Q1 IE411 HW#2

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a) $f(x_1,x_2) = (4x_1^2 - x_2)^2$

$= 16x_1^4 - 8x_1^2x_2 + x_2^2$

$$\nabla f(\underline{x}) = \begin{bmatrix} 64x_1^3 - 16x_1x_2 \\ -8x_1^2 + 2x_2 \end{bmatrix} = \begin{bmatrix} 16x_1(4x_1^2 - x_2) \\ -2(4x_1^2 - x_2) \end{bmatrix} = \underline{0}$$

$\Rightarrow$ All points in $\mathbb{R}^2$ which satisfy $4x_1^2 = x_2$ are stationary points.

$$\nabla^2 f(\underline{x}) = \begin{bmatrix} 192x_1^2 - 16x_2 & -16x_1 \\ -16x_1 & 2 \end{bmatrix},$$

$$\nabla^2 f(\underline{x}^*) = \begin{bmatrix} 128x_1^2 & -16x_1 \\ -16x_1 & 2 \end{bmatrix} \Rightarrow \text{p.s.d. due to leading determinants}$$

Since Hessian is p.s.d, stationary points can either be local mins or saddles. Notice that since f is a quadratic function, it is bounded by below with 0, hence points $[x_1, 4x_1^2]$ are nonstrict global minimums

b) $f(x_1,x_2,x_3) = 4x_1^4 - 2x_1^2 + x_2^2 + 2x_2x_3 + 2x_3^2$

$$\nabla f(\underline{x}) = \begin{bmatrix} 16x_1^3 - 4x_1 \\ 2x_2 + 2x_3 \\ 2x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} 4x_1(4x_1^2 - 1) \\ 2(x_2 + x_3) \\ 2(x_2 + 2x_3) \end{bmatrix} = \underline{0} \Rightarrow \begin{array}{l} x_1 = 0, \frac{1}{2}, -\frac{1}{2} \\ x_2 = x_3 = 0 \end{array}$$

stationary points

$$\nabla^2 f(\underline{x}) = \begin{bmatrix} 48x_1^2 - 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\nabla^2 f\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 4 \end{bmatrix} \rightarrow \text{indefinite due to eigenvalues } tr(A) = 2, \text{ but } det(A) = -16$$

$$\nabla^2 f\left(\begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}\right) = \nabla^2 f\left(\begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$\hookrightarrow$ p.d. due to leading determinants

$\underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a saddle point due to the necessary condition,

and $\underline{x} = \begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$ are strict local minimum points due to the sufficient condition.

They are also nonstrict global minimum points since $f\left(\begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}\right)$ and they bound f by below since any other value for $x_2$ & $x_3$ will increase the value.

c) $f(x_1,x_2) = 2x_2^3 - 6x_2^2 + 3x_1^2 x_2$

$$\nabla f(\underline{x}) = \begin{bmatrix} 6x_1 x_2 \\ 6x_2^2 - 12x_2 + 3x_1^2 \end{bmatrix} = \begin{bmatrix} 6x_1 x_2 \\ 6x_2(x_2-2) + 3x_1^2 \end{bmatrix} = \underline{0}$$

$\Rightarrow x_1 = 0$ & $x_2 = 0, 2$ stationary points

$$\nabla^2 f(\underline{x}) = \begin{bmatrix} 6x_2 & 6x_1 \\ 6x_1 & 12(x_2-1) \end{bmatrix}$$

$$\nabla^2 f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & -12 \end{bmatrix} \rightarrow \text{n.s.d due to eigenvalues}$$

$$\nabla^2 f\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} \rightarrow \text{p.d due to leading determinants}$$

$\underline{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ either can be a local max or saddle due to n.s.d. In $x_1$ direction, for $x_2 = 0$, In $x_2$ direction, it is a local maximum. In $x_1$ direction, function $f$ is 0, but for any infinitesimal positive $x_2$, in $x_1$ direction, function first decreases then increases. For a small neighborhood, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a local maximum point.

