IE 411: Introduction to Nonlinear Optimization

Fall 2022 - Homework Assignment 1
Due:

Question 1. Show that $\|\cdot\|_p$ for $p = \frac{1}{2}$ which is given by

$$\|x\|_{\frac{1}{2}} := \left(\sum_{i=1}^{n} \sqrt{|x_i|} \right)^2$$

is not a norm. (**Hint:** It is sufficient to find a counterexample.)

Question 2. In this question, you will prove the following statement step by step.
" *Let $\|\cdot\|$ be the Euclidean (ℓ_2) norm. For all $x, y \in \mathbb{R}^n$, we have*

$$|x^\top y| \leq \|x\| \, \|y\| . \tag{1}$$

Moreover, the equality holds if and only if $x = ky$ for some $k \in \mathbb{R}$."
The inequality given by (1) is called the **Cauchy-Schwarz inequality.**

a) Show the statement for $x = 0 \in \mathbb{R}^n$.

   For the remaining parts, assume that $x \neq 0$.

b) Show that the following equality holds for all $x, y \in \mathbb{R}^n$, $x \neq 0$:

$$\frac{1}{\|x\|^2} \left\| \|x\|^2 .y - (x^\top y).x \right\|^2 = \|x\|^2 \|y\|^2 - |x^\top y|^2 \tag{2}$$

c) Using equality (2), show that inequality (1) holds.

d) Assume that $\|x\| \, \|y\| = |x^\top y|$ holds. Using equality (2), show that $y = k.x$ for some $k \in \mathbb{R}$. (Write the value of k in terms of x, y.)

Question 3. Let $T \in \mathbb{R}^{2\times 2}$ be a linear operator defined such that for any $x = (x_1, x_2)^\top \in \mathbb{R}^2$, we have $Tx = (x_2, x_1)^\top$. Find all eigenvalues and eigenvectors of T.

Question 4. Let $A \in \mathbb{R}^{n\times n}$ be the matrix of all 1's, that is,

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

Find all the eigenvalues of A.

Question 5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined as $f(x,y) = x^2 + y^2 + 2x - 3y$. Find a global minimum point of f over the the unit ball $S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

Question 6. Find the global minimum and maximum points of the function $f(x,y) = 2x - 3y$ over the set $S = \{(x,y) : 2x^2 + 5y^2 \leq 1\}$.

Question 7. For each of the following functions, determine whether it is coercive or not:

a. $f(x_1, x_2) = 2x_1^2 - 8x_1x_2 + x_2^2$.

b. $f(x_1, x_2) = 4x_1^2 + 2x_1x_2 + 2x_2^2$.

c. $f(x_1, x_2) = x_1^4 + x_2^4$.

d. $f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$.