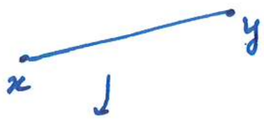
Convex Sets

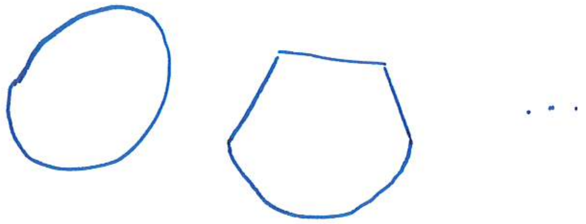
Defn: A set $C \subseteq \mathbb{R}^n$ is called convex if for any $x, y \in C$ and $\lambda \in [0,1]$, the point $\lambda x + (1-\lambda) y \in C$.

(The line segment between x, y is in C.



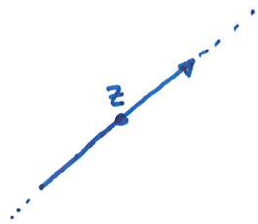
$\{\lambda x + (1-\lambda) y \mid \lambda \in [0,1]\}$)

e.g.



Examples:

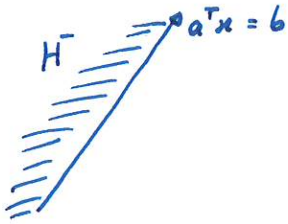
1. Let $z, d \in \mathbb{R}^n$, $d \neq 0$. $L = \{z + td \mid t \in \mathbb{R}\}$ - line



2.) Let $a \in \mathbb{R}^n$, $a \neq 0$, $b \in \mathbb{R}$.

$H = \{x \in \mathbb{R}^n \mid a^T x = b\}$ - hyperplane

$H^- = \{x \in \mathbb{R}^n \mid a^T x \leq b\}$ - halfspace



3.) Let $c \in \mathbb{R}^n$, $r > 0$ and $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

$\bar{B}(c, r) = \{x \in \mathbb{R}^n \mid \|x - c\| \leq r\}$

Let's show its convexity: Let $x, y \in \bar{B}(c,r)$, $\lambda \in [0,1]$.

$x, y \in \bar{B}(c,r) \Rightarrow \|x - c\| \leq r$, $\|y - c\| \leq r$.

$$\begin{aligned}
\|\lambda x + (1-\lambda) y - c\| = \|\lambda(x-c) + (1-\lambda)(y-c)\| &\leq \|\lambda(x-c)\| + \|(1-\lambda)(y-c)\| && \text{(Triangle ineq.)} \\
&= \lambda \|x-c\| + (1-\lambda)\|y-c\| && (\lambda \geq 0,\ 1-\lambda \geq 0) \\
&\leq \lambda r + (1-\lambda) r \\
&= r.
\end{aligned}$$

$\therefore\ \lambda x + (1-\lambda) y \in \bar{B}(c,r)$.

4.) Let $Q \in \mathbb{R}^{n \times n}$ be positive semidefinite, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$.

$E = \{x \in \mathbb{R}^n \mid x^T Q x + 2b^T x + c \leq 0\}$ - ellipsoid.

Lemma: Intersection of convex sets is convex. (finite or infinite collections)

Why? Let $C_i \subseteq \mathbb{R}^n$ be convex for all $i \in I$ (I: index set)

Let $C = \bigcap_{i \in I} C_i$. Let $x, y \in C$. Then, $x, y \in C_i$, $\forall i \in I$.

Since C_i is convex, $\forall \lambda \in [0,1)$: $\lambda x + (1-\lambda) y \in C_i$ for all $i \in I$.

Then, $\lambda x + (1-\lambda) y \in \bigcap_{i \in I} C_i$.

5.) Let $A^T \in \mathbb{R}^{m \times m}$, $b \in \mathbb{R}^m$.

$$\underset{\text{convex polytope}}{P} = \{x \in \mathbb{R}^n \mid A^T x \leq b\} = \underbrace{\bigcap_{i=1}^{m} \{x \in \mathbb{R}^n \mid a_i^T x \leq b_i\}}_{\text{intersection of halfspaces.}}$$

Defn: Let $x_1, \ldots, x_k \in \mathbb{R}^n$. A <u>convex combination</u> of $x_1, \ldots, x_k$ is a point $\lambda_1 x_1 + \cdots + \lambda_k x_k \in \mathbb{R}^n$ for some $\lambda_1, \ldots, \lambda_k \geq 0$ such that $\sum_{i=1}^{k} \lambda_i = 1$.

Defn: Let $S \subseteq \mathbb{R}^n$. The <u>convex hull</u> of S, conv S, is the set of all convex combinations of points from S, that is,

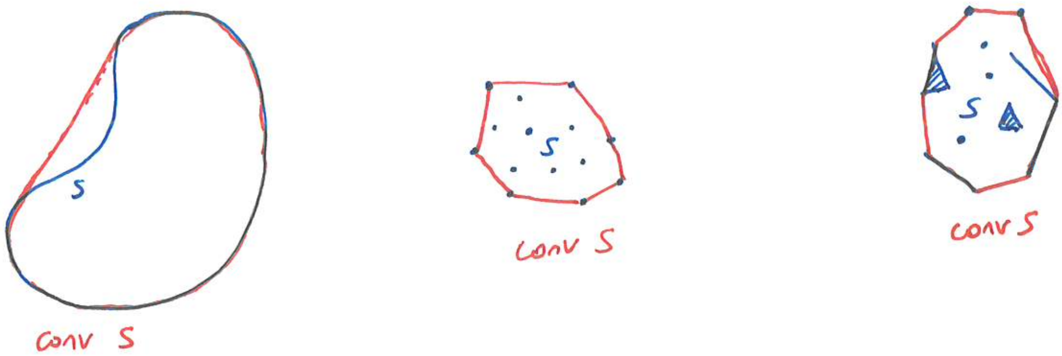
$$\text{conv } S = \left\{ \sum_{i=1}^{k} \lambda_i s_i \;\middle|\; k \in \mathbb{N}, \lambda_i \geq 0, \forall i, \sum_{i=1}^{k} \lambda_i = 1, s_i \in S \ \forall i \right\}.$$

Result: conv S is the smallest convex set containing S.