$\underline{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is a strict local minimum point due to p.d. (It cannot be global since $f$ is not bounded by below)

d) $f(x_1,x_2) = x_1^4 + 2x_1^2 x_2 + x_2^2 - 4x_1^2 - 8x_1 - 8x_2$

$$\nabla f(\underline{x}) = \begin{bmatrix} 4x_1^3 + 4x_1 x_2 - 8x_1 - 8 \\ 2x_1^2 + 2x_2 - 8 \end{bmatrix} = \underline{0}$$

$\Rightarrow x_2 = -x_1^2 + 4$

$\Rightarrow 4x_1^3 + 4x_1(-x_1^2+4) - 8x_1 - 8 = 0$

$\cancel{4x_1^3} - \cancel{4x_1^3} + 16x_1 - 8x_1 - 8 = 0$

$x_1 = 1 \Rightarrow x_2 = 3$

$\underline{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a stationary point

$$\nabla^2 f(\underline{x}) = \begin{bmatrix} 12x_1^2 + 4x_2 - 8 & 4x_1 \\ 4x_1 & 2 \end{bmatrix}$$

$$\nabla^2 f\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 16 & 4 \\ 4 & 2 \end{bmatrix} \rightarrow \text{p.d. due to eigenvalues (or leading determinants)}$$

$\Rightarrow \underline{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a strict local minimizer.

$(x_1^2 + x_2 - 4)^2 = x_1^4 + 2x_1^2 x_2 + x_2^2 - 8x_1^2 - 8x_2 + 16$

$f(\underline{x}) = (x_1^2 + x_2 - 4)^2 + 4x_1^2 - 16 - 8x_1$

$= (x_1^2 + x_2 - 4)^2 + 4(x_1 - 1)^2 - 20$

& $f\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = 1 + 6 + 9 - 4 - 8 - 24$

$= -20$

$f(\underline{x}) \geq -20 \Rightarrow \underline{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is also a strict global minimum point.

Q4 $S = \{ \underline{x} \in \mathbb{R}^2 \mid x_1^2 - x_2^2 + x_1 + x_2 \leq 4 \}$

$= \{ \underline{x} \in \mathbb{R}^2 \mid \underline{x}^T(A\underline{x} + \underline{b}) \leq 4,\ A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},\ \underline{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$

Consider, $\underline{x}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\underline{x}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

$\underline{x}_1^T(A\underline{x}_1 + \underline{b}) = -8 \leq 4 \Rightarrow \underline{x}_1 \in S$

$\underline{x}_2^T(A\underline{x}_2 + \underline{b}) = -18 \leq 4 \Rightarrow \underline{x}_2 \in S$

Let $\underline{x}_3 = \frac{1}{2}\underline{x}_1 + \frac{1}{2}\underline{x}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, i.e., a convex combination of $\underline{x}_1$ and $\underline{x}_2$.

$\underline{x}_3^T(A\underline{x}_3 + \underline{b}) = 12 > 4 \Rightarrow \underline{x}_3 \notin S$

$\Rightarrow S$ is not convex.

Q5 $S = \{ \underline{x} \in \mathbb{R}^2 \mid (x_1 - 1)^2 + x_2 = 1 \}$

Show that $A = \{ \underline{x} \in \mathbb{R}^2 \mid x_1 > 0 \} \cup \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$ is cone(S).

To prove, we need show two expressions:

①: $A \subseteq$ cone(S)

②: cone(S) $\subseteq A$.

Start by proving ② since it's easier to show:

Let $\underline{d} \in$ cone(S).

Then, by definition, $d_1 = \sum_i \underbrace{\lambda_i}_{\geq 0} \underbrace{x_{i_1}}_{\geq 0}$ because $\underline{x_i} \in S$.

$\Rightarrow d_1 \geq 0$

However, $d_1 = 0 \Leftrightarrow \sum_i \lambda_i x_{i_1} = 0$

$\Rightarrow \underline{d} = \underline{0}$ because $\lambda_i = 0$ or $\underline{x_i} = \underline{0}$ for $\forall i$.

If there are non-zero $\lambda_i$, then the corresponding vector must be $[0, 0]^T$ to satisfy $\underline{x_i} \in S$.

Hence, $\forall \underline{d} \in$ cone(S), $\underline{d} \in A \Rightarrow$ cone(S) $\subseteq A$.

Now, show ①:

Let $\underline{a} \in A$. For $\underline{a} = [0, 0]^T$, by conic hull definition it is obvious that $\underline{a} \in$ cone(S). For $a_1 > 0$, first, let's redefine S.

$(x_1 - 1)^2 + x_2 = 1 \Rightarrow x_1^2 + x_2^2 - 2x_1 + 1 = 1 \Rightarrow x_1^2 + x_2^2 - 2x_1 = 0$

$\Rightarrow S = \{ \underline{x} \in \mathbb{R}^2 \mid \underline{x}^T(\underline{x} + \underline{b}) = 0, \ \underline{b} = [-2, 0]^T \}$

We must show that $\forall \underline{a} \in A$, $\exists k \in \mathbb{R}, k \geq 0$, s.t. $\frac{1}{k} \underline{a} \in S$.

