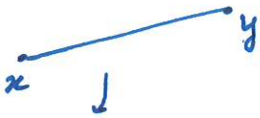
# Convex Sets

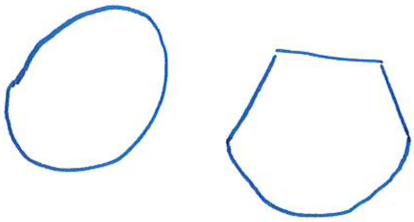
**Defn:** A set $C \subseteq \mathbb{R}^n$ is called convex if for any $x, y \in C$ and $\lambda \in [0,1]$, the point $\lambda x + (1-\lambda) y \in C$.

(The line segment between $x, y$ is in $C$.




$\{\lambda x + (1-\lambda) y \mid \lambda \in [0,1]\}$ )

e.g.



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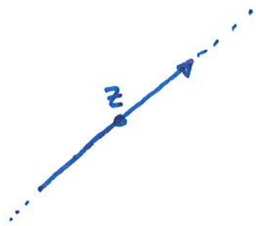
**Examples:**

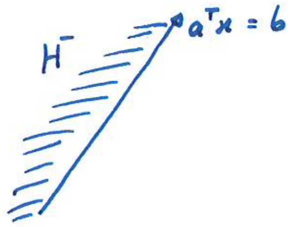
1. let $z, d \in \mathbb{R}^n$, $d \neq 0$. $L = \{ z + td \mid t \in \mathbb{R} \}$ - line




2.) let $a \in \mathbb{R}^n$, $a \neq 0$, $b \in \mathbb{R}$.

$H = \{ x \in \mathbb{R}^n \mid a^T x = b \}$ - hyperplane

$H^- = \{ x \in \mathbb{R}^n \mid a^T x \leq b \}$ - halfspace




3.) let $c \in \mathbb{R}^n$, $r > 0$ and $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

$$\bar{B}(c,r) = \{ x \in \mathbb{R}^n \mid \|x - c\| \leq r \}$$

Let's show its convexity: Let $x, y \in \bar{B}(c,r)$, $\lambda \in [0,1]$.

$x, y \in \bar{B}(c,r) \Rightarrow \|x-c\| \leq r$, $\|y-c\| \leq r$.

$$\begin{aligned}
\|\lambda x + (1-\lambda) y - c\| = \|\lambda(x-c) + (1-\lambda)(y-c)\| &\leq \|\lambda(x-c)\| + \|(1-\lambda)(y-c)\| \quad \text{(Triangle ineq.)} \\
&= \lambda \|x-c\| + (1-\lambda)\|y-c\| \quad (\lambda \geq 0,\ 1-\lambda \geq 0) \\
&\leq \lambda r + (1-\lambda) r \\
&= r.
\end{aligned}$$

$\therefore \lambda x + (1-\lambda) y \in \bar{B}(c,r)$.

4.) Let $Q \in \mathbb{R}^{n \times n}$ be positive semidefinite, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$.

$E = \{x \in \mathbb{R}^n \mid x^T Q x + 2b^T x + c \leq 0\}$ - ellipsoid.

Lemma: Intersection of convex sets is convex. (finite or infinite collections)

Why? Let $C_i \subseteq \mathbb{R}^n$ be convex for all $i \in I$ ($I$: index set)

Let $C = \bigcap_{i \in I} C_i$. Let $x, y \in C$. Then, $x, y \in C_i$, $\forall i \in I$.

Since $C_i$ is convex, $\forall \lambda \in [0,1]$: $\lambda x + (1-\lambda) y \in C_i$ for all $i \in I$.

Thus, $\lambda x + (1-\lambda) y \in \bigcap_{i \in I} C_i$.

5.) Let $A^T \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

$$P = \{x \in \mathbb{R}^n \mid A^T x \leq b\} = \underbrace{\bigcap_{i=1}^{m} \{x \in \mathbb{R}^n \mid a_i^T x \leq b_i\}}_{\text{intersection of halfspaces.}}$$

convex polytope

Defn: Let $x_1, \ldots, x_k \in \mathbb{R}^n$. A convex combination of $x_1, \ldots, x_k$ is a point $\lambda_1 x_1 + \cdots + \lambda_k x_k \in \mathbb{R}^n$ for some $\lambda_1, \ldots, \lambda_k \geq 0$ such that $\sum_{i=1}^{k} \lambda_i = 1$.

