Q1

IE411 HW #5

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minimize $x_1 - 4x_2 + x_3$

s.t. $x_1 + 2x_2 + 2x_3 = -2$

$x_1^2 + x_2^2 + x_3^2 \leq 1$

a) Notice that the feasible region is compact. Then, by Weierstrass theorem, there exists a solution. The solution is either regular (KKT point) or irregular.

$$\nabla f_1 = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}, \quad \nabla h_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Set of active constraints can only be linearly dependent iff $x_1 = x_2 = x_3$, which is not in the feasible region. $(0+0+0=-2)$.

$\Rightarrow$ All feasible points are regular.

$\Rightarrow$ Optimal solution is among the KKT points.

b) $L(\underline{x}, \lambda, \mu) = x_1 - 4x_2 + x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 1) + \mu(x_1 + 2x_2 + 2x_3 + 2)$

$\nabla_{\underline{x}} L = \underline{0} \Rightarrow$

$$\frac{\partial L}{\partial x_1} = 1 + 2\lambda x_1 + \mu = 0$$

$$\frac{\partial L}{\partial x_2} = -4 + 2\lambda x_2 + 2\mu = 0$$

$$\frac{\partial L}{\partial x_3} = 1 + 2\lambda x_3 + 2\mu = 0$$

$x_1 + 2x_2 + 2x_3 = -2$

$\lambda(x_1^2 + x_2^2 + x_3^2 - 1) = 0$, $\lambda \geq 0$, $\lambda \neq 0 \Rightarrow \|\underline{x}\|_2^2 = 1$

$\lambda = 0$ is infeasible due to μ, so

$$x_1 = \frac{-\mu - 1}{2\lambda}, \quad x_2 = \frac{-2\mu + 4}{2\lambda}, \quad x_3 = \frac{-2\mu - 1}{2\lambda}$$

$(\mu+1)^2 + (4-2\mu)^2 + (2\mu+1)^2 = 4\lambda^2$

$\mu^2 + 2\mu + 1 + 4\mu^2 - 16\mu + 16 + 4\mu^2 + 4\mu + 1 = 4\lambda^2$

$$\boxed{9\mu^2 - 10\mu + 18 = 4\lambda^2}$$

$$\frac{(-\mu-1)}{2\lambda} + \frac{(-4\mu+8)}{2\lambda} + \frac{(-4\mu-2)}{2\lambda} = -2$$

$$\boxed{-9\mu + 5 = -4\lambda}$$

$$\boxed{\mu = \frac{4\lambda + 5}{9}}$$

$\frac{\cancel{9}}{\cancel{81}\,9}(16\lambda^2 + 40\lambda + 25) - \frac{10}{9}(4\lambda + 5) + 18 = 4\lambda^2$

$\lambda^2\left(\frac{20}{9}\right) - \frac{137}{9} = 0$

$\rightarrow \boxed{\lambda = \sqrt{\frac{137}{20}}} \cong 2.617 \Rightarrow \mu = 1.71877793$

$\Rightarrow x_1 = -0.5193,\ x_2 = 0.1075,\ x_3 = -0.8478$

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is KKT & optimal. $f(\underline{x}) = -1.797$

Q2 minimize $x_1^2 - x_2^2 - x_3^2$

s.t. $x_1^4 + x_2^4 + x_3^4 \le 1$

a) $f(\underline{x}) = x_1^2 - x_2^2 - x_3^2$

$= x_1(x_1 + 0 + 0) + x_2(0 - x_2 + 0) + x_3(0 + 0 - x_3)$

$= \underline{x}^T A \underline{x}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow$ indefinite

$\Rightarrow$ f is not convex.

$\Rightarrow$ not a convex programming problem.

Compact domain set $\overset{WT}{\Rightarrow}$ solution

b) KKT conditions are not sufficient, but necessary.

