

Aşağıdaki integrallerin yakınsak olup olmadıklarını inceleyiniz.

$$1. \int_0^{\pi^2/4} \frac{\cos \sqrt{x}}{\sqrt{x}} dx$$

Çözüm.

$$= \lim_{a \rightarrow 0^+} \int_a^{\pi^2/4} \frac{\cos \sqrt{x}}{\sqrt{x}} dx, \quad u = \sqrt{x}, \quad du = \frac{1}{2\sqrt{x}} dx$$

$$x = a \Rightarrow u = \sqrt{a}$$

$$x = \pi^2/4 \Rightarrow u = \pi/2$$

$$= \lim_{a \rightarrow 0^+} \int_{\sqrt{a}}^{\pi/2} 2 \cos u du = \lim_{a \rightarrow 0^+} 2 \sin u \Big|_{\sqrt{a}}^{\pi/2}$$

$$= \lim_{a \rightarrow 0^+} 2 \left(\sin \frac{\pi}{2} - \sin \sqrt{a} \right)$$

$$= 2 \cdot (1 - 0)$$

$$= 2$$

$$2. \int_e^{\infty} \frac{dx}{x \ln x \sqrt{\ln x}}$$

$$I = \int_e^{\infty} (\ln x)^{-3/2} \cdot \frac{1}{x} dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$I = \lim_{b \rightarrow \infty} \int_e^b u^{-3/2} du = \lim_{b \rightarrow \infty} \left(-2 u^{-1/2} \Big|_{u=\ln x} \right)$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{2}{(\ln x)^{1/2}} \cdot \Big|_{x=e}^{x=b} \right)$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{2}{(\ln b)^{1/2}} + \frac{2}{1} \right) = 2$$

$$3. \int_0^{\infty} \frac{dx}{e^x + e^{-x}}$$

$$I = \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{e^x + e^{-x}} = \lim_{b \rightarrow \infty} \int_0^b \frac{e^x}{e^{2x} + 1} dx$$

$$e^x = u \Rightarrow e^x dx = du$$

$$= \lim_{b \rightarrow \infty} \int_0^b \frac{du}{u^2 + 1} = \lim_{b \rightarrow \infty} \arctan u = \lim_{b \rightarrow \infty} (\arctan e^x \Big|_0^b)$$

$$= \lim_{b \rightarrow \infty} (\arctan e^b - \arctan e^0)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \pi/4$$

$$4. \int_{-\infty}^2 \frac{dx}{(x-2)^{4/3}}$$

$$I = \underbrace{\int_{-\infty}^0 \frac{dx}{(x-2)^{4/3}}}_{I_1} + \underbrace{\int_0^2 \frac{dx}{(x-2)^{4/3}}}_{I_2}$$

$$I_1 = \lim_{b \rightarrow -\infty} \int_b^0 \frac{dx}{(x-2)^{4/3}} = \lim_{b \rightarrow -\infty} \left(-3 \frac{1}{(x-2)^{1/3}} \right) \Big|_b^0$$

$$= -3 \lim_{b \rightarrow -\infty} \left(-\frac{1}{2^{1/3}} - \frac{1}{(b-2)^{1/3}} \right)$$

$$= \frac{3}{\sqrt[3]{2}}$$

$$I_2 = \int_0^2 \frac{dx}{(x-2)^{4/3}} = \lim_{b \rightarrow 2^-} \int_0^b \frac{dx}{(x-2)^{4/3}}$$

$$= \lim_{b \rightarrow 2^-} \left(-3 \cdot \frac{1}{(x-2)^{1/3}} \right) \Big|_0^b$$

$$= -3 \lim_{b \rightarrow 2^-} \left(\frac{1}{(b-2)^{1/3}} + \frac{1}{2^{1/3}} \right) = -\infty$$

I_2 ıraksak $\Rightarrow I$ ıraksak olur.

$$5. \int_0^{\infty} \frac{x^2}{x^5+1} dx$$

$$I = \underbrace{\int_0^1 \frac{x^2}{x^5+1} dx}_{\text{belirli integral yakınsak!}} + \underbrace{\int_1^{\infty} \frac{x^2}{x^5+1} dx}_{?}$$

$$x^5 + 1 > x^3$$

$$\Rightarrow \frac{1}{x^5+1} < \frac{1}{x^3}$$

$$\Rightarrow \frac{x^2}{x^5+1} < x^2 \cdot \frac{1}{x^3} = \frac{1}{x}$$

$$\int_1^{\infty} \frac{1}{x^3} dx = \lim_{a \rightarrow \infty} \int_1^a \frac{1}{x^3} dx = \lim_{a \rightarrow \infty} \left[-\frac{1}{2x^2} \right]_1^a$$

$$= \lim_{a \rightarrow \infty} \left(-\frac{1}{2a} + \frac{1}{2} \right) = \frac{1}{2} \text{ yak.}$$

$$\Rightarrow \text{Kıyaslama testinden } \int_1^{\infty} \frac{x^2}{x^5+1} dx \text{ yakınsak olur.}$$

$$\Rightarrow \int_0^{\infty} \frac{x^2}{x^5+1} dx \text{ yak. olur.}$$

$$6. \int_{-\infty}^{\infty} \frac{2x dx}{(x^2+1)^2}$$

$$I = \int_{-\infty}^0 \frac{2x}{(x^2+1)^2} dx + \int_0^{\infty} \frac{2x}{(x^2+1)^2} dx$$

