Name Lastname	
Student ID	
Signature	
Classroom #	EE-

Q1 (20 pts)	
Q2 (15 pts)	
Q3 (30 pts)	
Q4 (25 pts)	
Q5 (10 pts)	
TOTAL	

EEE 473/573 – Spring 2015-2016
MIDTERM EXAM
27 March 2016, 16:00-18:30

- Open book, open notes.
- Provide appropriate explanations in your solution and show intermediate steps clearly. **No credit will be given otherwise**.

1) [20 points] Answer the following questions.

a) [5 points] Simplify the following expression:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(ax+b,y)dxdy$$

b) [5 points] Evaluate the following expression:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} rect\left(\frac{x}{2},\frac{y}{4}\right)\delta(y-2x)dxdy$$

c) [5 points] Suppose we have an imaging system with input $f(x,y)$ and output $g(x,y)$. If $G(u,v) = F(u,v) * H(u,v)$, is this an LSI system?

d) [5 points] Suppose we have a 1-D imaging system with input $f(x)$ and output $g(x)$. Below are sketches of the input and output in Fourier domain. Is this an LSI system? Explain your answer.

a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\frac{1}{|a|}\delta(x+\frac{b}{a},y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(-\frac{b}{a},0)\cdot\frac{1}{|a|}\cdot\delta(x+\frac{b}{a},y)dxdy$

$= \frac{1}{|a|}\cdot f(-\frac{b}{a},0)$

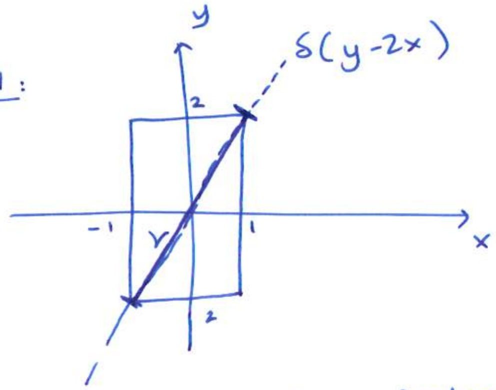
b) Two ways to solve.

First: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} rect\left(\frac{x}{2}\right).rect\left(\frac{y}{4}\right).\delta(y-2x)dxdy$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} rect\left(\frac{x}{2}\right).rect\left(\frac{2x}{4}\right)\delta(y-2x)dy\,dx$

$= \int_{-\infty}^{\infty} rect\left(\frac{x}{2}\right).rect\left(\frac{x}{2}\right)dx = \int_{-1}^{1}dx = 2$

Second:



$\underbrace{\delta(y-2x)}$

delta line at $y=2x$

$L = \sqrt{2^2+4^2} = 2\sqrt{5}$

So, this is a line integral! But we have write the delta as $\delta(x\cos\theta + y\sin\theta)$ (unit amplitude delta line)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{rect}\left(\frac{x}{2}, \frac{y}{4}\right) \delta\left(\sqrt{5}\left(y \cdot \frac{1}{\sqrt{5}} - x \cdot \frac{2}{\sqrt{5}}\right)\right) dx\,dy$$

$$= \frac{1}{\sqrt{5}} \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{rect}\left(\frac{x}{2}, \frac{y}{4}\right) \delta\left(\frac{y}{\sqrt{5}} - \frac{2x}{\sqrt{5}}\right) dx\,dy}_{= L = 2\sqrt{5}}$$

$$= \frac{2\sqrt{5}}{\sqrt{5}} = 2$$

c) $g(x,y) = f(x,y) \cdot h(x,y)$ in image domain

In general, this is NOT shift invariant. $\Rightarrow$ NOT LSI

insert $f(x-x_0, y-y_0)$ as input:

$$g'(x,y) = f(x-x_0, y-y_0) \cdot h(x,y) \neq g(x-x_0, y-y_0)$$

unless $h(x,y) = h(x-x_0, y-y_0)$ for all $\{x_0, y_0\}$

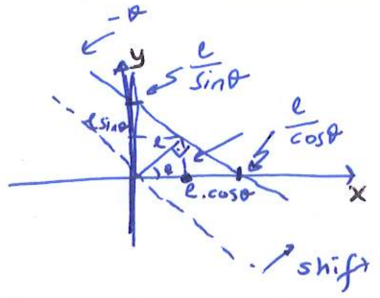
i.e., unless $h(x,y) = \text{constant}$

d) LSI systems cannot generate "new" frequencies.

$$G(u,v) = F(u,v) \cdot H(u,v)$$

if this is zero at a frequency, $G(u,v)$ is also zero at that frequency

So, not LSI.



