho\,d\theta$

ideal filter

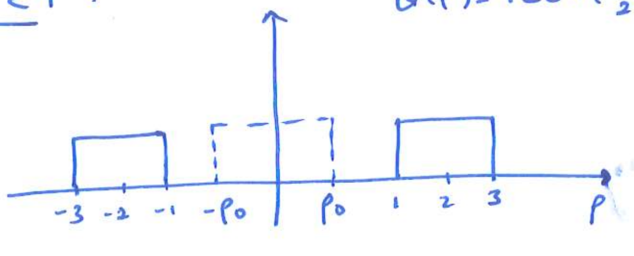
it is as if this is the 1DFT of the projection, which we will reconstruct with ideal $|\rho|$ filter

i.e., $\hat{G}(\rho) = \mathrm{rect}\left(\frac{\rho}{2\rho_0}\right)\cdot G(\rho)$

There are three different cases

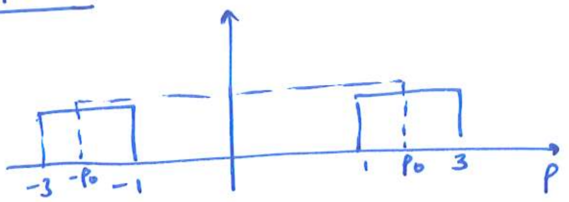
1) if $\rho_0 < 1$:

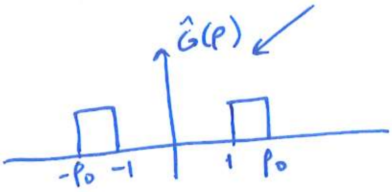
$$\hat{G}(\rho) = \mathrm{rect}\left(\frac{\rho}{2\rho_0}\right) \cdot G(\rho) = 0$$



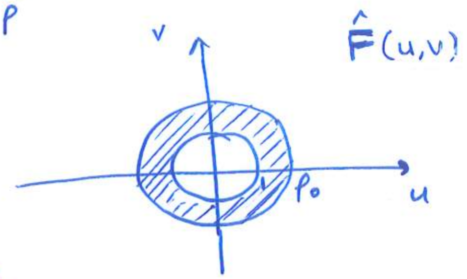
$$\Rightarrow \boxed{\hat{f}(r) = 0}$$

2) if $1 < \rho_0 < 3$





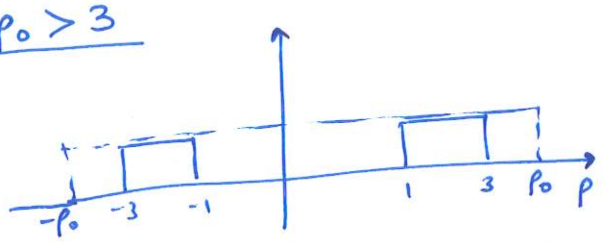
projection-slice theorem



$$\hat{F}(u,v) = \hat{F}(\rho) = \mathrm{rec}\left(\frac{\rho}{2\rho_0}\right) - \mathrm{rect}\left(\frac{\rho}{2}\right)$$

$$\boxed{\hat{f}(r) = 4\rho_0^2\, \mathrm{jinc}(2\rho_0 r) - 4\, \mathrm{jinc}(2r)}$$

3) if $\rho_0 > 3$



$$\hat{G}(\rho) = G(\rho) \Rightarrow \hat{F}(u,v) = F(u,v)$$

$$\Downarrow$$

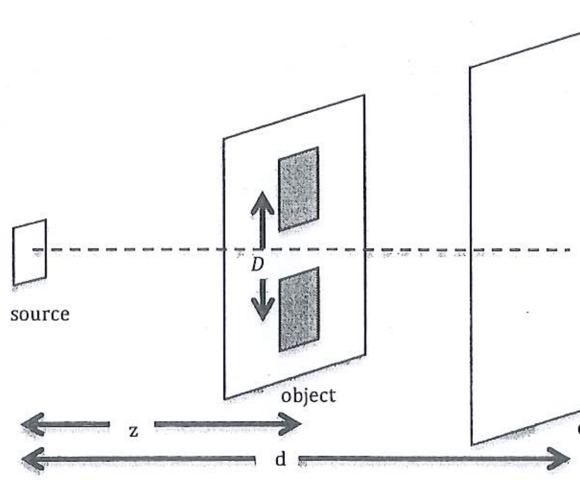
$$\boxed{\hat{f}(r) = f(r) = 36\, \mathrm{jinc}(6r) - 4\, \mathrm{jinc}(2r)}$$

4) [25 points] A square source of size L by L is used to image a planar object that contains two square holes (lesions), each size W by W. The rest of the planar object has zero transmittivity. The centers of the two holes are separated by a distance D along the x-direction. The exact depth of the planar object is not known, except that it is between $z = d/2$ and $z = 2d/3$.

a) [10 points] Find the largest source size, L, that ensures that the two lesions remain fully resolved, i.e., they remain not touching in the image, for all z within the range specified.

b) [5 points] What is the largest value of L if $D = 9W/4$?

c) [10 points] Using the value from part (b), find the value(s) of z (within the range specified) that maximize the image intensity at the center of the lesions.