$L(\underline{x}, \lambda) = x_1^2 - x_2^2 - x_3^2 + \lambda(x_1^4 + x_2^4 + x_3^4 - 1)$

$\nabla_{\underline{x}} L = \underline{0} \Rightarrow \frac{\partial L}{\partial x_1} = 2x_1 + 4\lambda x_1^3 = 0$ $\quad \lambda \ge 0$

$\frac{\partial L}{\partial x_2} = -2x_2 + 4\lambda x_2^3 = 0$ $\quad \lambda(x_1^4 + x_2^4 + x_3^4 - 1) = 0$

$\frac{\partial L}{\partial x_3} = -2x_3 + 4\lambda x_3^3 = 0$

~~Case 1: $\lambda = 0$~~

$\lambda = 0 \Rightarrow \underline{x} = \underline{0} \Rightarrow$ KKT point

~~Case 2: $\lambda \ne 0$:~~

$\lambda \ne 0 \Rightarrow x_1^4 + x_2^4 + x_3^4 = 1$

$2x_1(1 + 2\underset{\ge 0}{\lambda} x_1^2) = 0 \Rightarrow \boxed{x_1 = 0} \Rightarrow x_2^4 + x_3^4 = 1$

$x_2 = \mp \sqrt[4]{1 - x_3^4}$ (R)

$2x_2(-1 + 2\lambda x_2^2) = 0$

$x_2 = 0 \Rightarrow x_3 = \mp 1 \Rightarrow 2x_3(-1 + 2\lambda x_3^2) = 0 \Rightarrow \underline{x} = \begin{bmatrix} 0 \\ 0 \\ \mp 1 \end{bmatrix}$ is a KKT pt.

$-1 + 2\lambda \Rightarrow \lambda > 0$

$\lambda = \frac{1}{2x_2^2} = \frac{1}{2x_3^2} \Rightarrow x_2 = \mp x_3$ $\quad 2x_2^4 = 1 \Rightarrow x_2 = \mp \frac{1}{\sqrt[4]{2}}$

$x_3 = \mp \frac{1}{\sqrt[4]{2}}$

KKT pts: $\left\{ \begin{bmatrix} 0 \\ 1/\sqrt[4]{2} \\ 1/\sqrt[4]{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt[4]{2} \\ -1/\sqrt[4]{2} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt[4]{2} \\ 1/\sqrt[4]{2} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt[4]{2} \\ -1/\sqrt[4]{2} \end{bmatrix} \right\}$

c) 4 KKT points satisfy the problem, where $f(\underline{x}) = -\sqrt{2}$.

$\underline{x} = [0, 0, \mp 1]$ is not optimal. $(f(\underline{x}) = -1)$.

Q3

minimize $x_1^4 - x_2^2$

s.t. $x_1^2 + x_2^2 \leq 1$

$2x_2 + 1 \leq 0$

Notice that the feasible region is compact. Then, by Weierstrass Theorem, there exists a solution. The solution is either regular (KKT point) or irregular.

$$\nabla f = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}, \quad \nabla h = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Set of active constraints can only be linearly dependent iff $x_1 = x_2$ which is not in the feasible region $(0+1 \leq 0)$

$\Rightarrow$ All feasible points are regular.

$\Rightarrow$ Optimal solution is among the KKT points.

$$L(\underline{x}, \underline{\lambda}) = x_1^4 - x_2^2 + \lambda_1(x_1^2 + x_2^2 - 1) + \lambda_2(2x_2 + 1)$$

$$\nabla_{\underline{x}} L = \underline{0} \Rightarrow \frac{\partial L}{\partial x_1} = 4x_1^3 + 2\lambda_1 x_1 = 0$$

$$\frac{\partial L}{\partial x_2} = -2x_2 + 2\lambda_1 x_2 + 2\lambda_2 = 0$$

$$\lambda_1, \lambda_2 \geq 0$$

$$\lambda_1(x_1^2 + x_2^2 - 1) = 0, \quad \lambda_2(2x_2 + 1) = 0$$

Case 1: $\lambda_1 = \lambda_2 = 0$:

$x_1 = x_2 = 0$, not feasible.