2) [15 points] What is the 2D Fourier Transform of the following function?

$$f(x,y) = \delta(x\cos\theta + y\sin\theta - l)$$

Sketch $f(x,y)$ and $|F(u,v)|$ for $\theta = 30°$.

Start with $f_1(x,y) = \delta(x)$ $\rightarrow$ $F_1(u,v) = \delta(v)$

Rotation : (by $-\theta$)

$f_2(x,y) = f_1(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$ $\Rightarrow$ $F_2(u,v) = F_1(u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta)$

$= \delta(x\cos\theta + y\sin\theta)$ $\Rightarrow$ $F_2(u,v) = \delta(-u\sin\theta + v\cos\theta)$

shifting x by $x_0 = l\cos\theta$, y by $y_0 = l\sin\theta$

$f(x,y) = f_2(x - l\cos\theta, y - l\sin\theta)$

$= \delta((x - l.\cos\theta)\cos\theta + (y - l\sin\theta)\sin\theta)$

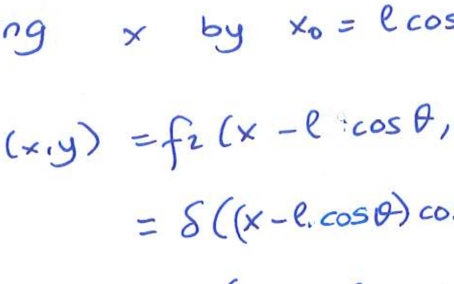
$= \delta(x\cos\theta + y\sin\theta - l)$

$F(u,v) = F_2(u,v).e^{-j2\pi(u.l\cos\theta + v.l\sin\theta)}$

$= \delta(-u\sin\theta + v\cos\theta).e^{-j2\pi l(u.\cos\theta + v.\sin\theta)}$

(OR) $= \delta(-u\sin\theta + v\cos\theta).e^{-j2\pi l\frac{v}{\sin\theta}}$

(OR) $= \delta(-u\sin\theta + v\cos\theta).e^{-j2\pi l\frac{u}{\cos\theta}}$



delta line @ 30°, perpendicular to the image domain line

line at $\frac{u}{\cos\theta} = \frac{v}{\sin\theta}$

alternative :

$$F(u,v) = \iint_{-\infty}^{\infty} \delta(x\cos\theta + y\sin\theta - \ell)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$= \iint_{-\infty}^{\infty} \frac{1}{|\cos\theta|}\, \delta\left(x + \frac{y\sin\theta}{\cos\theta} - \frac{\ell}{\cos\theta}\right) e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{|\cos\theta|} \cdot e^{-j2\pi\left(u\left(\frac{\ell}{\cos\theta} - \frac{y\sin\theta}{\cos\theta}\right) + vy\right)} dy$$

$$= \frac{1}{|\cos\theta|} \cdot e^{-j2\pi u \frac{\ell}{\cos\theta}} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-j2\pi y\left(v - \frac{u\sin\theta}{\cos\theta}\right)} dy}_{\delta\left(v - \frac{u\sin\theta}{\cos\theta}\right)}$$

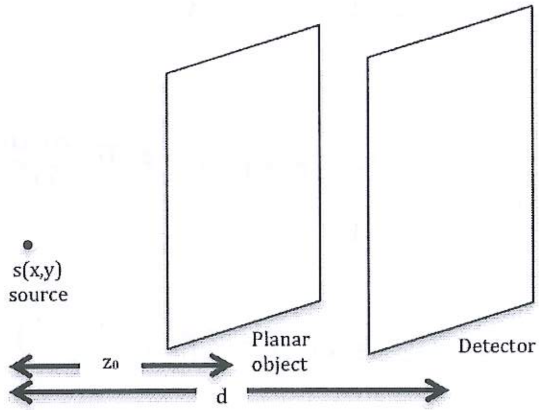
$$= e^{-j2\pi \frac{u\ell}{\cos\theta}} \cdot \delta(v\cos\theta - u\sin\theta)$$

