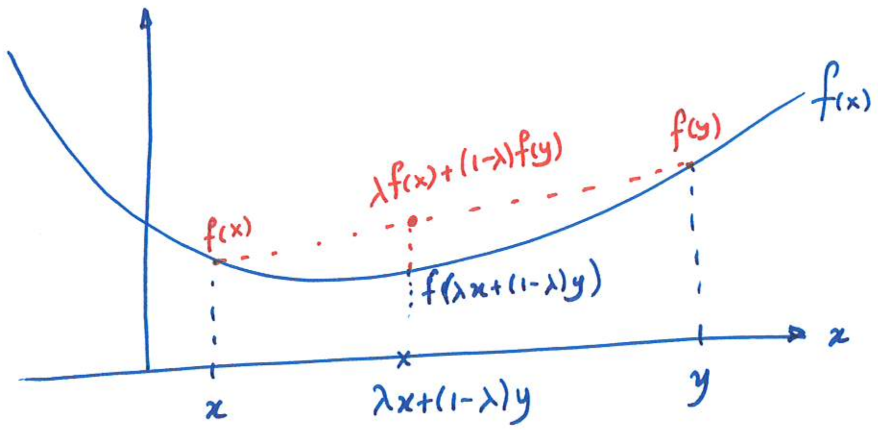
Convex Functions

Defn: A function $f : C \to \mathbb{R}$ defined on a convex set $C \subseteq \mathbb{R}^n$ is called convex if $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$ for any $x, y \in C$, $\lambda \in [0,1]$.



f is called strictly convex if $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$ for all $x, y \in C$ with $x \neq y$ and $\lambda \in (0,1)$.

e.g. Any norm is convex. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$; $x, y \in \mathbb{R}^n$, $\lambda \in [0,1]$.

$$\|\lambda x + (1-\lambda)y\| \le \|\lambda x\| + \|(1-\lambda)y\| \quad \text{(triangle ineq.)}$$
$$= \lambda \|x\| + (1-\lambda)\|y\| \quad \text{since } \lambda \ge 0,\ (1-\lambda) \ge 0.$$

Jensen's inequality: Let $f : C \to \mathbb{R}$ be a convex function over a convex set $C \subseteq \mathbb{R}^n$, $x_1, \dots, x_k \in C$, $\lambda_1, \dots, \lambda_k \ge 0$ with $\sum_{i=1}^k \lambda_i = 1$. Then,

$$f\left(\sum_{i=1}^k \lambda_i x_i\right) \le \sum_{i=1}^k \lambda_i f(x_i)$$

This can be proven by induction!

Extended Real Valued Functions

$\tilde{f}: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$

The (effective) domain of $\tilde{f}$ is $\operatorname{dom} \tilde{f} = \{x \in \mathbb{R}^n \mid f(x) \in \mathbb{R}\}$

Note that if $f: \mathbb{R}^n \to \mathbb{R}$ is a function which is not well-defined on the whole space, then it's possible to assign value "$\pm\infty$" for those values. When we talk about convex functions we only consider $+\infty$.

The extended value extension of f is:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{dom} f \\ +\infty & \text{otherwise.} \end{cases}$$

Recall that for convexity of f, we need to check two properties:

1.) $\operatorname{dom} f \subseteq \mathbb{R}^n$ is convex.

2.) $\forall x, y \in \mathbb{R}^n, \forall \theta \in [0,1]: f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y)$.

When we consider the extended value extension, 2.) implies 1.).

Why? If $x, y \in \operatorname{dom} f \Rightarrow f(x), f(y) \in \mathbb{R}$ and $\theta f(x) + (1-\theta) f(y) \in \mathbb{R}$.

2 implies that $f(\theta x + (1-\theta) y) \in \mathbb{R}$ as well. Then, $\theta x + (1-\theta) y \in \operatorname{dom} f$.

This proves that the domain is a convex set.

- An important example of an extended real valued function is the "indicator function" of a set $C \subseteq \mathbb{R}^n$: (Notation: $I_C(x)$ or $\delta_C(x)$)

$$I_C(x) = \begin{cases} 0 & x \in C \\ +\infty & x \notin C. \end{cases}$$

Note that if the set C is convex, then I_C is a convex function.

Clearly $\operatorname{dom} I_C = C$ is convex. Moreover the function is constant on its domain.