(We need to show this because if $\underline{a} = k\underline{s}$, $\underline{s} \in S$, then it is obvious that $\underline{a} \in$ cone(S) for $k \geq 0$.)

$\frac{1}{k} \underline{a} \in S \Rightarrow \frac{1}{k} \underline{a}^T \left( \frac{1}{k} \underline{a} + \underline{b} \right) = 0$

$\frac{1}{k^2} \|\underline{a}\|_2^2 + \frac{1}{k} \underline{a}^T \underline{b} = 0$

For $k > 0$: ($k \neq 0$ is arbitrary due to $a_1 > 0$)

$\frac{1}{k} \|\underline{a}\|_2^2 + \underline{a}^T \underline{b} = 0$

$\frac{1}{k} \|\underline{a}\|_2^2 - 2a_1 = 0 \Rightarrow k = \frac{\|\underline{a}\|_2^2}{2a_1} \geq 0$

Notice that $\|\underline{a}\|_2^2 \geq 0$ and by initial assumption, $a_1 > 0$. Hence $k \geq 0$.

Hence, $\forall \underline{a} \in A$, $\underline{a} = \sum_i \lambda_i \underline{x_i}$, $\lambda_i \geq 0$, $\forall \underline{x_i} \in S$

$= k\underline{s}$, $\underline{s} \in S$, $k \geq 0$

$\forall \underline{a} \in A$, $\underline{a} \in$ cone(S) $\Rightarrow A \subseteq$ cone(S)

①&② $\Rightarrow$ A is equivalent to cone(S).

Q6 Let $\underline{a}, \underline{b} \in \mathbb{R}^n$, $\underline{a} \neq \underline{b}$. For what values of $\mu$ is the set

$$S_\mu = \{ \underline{x} \in \mathbb{R}^n \mid \|\underline{x}-\underline{a}\|_2 \leq \mu \|\underline{x}-\underline{b}\|_2 \}$$ convex?

First, consider $\mu<0$:

$\forall \mu<0$, $S_\mu=\emptyset$ because $\|\underline{x}-\underline{a}\|_2 < 0$ cannot be satisfied for $\forall \underline{a} \in \mathbb{R}^n$ due to definition of norm. So, we must check if empty set is convex or not.

Consider two convex sets, A and B. Let $\underline{x}, \underline{y} \in A \cap B$.

Notice that $\lambda \underline{x} + (1-\lambda)\underline{y} \in A\cap B$, $\forall \lambda \in (0,1)$ since $\underline{x},\underline{y} \in A\cap B \Rightarrow \underline{x}\in A, \underline{x} \in B, \underline{y}\in A, \underline{y} \in B$

$\Rightarrow \lambda \underline{x} + (1-\underline{y}) \in A$, $\lambda \underline{x} + (1-\underline{y}) \in B$ (A & B are convex)

$\Rightarrow \lambda \underline{x} + (1-\underline{y}) \in A\cap B$

$\Rightarrow A \cap B$ is convex.

So, for any A, B, $A\cap B$ is also convex if A and B are convex. If A and B are disjoint sets, i.e., $A\cap B = \emptyset$, then $\emptyset$ also must be convex w.l.o.g.

Hence $\mu<0$, $S_\mu$ is convex.

Now consider $\mu=0$:

$\|\underline{x}-\underline{a}\|_2 = 0 \Rightarrow \underline{x} = \underline{a}$, $S_\mu = \{\underline{a}\} \Rightarrow$ convex

$\lambda \underline{a} + (1-\lambda)\underline{a} = \underline{a}$

For $\mu>0$, things are more complex. First, consider $\mu=1$, where $S_\mu$ is a half plane, i.e., convex. Then, for $\mu>0$:

$$\|\underline{x}-\underline{a}\|_2 \leq \mu \|\underline{x}-\underline{b}\|_2 \Rightarrow \|\underline{x}-\underline{a}\|_2^2 \leq \mu^2 \|\underline{x}-\underline{b}\|_2^2 \quad (\mu>0)$$

$$\rightarrow \|\underline{x}\|_2^2 + \|\underline{a}\|_2^2 - 2\underline{x}^T\underline{a} \leq \mu^2\|\underline{x}\|_2^2 + \mu^2\|\underline{b}\|_2^2 - 2\mu^2\underline{x}^T\underline{b}$$

$$\|\underline{x}\|_2^2(1-\mu^2) - 2\underline{x}^T(\underline{a}-\mu^2\underline{b}) + (\|\underline{a}\|_2^2 - \mu^2\|\underline{b}\|_2^2) \leq 0$$