$$f(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta) \Rightarrow F(u\cos\theta - v\sin\theta,\ u\sin\theta + v\cos\theta)$$

rotated the same way

3) [30 points] An x-ray source, $s(x,y)$, is placed at a distance d from the detector. It is used to image a planar object at a depth z_0, with transmittivity:

$$t(x,y) = a + b\cos(2\pi u_0 x)$$



a) [5 points] If the source is a point source, what is the intensity at the detector plane, $I_d(x,y)$?

b) [5 points] What is the contrast in $I_d(x,y)$? If possible, write the contrast as a function of z_0, the positioning of the planar object. How should you position the object to get the best contrast?

Note: Here, use the modulation of the image as the contrast metric.

c) [10 points] Repeat part (a) if the source is no longer a point source, but is given as $s(r) = exp\left(-\pi\frac{r^2}{\beta^2}\right)$.

d) [10 points] What is the contrast in $I_d(x,y)$ from part (c)? If possible, write the contrast as a function of z_0, the positioning of the planar object. How should you position the object to get the best contrast?

<u>Ignore all obliquities for this question</u> (i.e., assume $\cos\theta \approx 1$).

a) for $s(x,y) = I_s.\delta(x,y)$

$$I_d(x,y) = \frac{1}{4\pi d^2}\cdot\frac{1}{m^2}\, s\left(\frac{x}{m},\frac{y}{m}\right) * t\left(\frac{x}{M},\frac{y}{M}\right) \quad , \text{where } M = \frac{d}{z_0}, \; m = -\frac{d-z_0}{z_0}$$

$$= \frac{1}{4\pi d^2}\,\frac{I_s}{m^2}\,\underbrace{\delta\left(\frac{x}{m},\frac{y}{m}\right)}_{m^2\delta(x,y)} * \left[a + b.\cos\left(2\pi u_0\frac{x}{M}\right)\right]$$

$$= \frac{I_s}{4\pi d^2}\cdot\delta(x,y) * \left[a + b.\cos\left(2\pi u_0\frac{x}{M}\right)\right]$$

$$= \frac{I_s}{4\pi d^2}\cdot\left[a + b.\cos\left(2\pi u_0\frac{x.z_0}{d}\right)\right]$$

b) $$m = \frac{I_{dmax} - I_{dmin}}{I_{dmax} + I_{dmin}} = \frac{(a+b)-(a-b)}{(a+b)+(a-b)} = \frac{b}{a}$$

m is <u>independent</u> of z_0.

c) $$I_d(x,y) = \frac{1}{4\pi d^2}\cdot\frac{1}{m^2}\cdot s\left(\frac{x}{m},\frac{y}{m}\right) * \left[a + b.\cos\left(2\pi u_0\frac{x}{M}\right)\right]$$

$$= \frac{1}{4\pi d^2 m^2}\cdot\left[a.m^2.S(0,0) + b.m^2.S\left(\frac{m u_0}{M},0\right).\cos\left(2\pi\frac{u_0 x}{M}\right)\right]$$

where $S(u,v) = \mathcal{F}_{2D}\left\{\exp\left(-\pi\frac{r^2}{\beta^2}\right)\right\} = \beta^2.\exp\left\{-\pi\beta^2(u^2+v^2)\right\}$

so, $$I_d(x,y) = \frac{\beta^2}{4\pi d^2}\left[a + b.\exp\left\{-\pi\beta^2\frac{m^2 u_0^2}{M^2}\right\}.\cos\left(2\pi\frac{u_0}{M}x\right)\right] \rightarrow$$

d) $m = \frac{b}{a} \cdot \exp\left\{-\beta^2 \pi \frac{m^2 u_0^2}{M^2}\right\}$, where $m = -\frac{d - z_0}{z_0}$

$M = \frac{d}{z_0}$

$$m = \frac{b}{a} \cdot \exp\left\{-\pi \beta^2 \left(\frac{d - z_0}{d}\right)^2 u_0^2\right\}$$

This is maximum (i.e, $m_{max} = \frac{b}{a}$) when $\boxed{d = z_0}$

i.e., place the object right in front of the detector.

