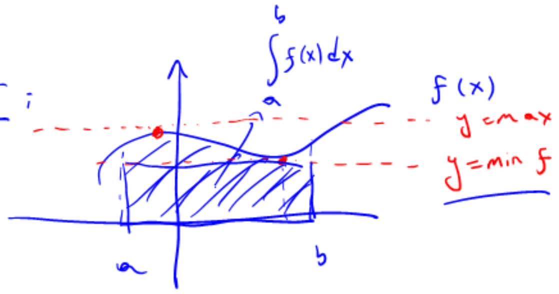
MAT 122 Uygulama 2 (istatistik) 09.03.2023

1-) $f(x)$ $[a,b]$ aralığında integrallenebilir bir fonk. olsun. Buna göre

$\forall x \in [a,b]$ için $f(x) \geq 0 \Rightarrow \int_a^b f(x)\,dx \geq 0$ old. gösteriniz.

İspat:



$0 \leq (b-a)\min f \leq \int_a^b f(x)\,dx \leq (b-a)\max f$ biliyoruz.

$\forall x \in [a,b]$ için $f(x) \geq 0 \Rightarrow \min f \geq 0$. Ayrıca $(b-a) \geq 0$ olduğu düşünülürse $\min f \cdot (b-a) \geq 0$ ve ispat tamamlanır.

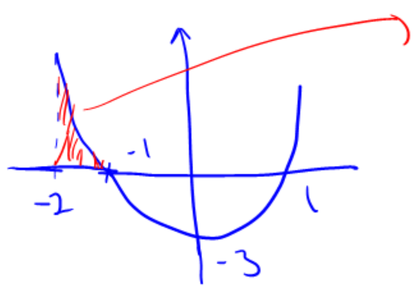
2-) Verilen fonksiyonların grafiklerini çizin ve ortalama değerlerini bulunuz.

a-) $f(x) = 3x^2 - 3$, $x \in [-2,-1]$

b-) $g(x) = |x| - 1$, $x \in [-1,1]$

Çözüm: $\text{ort}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

a-)



$$\frac{1}{-1-(-2)}\int_{-1}^{-2}(3x^2-3)\,dx = 1\left\{x^3 - 3x \Big|_{-1}^{-2}\right\}$$

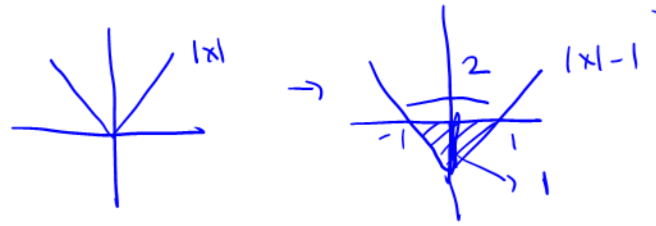
$$= -8+6-\{-1+3\}$$

$$= -2-2 = -4$$

b-) $\text{ort}(f) = \frac{1}{1-(-1)} \int_{-1}^{1} (|x|-1)\,dx = \frac{1}{2} \Big[\underbrace{\int_{-1}^{0} (-x-1) + \int_{0}^{1} x-1}_{1} \Big]$ hesaplanabilir.

$= 1/2$

2. yol:



3-) $f(x)$ $[a,b]$ aralığında sürekli bir fonksiyon olsun. Buna göre

$$\int_a^b \text{ort}(f)\,dx = \int_a^b f(x)\,dx$$ old. ispatlayınız.

Bu sonucu geometrik olarak yorumlayınız.

ispat: $\text{ort}(f) = \frac{1}{b-a} \int_a^b f(x)\,dx$ old. biliyoruz. Yani $\text{ort}(f) = M$ ve

$M \in \mathbb{R}$ old. söyleyebiliriz. Öyleyse

$$\underbrace{\int_a^b \text{ort}(f)\,dx} = \int_a^b \underbrace{\frac{1}{b-a} \int_a^b f(x)\,dx} = \int_a^b M.\,dx = M \int_a^b 1\,dx$$

$$= M \left\{ x \Big|_a^b \right\}$$

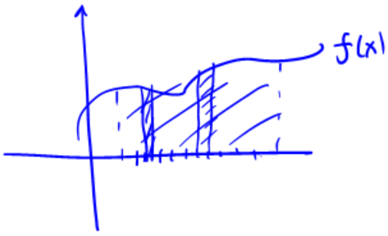
$$= M.(b-a)$$

$$= \left\{ \frac{1}{\cancel{b-a}} \int_a^b f(x)\,dx \right\} . \cancel{b-a}$$

$$= \int_a^b f(x)\,dx \qquad \square$$

4-) Kalkülüsün Temel Teoremini kullanarak aşağıdaki integralleri hesaplayınız.