Case 2: $\lambda_1 = 0, \lambda_2 \neq 0$:

$\Rightarrow 2x_2 + 1 = 0 \Rightarrow \boxed{x_2 = -\frac{1}{2}}$, $4x_1^3 = 0 \Rightarrow \boxed{x_1 = 0}$

$1 + 0 + 2\lambda_2 = 0 \Rightarrow \lambda_2 = -\frac{1}{2} < 0 \Rightarrow \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$ is NOT a KKT point.

Case 3: $\lambda_1 \neq 0, \lambda_2 = 0$:

$\boxed{x_1^2 + x_2^2 = 1}$, $2\lambda_1 x_2 = 2x_2 \Rightarrow \lambda_1 = 1$

$4x_1^3 + 2x_1 = 0$, $2x_1(2x_1^2 + 1) = 0$

$\rightarrow x_1 = 0 \Rightarrow x_2 = 1$, $2(1) + 1 > 0 \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is NOT a KKT point.

$x_2 = -1$, $2(-1) + 1 \leq 0 \Rightarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is a KKT point.

Case 4: $\lambda_1 \neq 0$, $\lambda_2 \neq 0$:

$x_1^2 + x_2^2 = 1$, $\quad 2x_2 + 1 = 0 \Rightarrow \quad x_2 = -\frac{1}{2}$, $x_1 = \mp \frac{\sqrt{3}}{2}$

for $x_1 = +\frac{\sqrt{3}}{2}$,

$$2x_1(\lambda_1 + 2x_1^2) = 0$$

$$\sqrt{3}(\lambda_1 + \tfrac{3}{2}) \Rightarrow \quad \lambda_1 = -\tfrac{3}{2}, \text{ infeasible}$$

for $x_1 = -\frac{\sqrt{3}}{2}$,

$$2x_1(\lambda_1 + 2x_1^2) = 0$$

$$-\sqrt{3}(\lambda_1 + \tfrac{3}{2}) = 0 \Rightarrow \quad \text{infeasible}$$

Hence, the sole KKT point is the optimal point

$$\boxed{\underline{x}^* = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad f(\underline{x}^*) = -1}$$

Q4 minimize $(x_1-3)^2+(x_2-2)^2 = f(\underline{x})$

s.t. $x_1+x_2=1$

$x_1, x_2 \geq 0$

a) Domain is compact $\overset{WT}{\Longrightarrow}$ There exists a solution(s).

Also, problem is convex, since f is a convex function and $D := \{\underline{x} \in \mathbb{R}^2 \mid x_1+x_2=1,\ x_1 \geq 0,\ x_2 \geq 0\}$ is a convex set. $\Rightarrow$ KKT conditions are sufficient conditions.

$L(\underline{x}, \lambda, \mu) = (x_1-3)^2+(x_2-2)^2+\lambda_1(-x_1)+\lambda_2(-x_2)+\mu(x_1+x_2-1)$

$\nabla_{\underline{x}} L = \underline{0} \Rightarrow \frac{\partial L}{\partial x_1} = 2(x_1-3)-\lambda_1+\mu=0$

$\frac{\partial L}{\partial x_2} = 2(x_2-2)-\lambda_2+\mu=0$

$\lambda_1, \lambda_2 \geq 0,\ \lambda_1(-x_1)=0,\ \lambda_2(-x_2)=0,\ x_1+x_2=1$

Case 1: $\lambda_1=0,\ \lambda_2=0$

$2x_1-6+\mu=0$, $2x_2-4+\mu=0$ $\Rightarrow$ $x_1=x_2+1$, $x_1=1-x_2$ $\Rightarrow$ $\boxed{x_2=0} \Rightarrow \boxed{x_1=1}$, $\mu=4$

$\underline{x}=\begin{bmatrix}1\\0\end{bmatrix}$ is a KKT point.