4) [25 points] A 2D function f(x, y) generates the following sinogram:

$$g(l, \theta) = \frac{1}{a}\cos(2\pi a l) + \frac{1}{b}\cos(2\pi b l) \,.$$

a) [10 points] Determine the 2D function $f(x, y)$ (or $f(r, \emptyset)$).

b) [8 points] If the CT image reconstruction is performed with a "naive" backprojection reconstruction (i.e., without filtering), determine the reconstructed image.

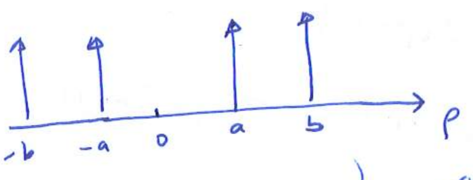
Then, simplify your answer for the case of $b \gg a$. Let's call this result $f_b(x, y)$.

c) [7 points] You want to get a better result than $f_b(x, y)$, so you use filtered backprojection with a windowed filter, $|\rho|\text{rect}\left(\frac{\rho}{2\rho_0}\right)$. However, for some reason, the new image still looks like $f_b(x, y)$, only scaled by a constant (i.e., $f_{new}(x, y) = \beta\, f_b(x, y)$, for some constant β).

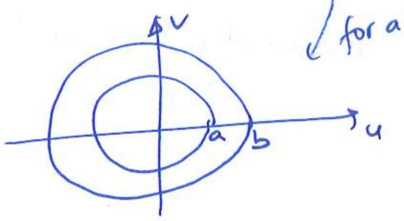
What do you think is the source of this problem? How would you fix this problem, so that you can perfectly reconstruct $f(x, y)$?

a) $G(\rho,\theta) = \frac{1}{a}\cdot\frac{1}{2}\left[\delta(\rho-a)+\delta(\rho+a)\right] + \frac{1}{2b}\left[\delta(\rho-b)+\rho(\rho+b)\right]$

independent of θ



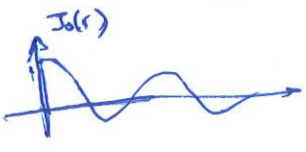
for all θ



Using projection-slice theorem,
for $\rho \geq 0$ always (polar coordinates)

$F(\rho) = \underbrace{\frac{1}{2a}\delta(\rho-a) + \frac{1}{2b}\delta(\rho-b)}_{\text{delta ring}}$

$f(r) = \frac{1}{2a}\cdot 2\pi a . J_0(2\pi a r) + \frac{1}{2b} 2\pi b J_0(2\pi b r)$



$\boxed{f(r) = \pi\, J_0(2\pi a r) + \pi . J_0(2\pi b r)}$

b) $f_{ideal}(x,y) = \int_0^{\pi}\left[\int_{-\infty}^{\infty}|\rho|\, G(\rho,\theta)e^{j2\pi\rho l}\, d\rho\right]_{l = x\cos\theta + y\sin\theta} d\theta$

with ramp filter

$f'(x,y) = \int_0^{\pi}\left[\left[\int_{-\infty}^{\infty} G(\rho,\theta)e^{j2\pi\rho l}\, d\rho\right]\right]_{l = x\cos\theta+y\sin\theta} d\theta$ ← without filter

$\ldots \int_{-\infty}^{\infty}|\rho|\cdot\left(\underbrace{\frac{1}{|\rho|}G(\rho,\theta)}\right)\ldots$

So, we will find inverse F.T of $\frac{1}{|\rho|}.F(\rho)$

$f'(x,y) = \mathcal{F}_{2D}^{-1}\left\{\frac{1}{|\rho|}F(\rho)\right\} = \frac{1}{2a}\cdot\frac{1}{|\rho|}.\delta(\rho-a) + \frac{1}{2b}\frac{1}{|\rho|}\delta(\rho-b)$

$= \frac{1}{2a^2}\delta(\rho-a) + \frac{1}{2b^2}\delta(\rho-b)$

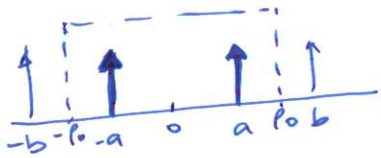
$$f'(x,y) = \frac{\pi}{a} J_0(2\pi a r) + \frac{\pi}{b} J_0(2\pi b r)$$

if $b >> a$, $\quad f_b(x,y) = \frac{\pi}{a} J_0(2\pi a r)$

high-frequency part is lost! due to blur.

c) If we are not seeing the "b" part of the image. we must be filtering it out by mistake.