A use of indicator function:

$$\left(\underset{x \in C}{\text{minimize}} \; f(x) \right) \text{ is equivalently } \left(\underset{x \in \mathbb{R}^n}{\text{minimize}} \left(f(x) + I_C(x) \right) \right).$$

- Another example is the <u>support function</u> of S for $S \subseteq \mathbb{R}^n$.

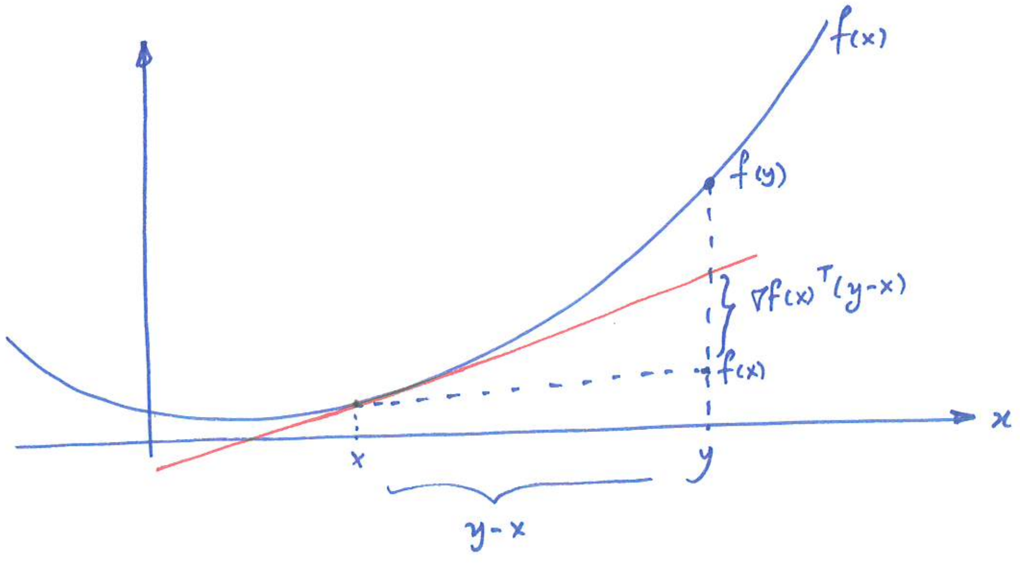
$$\sigma_S(x) = \sup_{y \in S} x^T y \quad \left(= \max_{y \in S} x^T y \right)$$

We will see later that this is a convex extended real valued function.

First order condition for convexity

<u>Theorem:</u> Let $f : C \to \mathbb{R}$ be a continuously differentiable function defined on a convex set $C \subseteq \mathbb{R}^n$. f is convex if and only if

$$\forall\, x, y \in C : \quad f(y) \geq f(x) + \nabla f(x)^T (y - x). \qquad (\star)$$



Proof: $\Rightarrow$ Assume f is convex. Let's prove ⊛. Let $x, y \in C$ be arbitrary.

If $x = y$, then ⊛ holds. Assume $x \neq y$. Let $\lambda \in (0,1]$. Then,

$$f(\lambda y + (1-\lambda)x) \leq \lambda f(y) + (1-\lambda) f(x) \quad \text{holds.}$$

$$\Rightarrow \quad f(x + \lambda(y-x)) - f(x) \leq \lambda f(y) - \lambda f(x)$$

$$\Rightarrow \quad \frac{f(x+\lambda(y-x))}{\lambda} \leq f(y) - f(x) \qquad (\text{as } \lambda > 0)$$

Taking the limit as $\lambda \downarrow 0$, we obtain:

$$\underbrace{f'(x; y-x)}_{\nabla f(x)^T (y-x)} \leq f(y) - f(x). \implies ⊛ \text{ holds.}$$

as f is conts. diff.

$\Leftarrow$ Assume ⊛ holds. Let $x, y \in C$, $\lambda \in (0,1)$. We want to show

that $$f(\underbrace{\lambda x + (1-\lambda) y}_{u}) \leq \lambda f(x) + (1-\lambda) f(y)$$

Note that as C is convex, $u \in C$, where $u := \lambda x + (1-\lambda) y$.

Let's apply ⊛ to x & u and y & u separately.