Case 2: $\lambda_1=0,\ \lambda_2\neq 0$:

$x_2=0 \Rightarrow x_1=1$, $\mu=4 \Rightarrow$

Case 3: $\lambda_1\neq 0,\ \lambda_2=0$:

$x_1=0 \Rightarrow x_2=1$

$-2+\mu=0 \Rightarrow \mu=2$, $-6-\lambda_1+2=0 \Rightarrow \lambda_1=-4$

$\underline{x}=\begin{bmatrix}0\\1\end{bmatrix}$ is NOT a KKT point

Case 4: $\lambda_1\neq 0,\ \lambda_2\neq 0$:

$x_1=x_2=0 \Rightarrow x_1+x_2\neq 1$, $\underline{x}=\begin{bmatrix}0\\0\end{bmatrix}$ is NOT a KKT point.

$\Rightarrow \boxed{\underline{x}^*=\begin{bmatrix}1\\0\end{bmatrix},\ f(\underline{x}^*)=8}$

b)

b) Lagrangian: $L(\underline{x}, \underline{z}, \mu) = (x_1-3)^2 + (x_2-2)^2 + \mu(x_1+x_2-1)$
Integrate non-negativity into domain.

Dual: $g(\underline{z}, \mu) = \inf_{\underline{x} \in \mathbb{R}^2_{++}} L(\underline{x}, \underline{z}, \mu)$

$$= \inf_{x_1 \geq 0} \left\{ x_1^2 + x_1(\mu-6) + 9 \right\} + \inf_{x_2 \geq 0} \left\{ x_2^2 + x_2(\mu-4) + 4 \right\} - \mu$$

parabola eq.

$$= \frac{-(\mu^2 - 12\mu + 36 - 36)}{4} - \frac{(\mu^2 - 8\mu + 16 - 16)}{4} - \mu$$

$$= -\frac{\mu^2}{2} + 4\mu$$

Dual problem: maximize $-\frac{\mu^2}{2} + 4\mu \rightarrow$ convex

From primal problem, we can say that strong duality exists due to Slater's condition.

c) maximize $-\frac{\mu^2}{2} + 4\mu$

No constraint $\xrightarrow{\text{derivative}} -\mu + 4 = 0 \Rightarrow \mu = +4 \Rightarrow \boxed{g(\mu) = 8}$

strong duality verified.

Also, $\mu = 4$ satisfies infimum condition since optimal x values are non-negative for $\mu = 4$

Q5

$$\text{minimize } x_1^2+2x_2^2+2x_1x_2+x_1-x_2-x_3 = f(\underline{x})$$
$$\text{s.t. } x_1+x_2+x_3 \le 1$$
$$x_3 \le 3$$

a) $f(\underline{x}) = x_1(x_1+x_2+0)+x_2(x_1+2x_2+0)+x_3(0+0+0)+x_1-x_2-x_3$

$= \underline{x}^T A \underline{x} + \underline{b}^T \underline{x}, \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{psd}$

$\Rightarrow f$ is a convex function.

$D := \{\underline{x} \in \mathbb{R}^3 \mid x_1+x_2+x_3 \le 1, x_3 \le 3\}$ is a convex set.

(D looks like a plane with a starting line in space).

$\Rightarrow$ Problem is convex.

b) KKT points are sufficient.

$L(\underline{x}, \underline{\lambda}) = x_1^2+2x_2^2+2x_1x_2+x_1-x_2-x_3+\lambda_1(x_1+x_2+x_3-1)+\lambda_2(x_3-3)$

$\nabla_{\underline{x}} L = \underline{0} \Rightarrow \quad \frac{\partial L}{\partial x_1} = 2x_1+2x_2+1+\lambda_1 = 0$

$\frac{\partial L}{\partial x_2} = 4x_2+2x_1-1+\lambda_1 = 0$

$\frac{\partial L}{\partial x_3} = -1+\lambda_1+\lambda_2 = 0$

$\lambda_1, \lambda_2 \ge 0, \quad \lambda_1(x_1+x_2+x_3-1)=0, \quad \lambda_2(x_3-3)=0$

Case 1: $\lambda_1=0, \lambda_2=0$:

$\frac{\partial L}{\partial x_3}=0$ does not hold.

