(20

Ex 2: (a) $\frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\cdots=\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2}=\sum_{1}^{\infty}\frac{2n+1}{n^2+2n+1}$

$a_n=\frac{2n+1}{n^2+2n+1}$, choose $b_n=1/n \Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{2n+1}{n^2+2n+1}}{1/n}=2>0$.

and $\sum_{1}^{\infty}1/n$ diverges, hence (by comp. lim test) $\sum_{1}^{\infty}\frac{2n+1}{(n+1)^2}$ diverges

b) $1+\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\cdots+\quad=\sum_{1}^{\infty}\frac{1}{2^n-1}$, $a_n=\frac{1}{2^n-1}$, choose $b_n=\frac{1}{2^n}\geq 0$

$\Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1/2^n-1}{1/2^n}=\lim_{n\to\infty}\frac{2^n}{2^n-1}\overset{\text{l'Hosp.}}{=}\lim_{n\to\infty}\frac{2^n\ln 2}{2^n\ln 2}=1>0$. and

since $\sum_{1}^{\infty}\frac{1}{2^n}$ converges, $\sum_{1}^{\infty}\frac{1}{2^n-1}$ converges (by Lim. comp. test.)

c) $\frac{1+2\ln 2}{9}+\frac{1+3\ln 3}{14}+\frac{1+4\ln 4}{21}+\cdots=\sum_{n=2}^{\infty}\frac{1+n\ln n}{n^2+5}$

$a_n=\frac{1+n\ln n}{n^2+5}>0$ (for $n\geq 2$) and choose $b_n=\frac{1}{n}>0$

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{1+n\ln n}{n^2+5}}{1/n}=\lim_{n\to\infty}\frac{n+n^2\ln n}{n^2+5}=\lim_{n\to\infty}\frac{\cancel{n^2}\left(\frac{1}{n}+\ln n\right)}{\cancel{n^2}\left(1+\frac{5}{n^2}\right)}=\infty$

and since $\sum_{2}^{\infty}\frac{1}{n}$ diverges, $\sum_{2}^{\infty}a_n=\sum_{2}^{\infty}\frac{1+n\ln n}{n^2+5}$ diverges.

Ex. 3: Does $\sum_{1}^{\infty}\frac{\ln n}{n^{3/2}}$ converge? $\left(\frac{\ln n}{n^{3/2}}<\frac{n^{1/4}}{n^{3/2}}=\frac{1}{n^{5/4}}=b_n\right.$

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\ln n/n^{3/2}}{1/n^{5/4}}=\lim_{n\to\infty}\frac{\ln n}{n^{1/4}}=\left(\frac{\infty}{\infty}\right)=\lim_{n\to\infty}\frac{1/n}{\left(\frac{1}{4}\right).n^{-3/4}}=\lim_{n\to\infty}\frac{4}{n^{1/4}}=0$.

and since $\sum_{1}^{\infty}b_n=\sum_{1}^{\infty}\frac{1}{n^{5/4}}$ converges, $\sum_{1}^{\infty}\frac{\ln n}{n^{3/2}}$ converges. (by Lim. com. test)

11.5. The Ratio and Root Tests:

Theorem 11 (Ratio Test): Let $\sum_{1}^{\infty}a_n$ be a series with positive terms and suppose

that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=A$, Then
(i) the series converges if $A<1$
(ii) " " diverges if $A>1$
(iii) the test is ~~inclusive~~ inconclusive if $A=1$

Ex-1. (a) $\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$ (b) $\sum_{n=1}^{\infty}\frac{(2n)!}{n!\,n!}$ (c) $\sum_{1}^{\infty}\frac{4^n.n!\,n!}{(2n)!}$

a) $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2^{n+1}+5}{3^{n+1}}.\frac{3^n}{2^n+5}=\lim_{n\to\infty}\frac{\cancel{2^n}\left(2+\frac{5}{2^n}\right)}{3\cdot\cancel{2^n}\left(1+\frac{5}{2^n}\right)}=2/3<1 \overset{\text{Ratio Test}}{\Longrightarrow}$

The series $\sum_{0}^{\infty}\frac{2^n+5}{3^n}$ converges. $\left(=\sum\left(\frac{2}{3}\right)^n+5\sum\frac{1}{3^n}=\frac{1}{1-\frac{2}{3}}+5.\frac{1}{1-\frac{1}{3}}=3+\frac{15}{2}=\frac{21}{2}\right.$

lim. conv. $\to b_n=\left(\frac{2}{3}\right)^n$

b) $a_n=\frac{(2n)!}{n!\,n!}\Rightarrow \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(2n+2)!}{(n+1)!\,(n+1)!}.\frac{n!\,n!}{(2n)!}=\lim_{n\to\infty}\frac{(2n+2)(2n+1).\cancel{(2n)!}\,\cancel{n!}\,\cancel{n!}}{(n+1)(n+1).\cancel{n!}\,\cancel{n!}\,\cancel{(2n)!}}=4>1$