① $f(x) \geq f(u) + \nabla f(u)^T \underbrace{(x-u)}_{\substack{x - \lambda x - (1-\lambda) y \\ \shortparallel \\ (1-\lambda)(x-y)}}$ ② $f(y) \geq f(u) + \nabla f(u)^T \underbrace{(y-u)}_{\substack{y - \lambda x - (1-\lambda) y \\ \shortparallel \\ \lambda (y-x)}}$

So, we have

$$① \quad f(x) \geq f(u) + (1-\lambda)\nabla f(u)^T(x-y) \qquad \Big/ \text{ Multiply with } \frac{\lambda}{1-\lambda}$$

$$② \quad f(y) \geq f(u) + (-\lambda)\nabla f(u)^T(x-y) \qquad \Big/ \text{ add}$$

$$\Rightarrow \quad \frac{\lambda}{1-\lambda} f(x) + f(y) \geq \frac{\lambda}{1-\lambda} f(u) + f(u) + \lambda \nabla f(u)^T(x-y) - \lambda \nabla f(u)^T(x-y)$$

$$\Rightarrow \quad \frac{\lambda f(x) + (1-\lambda) f(y)}{1-\lambda} \geq \frac{\lambda f(u) + (1-\lambda) f(u)}{1-\lambda}$$

$$\Rightarrow \quad \lambda f(x) + (1-\lambda) f(y) \geq \lambda f(u) + (1-\lambda) f(u) = f(u) = f(\lambda x + (1-\lambda)y).$$

$\blacksquare$

Second Order Conditions

Theorem: Let $f: C \to \mathbb{R}$ be twice continuously differentiable over C, $C \subseteq \mathbb{R}^n$ is an open convex set. f is convex if and only if $\nabla^2 f(x) \succcurlyeq 0$ for any $x \in C$.

Proof: $\Leftarrow$: Assume that $\nabla^2 f(x) \succcurlyeq 0$ for any $x \in C$.

Let $x, y \in C$ be arbitrary. By the linear approximation theorem, $\exists z \in [x,y]$ s.t.

$$f(y) = f(x) + \nabla f(x)^T(y-x) + \frac{1}{2}(y-x)^T \nabla^2 f(z)(y-x).$$

Note that $z \in C$, hence $\nabla^2 f(z) \succcurlyeq 0$ holds, which implies

$\frac{1}{2}(y-x)^T \nabla^2 f(z)(y-x) \geq 0$ for all $x,y \in C$. Then,

$f(y) \geq f(x) + \nabla f(x)^T (y-x)$ holds for all $x,y \in C$.

This is the first order condition for convexity. Hence, f is convex.

$\Longrightarrow$: Assume that f is convex over C. Let $x \in C$ be arbitrary, $y \in \mathbb{R}^n$.

Since C is an open set, $\exists\, \varepsilon > 0$ s.t. $x + \lambda y \in C$ for all $\lambda \leq \varepsilon$.

By the first order condition, we have

$$f(x+\lambda y) \geq f(x) + \nabla f(x)^T [x + \lambda y - x]$$

$$\Rightarrow \quad f(x+\lambda y) \geq f(x) + \lambda \nabla f(x)^T y. \quad (*)$$

By quadratic approximation theorem we have

$$f(x+\lambda y) = f(x) + \nabla f(x)^T (x+\lambda y - x) + \frac{1}{2}(x+\lambda y - x)^T \nabla^2 f(x)(x+\lambda y - x) + o\left(\|x+\lambda y - x\|^2\right)$$

$$\Rightarrow f(x+\lambda y) = f(x) + \lambda \nabla f(x)^T y + \frac{\lambda^2}{2} y^T \nabla^2 f(x) y + \underbrace{o(\lambda^2 \|y\|^2)}_{\text{A function } h \text{ satisfying } \lim_{\lambda \downarrow 0} \frac{h(\lambda)}{\lambda^2} = 0.}$$

$$\overset{\text{by } (*)}{\geq} f(x) + \lambda \nabla f(x)^T y.$$

Hence, $\frac{\lambda^2}{2} y^T \nabla^2 f(x) y + o(\lambda^2 \|y\|^2) \geq 0$ holds for any $0 < \lambda \leq \varepsilon$.

Let's divide everything by $\lambda^2 > 0$: $\quad \frac{1}{2} y^T \nabla^2 f(x) y + \frac{o(\lambda^2 \|y\|^2)}{\lambda^2} \geq 0$.

Taking the limit as $\lambda \downarrow 0$ yields that

$$\frac{1}{2} y^T \nabla^2 f(x) y \geq 0.$$

Recall that $x \in C, y \in \mathbb{R}^n$ are arbitrary. Hence, the inequality holds for all such x, y. This implies that $\nabla^2 f(x) \succeq 0$ for all $x \in C$. ▨

Theorem: Let $f: C \to \mathbb{R}^n$ be twice diff. continuous over $C \subseteq \mathbb{R}$. C: open, convex. If $\nabla^2 f(x) \succ 0$ for any $x \in C$, then f is strictly convex over C.

