Research Proposal:
Garside structures for the Yang–Baxter equation beyond the Artin–Tits case

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1 Current state of research

The Yang–Baxter equation (YBE) is a fundamental equation in physics, appearing for instance in statistical mechanics, and quantum mechanics. Finding solutions to this equation is, in a sense, an intermediate step to solving more complex problems. However, the search for its solution is already a particularly hard task.

To simplify the problem, in 1992, Drinfeld suggested studying the case of set-theoretical solutions, that is pairs (X, r) with X a set and r a bijection $X \times X \xrightarrow{\sim} X \times X$ such that, on $X \times X \times X$ we have $(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r)$.

However, the classification of set-theoretical solutions is still quite difficult, and thus further hypotheses are needed. This was done in a fundamental article in 1999 by Etingof–Schedler–Soloviev ([13]). In this paper, the authors propose to study set-theoretical solutions that, when written as $r(x, y) = (\lambda_x(y), \rho_y(x))$, are: finite ($|X| < \infty$), involutive ($r^2 = \mathrm{id}_{X\times X}$) and non-degenerate ($\lambda_x$ and ρ_x are bijective for all $x \in X$). For simplicity, we will call "solution" a set-theoretical solution satisfying these conditions.

The same authors also defined the structure group associated to a solution by the presentation $G(X, r) = \langle X \mid xy = \lambda_x(y)\rho_y(x)\rangle$. This family of groups has many known interesting properties. Let $G(X, r)$ be a structure group, then:

- It embeds as a subgroup of $\mathbb{Z}^X \rtimes \mathfrak{S}_X$, with the property that projecting on the first coordinate is bijective. In other words, we have a bijective set map $\Pi\colon \mathbb{Z}^X \to G(X, r)$. We will denote by $\psi(g) \in \mathfrak{S}_X$ the permutation associated to an element $g \in G(X, r)$.
- It is a brace (first introduced by Rump in [19]), a structure similar to a ring but where distributivity is replaced by $a(b + c) = ab - a + ac$. The brace approach is currently one of the main tools to study solutions, and a very active field of research in (quantum) algebra.
- It is a Garside group (in the sense of [11]). The definition is technical, but this automatically gives many results such as solutions to the word problem and the conjugacy problem.

This last point, obtained by Chouraqui in 2010 ([8]), started a parallel between structure groups of solutions and spherical Artin–Tits groups (for instance the braid groups Br_n). This comparison is the basis of the author's work.

It is well-known that any spherical Artin–Tits group A has an associated finite Coxeter group W obtained by adding a quadratic relation for every generator: $W = A/\langle s^2\rangle$. For instance, the Coxeter group associated to the braid group Br_n is the symmetric group $\mathfrak{S}_n$. Moreover, the Coxeter group can be used to recover the Artin–Tits group and its Garside structure ([11]).

In 2015, Dehornoy ([10]) constructed a finite quotient of each structure group in the following form: $\overline{G}(X, r) = G(X, r)/\langle x^{[d]}\rangle$. Here, $x^{[d]}$ is the element corresponding to $(0, \ldots, k, \ldots, 0) \in \mathbb{Z}^X$ where d is the minimal integer such that the permutation $\psi(x^{[d]})$ is trivial for all $x \in X$, i.e. $x^{[d]} \mapsto (kx, 1) \in \mathbb{Z}^X \rtimes \mathfrak{S}_X$. This group is often called "Coxeter-like" as it also possesses the property that one can recover $G(X, r)$ and its Garside structure from it. The integer d, which depends on the solution, is called "Dehornoy's class" and is a fundamental combinatorial invariant for the classification of solutions.

Moreover, Dehornoy also constructed a faithful monomial linear representation of the structure group $\Theta\colon G(X, r) \to \mathrm{GL}_{|X|}\left(\mathbb{C}[z^{\pm 1}]\right)$ defined by sending $x \in X$ to $D_x P_x$, where $D_x = \mathrm{diag}(1, \cdots, z, \cdots, 1)$ and P_x is the permutation matrix of λ_x. This representation then specializes to a faithful monomial representation of the Coxeter-like quotient $\overline{\Theta}\colon \overline{G}(X,r) \to \mathrm{Gl}_{|X|}(\mathbb{C})$ by taking $z = \exp(\frac{2i\pi}{d})$. These representations can be used to characterize some combinatorial properties of solutions ([12]) but are also practical when working with computer algebra systems to obtain and verify conjectures.