Defn: Let $S \subseteq \mathbb{R}^n$. The convex hull of $S$, conv $S$, is the set of all convex combinations of points from $S$, that is,

$$\text{conv } S = \left\{ \sum_{i=1}^{k} \lambda_i s_i \;\middle|\; k \in \mathbb{N}, \lambda_i \geq 0, \forall i, \sum_{i=1}^{k} \lambda_i = 1, s_i \in S \ \forall i \right\}.$$

Result: conv $S$ is the smallest convex set containing $S$.

Corollary: If $S$ is convex, then conv $S = S$.

If $S$ is convex, $T \subseteq S$, then conv $T \subseteq$ conv $S = S$.

Proof of the previous result: First note that $S \subseteq \operatorname{conv}(S)$. Indeed,

for any $s \in S$, taking $k=1$, $\lambda_1 = 1$ in conv $S$ shows that $s \in \operatorname{conv}(S)$.

Moreover, conv $(S)$ is a convex set.

Let $\sum_{i=1}^{k} \lambda_i s_i$, $\sum_{j=1}^{\ell} \mu_j t_j \in \operatorname{conv} S$ where $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_\ell \geq 0$

$s_1, \dots, s_k, t_1, \dots, t_\ell \in S$

$\sum_{i=1}^{k} \lambda_i = 1$, $\sum_{j=1}^{\ell} \mu_j = 1$.

Let $\theta \in [0,1]$ be arbitrary.

To show: $\theta \sum_{i=1}^{k} \lambda_i s_i + (1-\theta) \sum_{j=1}^{\ell} \mu_j t_j \in \operatorname{conv} S$

$$\sum_{i=1}^{k} \theta \lambda_i s_i + \sum_{j=1}^{\ell} (1-\theta) \mu_j t_j = \sum_{i=1}^{k+\ell} \alpha_i s_i$$ where $\alpha_i = \lambda_i \cdot \theta$, $i = 1, \dots, k$

$\alpha_i = (1-\theta)\mu_{i-k}$, $i = k+1, \dots, k+\ell$

$s_i = t_{i-k}$, $i = k+1, \dots, k+\ell$

This would be in conv $S$ if $\sum_{i=1}^{k+1} \alpha_i = 1$.

But $\sum_{i=1}^{k+\ell} \alpha_i = \theta \sum_{i=1}^{k} \lambda_i + (1-\theta) \sum_{j=1}^{\ell} \mu_j = \theta + (1-\theta) = 1$. ✓

Finally, we will show that any convex set $T$ containing $S$ also contains conv $S$.

Let $T \supseteq S$, $T$: convex. To show: $\operatorname{conv}(S) \subseteq T$.

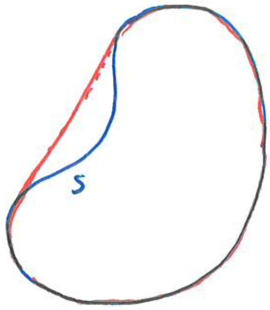
Let $z \in \operatorname{conv}(S)$. Then, $z = \sum_{i=1}^{k} \lambda_i s_i$ for some $k \in \mathbb{N}$, $\lambda_i \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1$, $s_i \in S$.

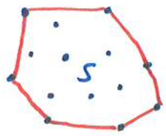
As $s_i \in S$, we have $s_i \in T$. Moreover, $T$ is a convex set. Then

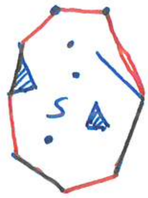
$z \in T$ also holds.

∎

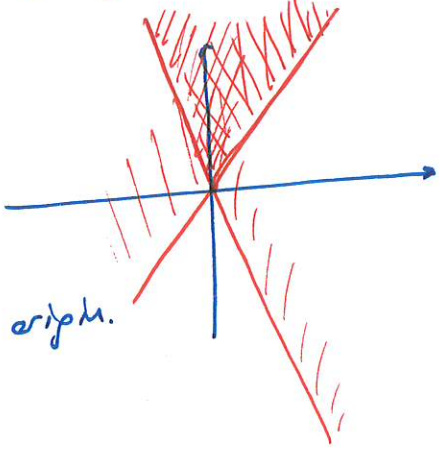
e.g.




**Defn:** A set $S \subseteq \mathbb{R}^n$ is called a <u>cone</u> if for any $x \in S$, $\lambda \geq 0$ we have $\lambda x \in S$.



e.g. a ray passing through the origin.
a halfspace passing through the origin
intersection of halfspaces passing through the origin.