- Note that the reverse of this statement is not necessarily correct.

Consider $f(x) = x^4$ over $\mathbb{R}$. f is strictly convex over $\mathbb{R}$.

$f'(x) = 4x^3$, $f''(x) = 12x^2$.

Note that $f''(0) = 0 \not\succ 0$, even though f is strictly convex.

Example: Let $f: \mathbb{R} \times \mathbb{R}_{++} \to \mathbb{R}$, $\mathbb{R} \times \mathbb{R}_{++} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$

$f(x) = \frac{x_1^2}{x_2}$. Let's show that f is convex by checking its hessian.

$$\nabla f(x) = \begin{bmatrix} \frac{2x_1}{x_2} \\ -\frac{x_1^2}{x_2^2} \end{bmatrix}, \quad \nabla^2 f(x) = \begin{bmatrix} \frac{2}{x_2} & \frac{-2x_1}{x_2^2} \\ \frac{-2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix} = 2 \begin{bmatrix} \frac{1}{x_2} & \frac{-x_1}{x_2^2} \\ \frac{-x_1}{x_2^2} & \frac{x_1^2}{x_2^3} \end{bmatrix}$$

We want to check if $\nabla^2 f(x) \succeq 0$ for all $x \in \mathbb{R} \times \mathbb{R}_{++}$.

This will be the case if both eigenvalues are non-negative. As this is a 2×2 matrix, it's sufficient to show that the trace & determinant are non-negative. (Recall that $tr\,\nabla^2 f(x) = \lambda_1 + \lambda_2$, $\det \nabla^2 f(x) = \lambda_1 \cdot \lambda_2$)

$$tr\,\nabla^2 f(x) = 2\left(\frac{1}{x_2} + \frac{x_1^2}{x_2^3}\right) = \frac{2(x_2^2 + x_1^2)}{x_2^3} > 0 \quad \text{over} \quad x \in \mathbb{R}\times\mathbb{R}_{++}.$$
(as $x_2 > 0$)

$$\det \nabla^2 f(x) = 4\left(\frac{1}{x_2}\cdot\frac{x_1^2}{x_2^3} - \frac{x_1^2}{x_2^4}\right) = 0.$$

Then, we conclude that one of the eigenvalues is zero and the other one is strictly positive for all $x = (x_1, x_2) \in \mathbb{R}\times\mathbb{R}_{++}$. Hence, $\nabla^2 f(x) \succeq 0$ for $x \in \mathbb{R}\times\mathbb{R}_{++}$, f is convex on this domain.

Example: (log-sum-exp)

let $f(x) = \log\left(\sum_{i=1}^{n} e^{x_i}\right)$ defined over $\mathbb{R}^n$. The function $f: \mathbb{R}^n \to \mathbb{R}$ is convex for all $n \in \mathbb{N}$. We will show this for $n=2$.

$f(x) = \log(e^{x_1} + e^{x_2})$, over $\mathbb{R}^2$.

$$\nabla f(x) = \begin{bmatrix} \frac{e^{x_1}}{e^{x_1}+e^{x_2}} \\ \frac{e^{x_2}}{e^{x_1}+e^{x_2}} \end{bmatrix} = \frac{1}{e^{x_1}+e^{x_2}}\begin{bmatrix} e^{x_1} \\ e^{x_2} \end{bmatrix}.$$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{e^{x_1}(e^{x_1}+e^{x_2}) - e^{x_1}e^{x_1}}{(e^{x_1}+e^{x_2})^2} & \frac{-e^{x_1}e^{x_2}}{(e^{x_1}+e^{x_2})^2} \\ \frac{-e^{x_1}e^{x_2}}{(e^{x_1}+e^{x_2})^2} & \frac{e^{x_2}(e^{x_1}+e^{x_2}) - e^{x_2}e^{x_2}}{(e^{x_1}+e^{x_2})^2} \end{bmatrix}$$

For simplicity, let $a = e^{x_1}$, $b = e^{x_2}$. Note that $a, b > 0$ for all x.