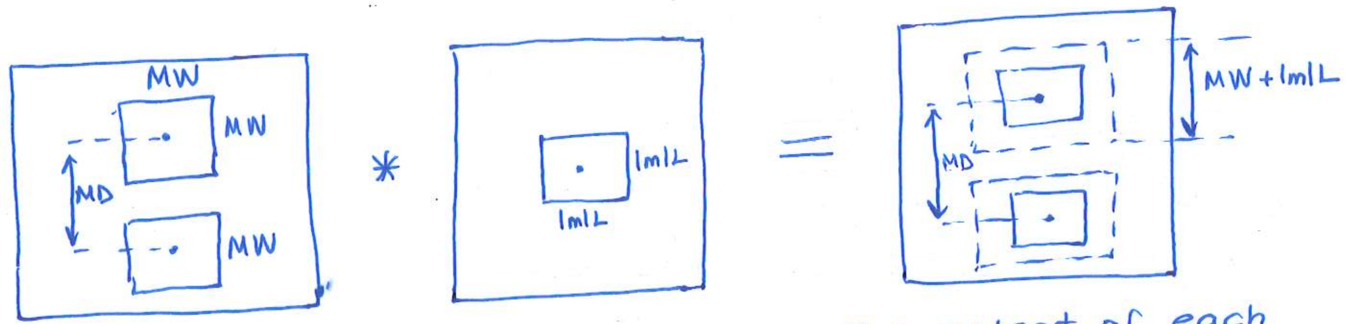
Ignore all obliquities (i.e., ignore $\cos\theta$ terms).
Hint: There is no need to fully calculate $I_d(x, y)$ for this question.

a) $s(x,y) = \text{rect}\left(\frac{x}{L}, \frac{y}{L}\right)$ and $t_z(x,y) = \text{rect}\left(\frac{x - D/2}{W}, \frac{y}{W}\right) + \text{rect}\left(\frac{x + D/2}{W}, \frac{y}{W}\right)$

$$I_d(x,y) = \frac{1}{4\pi d^2 m^2} \cdot s\left(\frac{x}{m}, \frac{y}{m}\right) * t_z\left(\frac{x}{M}, \frac{y}{M}\right)$$

$$= \frac{1}{4\pi d^2 m^2} \, \text{rect}\left(\frac{x}{mL}, \frac{y}{mL}\right) * \left[\text{rect}\left(\frac{x - MD/2}{MW}, \frac{y}{MW}\right) + \text{rect}\left(\frac{x - MD/2}{MW}, \frac{y}{MW}\right)\right]$$

So,



After convolution at the detector plane, the extent of each lesion will increase from MW to $MW + |m|L$ (this is basic knowledge about convolution).

we want: $MD > MW + |m| \cdot L$

$$\frac{d}{z} D > \frac{d}{z} W + \frac{(d-z)}{z} L \Rightarrow \boxed{L < \frac{d\,(D - W)}{d - z}}$$

* Knowing that this must be satisfied for all $\frac{d}{2} \leq z \leq \frac{2d}{3}$, worst case is at $z = d/2$:

$$L < \frac{d\,(D - W)}{d - d/2} = 2\,(D - W)$$

* So, largest source size is $\boxed{L = 2\,(D - W)}$

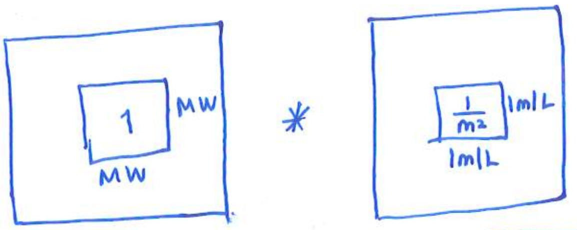
b) if $D = \frac{9W}{4} \Rightarrow L = 2\left(\frac{9W}{4} - W\right) = \boxed{\frac{5}{2} W}$

c) It suffices to look at one lesion only.
Because obliquity is ignored, it may be easier to just look at one centered lesion:

so, consider $t(x,y) = \text{rect}\left(\frac{x}{W}, \frac{y}{W}\right)$

$$I_d(x,y) = \frac{1}{4\pi d^2 m^2} \cdot \text{rect}\left(\frac{x}{mL}, \frac{y}{mL}\right) * \text{rect}\left(\frac{x}{MW}, \frac{y}{M}\right)$$