**Lemma:** A set $S$ is a convex cone if and only if

a) $x, y \in S \Rightarrow x + y \in S$

b.) $x \in S, \lambda \geq 0 \Rightarrow \lambda x \in S$

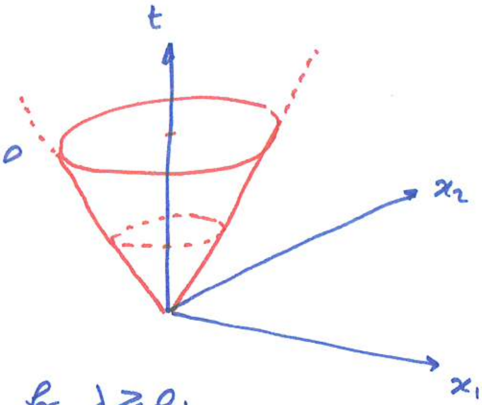
**Proof:** Assume $S$ is a convex cone. b.) holds by definition. Moreover for any $x, y \in S$, since $S$ is a cone $2x, 2y \in S$. Using that $S$ is convex, we have $\frac{1}{2}(2x) + \frac{1}{2}(2y) = x + y \in S$.

Now, let's assume a), b) hold. From b.), we know that $S$ is a cone.
To show convexity, let $x, y \in S$ be arbitrary, $\lambda \in [0,1]$.
Using b.), we know that $\lambda x \in S$ and $(1-\lambda) y \in S$.
Using (a), we obtain $\lambda x + (1-\lambda) y \in S$. $\blacksquare$

**Example:** Ice cream cone (Lorentz cone)

$$L^n = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; \|x\| \leq t,\ x \in \mathbb{R}^n,\ t \in \mathbb{R} \right\}$$

For $n=2$, $\|x\| \leq t$ holds iff $x_1^2 + x_2^2 \leq t^2$, $t \geq 0$




Let's show that $L^n$ is a convex cone.

First, note that if $\begin{pmatrix} x \\ t \end{pmatrix} \in L^n$, then $\lambda \begin{pmatrix} x \\ t \end{pmatrix} \in L^n$ for $\lambda \geq 0$.

Indeed, $\|\lambda x\| = |\lambda| \, \|x\| = \lambda \|x\| \leq \lambda t$ as $\|x\| \leq t$.

Let $\begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} y \\ s \end{pmatrix} \in L^n$. To show: $\begin{pmatrix} x+y \\ t+s \end{pmatrix} \in L^n$.

$$\|x+y\| \underset{\substack{\downarrow \\ \text{triangle} \\ \text{ineq.}}}{\leq} \|x\| + \|y\| \leq t+s \quad \checkmark$$

## Convex Cones constructed from sets

**Dual Cone:** Let $K$ be a cone in $\mathbb{R}^n$. The set

$$K^+ = \{ y \in \mathbb{R}^n \mid x^T y \geq 0 \text{ for all } x \in K \}$$

is called the (positive) dual cone of $K$.

Let's show that $K^+$ is always a convex cone. (even if $K$ is not convex nor cone)

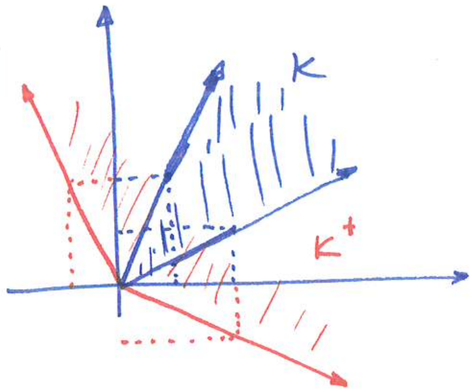
Let $y \in K^+$, $\lambda \geq 0$ $\Rightarrow$ $x^T(\lambda y) = \lambda x^T y \geq 0$ holds for all $x \in K$ as $y \in K^+$, $\lambda \geq 0$.

Let $y_1, y_2 \in K^+ \Rightarrow x^T y_1 \geq 0$, $x^T y_2 \geq 0$ hold for all $x, y \in K$.

$$x^T(y_1 + y_2) = x^T y_1 + x^T y_2 \geq 0 \quad \checkmark$$

(we haven't used the fact that $K$ is a cone!)