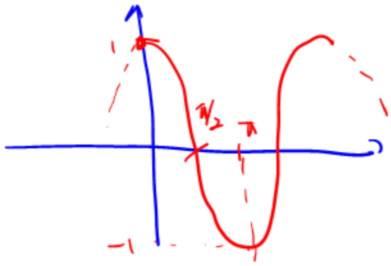
a-) $\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s}\,ds$ b-) $\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|) = I$ c-) $\int_0^{\pi/4} \tan x\,dx$



$$\int_a^b f(x)\,dx = F(b) - F(a) \quad ; \quad F'(x) = f(x)$$

a-) $$\int_1^{\sqrt{2}} \left(\frac{s^2}{s} + \frac{\sqrt{s}}{s}\right) ds = \int_1^{\sqrt{2}} \left(s + \overset{s^{-1/2}}{\frac{1}{\sqrt{s}}}\right) ds = \frac{s^2}{2} + \frac{s^{+1/2}}{1/2}\Bigg|_1^{\sqrt{2}} = \left[1 + \frac{\sqrt{\sqrt{2}}}{1/2}\right] - \left[\frac{1}{2} + 2\right]$$

b-)



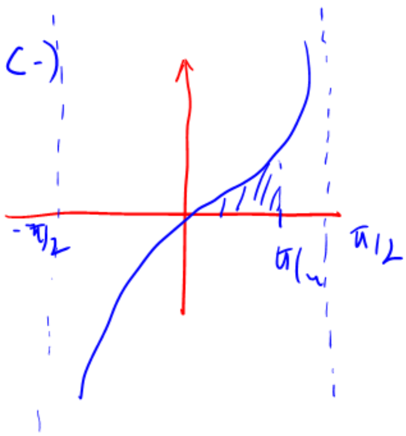
her $x \in [0, \pi/2]$ için $\cos x \geq 0$

" $x \in [\pi/2, \pi]$ için $\cos x \leq 0$.

$$I = \frac{1}{2}\left\{ \int_0^{\pi/2} \cos x + \cos x + \underbrace{\int_{\pi/2}^{\pi} [\cos x + (-\cos x)]\,dx}_{0} \right\}$$

$$= \sin x \Big|_0^{\pi/2} = \sin \pi/2 - \sin 0 = 1 - 0 = 1$$

c-)



$$\int_0^{\pi/4} \tan x\,dx = -\ln|\cos x| \Big|_0^{\pi/4}$$

$$= -\ln \frac{\sqrt{2}}{2} - \ln 1$$

$$= -\ln\left(2^{-1/2}\right) - 0$$

$$= \frac{1}{2} \ln 2 .$$

$\tan x = \frac{\sin x}{\cos x}$

$(\ln(\cos x))' = \frac{-\sin x}{\cos x}$

$= -\tan x$

5-) Aşağıdaki türevleri hesaplayınız.

a-) $\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}} \cos t \, dt$

b-) $\dfrac{d}{d\theta}\displaystyle\int_0^{\tan\theta} \sec^2 y \, dy$

c-) $\dfrac{d}{dx}\displaystyle\int_{\sqrt{x}}^{0} \sin(t^2)\, dt$

d-) $\dfrac{d}{dt}\displaystyle\int_{\tan x}^{0} \dfrac{dt}{1+t^2}$ (Ödev)

Çözüm: Soruları 2 farklı metod ile çözebiliriz. Birincisi önce integral almak ve sonra türev almak. İkinci yol ise doğrudan türev almak.

a-) 1. yol: $\displaystyle\int_0^{\sqrt{x}} \cos t \, dt = \sin t \Big|_0^{\sqrt{x}} = \sin\sqrt{x} - \cancel{\sin 0}$

$$\frac{d}{dx}\int_0^{\sqrt{x}} \cos t \, dt = \frac{d}{dx}\sin\sqrt{x} = \frac{\cos\sqrt{x}}{2\sqrt{x}}.$$

2. yol: $\sqrt{x} = u$ olsun.

$$\frac{d}{dx}\int_0^{u} \cos t \, dt \overset{K.T.T.}{=} \frac{d}{dx} F(u)$$

$$= F'(u) \cdot \frac{du}{dx}$$

$$= \cos u \cdot \frac{1}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{2\sqrt{x}}.$$

$\displaystyle\int_0^{x} f(t)\,dt = F(x)$ ve burada $F'(x) = f(x)$

$F \to u \to x$ $\quad F'$

$\displaystyle\int_0^{u} \underbrace{\cos t}_{f} \, dt = F(u)$

$F'(u) = f(u) = \cos u$

$(\tan\theta)' = \left(\dfrac{\sin\theta}{\cos\theta}\right)' = \dfrac{1}{\cos^2\theta} = \sec^2\theta$

b-) 1. yol

$$\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 t \, dt = \frac{d}{d\theta}\left\{ \tan t \Big|_0^{\tan\theta} \right\}$$

$$= \frac{d}{d\theta} \tan(\tan\theta) = \sec^2(\tan\theta) \cdot \sec^2\theta$$

2. yol: $\underline{\tan\theta = u}$ olsun. Buna göre $\int_0^u \sec^2 t \cdot dt = F(u)$ olsun. $F'(u) = \sec^2 u$.