2 Proposed projects

In general, the author's major interests lie in algebra: group theory, representation theory, combinatorics, homology theory and category theory. Our strategy is to try to use the most suitable techniques for a given problem. The author's work is then divided along the following directions:

1. Understanding the possible values of Dehornoy's class depending on the size of the solution (bounds, divisibility, etc.).
2. Continue to develop the parallel between spherical Artin–Tits groups and structure groups of solutions.
3. Studying Garside structures from and for solutions (constructing new Garside structures and Garside groups).

For all of these three points, the algebraic properties of braces are a fundamental tool. Apart from a "direct" approach to the classification of solutions, another approach is the classification of braces (based on [2]). For this classification, any algebraic or combinatorial properties of braces are interesting, and such properties are often obtained as a byproduct of new constructions. This justifies directions 1 and 2, but also the need for a more "abstract" approach to group theory (compared to a combinatorial one).

2.1 Dehornoy's class

Given a solution (X,r) of size $|X| = n$, the best known upper bound is $d_X \leq n!$. However, numerical values are much lower, for instance by checking all solutions of size ten (given by [1]) we find $|X| = 10 \Rightarrow d_X \leq 30$.

Following [13], we will say that a solution is indecomposable if there does not exist a proper partition $X = Y \sqcup Z$ such that $Y \times Y$ and $Z \times Z$ are stable under r.

This leads to the following conjectures:

Conjecture 1 ([14])**.** *Let (X,r) be a solution of size $|X| = n$ and Dehornoy's class d.*

(a) d is bounded by the "maximum of different products of partitions of n into distinct parts":

$$d \leq \max\left(\left\{\prod_{i=1}^{k} n_i \,\middle|\, k \in \mathbb{N}, 1 \leq n_1 \leq \cdots \leq n_k, n_1 + \cdots + n_k = n\right\}\right).$$

(b) If (X,r) is indecomposable, then $d \leq n$.

This conjecture was obtained from computer assisted computations over the full list of solutions of size ≤ 10, and is known to hold for particular families of solutions satisfying some algebraic conditions ([14, 6]). In regards to the classification, the conjecture is also interesting, as its proof would imply a better understanding of the combinatorial behaviour of solutions.

Instead of aiming for a general proof of this conjecture, we decompose it into several steps. First, we intend to show that the conjecture holds for new families of solutions:

Problem 1. Prove that Conjecture 1 holds when the structure group $G(X,r)$ is a p-group.

For indecomposable solutions these families are interesting, as in [14] it is shown that solutions whose size and Dehornoy's class are powers of the same prime can be considered as "elementary solutions" (building blocks to build any solution).

The clear generalization that would follow is:

Problem 2. Prove that Conjecture 1 holds when the structure group $G(X,r)$ is nilpotent.

These cases are the next step to the already known cases, especially for indecomposable solutions (involving the abelianity/cyclicity of the permutation group, as in [14, 6]). We know that the Dehornoy's class of a solution is equal to the exponent of the additive permutation group (the projection of $G < \mathbb{Z}^X \rtimes \mathfrak{S}_X$ on $\mathfrak{S}_X$). Thus, this approach involves an algebraical and combinatorial study of the permutation brace of a solution.

To go from indecomposable solutions to general ones, we want to investigate the following:

Problem 3. Prove that Conjecture $1(b)$ implies Conjecture $1(a)$.

The motivation for this implication is the fact that decomposing a solution yields an integer partition of its size, and partitions of integers appear in Conjecture $1(a)$. When decomposing a solution X as $Y \sqcup Z$, we obtain two solutions Y and Z with their own Dehornoy's classes, but also an "action" of each part on the other. The key step is to understand how Dehornoy's class behaves under these "actions", which will also unveil how decomposable solutions are constructed from indecomposable solutions.

2.2 Hecke algebras of a solution

It is important to note that Dehornoy's construction of the Coxeter-like group works for any multiple of Dehornoy's class, i.e. for any $m \in \mathbb{N}$ the quotient $\overline{G}_m(X,r)$ is also a "Coxeter-like" group.