Ex: let $K = \text{cone}\left(\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}\right)$
$= \left\{ \lambda_1 \begin{pmatrix}1\\2\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\end{pmatrix} \in \mathbb{R}^2 \mid \lambda_1, \lambda_2 \geq 0 \right\}$




$K^+ = \{ y \in \mathbb{R}^2 \mid x^T y \geq 0 \text{ for } x \in K \}$

$$= \bigcap_{x \in K} \{ y \in \mathbb{R}^2 \mid x^T y \geq 0 \}$$

$$= \text{cone}\left(\left\{\begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix}\right\}\right) = \left\{ \lambda_1 \begin{pmatrix}-1\\2\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\-1\end{pmatrix} \in \mathbb{R}^2 \mid \lambda_1, \lambda_2 \geq 0 \right\}$$

In general, computing the dual cone is a difficult problem!

## Normal Cone

Let $C \subseteq \mathbb{R}^n$ be a set and $x_0 \in \text{bd}\, C$ (a boundary point). The set

$$N_C(x_0) = \{ y \in \mathbb{R}^n \mid \forall x \in C : y^T(x - x_0) \leq 0 \}$$

is the <u>normal cone of $C$ at $x_0$</u>. (The set of vectors that define a supporting hyperplane to $C$ at $x_0$.)

For any set $C \subseteq \mathbb{R}^n$, $x_0 \in \text{bd}\, C$, $N_C(x_0)$ is a convex cone.

1.) let $y \in N_C(x_0) \Rightarrow y^T(x - x_0) \leq 0$ for all $x \in C$

let $\lambda \geq 0 \Rightarrow (\lambda y)^T (x - x_0) \leq 0$ for all $x \in C$. ✓

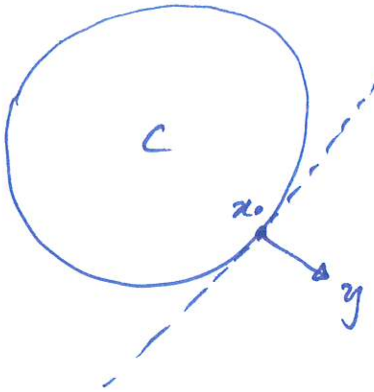
2.) let $y_1, y_2 \in N_C(x_0) \Rightarrow y_1^T(x - x_0) \leq 0$, $y_2^T(x - x_0) \leq 0$ for all $x \in C$.

let $\theta \in [0,1]$, $y = \theta y_1 + (1-\theta) y_2$.

$(\theta y_1 + (1-\theta) y_2)^T (x - x_0) = \theta\, y_1^T (x - x_0) + (1-\theta)\, y_2^T (x - x_0) \leq 0$ for all $x \in C$.

Hence $\theta y_1 + (1-\theta) y_2 \in N_C(x_0)$.

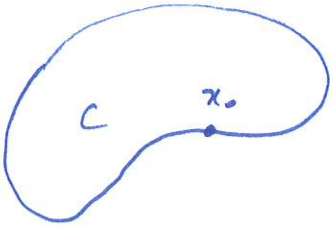
e.g.




$H := \{x \in \mathbb{R}^n \mid y^T x = y^T x_0\}$ – supporting hyperplane.

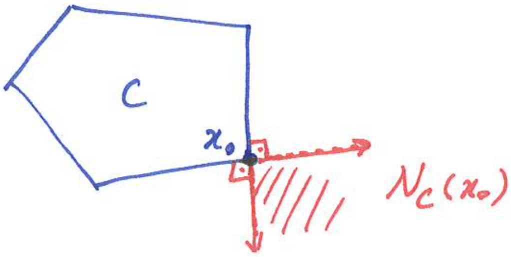
Note that $C \subseteq \{x \in \mathbb{R}^n \mid y^T x \leq y^T x_0\}$

$$N_C(x_0) = \text{cone}\{y\}.$$




There's no such hyperplane at $x_0$!

Indeed, $N_C(x_0) = \{0\}$ (a convex cone)

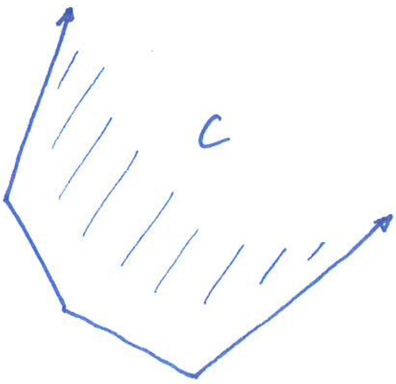


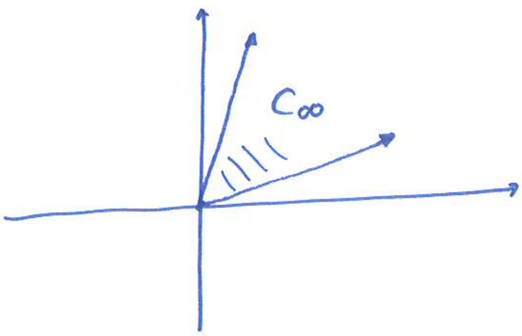

## Recession Cone

Let $C \subseteq \mathbb{R}^n$. A <u>recession direction</u> of $C$ is a vector $d \in \mathbb{R}^n$ such that $c + \lambda d \in C$ for all $c \in C, \lambda \geq 0$.

