IE 411: Introduction to Nonlinear Optimization

Fall 2022 - Homework Assignment 4
Due: December 5, 2022

Question 1. Let $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^n$. Show that exactly one of the following two systems is feasible:

a) $Ax \geq 0, x \geq 0, c^\mathsf{T} x > 0$.

b) $A^\mathsf{T} y \geq c, y \leq 0$.

Hint: Rewrite system given by (a) in the form of $\tilde{A}\tilde{x} \geq 0, \tilde{c}^\mathsf{T}\tilde{x} > 0$ and apply Farkas' Lemma. How would you define $\tilde{A}, \tilde{x}, \tilde{c}$ in terms of A, x, c?

Question 2. Consider the maximization problem

$$\begin{array}{rl} \text{maximize} & x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2 \\ \text{subject to} & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{array}$$

a) Is the problem convex?

b) Find all the KKT points of the problem.

c) Find the optimal solution of the problem.

Question 3. Consider the problem

$$\begin{array}{rl} \text{minimize} & -x_1x_2x_3 \\ \text{subject to} & x_1 + 3x_2 + 6x_3 \leq 48 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

a) Write the KKT conditions for the problem.

b) Find the optimal solution of the problem.

Question 4. Consider the problem

$$\begin{aligned} \text{minimize} \quad & x_1^2 + x_2^2 + x_1 \\ \text{subject to} \quad & x_1 + x_2 \leq a, \end{aligned}$$

where $a \in \mathbb{R}$ is a parameter.

a) Solve the problem using KKT conditions. (The solution will be in terms of the parameter a. You may need to consider different cases for a.)

b) Let $h(a)$ be the optimal value of the problem with parameter a. Write an explicit expression for h.

c) Show that $h : \mathbb{R} \to \mathbb{R}$ is a convex function.

Question 5. Use the KKT conditions to solve the problem

$$\begin{aligned} \text{minimize} \quad & x_1^2 + x_2^2 \\ \text{subject to} \quad & -2x_1 - x_2 + 10 \leq 0 \\ & x_2 \geq 0 \end{aligned}$$