IE 411: Introduction to Nonlinear Optimization

Fall 2022 - Homework Assignment 2
Due: October 20, 2022

Question 1. For each of the following functions, find all the stationary points and classify them according to whether they are saddle points, strict/nonstrict local/global minimum/maximum points:

a. $f(x_1, x_2) = (4x_1^2 - x_2)^2$.

b. $f(x_1, x_2, x_3) = 4x_1^4 - 2x_1^2 + x_2^2 + 2x_2x_3 + 2x_3^2$.

c. $f(x_1, x_2) = 2x_2^3 - 6x_2^2 + 3x_1^2x_2$

d. $f(x_1, x_2) = x_1^4 + 2x_1^2x_2 + x_2^2 - 4x_1^2 - 8x_1 - 8x_2$.

Question 2. Generate thirty points $(x_i, y_i), i = 1, \ldots, 30$, by the code:

```
rand('seed',314);
x=linspace(0,1,30)';
y=2*x.^2-3*x+1+0.05*randn(size(x));
```

Find the quadratic function $y = ax^2 + bx + c$ that best fits the points in the least square sense. Indicate what are the parameters a, b, c found by the solution and plot the points along with the derived quadratic function. (Print out the Command Window of your MATLAB code together with the plot and attach it to your solution.)

Question 3. Consider the quadratic minimization problem

$$\text{minimize} \quad x^{\mathsf{T}}Ax, \quad x \in \mathbb{R}^5,$$

where A is the 5×5 Hilbert matrix defined by $A_{ij} = \frac{1}{i+j-1}$, for $i, j = 1, \ldots, 5$. This matrix can be constructed via the MATLAB command `A=hilb(5)`. Implement and run the following methods and compare the number of iterations required by each of the methods when the initial vector is $x_0 = (1, 2, 3, 4, 5)^{\mathsf{T}}$ to obtain a solution with $\|\nabla f(x)\| \leq 10^{-4}$:

- gradient method with backtracking stepsize rule and parameters $\alpha = 0.5, \beta = 0.5, s = 1$;
- gradient method with backtracking stepsize rule and parameters $\alpha = 0.1, \beta = 0.5, s = 1$;
- gradient method with exact line search.

Question 4. Show that the following set is not convex:

$$S = \{x \in \mathbb{R}^2 \mid x_1^2 - x_2^2 + x_1 + x_2 \leq 4\}.$$

Question 5. Show that the conic hull of the set

$$S = \{x \in \mathbb{R}^2 \mid (x_1 - 1)^2 + x_2^2 = 1\}$$

is the set

$$\{x \in \mathbb{R}^2 \mid x_1 > 1\} \cup \{(0, 0)^\mathsf{T}\}.$$

(Note that even if the set S is closed its conic hull $\operatorname{cone} S$ is not a closed set.)

Question 6. Let $a, b \in \mathbb{R}^n$ with $a \neq b$. For what values of μ is the set

$$S_\mu = \{x \in \mathbb{R}^n \mid \|x - a\|_2 \leq \mu \|x - b\|_2\}$$

convex?