We want to maximize $I_d(0,0)$ in this case:



← magnified source

↑ magnified lesion

the area under is L^2, independent of m (i.e., independent of source magnification)

$I_d(0,0)$ is maximized if the magnified source fits inside magnified lesion during convolution.

i.e., we want $MW \geq |m| L$ insert $L = \frac{5}{2} W$

$$\frac{d}{z} W \geq \frac{(d-z)}{z} \frac{5}{2} W$$

$$2d \geq (d-z)5$$

$$5z \geq 3d \Rightarrow \boxed{z \geq \frac{3d}{5}}$$

Also knowing that $\frac{d}{2} \leq z \leq \frac{2d}{3}$,

we must have $\boxed{\frac{3d}{5} \leq z \leq \frac{2d}{3}}$ to maximize intensity at centers of lesions

* Alternative explanation for part (c)

For a centered lesion, $t(x,y) = \mathrm{rect}\left(\frac{x}{W}, \frac{y}{W}\right)$

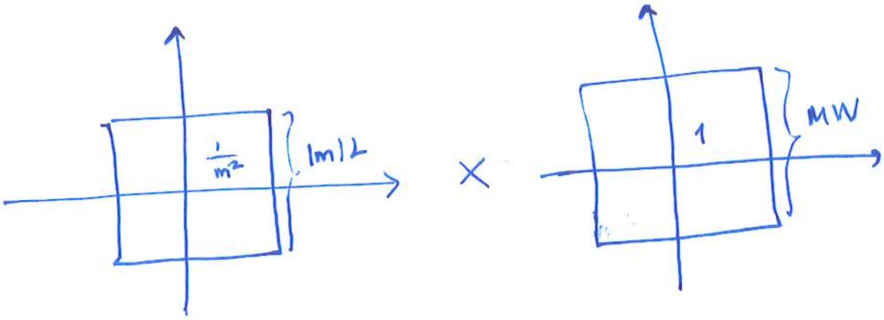
$$I_d(x,y) = \frac{1}{4\pi d^2} \cdot \frac{1}{m^2}\,\mathrm{rect}\left(\frac{x}{mL}, \frac{y}{mL}\right) * \mathrm{rect}\left(\frac{x}{MW}, \frac{y}{MW}\right)$$

$$= \frac{1}{4\pi d^2} \iint \frac{1}{m^2}\,\mathrm{rect}\left(\frac{\xi}{mL}, \frac{\eta}{mL}\right) \mathrm{rect}\left(\frac{x-\xi}{MW}, \frac{y-\eta}{MW}\right) d\xi\, d\eta$$

$$I_d(0,0) = \frac{1}{4\pi d^2} \iint \frac{1}{m^2}\,\mathrm{rect}\left(\frac{\xi}{mL}, \frac{\eta}{mL}\right) \cdot \mathrm{rect}\left(\frac{-\xi}{MW}, \frac{-\eta}{MW}\right) d\xi\, d\eta$$

from symmetry

$$= \frac{1}{4\pi d^2} \iint \underbrace{\frac{1}{m^2}\,\mathrm{rect}\left(\frac{\xi}{mL}, \frac{\eta}{mL}\right) \cdot \mathrm{rect}\left(\frac{\xi}{MW}, \frac{\eta}{MW}\right) d\xi\, d\eta}_{\text{area under multiplication of two 2D functions}}$$



* if $|m|L \leq MW$,

$$I_d(0,0) = \frac{1}{4\pi d^2} \cdot \frac{1}{m^2} \cdot (|m| \cdot L)^2 = \frac{L^2}{4\pi d^2}$$

* if $|m|L > MW$

$$I_d(0,0) = \frac{1}{4\pi d^2} \cdot \frac{1}{m^2}(MW)^2 < \frac{1}{4\pi d^2} \cdot \frac{1}{m^2}(|m|L)^2 = \frac{L^2}{4\pi d^2}$$

So, the magnified source must fit inside the magnified object.
In other words, we must keep the source blur below a level, otherwise blurring causes a reduction in signal.

So, $|m| \cdot L \leq MW$, with $L = \frac{5}{2} W$

gives $z \geq \frac{3d}{5}$ (see previous page)