Case 2: $\lambda_1=0, \lambda_2 \ne 0$:

$x_3=3, \quad \lambda_2=1, \quad \begin{matrix} 2x_1+2x_2=-1 \\ 2x_1+4x_2=1 \end{matrix} \Rightarrow x_2=1 \Rightarrow x_1=-\frac{3}{2}$

$x_1+x_2+x_3 = 3+1-\frac{3}{2} > 1, \quad \underline{x} = \begin{bmatrix} -3/2 \\ 1 \\ 3 \end{bmatrix}$ is NOT a KKT point

Case 3: $\lambda_1 \ne 0, \lambda_2 = 0$:

$x_1+x_2+x_3=1, \quad \lambda_1=1, \quad \begin{matrix} 2x_1+2x_2=-2 \\ 2x_1+4x_2=0 \end{matrix} \Rightarrow x_2=1, x_1=-2$

$x_1+x_2=-1 \Rightarrow x_3 = 2 \le 3$

$\underline{x} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is a KKT point.

$$\Rightarrow \boxed{\underline{x}^* = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \quad f(\underline{x}^*) = -3}$$

c) $g(\underline{\lambda}) = \inf_{\underline{x}} L(\underline{x}, \underline{\lambda})$

$$= \inf_{x_1,x_2} \left\{ x_1^2 + 2x_2^2 + 2x_1x_2 + x_1 - x_2 + \lambda_1(x_1+x_2) \right\} + \inf_{x_3} \left\{ -x_3 + \lambda_1 x_3 + \lambda_2 x_3 \right\} - \lambda_1 - 3\lambda_2$$

$$= \inf_{x_1,x_2} \underbrace{\left\{ [x_1 \; x_2] \underset{\text{pos def}}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [1+\lambda_1, \lambda_1 - 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\}}_{\text{gradient}} + \underbrace{\inf_{x_3} \left\{ x_3(\lambda_1 + \lambda_2 - 1) \right\}}_{\substack{0 \text{ if } \lambda_1+\lambda_2 = 1 \\ -\infty \text{ o.w.}}} - \lambda_1 - 3\lambda_2$$

$$2A\underline{x} + \underline{b} = 0$$

$$\underline{x} = -\frac{1}{2} A^{-1} \underline{b}$$

$$\underline{x} = \left(-\frac{1}{2}\right) \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1+\lambda_1 \\ \lambda_1 - 1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} \lambda_1 + 3 \\ -2 \end{bmatrix}$$

$$x_1 = -\frac{(\lambda_1+3)}{2}, \; x_2 = 1$$

for $x_1 = -\frac{(\lambda_1+3)}{2}, x_2 = 1$

$$= \frac{\overset{(\lambda_1+3)^2}{\lambda_1^2 + 6\lambda_1 + 9}}{4} + \cancel{2}^{1} - (\lambda_1 + 3) - \frac{(\lambda_1+3)}{2} - \cancel{1} + \cancel{\lambda_1} - \frac{\overset{2\lambda_1(\lambda_1+3)}{(\lambda_1^2 + 3\lambda_1)}}{\underset{(2)}{\cancel{4}}} - \cancel{\lambda_1} - 3\lambda_2$$

$$= \frac{(\lambda_1+3)(3-\lambda_1)}{4} + 1 + \left(-\frac{3}{2}\right)\frac{(\lambda_1+3)}{\underset{(4)}{1}} - 3\lambda_2$$

$$= \begin{cases} \frac{(\lambda_1+3)(-\lambda_1-3)}{4} - 3\lambda_2 + 1, & \lambda_1 + \lambda_2 = 1 \\ -\infty, & \text{o.w.} \end{cases}$$

Dual problem: minimize $\quad \frac{(\lambda_1+3)^2}{4} + 3\lambda_2 - 1$

s.t. $\quad \lambda_1 + \lambda_2 = 1$