Convex Optimization

A convex optimization problem is of the form

$$\begin{aligned} &\text{minimize} \quad f_0(x) \\ &\text{subject to} \quad x \in C, \end{aligned}$$

where $C \subseteq \mathbb{R}^n$ is a convex set and $f_0 : C \to \mathbb{R}$ is a convex function.

<u>Functional form:</u> Let $f_0, f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ be convex and $h_1, \ldots, h_m : \mathbb{R}^n \to \mathbb{R}$ be affine (linear) functions. A general form of convex optimization problem is given by

$$(P) \quad \begin{aligned} &\text{minimize} \quad f_0(x) \\ &\text{subject to} \quad f_i(x) \leq 0 \quad i = 1, \ldots, m \\ &\qquad\qquad\quad h_j(x) = 0 \quad j = 1, \ldots, p. \end{aligned}$$

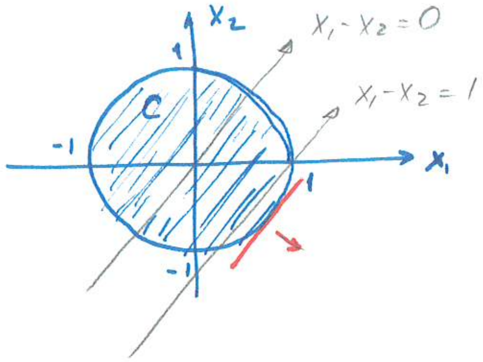
Note that the feasible region is a convex set. Indeed,

$$C := \{ x \in \mathbb{R}^n \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_j(x) = 0,\ j = 1, \ldots, p \}$$

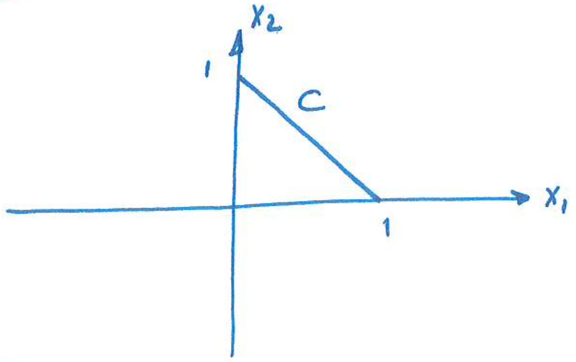
$$= \underbrace{\bigcap_{i=1}^{m} \underbrace{\{ x \in \mathbb{R}^n \mid f_i(x) \leq 0 \}}_{\substack{\text{sublevel set of} \\ \text{convex function } f_i \\ \downarrow \\ \text{convex set}}} \cap \bigcap_{j=1}^{p} \underbrace{\{ x \in \mathbb{R}^n \mid h_j(x) = 0 \}}_{\substack{\text{hyperplane for} \\ \text{each } j \\ \downarrow \\ \text{convex set}}}}_{\text{convex set (intersection of convex sets)}}$$

Example:

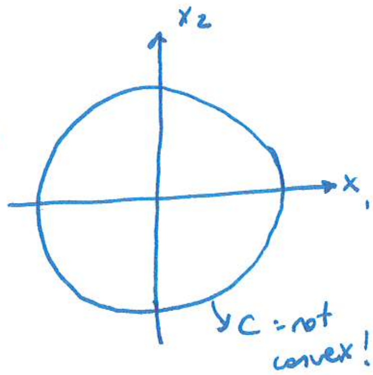
$$\left.\begin{array}{ll}\text{minimize} & x_1 - x_2 \\ \text{subject to} & x_1^2 + x_2^2 \le 1\end{array}\right\} \text{ convex program}$$



$$\left.\begin{array}{ll}\text{minimize} & x_1 - x_2 \\ \text{subject to} & x_1^2 + x_2^2 \le 1 \\ & x_1 + x_2 = 1\end{array}\right\} \text{ convex program}$$



$$\left.\begin{array}{ll}\text{minimize} & x_1 - x_2 \\ \text{subject to} & x_1^2 + x_2^2 = 1\end{array}\right\} \text{ NOT a convex program}$$



Examples of Convex Optimization Problems

1-) Linear programming.

$$\left(\begin{array}{ll}\text{minimize} & c^T x \\ \text{s.t.} & Ax \le b \\ & Bx = d\end{array}\right)$$

Linear (affine) functions are convex!

2-) Convex quadratic programming

$$\left(\begin{array}{ll}\text{minimize} & x^T Q x + 2b^T x \\ \text{s.t.} & Ax \le c\end{array}\right)$$

Let $Q \in \mathbb{R}^{n \times n}$, positive semidefinite

$A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^m$.

Clearly $f_0(x) = x^T Q x + 2b^T x$ is convex quadratic.

3-) Convex Quadratically Constrained Quadratic Programming (CQCQP's)

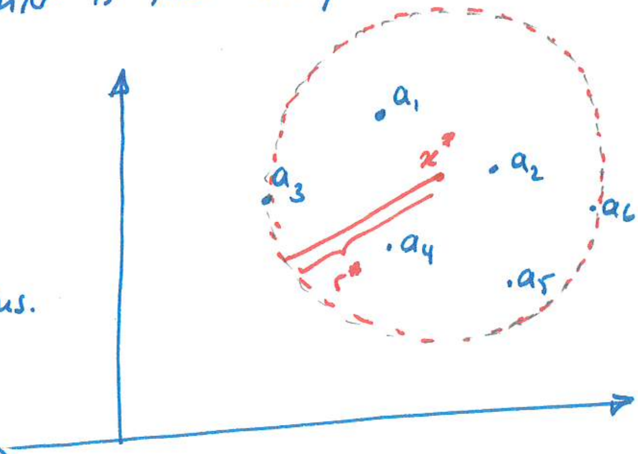
$$\text{minimize} \quad x^T Q_0 x + 2 b_0^T x + c_0$$

$$\text{subject to} \quad x^T Q_i x + 2 b_i^T x + c_i \leq 0, \quad i = 1, \dots, m$$

If $Q_0, Q_1, \dots, Q_m$ are positive semidefinite, then this is a convex program.

4-) Chebyshev Center of a set of points

Assume we are given m points $a_1, \dots, a_m \in \mathbb{R}^n$ and want to compute the centr of the min. radius closed ball containing all the points. This ball is called the Chebyshev ball & its centr is the Chebyshev center.



We look for $x^* \in \mathbb{R}^n$, $r^* > 0$ s.t

$\| x^* - a_i \| \leq r^*$ for all i,

where r^* is the minimum such radius.

$$\left(\begin{array}{ll} \text{minimize} & r \\ \text{subject to} & \| x - a_i \| \leq r, \quad i = 1, \dots, m \\ & x \in \mathbb{R}^n, \ r \in \mathbb{R}. \end{array} \right)$$

Note that $f_0(x, r) = r$ is an affine, hence convex function, and

$f_i(x, r) = \| x - a_i \| - r$ is a convex function for all i.

* Convex Optimization Solver: CVX (cvxr.com)