① If $(1-\mu^2)>0$, i.e., $\mu<1$ (or $0<\mu<1$ due to initial condition)

$$\|\underline{x}\|_2^2 - 2\underline{x}^T\left(\frac{\underline{a}-\mu^2\underline{b}}{1-\mu^2}\right) + \left(\frac{\|\underline{a}\|_2^2-\mu^2\|\underline{b}\|_2^2}{1-\mu^2}\right) \leq 0$$

Let $\underline{m} = \frac{\underline{a}-\mu^2\underline{b}}{1-\mu^2}$, $n = \frac{\|\underline{a}\|_2^2-\mu^2\|\underline{b}\|_2^2}{1-\mu^2}$. Then,

$\underline{x}^T\underline{x} - 2\underline{m}^T\underline{x} + n \leq 0 \Rightarrow$ We have a quadratic term where $A=I$.

Let $S = \{\underline{x}\in\mathbb{R}^n \mid \underline{x}^T\underline{x} - 2\underline{m}^T\underline{x} + n \leq 0, \underline{m}\in\mathbb{R}^n, n \in \mathbb{R}\}$. If $S$ is convex, then $S_\mu$ will be convex for $0<\mu<1$.

Also, let's examine it for $\forall A$, i.e., $S = \{\underline{x}\in\mathbb{R}^n \mid \underline{x}^TA\underline{x} - 2\underline{m}^T\underline{x} + n \leq 0\}$

If $S_-$ is convex, then any line segment through it should also be convex. Let $L=\{t\underline{x}+\underline{x}_0 \in \mathbb{R}^n \mid \underline{x}_0 \in \mathbb{R}^n,\ t \in \mathbb{R}\}$. Then,

$$S_- \cap L = \{\underline{x} \in \mathbb{R}^n \mid (t\underline{x}+\underline{x}_0)^T A (t\underline{x}+\underline{x}_0) - 2\underline{m}^T(t\underline{x}+\underline{x}_0) + n \leq 0\}$$

$$(t\underline{x}+\underline{x}_0)^T A (t\underline{x}+\underline{x}_0) - 2\underline{m}^T(t\underline{x}+\underline{x}_0) + n \leq 0$$

$$\Rightarrow (\underline{x}^T A \underline{x})t^2 + (2\underline{x}_0^T A \underline{x} - 2\underline{m}^T\underline{x})t + (\underline{x}_0^T A \underline{x}_0 - 2\underline{m}^T\underline{x}_0 + n) \leq 0$$

Notice that if $\underline{x}^T A \underline{x} \geq 0, \forall \underline{x} \neq \underline{0}$ then this function will always increase moving away from the minimum w.r.t. parameter $t$. The last expression is bounded above by zero, and is valid for $\forall t \in [r_1, r_2]$ where $r_1$ & $r_2$ are the roots of the expression. Since the arbitrary line segment is convex due to $t$ being continuous in a range, we can infer that $S_-$ is convex if $A$ is positive semidefinite. In our problem, $A=I$, which satisfies the condition. Hence, for $0<\mu<1$ $S_\mu$ is convex, since even if the minimum of the expression is positive, set will be empty set, which is also convex.

② If $\mu>1$, then $\underline{x}^T\underline{x} - 2\underline{m}^T\underline{x} + n \geq 0 \Rightarrow -\underline{x}^T\underline{x} + 2\underline{m}^T\underline{x} - n \leq 0$.

Using the previous line argument, if the maximum of parametric function is positive, then $S_\mu$ will be non-convex

$$(-\underline{x}^T A \underline{x})t^2 + (-2\underline{x}_0^T A \underline{x} - 2\underline{m}^T\underline{x})t + (-\underline{x}_0^T A \underline{x}_0 - 2\underline{m}^T\underline{x}_0 + n) \leq 0$$

$$\downarrow$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Always decreasing while moving away from the maximum. Expression is satisfied if $t \in (-\infty, r_1] \cup [r_2, \infty)$, where $r_1$ & $r_2$ are the roots, which isn't continuous necessarily.

Assuming parametric functions have real roots and empty sets are convex, then $S_\mu$ is convex for $\mu \leq 1$.

(For $\mu=1$, $S_\mu$ is a half space, which is convex).