$$\nabla^2 f(x) = \begin{bmatrix} \frac{a}{a+b} - \frac{a^2}{(a+b)^2} & \frac{-ab}{(a+b)^2} \\ \frac{-ab}{(a+b)^2} & \frac{b}{a+b} - \frac{b^2}{(a+b)^2} \end{bmatrix}$$

$$\operatorname{tr} \nabla^2 f(x) = \underbrace{\frac{a}{a+b} + \frac{b}{a+b}} - \frac{a^2}{(a+b)^2} - \frac{b^2}{(a+b)^2}$$

$$= 1 - \underbrace{\frac{a^2 + 2ab + b^2}{(a+b)^2}}_{1} + \frac{2ab}{(a+b)^2} = \frac{2ab}{(a+b)^2} > 0 \text{ for all } x.$$

$$\det \nabla^2 f(x) = \left(\frac{a}{a+b} - \frac{a^2}{(a+b)^2}\right)\left(\frac{b}{a+b} - \frac{b^2}{(a+b)^2}\right) - \frac{a^2 b^2}{(a+b)^2}$$

$$= \frac{ab}{(a+b)^2} - \frac{ab^2}{(a+b)^3} - \frac{a^2 b}{(a+b)^3} + \cancel{\frac{a^2 b^2}{(a+b)^4}} - \cancel{\frac{a^2 b^2}{(a+b)^4}}$$

$$= \frac{ab(a+b) - ab^2 - a^2 b}{(a+b)^3} = 0.$$

Since $\det \nabla^2 f(x) = 0$ $(= \lambda_1 \lambda_2)$ and $\operatorname{tr} \nabla^2 f(x) > 0$ $(= \lambda_1 + \lambda_2)$, one of the eigenvalues is zero and the other one is positive. Hence $\nabla^2 f(x) \succeq 0$ holds for all $x \in \mathbb{R}^2$, as expected.

Operations Preserving Convexity

1.) Let $f: C \to \mathbb{R}$ be a convex function over a convex set C, $\alpha \geq 0$.

Then $\alpha f : C \to \mathbb{R}$ is also convex. $((\alpha f)(x) = \alpha . f(x).)$

2.) Let $f_i : C \to \mathbb{R}$ be convex functions over convex set C, for $i = 1, \dots, p$.

Then, $(f_1 + \dots + f_p)(x) = f_1(x) + \dots + f_p(x)$ is also convex.

- Using 1 & 2, one can show that if $f_1, \dots, f_p$ are convex and $\alpha_1, \dots, \alpha_p \geq 0$

then $f = \sum_{i=1}^{p} \alpha_i f_i$ is also convex.

3.) (Linear change of variables) Let $f : C \to \mathbb{R}$ be convex over convex set $C \subseteq \mathbb{R}^n$. Let $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^m$.

Let $D := \{ y \in \mathbb{R}^m \mid Ay + b \in C \}$, define $g(y) := f(Ay + b)$.

Then $g : D \to \mathbb{R}$ is convex.

Example: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x_1^2 + 2x_1x_2 + 3x_2^2 + 2x_1 - 3x_2 + e^{x_1}$

Note that $f(x) = f_1(x) + f_2(x)$ where $f_1(x) = x_1^2 + 2x_1x_2 + 3x_2^2 + 2x_1 - 3x_2$
$f_2(x) = e^{x_1}$

$$\nabla f_1(x) = \begin{bmatrix} 2x_1 + 2x_2 + 2 \\ 2x_1 + 6x_2 - 3 \end{bmatrix}, \quad \nabla^2 f_1(x) = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}$$

$\mathrm{tr}\, \nabla^2 f_1 = 8 > 0$

$\det \nabla^2 f_1 = 12 - 4 = 8 > 0$

Hence, $\nabla^2 f_1 \succeq 0$. f_1 is convex.

On the other hand, e^t is a convex function over $\mathbb{R}$ and $(x_1, x_2) \to x_1$ is a linear function over $\mathbb{R}^2$. From (3), f_2 is also convex.

From (2), $f = f_1 + f_2$ is convex.

Example: $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x) = \underbrace{e^{x_1 - x_2 + x_3}}_{f_1(x)} + \underbrace{e^{2x_2}}_{f_2(x)} + \underbrace{x_1}_{f_3(x)}$.

Again, $t \to e^t$ is a convex function from $\mathbb{R}$ to $\mathbb{R}$.

Moreover, $(x_1, x_2, x_3) \longrightarrow x_1 - x_2 + x_3$ is a linear function from $\mathbb{R}^3$ to $\mathbb{R}$

$(x_1, x_2, x_3) \longrightarrow 2x_2$ " " " " " "

$(x_1, x_2, x_3) \longrightarrow x_1$ is also linear.

These imply f_1, f_2, f_3 are convex & the sum is also convex.