Recall that for a finite Coxeter group W, generated by S and with associated Artin–Tits group A, the generic Iwahori–Hecke algebra is a one-parameter deformation of the group ring $\mathbb{Z}[W] = \mathbb{Z}[A]/\langle s^2\rangle$ as $\mathcal{H}_q(W) = \mathbb{Z}[q^{\pm 1}][A]/\langle s^2 = (q-1)s + q\rangle$. This algebra has numerous interesting properties. In particular it is a free $\mathbb{Z}[q^{\pm 1}]$-module of rank $|W|$. These algebras can, for instance, be used to construct invariants of knots or to understand the representation theory of groups of Lie type.

In [15] analogous objects in the context of solutions to the YBE are introduced and studied. The analogue of the generic Iwahori–Hecke algebra for a solution is

$$\mathcal{H}_q(X,r) = \mathbb{Z}[q^{\pm 1}][G(X,r)]/\langle (x^{[d]})^2 = (q-1)x^{[d]} + q\rangle.$$

This algebra then has the expected properties. In particular it is a free $\mathbb{Z}[q^{\pm 1}]$-module of rank $|\overline{G}_2(X,r)|$.

For Coxeter groups, we are mostly limited to quadratic relations, but surprisingly this is not the case here. For any commutative domain R, and any monic polynomial $P \in R[t]$ of degree $m > 1$, we define

$$\mathcal{H}((X,r),P) = R[G(X,r)]/(P(x^{[d]})).$$

Under a few appropriate hypotheses this algebra retains the expected properties. In particular, it always is a free R-module of rank $|\overline{G}_m(X,r)|$. Thus we have a family of Hecke algebras for any solution (one for each choice of a polynomial).

Moreover, just like for a Hecke algebra of a finite Coxeter group, Tits' deformation theorem does hold. This means that under a suitable extension of the base ring, the irreducible characters of the Hecke algebra are in bijection with the irreducible characters of the Coxeter-like group. Understanding either characters and the separability of the algebra will thus be very useful in providing a new algebraic framework to study solutions.

In general, the development and understanding of new algebraic objects related to the Yang–Baxter equation comes with new knowledge on the combinatorial properties of solutions. Our work on Hecke algebras implies some combinatorially difficult questions, which can be answered by studying and improving our comprehension of the permutation groups and Coxeter-like groups associated to solutions. Thus, we are interested in progressing further the theory of Hecke algebras for solutions, also in a way to keep developing parallels with spherical Artin–Tits groups.

Similarly to Coxeter groups, these algebras have over $R = \mathbb{Z}[q^{\pm 1}]$ a natural basis $(T_g)_{g \in \overline{G}_l(X,r)}$. They also have an anti-involution ι defined by $q \mapsto q^{-1}$ and $T_g \mapsto T_{g^{-1}}$. The construction of the Kazhdan–Lusztig polynomials, which form a basis of the Iwahori–Hecke algebra, is well-known for Coxeter groups ([16]). These polynomials were used in [17] to approach the representation theory of Iwahori–Hecke algebras. We intend to develop a similar construction for structure groups of solutions:

Problem 4. Construct the Kazhdan–Lusztig polynomials for $\mathcal{H}((X,r),P)$ over $\mathbb{Z}\left[q^{\pm\frac{1}{2}}\right]$ that are invariant under ι and form a basis of $\mathcal{H}((X,r),P)$.

The classical techniques for Coxeter groups can certainly be mimicked. However in [15] a mix of techniques from Coxeter groups and the Yang–Baxter equation are used to obtain simple proofs, so the same is expected to happen here. A first approach using computer algebra systems (`GAP` or `SageMath`) is certainly helpful to get some intuition and verify some expected results.

Moreover, Kazhdan–Lusztig bases are still not known in the context of Hecke algebras of complex reflection groups ([18]). The Dehornoy's representation of the Coxeter-like quotients yields monomial matrices with roots of unity, which are subgroups from the infinite family of complex reflection groups denoted $G(de,d,r)$. This means that, computing a Kazhdan–Lusztig basis for Hecke algebras of solutions may lead to new approaches to study the Hecke algebras of complex reflection groups.

It was proven in [12] that the indecomposability of a solution is equivalent to the irreducibility of Dehornoy's monomial representation of $\overline{G}_l$ for $l \geq 2$. As we are interested in the representation theory of Hecke algebras of a solution, it is important to construct some of their irreducible representations:

Problem 5. Construct new irreducible representations of $\mathcal{H}((X,r),P)$.