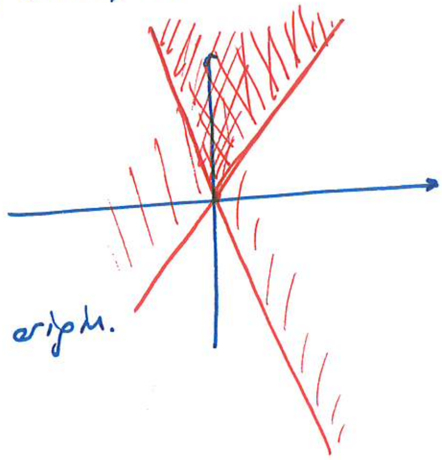
If S is convex, then conv $S = S$.

If S is convex, $T \subseteq S$, then conv $T \subseteq$ conv $S = S$.

e.g.



<u>Defn</u>: A set $S \subseteq \mathbb{R}^n$ is called a <u>cone</u> if for any $x \in S$, $\lambda \geq 0$ we have $\lambda x \in S$.



e.g. a ray passing through the origin.

a halfspace passing through the origin

intersection of halfspaces passing through the origin.

<u>Lemma</u>: A set S is a convex cone if and only if

a) $x, y \in S \Rightarrow x + y \in S$

b.) $x \in S$, $\lambda \geq 0 \Rightarrow \lambda x \in S$

<u>Proof</u>: Assume S is a convex cone. b.) holds by definition. Moreover for any $x, y \in S$, since S is a cone $2x, 2y \in S$. Using that S is convex, we have $\frac{1}{2}(2x) + \frac{1}{2}(2y) = x + y \in S$.

Now, let's assume a), b) hold. From b.), we know that S is a cone.

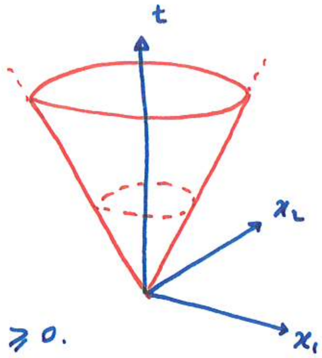
To show convexity, let $x, y \in S$ be arbitrary, $\lambda \in [0,1]$.

Using b.), we know that $\lambda x \in S$ and $(1-\lambda) y \in S$.

Using (a), we obtain $\lambda x + (1-\lambda) y \in S$. $\blacksquare$

Example: Ice cream cone (Lorenz cone)

$$L^n = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; \|x\| \leq t,\ x \in \mathbb{R}^n,\ t \in \mathbb{R} \right\}$$



For $n=2$, $\|x\| \leq t$ holds iff $x_1^2 + x_2^2 \leq t^2$, $t \geq 0$.

Let's show that L^n is a convex cone.

First, note that if $\begin{pmatrix} x \\ t \end{pmatrix} \in L^n$, then $\lambda \begin{pmatrix} x \\ t \end{pmatrix} \in L^n$ for $\lambda \geq 0$.

Indeed, $\|\lambda x\| = |\lambda| \|x\| = \lambda \|x\| \leq \lambda t$ since $\|x\| \leq t$.

Let $\begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} y \\ s \end{pmatrix} \in L^n$. To show: $\begin{pmatrix} x+y \\ t+s \end{pmatrix} \in L^n$.

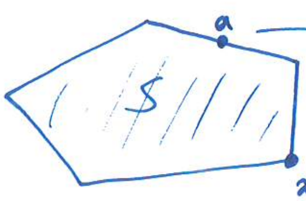
$$\|x+y\| \underset{\text{triangle ineq.}}{\leq} \|x\| + \|y\| \leq t+s. \checkmark$$

Defn: Given k points $x_1, \ldots, x_k \in \mathbb{R}^n$, a conic combination of $x_1, \ldots, x_k$ is a point of the form $\lambda_1 x_1 + \cdots + \lambda_k x_k$ for some $\lambda_1, \ldots, \lambda_k \geq 0$.

The conic hull of a set is the smallest convex cone containing that set.

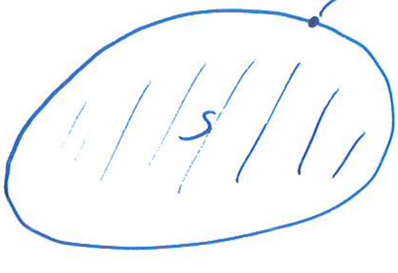
Defn: Let $S \subseteq \mathbb{R}^n$ be a convex set. A point $x \in S$ is an extreme point of S if there's no $x_1, x_2 \in S$, $x_1 \neq x_2$ and $\lambda \in (0,1)$ such that

$$x = \lambda x_1 + (1-\lambda) x_2.$$



Consider $a \in S$. Not an extreme point

$x \to$ an extreme point.



Any point on the bd S is an extreme point.