The set of all $d \in \mathbb{R}^n$ satisfying this is the recession cone of $C$.

$$C_\infty := \{d \in \mathbb{R}^n \mid c + \lambda d \in C \text{ for all } c \in C, \lambda \geq 0\}.$$



Recession cone of a convex set is a convex cone.

Let $C \subseteq \mathbb{R}^n$ be convex.

1.) Let $d \in C_\infty$, $\mu \geq 0$ be arbitrary. Check if $\mu d \in C_\infty$, that is, $c + \lambda(\mu d) \overset{?}{\in} C$ for all $c \in C$, $\lambda \geq 0$. Note that $c + (\lambda\mu)d \in C$ holds as $d \in C_\infty$, $\lambda\mu \geq 0$.

2.) Let $d_1, d_2 \in C_\infty$. Check if $d_1 + d_2 \in C_\infty$.

For arbitrary $c \in C$, $\lambda \geq 0$: $c + \lambda(d_1 + d_2) \overset{?}{\in} C$.

$$c + \lambda(d_1+d_2) = \frac{1}{2}\underbrace{(c + 2\lambda d_1)}_{\substack{\in C \\ \text{since } d_1 \in C_\infty \\ c \in C,\ 2\lambda \geq 0}} + \frac{1}{2}\underbrace{(c + 2\lambda d_2)}_{\substack{\in C \\ \text{since } d_2 \in C_\infty \\ c \in C,\ 2\lambda \geq 0}} \in C \text{ as } C \text{ is a convex set.}$$

Note that for a bounded set, the recession cone is $\{0\}$. The reverse also holds.

Example: Let $S = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

Show that $S_\infty = \{x \in \mathbb{R}^n \mid Ax \leq 0\}$.

We will show ① $S_\infty \subseteq \{x \in \mathbb{R}^n \mid Ax \leq 0\}$ and ② $S_\infty \supseteq \{x \in \mathbb{R}^n \mid Ax \leq 0\}$.

① Let $d \in \mathbb{R}^n$ be a recession direction, i.e., $d \in S_\infty$. Then, for all $x \in S$, $\lambda \geq 0$,

$A(x + \lambda d) \leq b$ holds. In particular, let $x \in S$ be fixed.

This implies $\lambda . Ad \leq \underbrace{b - Ax}_{\in \mathbb{R}^m}$ holds for all $\lambda \geq 0$.

If $Ad \not\leq 0$, then there would be a component $(Ad)_i > 0$. Taking the limit as $\lambda \nearrow +\infty$, we would obtain $+\infty \leq \underbrace{(b - Ax)_i}_{\in \mathbb{R}}$. A contradiction.

Hence, it must be true that $Ad \leq 0. \Rightarrow d \in \{x \in \mathbb{R}^n \mid Ax \leq 0\}$.

② Let $\bar{x} \in \mathbb{R}^n$ with $A\bar{x} \leq 0$. To show: $\bar{x}$ is a recession direction.

Let $x \in S$, $\lambda \geqslant 0$ be arbitrary. Consider $x + \lambda \bar{x}$.

$$A(x+\lambda\bar{x}) = \underbrace{Ax}_{\leq b} + \lambda \underbrace{A\bar{x}}_{\leq 0} \leq b$$ holds. Hence $x + \lambda\bar{x} \in S$. ✓

**Defn:** Given $k$ points $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$, a conic combination of $x_1, \ldots, x_k$ is a point of the form $\lambda x_1 + \ldots + \lambda_k x_k$ for some $\lambda_1, \ldots, \lambda_k \geqslant 0$.

The conic hull of a set $S$ is the set of all conic combinations of points from that $S$. It's denoted by cone$(S)$.

Result: cone$(S)$ is the smallest "convex cone" containing set $S$.

**Defn.** Let $S \subseteq \mathbb{R}^n$ be a convex set. A point $x \in S$ is an extreme point of $S$ if there's no $x_1, x_2 \in S$, $x_1 \neq x_2$ and $\lambda \in (0,1)$ such that $x = \lambda x_1 + (1-\lambda) x_2$.