For Coxeter groups, this can be done using the Kazhdan–Lusztig polynomials ([16]), so we aim at adapting those methods to the context of solutions to the YBE.

Notice that our definition of Hecke algebras of a solution is purely algebraic. On the other hand, one can define Hecke algebras of Coxeter groups through a geometric approach. This approach starts from the Bruhat decomposition $\mathrm{GL}_n(K) = \bigsqcup_{w \in \mathfrak{S}_n} BwB$ (where B is the Borel subgroup of upper triangular matrices). It would then be interesting to have a geometric interpretation of our definition:

Problem 6. Provide a geometrical definition of Hecke algebras for structure groups of solutions.

In particular, we hope to be able to find a "Bruhat-like" decomposition for the faithful monomial representation inside $\mathrm{GL}_{|X|}\left(\mathbb{C}[z^{\pm 1}]\right)$. This will also provide a better understanding of the permutation group of solutions.

2.3 New Garside structures

Recall that a regular subgroup of a semi-direct product $G \rtimes H$ is a subgroup such that the projection on G is bijective.

We know that regular subgroups of $\mathbb{Z}^n \rtimes \mathfrak{S}_n$ are Garside groups ([8]). The Garside structure comes from that of $\mathbb{Z}^n$ deformed by the action of the Weyl group $\mathfrak{S}_n$. This leads to the question: can we generalize this to other Weyl groups acting on their maximal torus? That is, taking a Weyl group W and a "discrete" maximal torus T (just like $\mathbb{Z}^n$ is the discrete version of $\mathbb{R}^n$, the maximal torus of $\mathfrak{S}_n$), can we retrieve all the results mentioned above for regular subgroups of $T \rtimes W$?

Problem 7. Show that, given a Weyl group W with maximal torus of rank $r(W)$, a regular subgroup of $\mathbb{Z}^{r(W)} \rtimes W$ is a Garside group.

We strongly believe that these regular subgroups are Garside groups for all Weyl groups, i.e. for all crystallographic finite Coxeter groups. For the Weyl groups of type B_n, we can prove that such subgroups are Garside groups. This is done by adapting the representation theoretical techniques used in [14]. The other cases should then be studied with the same methodology, each independently or under a common theory. The starting point would be to use the algebraic tools coming from the fact that these subgroups are braces ([7]).

Then, the question is to adapt what we know from structure groups of solutions: "Coxeter-like" quotients, to find presentations that generalize the structure group of solutions, and even to construct Hecke algebras.

Problem 8. Given a Garside regular subgroup of $\mathbb{Z}^{r(W)} \rtimes W$: construct Coxeter-like quotients, find presentations that generalize the structure group of solutions, construct Hecke algebras.

From a Garside perspective, this would be an important construction, as it would "unite" the two large families of Garside groups: spherical Artin–Tits groups and structure groups of solutions.

After this, the next step would be to study regular subgroups of the holomoph of a Garside group G, i.e. regular subgroups of $G \rtimes \mathrm{Aut}(G)$:

Problem 9. Given a Garside group G, find a condition for a regular subgroup of $G \rtimes \mathrm{Aut}(G)$ to be Garside.

On the other hand, it is a major goal to get new Garside structures on the structure groups of solutions. A way to achieve this is to study dual Garside structure ([5, 3, 4]):

Problem 10. Construct a dual Garside structure for structure groups of solutions.

To do so, we will study the lattice structure of an analogue of the set of reflections in a Coxeter group, that is $T = \{gxg^{-1} \mid x \in X, g \in \overline{G}(X,r)\}$. First looking at explicit examples with a computer algebra system would be insightful, to find an analogue of the Coxeter element inside the Coxeter-like group.

Lastly, we are interested in the study of parabolic subgroups, that is conjugate of subgroups generated by a subset of X. For the structure groups, this was covered by [9], where it is shown that the parabolics are also structure groups. Thus our objective is the following:

Problem 11. Understand the parabolic subgroups of the Coxeter-like quotients.

We expect that parabolic subgroups are again Coxeter-like quotients, which would be shown by adapting transposing the techniques of [9].

In general, Garside structures are very rare. This project with the idea of "deforming Garside structures" through automorphisms aims at providing both a new way to construct Garside groups but also a systematic approach to natural constructions such as the Coxeter-like quotient.

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