$\overset{\text{Ratio Test}}{\Longrightarrow}\sum_{1}^{\infty}\frac{(2n)!}{n!\,n!}$ diverges.

c) $a_n = \dfrac{4^n (n!)^2}{(2n)!}$ ise $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \dfrac{4^{n+1}((n+1)!)^2}{(2n+2)!} \cdot \dfrac{(2n)!}{4^n (n!)^2} = \lim\limits_{n\to\infty} \dfrac{4 \cdot (n+1)^2}{2(n+1)(2n+1)} = 1$

olur ki bu testle sonuca ulaşılmaz.

Ancak $\dfrac{a_{n+1}}{a_n} = \dfrac{2n+2}{2n+1}$ ve $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1$ old. dan $a_{n+1} \geq a_n$ dır; çünkü $\dfrac{a_{n+1}}{a_n} = \dfrac{2n+2}{2n+1} \geq 1$ dir. O halde tüm terimler $a_1 = 2$ den büyük veya eşittir.

Dolayısıyla $\lim\limits_{n\to\infty} a_n \neq 0$ olur ki, genel terim testinden, $\sum\limits_{n=1} a_n$ serisi ıraksak olur.

2) Kök Testi: $\sum\limits_{n=1}^{\infty} a_n$ serisi ($\forall n \geq N$ için $a_n \geq 0$) biçiminde olsun ve $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = A$ verilsin. Bu durumda, (i) $A<1 \Rightarrow$ seri yakınsak,
(ii) $A>1 \Rightarrow$ seri ıraksaktır,
(iii) $A=1$ ise bu test sonuç vermez.

Kanıt: (i) $A<1$ olsun ve $\varepsilon>0$ seçelim öyle ki $A+\varepsilon<1$ olsun.
$\sqrt[n]{a_n} \to A$ old. dan $\exists M \geq N$; $\forall n \geq M \Rightarrow \sqrt[n]{a_n} < A+\varepsilon$ dir.
$\forall n \geq M$ için $\Rightarrow a_n < (A+\varepsilon)^n$ ve $\sum\limits_{n=M}^{\infty} (A+\varepsilon)^n$ bir geometrik seri ve $|A+\varepsilon|=r|<1$ old. dan yakınsak olur ki sınırdan karşılaştırma testinden $\sum\limits_{n=M}^{\infty} a_n$ de yakınsak olur; dolayısıyla da $\sum\limits_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_{M-1} + \sum\limits_{n=M}^{\infty} a_n$ de yak. olur.
(ii) $1 < A \leq \infty$ ise $\sqrt[n]{a_n} \to A$ old. dan $\exists M$; $\forall n > M$ için $a_n \geq 1$ dir.
$\Rightarrow \lim\limits_{n\to\infty} a_n \neq 0 \Rightarrow \sum\limits_{n=M}^{\infty} a_n$ ıraksak $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ ıraksak olur.
(iii) $A=1$ olursa $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ ıraks. $\to \lim\limits_{n\to\infty} \sqrt[n]{1/n} = 1$; $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ yak $\to \lim\limits_{n\to\infty} \sqrt[n]{1/n^2} = 1$. eşitlik.

Örnek a) $\underbrace{\sum\limits_{n=1}^{\infty} \dfrac{n^2}{2^n}}_{\text{yak}}$ b) $\underbrace{\sum\limits_{n=1}^{\infty} \dfrac{2^n}{n^2}}_{\text{ırak}}$ c) $\underbrace{\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{1+n}\right)^n}_{\text{yak}}$.

Örnek. $a_n = \begin{cases} \dfrac{n}{2^n}, & n \text{ tek ise} \\ \dfrac{1}{2^n}, & n \text{ çift} \end{cases}$ olmak üzere $\sum\limits_{n=1}^{\infty} a_n$ serisinin yak. veya ırak. inceleyin.

Bu seri için oran testiyle sonuca ulaşılmaz, çünkü: $\dfrac{a_{n+1}}{a_n} = \begin{cases} \dfrac{1}{2n}, & n \text{ tek} \\ \dfrac{n+1}{2}, & n \text{ çift} \end{cases}$ old. dan $n\to\infty$ $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ yoktur.

n tek $\Rightarrow n+1$ çift $\dfrac{a_{n+1}}{a_n} = \dfrac{\frac{1}{2^{n+1}}}{\frac{n}{2^n}} = \dfrac{1}{2n}$

n çift $\Rightarrow n+1$ tek $\dfrac{a_{n+1}}{a_n} = \dfrac{\frac{n+1}{2^{n+1}}}{1/2^n} = \dfrac{n+1}{2}$ dir.