$$\Rightarrow \frac{d}{d\theta} \int_0^u \sec^2 t\, dt = \frac{d}{d\theta} F(u) = F'(u) \cdot \frac{du}{d\theta} = \sec^2 u \cdot \sec^2\theta$$
$$= \sec^2(\tan\theta) \cdot \sec^2\theta .$$

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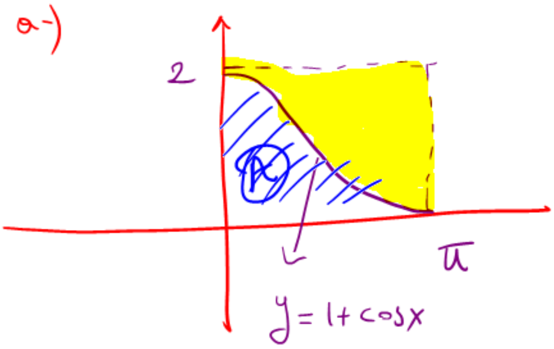
c-) $\sqrt{x} = u$

$$\frac{d}{dx} \int_{\sqrt{x}}^{0} \sin(t^2)\, dt \overset{\uparrow}{=} \frac{d}{dx} \int_u^0 \sin(t^2)\, dt = \frac{d}{dx}\left(- \int_0^u \sin(t^2)\, dt\right)$$

$$= \frac{d}{dx}\left(-F(u)\right) = -F'(u) \cdot \frac{du}{dx} = -\sin(u^2) \cdot \frac{1}{2\sqrt{x}} = \frac{-\sin x}{2\sqrt{x}}$$

6-) Aşağıdaki bölgelerin alanlarını hesaplayınız.

a-)



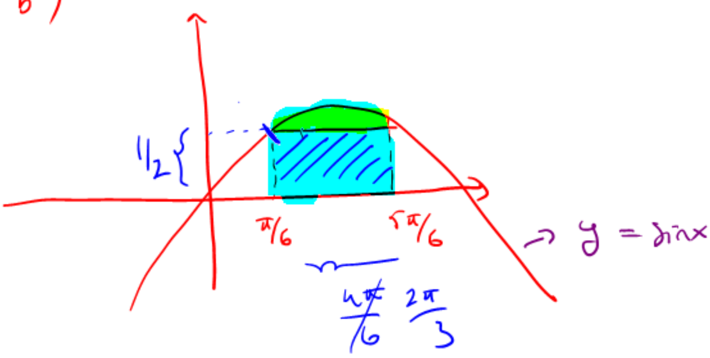
$$\int_0^\pi (1+\cos x)\, dx = A = x + \sin x \Big|_0^\pi = \pi - 0 - [0 - 0]$$

A (Dikdörtgen) − A = A(Sarı)

$2\pi - \pi = A(\text{Sarı})$

$\pi =$

b-)



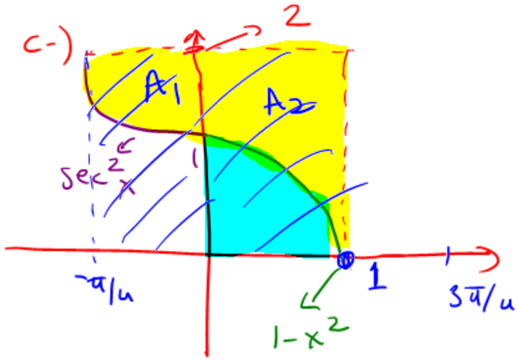
$$\int_{\pi/6}^{5\pi/6} \sin x\, dx = \text{Mavi} = -\cos x \Big|_{\pi/6}^{5\pi/6}$$

$$A(\text{Dikdörtgen}) = \frac{1}{2} \cdot \frac{2\pi}{3} = \frac{\pi}{3}$$

$$= \left[-\cos\left(\frac{5\pi}{6}\right) - \left(-\cos\frac{\pi}{6}\right)\right]$$

$$\boxed{\text{Sonuç} = A(\text{Mavi}) - A(\text{dikdörtgen})}$$

c-)



$\frac{1}{\cos^2 x} = \sec^2 x = 2$ olsun.