$$= \lim_{a \rightarrow -\infty} \int_a^0 \frac{2x}{(x^2+1)^2} dx + \lim_{b \rightarrow \infty} \int_0^b \frac{2x}{x^2+1} dx$$

$$= \lim_{a \rightarrow -\infty} \int_a^0 \frac{du}{u^2} + \lim_{b \rightarrow \infty} \int_0^b \frac{du}{u^2}$$

$$= \lim_{a \rightarrow -\infty} \left[-\frac{1}{u} \right]_{u=a}^{u=0} + \lim_{b \rightarrow \infty} \left[-\frac{1}{u} \right]_{u=0}^{u=b}$$

$$= \lim_{a \rightarrow -\infty} \left[-\frac{1}{x^2+1} \right]_a^0 + \lim_{b \rightarrow \infty} \left[-\frac{1}{x^2+1} \right]_0^b$$

$$= \lim_{a \rightarrow -\infty} \left[-1 + \frac{1}{a^2+1} \right] + \lim_{b \rightarrow \infty} \left[-\frac{1}{b^2+1} + 1 \right]$$

$$= -1 + 1 = 0 \quad (\text{yakınsak})$$

$$7. \int_0^{\infty} \frac{dx}{(x^5 + x^2)^{1/3}}$$

$$I = \underbrace{\int_0^1 \frac{dx}{(x^5 + x^2)^{1/3}}}_{I_1} + \underbrace{\int_1^{\infty} \frac{dx}{(x^5 + x^2)^{1/3}}}_{I_2}$$

$$0 < x \leq 1 \Rightarrow \frac{1}{(x^5 + x^2)^{1/3}} < \frac{1}{(x^2)^{1/3}} = x^{-2/3}$$

$$\int_0^1 x^{-2/3} dx = \lim_{a \rightarrow 0^+} \int_a^1 x^{-2/3} dx = \lim_{a \rightarrow 0^+} 3x^{1/3} \Big|_a^1$$

$$= \lim_{a \rightarrow 0^+} (3 - a^{1/3}) = 3$$

Kıyaslama testinden I_1 yakınsak olur.

I_2 için :

$$1 \leq x \Rightarrow \frac{1}{(x^5 + x^2)^{1/3}} < \frac{1}{x^{5/3}}$$

$$\int_1^{\infty} \frac{dx}{x^{5/3}} = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^{5/3}} = \lim_{b \rightarrow \infty} \left[-\frac{3}{2} x^{-2/3} \right]_1^b$$

$$= \lim_{b \rightarrow \infty} \left[-\frac{3}{2} b^{-2/3} + \frac{3}{2} \right] = 3/2 \Rightarrow I_2 \text{ yak} \Rightarrow I \text{ yak.}$$

$$8. \int_1^{\infty} \left(1 - \cos \frac{2}{x}\right) dx$$

$$1 - \cos \frac{2}{x} = 2 \sin^2 \left(\frac{1}{x} \right) = 2 \cdot \left(\sin \frac{1}{x} \right)^2 < 2 \cdot \frac{1}{x^2} \\ \left(|\sin x| \leq |x| \right)$$

$$\Rightarrow \int_1^{\infty} \frac{dx}{x^2} = \lim_{a \rightarrow \infty} \int_1^a \frac{dx}{x^2} = \lim_{a \rightarrow \infty} \left[-\frac{1}{a} + 1 \right] = 1$$

$$\Rightarrow \int_1^{\infty} \left(1 - \cos \frac{2}{x}\right) dx \text{ yakınsak olur.}$$

$$9. \int_1^{\infty} \frac{dx}{x^2 + \ln x}$$

$$x \geq 1 \Rightarrow 0 < \frac{1}{x^2 + \ln x} \leq \frac{1}{x^2} \quad (\ln x \geq 0)$$

$$\int_1^{\infty} \frac{1}{x^2} dx \text{ yak.}$$

$$\text{Kıyaslama testinden} \int_1^{\infty} \frac{dx}{x^2 + \ln x} \text{ yak. olur.}$$

$$10. \int_0^{\infty} \frac{dx}{xe^x}$$

$$\int_0^{\infty} \frac{dx}{xe^x} = \underbrace{\int_0^1 \frac{dx}{xe^x}}_{I_1} + \underbrace{\int_1^{\infty} \frac{dx}{xe^x}}_{I_2}$$

$$0 < x \leq 1 \quad \text{iken} \quad \frac{1}{xe^x} \geq \frac{1}{xe} > 0 \quad \left[e^x \text{ artan old. dan} \right. \\ \left. x \leq 1 \Rightarrow e^x \leq e \right]$$

$$\int_0^1 \frac{1}{xe} dx = \frac{1}{e} \int_0^1 \frac{1}{x} dx = \frac{1}{e} \lim_{u \rightarrow 0^+} \int_u^1 \frac{1}{u} du$$

$$= \frac{1}{e} \lim_{u \rightarrow 0^+} [\ln 1 - \ln u] = \infty \quad (\text{ıraksak})$$

Kıyaslama testinden I_1 ıraksak olur $\Rightarrow I$ ıraksaktır.