Ancak Kök testi ile $\sqrt[n]{a_n} = \begin{cases} \dfrac{\sqrt[n]{n}}{2}, & n \text{ tek} \\ 1/2, & n \text{ çift} \end{cases}$ olur.

$\dfrac{1}{2} \leq \sqrt[n]{a_n} \leq \dfrac{\sqrt[n]{n}}{2}$ olur ve $\lim\limits_{n\to\infty} 1/2 = \lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n}}{2} = 1/2$ old. dan sandviç teor. den $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1/2 < 1$ olup seri yakınsak bulunur.

Not: $\boxed{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = A \Rightarrow \lim_{n\to\infty} \sqrt[n]{a_n} = A}$ dır..

Soru 1. $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ olduğunu gösteriniz; Bunun için genel terimi $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ ni yakınsak olduğunu göstermek yeterlidir, çünkü bu durumda genel terim testinden $\lim_{n\to\infty} a_n = 0$ bulunur. Gerçekten, oran testinden $\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)\cancel{n!} \cdot n^n}{(n+1)\cdot(n+1)^n}$

$= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = 1/e < 1$ old. dan seri $\left(\sum_{n=1}^{\infty} \frac{n!}{n^n}\right)$ yakınsak $\xRightarrow{\text{G.T.T.}}$

$\lim_{n\to\infty} \frac{n!}{n^n} = 0$ bulunur.

Soru 2: $\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$ old. göst. Bunun için de $a_n = \frac{n}{\sqrt[n]{n!}}$ olmak üzere $b_n = (a_n)^n$ denirse, $b_n = \frac{n^n}{n!}$ ve

$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{\underset{(n+1).\cancel{n!}}{(n+1)!}} \cdot \frac{\cancel{n!}}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = e$ olur $\overset{\text{(Not)}}{\Rightarrow} \lim_{n\to\infty} \sqrt[n]{b_n} = \lim_{n\to\infty} a_n$

$= \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$ bulunur.

Soru 3: $\lim_{n\to\infty} \frac{\sqrt[n]{(n+1)(n+2)\cdots(2n)}}{n} = ?$, Yine $a_n = \frac{\sqrt[n]{(n+1)(n+2)\cdots(2n)}}{n}$ olmak üzere

ve $b_n = (a_n)^n = \frac{(n+1).(n+2)\cdots(2n)}{n^n}$ dir $= \frac{(1.2.3\cdots n).(n+1).(n+2)\cdots(2n)}{(1.2.3\cdots n).n^n} = \frac{(2n)!}{n!\,n^n}$

Olur. Buradan da $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{\overset{2(n+2)(2n+1)\cancel{(2n)!}}{(2n+2)!}}{\underset{(n+1).\cancel{n!}}{(n+1)!}\,\underset{(n+1).(n+1)^n}{(n+1)^{n+1}}} \cdot \frac{\cancel{n!}\,n^n}{\cancel{(2n)!}} = \lim_{n\to\infty} \frac{2(2n+1).n^n}{(n+1)(n+1)^n}$

$= \lim_{n\to\infty} \frac{2(2n+1)}{n+1} . \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{4}{e}$ bulunur $\overset{\text{(Not)}}{\Rightarrow} \lim_{n\to\infty} \sqrt[n]{b_n} = \lim_{n\to\infty} a_n = \frac{4}{e}$ olur.

Alıştırmalar: (11) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ (20) $\sum_{n=1}^{\infty} \frac{n.2^n.(n+1)!}{3^n.n!}$ 26) $\sum_{n=1}^{\infty} \frac{3^n}{n^3.2^n}$

30) $a_1 = 3$, $a_{n+1} = \frac{n}{n+1}.a_n \to a_n = \frac{3}{n}$ ($\sum_{n=1}^{\infty} \frac{3}{n}$ ıraksak), 41) $\sum_{n=1}^{\infty} \frac{n^n}{2^{(n^2)}}$

(44) $\sum_{n=1}^{\infty} \frac{1.3\cdots(2n-1)}{[2.4\cdots 2n].(3^n+1)}$

(30) $\left. \begin{array}{l} a_1 = 3,\ a_2 = \frac{1}{2}.3 = \frac{3}{2},\ a_3 = \frac{2}{3}.\frac{1}{2}.3 = \frac{3}{3} \\ a_4 = \frac{3}{4}.\frac{2}{3}.\frac{1}{2}.3 = \frac{3}{4},\ a_5 = \frac{4}{5}.\frac{3}{4}.\frac{2}{3}.\frac{1}{2}.3 = \frac{3}{5} \end{array} \right\} \Rightarrow a_n = \frac{3}{n}$ dır

$\Rightarrow \sum_{n=1}^{\infty} \frac{3}{n}$ serisi ıraksaktır.