$\swarrow \text{rect}\left(\frac{\rho}{2\rho_0}\right)$



we are reconstructing.

$$f'(x,y) = \mathcal{F}_{2D}^{-1}\left\{ \text{rect}\left(\frac{\rho}{2\rho_0}\right) F(\rho) \right\}$$

if $a < \rho_0 < b$,

then this gives

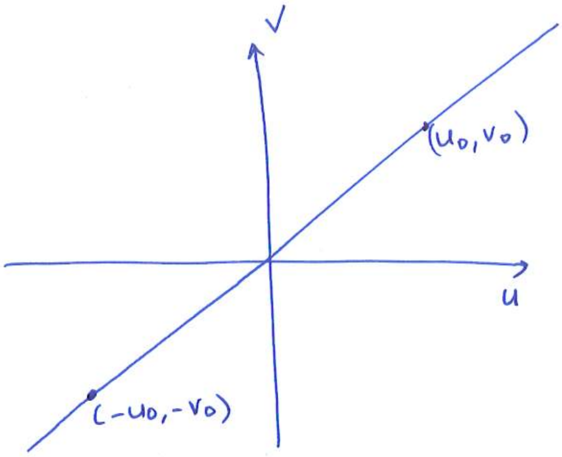
$$f'(x,y) = \mathcal{F}_{2D}^{-1}\left\{ \frac{1}{2a} \delta(\rho - a) \right\}$$

$$= \pi J_0(2\pi a r) = a \cdot f_b(x,y)$$

We should choose $\rho_0 > b$ to reliably reconstruct the actual object.

5) [10 points] If $f(x,y)$ is a real-valued function, its 2D Fourier Transform has conjugate symmetry, i.e., $F(u,v) = F^*(-u,-v)$. Since the linear attenuation coefficient as a function of space, $\mu(x,y)$, is real-valued, CT images have this conjugate symmetry property, as well.

Porof. Zihni Sinir claims that he can take advantage of this conjugate symmetry property for CT to use half as many projections, and thereby reduce the radiation dose by a factor of 2. Explain his claim. Do you agree? Explain your answer.



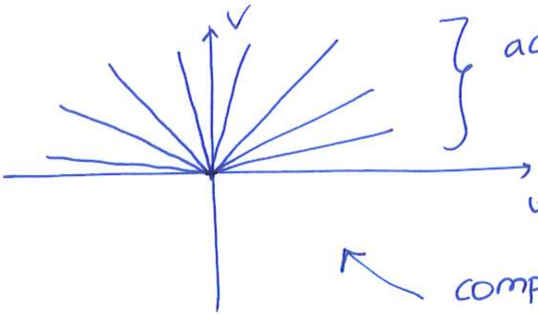
Conjugate symmetry

if we know $F(u_0, v_0)$,

we can calculate $F(-u_0, -v_0)$.

$F(-u_0, -v_0) = F^*(u_0, v_0)$.

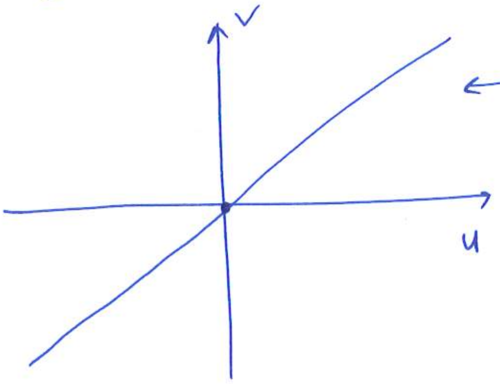
So, Porof. Zihni Sinir wants this:



acquire this portion

compute this portion

* This is NOT possible in CT. By the projection-slice theorem, every projection gives us the full line in Fourier domain.



one projection gives us this line.

* So, conjugate symmetry is irrelevant here. Porof. Sinir is wrong..