- In general, compositions of two convex functions is not convex.

e.g. let $g(t) = t^2$, $h(t) = t^2 - 4$ — both convex.

Consider $f(t) = g(h(t)) = g(t^2 - 4) = (t^2 - 4)^2$.

$f'(t) = 2(t^2 - 4) \cdot 2t = 4t(t^2 - 4)$

$f''(t) = 4(t^2 - 4 + t \cdot (2t)) = 4(3t^2 - 4)$

Note that $f''(0) = -16 < 0$.

- Composition with a "non-decreasing" convex function is convex!

Theorem: $f: C \to \mathbb{R}$ convex over a convex set $C \subseteq \mathbb{R}^n$.
$g: I \to \mathbb{R}$ be non-decreasing convex over interval $I \subseteq \mathbb{R}$.

Assume that $f(C) := \{f(x) \mid x \in C\} \subseteq I$. Then, $h := g \circ f$ is convex over C. $(h: C \to \mathbb{R},\ h(x) = g(f(x)))$

Proof: Let $x, y \in C$, $\lambda \in [0,1]$.

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &= g(f(\lambda x + (1-\lambda)y)) \\ &\leq g(\lambda f(x) + (1-\lambda) f(y)) \quad \text{as } f \text{ is convex and } g \text{ is nondecreasing} \\ &\leq \lambda g(f(x)) + (1-\lambda) g(f(y)) \quad \text{as } g \text{ is convex} \\ &= \lambda h(x) + (1-\lambda) h(y). \end{aligned}$$

$\blacksquare$

Example: Let $h(x) = (\|x\|^2 + 1)^2$.

Note that $h(x) = g(f(x))$ for $f(x) = \|x\|^2 + 1$, $g(t) = t^2$.

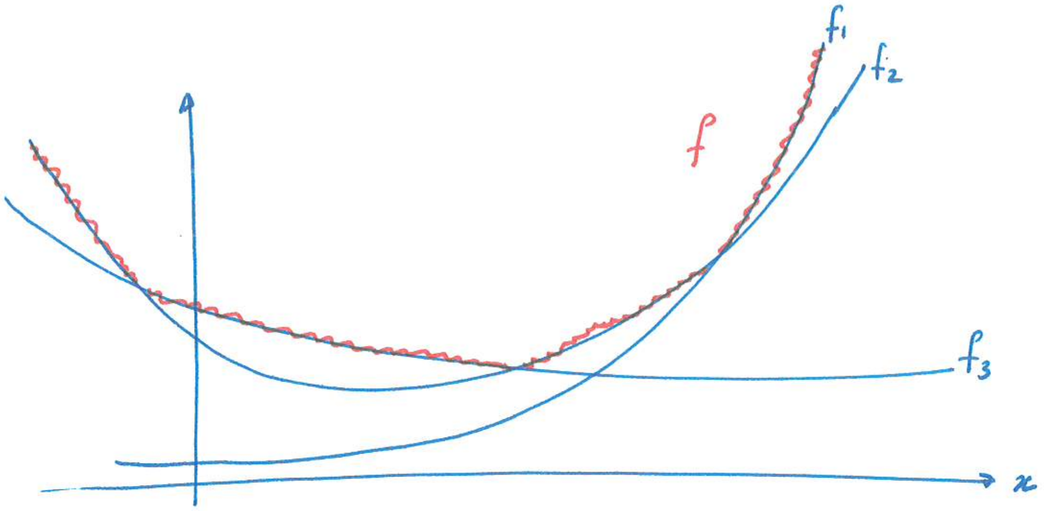
Both f & g are convex. Moreover, g is increasing over interval $[0, \infty)$.

The image of $\mathbb{R}^n$ under f is $[1, \infty) \subseteq [0, \infty)$.

By the previous result, h is convex.

Theorem: (Pointwise max. of convex functions)
Let $f_1, \dots, f_p : C \to \mathbb{R}$ be convex functions over convex set $C \subseteq \mathbb{R}^n$.
Then, the maximum function $f(x) = \max_i f_i(x)$ is also convex.

* Note that this holds also for infinite even uncountable collection of functions.