$\Rightarrow \cos^2 x = \frac{1}{2} \Rightarrow \cos x = \mp \frac{1}{\sqrt{2}} = \mp \frac{\sqrt{2}}{2}$

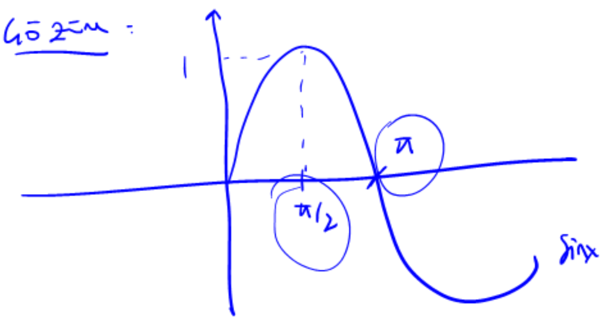
$-\pi/4, \ \frac{3\pi}{4}$

$$A_1 = \frac{\pi}{4} \cdot 2 - \int_{-\pi/4}^{0} \sec^2 x \, dx = \frac{\pi}{2} - \tan x \Big|_{-\pi/4}^{0} = \frac{\pi}{2} - \Big[\tan 0 - \underbrace{\tan(-\pi/4)}_{1}\Big] = \frac{\pi}{2} - 1$$

$$A_2 = 1 \cdot 2 - \int_0^1 (1-x^2)\,dx = 2 - \Big[x - \frac{x^3}{3} \Big|_0^1\Big] = 2 - \Big[1 - \frac{1}{3} - 0\Big] = 2 - \frac{2}{3} = \frac{4}{3}.$$

7-) $k \in \mathbb{R}^+$ olmak üzere, x-ekseni ile $y = \sin(kx)$ eğrisinin bir yayı arasında kalan alanın $2/k$ old. gösteriniz.

Çözüm:



$y = \sin(kx)$

$y = \sin\left(k \cdot \frac{\pi}{2k}\right)$

$$A = \int_0^{\pi/k} \sin(kx)\,dx = -\frac{\cos kx}{k} \Big|_0^{\pi/k} = -\frac{\cos \cancel{k} \cdot \frac{\pi}{\cancel{k}}}{k} - \left(-\frac{\cos 0}{k}\right)$$

$$= \frac{1}{k} + \frac{1}{k} = \frac{2}{k} \qquad \square$$

8-) Aşağıdaki integralleri değişken dönüşümü yaparak hesaplayınız.

a-) $\int \frac{1}{\theta^2} \cdot \sin\left(\frac{1}{\theta}\right) \cdot \cos\left(\frac{1}{\theta}\right) d\theta = I$

$\int f(g(x)) \cdot g'(x)\,dx$

$u = \frac{1}{\theta}$ olsun.

$\Rightarrow du = \left(-\frac{1}{\theta^2}\right) \cdot d\theta \Rightarrow \frac{1}{\theta^2}\, d\theta = -du$

$$\boxed{g(x) = u \quad g'(x) \cdot dx = du \Rightarrow \int f(u) \cdot du}$$

$$I = \int \sin u \cdot \cos u \cdot (-du) = -\frac{1}{2}\int \underbrace{2 \cdot \sin u \cos u}_{\sin 2u}\, du = -\frac{1}{2} \cdot \left(\frac{-\cos 2u}{2}\right)$$

$$= \frac{\cos 2u}{4} = \frac{\cos\left(2 \cdot \frac{1}{\theta}\right)}{4}$$

b-) $\int \sqrt{\frac{x-1}{x^5}}\, dx = ? = I$

$$I = \int \sqrt{\frac{x-1}{x^4 \cdot x}} = \int \frac{1}{x^2} \sqrt{\frac{x-1}{x} \cdot 1}\, dx$$

$\frac{1}{x}$

, $\frac{1}{x} = u$ olsun.

$\Rightarrow -\frac{1}{x^2} dx = du$

$$= \int \sqrt{1 - \frac{1}{x}} \cdot (-du)$$

$$= -\int \sqrt{1-u} \cdot du$$

, $1 - u = p$ olsun.

$-du = dp$

$$= -\int \sqrt{p} \cdot (-dp) = \int \sqrt{p}\, dp = \frac{p^{3/2}}{3/2}$$

$$= \frac{(1-u)^{3/2}}{3/2}$$

$$= \frac{\left(1 - \frac{1}{x}\right)^{3/2}}{3/2} \quad \square$$

c-) $\int 3x^5 \cdot \sqrt{x^3+1}\, dx = I = ?$

$x^3 + 1 = u^2$ olsun.

$3x^2 \cdot dx = 2 \cdot u \cdot du$

$$\Rightarrow \int 3x^2 \cdot x^3 \cdot \sqrt{x^3+1}\, dx$$

$$\Rightarrow \int 2u \cdot (u^2-1) \cdot |u| \cdot du .$$

(Yeni bir çözüm bulun ödev).