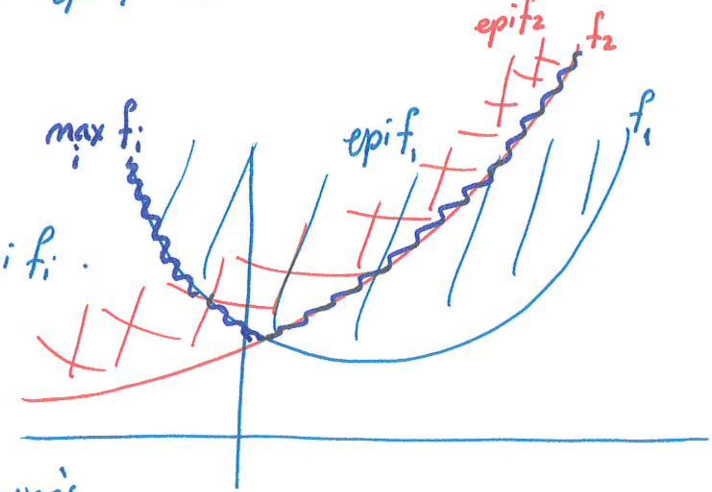


Defn: Let $f: C \to \mathbb{R}$ be a function over $C \subseteq \mathbb{R}^n$. The "epigraph" of f is defined by

$$\text{epi } f = \{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \le t \} \subseteq \mathbb{R}^{n+1}.$$

Result: f is convex if and only if epi f is a convex set in $\mathbb{R}^{n+1}$.

Proof:
(exercise)



Note that $\text{epi}\left(\max_i f_i\right) = \bigcap_i \text{epi } f_i$.

As (f_i) are convex, epi f_i are convex sets in $\mathbb{R}^{n+1}$. Then, their intersection is also convex. This proves the previous result.

Theorem: Let $f: C \times D \to \mathbb{R}$ be a convex function over set $C \times D$, where $C \subseteq \mathbb{R}^n$, $D \subseteq \mathbb{R}^m$ are convex sets. Let $g(x) := \min_{y \in D} f(x,y)$, $x \in C$.

Then, $g: C \to \mathbb{R}$ is a convex function.

Example: (Distance function) Let $C \subseteq \mathbb{R}^n$ be convex. Let,

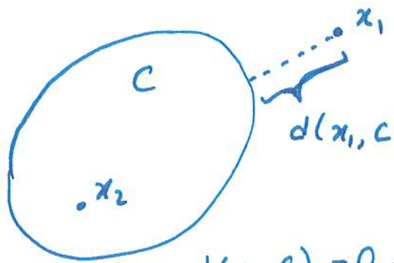
$$d(x, C) := \min \{ \|x-y\| \mid y \in C \}.$$

Note that $f(x,y) := \|x-y\|$ is a convex function. (Norm is convex and $(x,y) \to x-y$ is affine (linear).)

Here, $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and $C \subseteq \mathbb{R}^n$ is convex.

Then, $d(\cdot, C) : \mathbb{R}^n \to \mathbb{R}$ defined as above is also a convex function.

e.g.

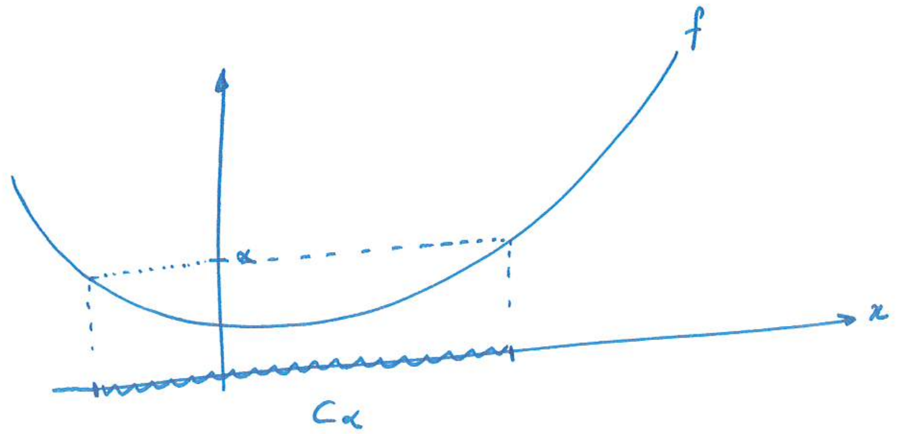


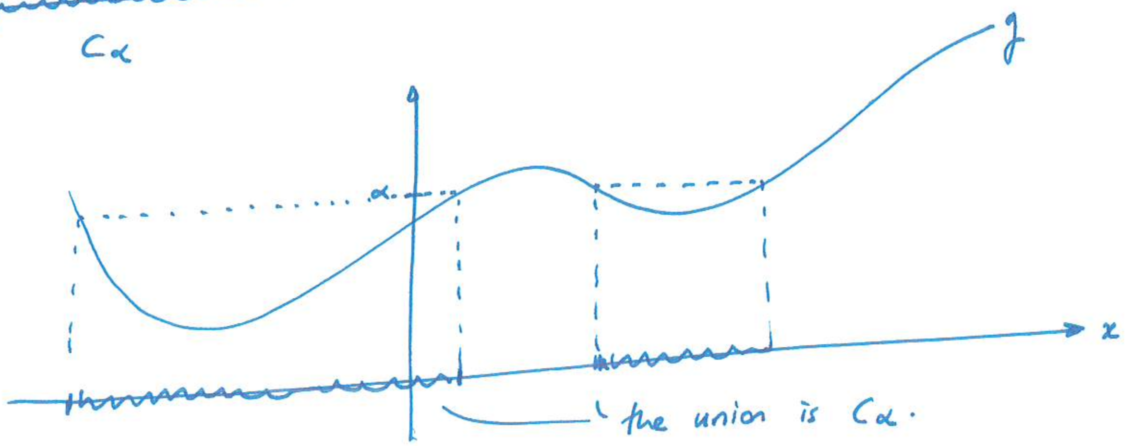
$d(x_1, C)$ (if the norm is the Euclidean norm)

$d(x_2, C) = 0$.

Defn: (sublevel sets) Let $f: S \to \mathbb{R}$ be a function over set $S \subseteq \mathbb{R}^n$.

α sublevel set of f is $C_\alpha = \{ x \in S \mid f(x) \le \alpha \}$.





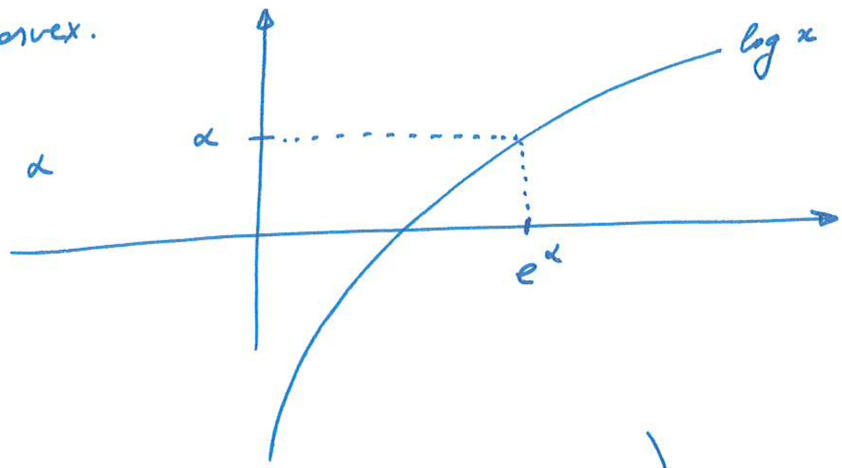
the union is C_α.

Result: If $f: C \to \mathbb{R}$ is a convex function, C_α is a convex set for any α.

(Proof: exercise)

Note that the reverse is not true. Consider the logarithm function.

$\log: \mathbb{R}_{++} \to \mathbb{R}$ is not convex.



But α sublevel set for any α is an interval, indeed,

$$C_\alpha = (0, e^\alpha].$$

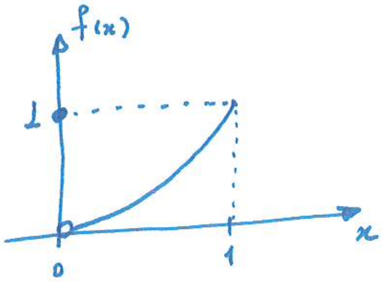
(Indeed, any monotone function would satisfy this property!)

Defn: A function $f: C \to \mathbb{R}$ defined over convex set $C \subseteq \mathbb{R}^n$ is called <u>quasiconvex</u> if α sublevel sets are convex for all α.

Continuity of Convex Functions

Result - without proof: Convex functions are continuous over the <u>interior</u> of their domain. If the function is defined over $\mathbb{R}^n$ and convex, then it's continuous over $\mathbb{R}^n$.

e.g. $f(x) = \begin{cases} 1, & x = 0 \\ x^2, & 0 < x \leq 1 \end{cases}$, $\operatorname{dom} f = [0, 1]$.



f is convex over its domain. However, not continuous at $x = 0$. It's continuous over $(0, 1)$ though!