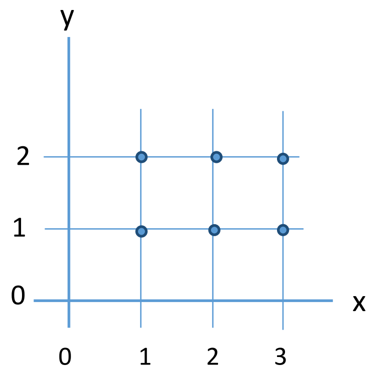
UYGULAMA IX
(Koşullu Dağılımlar)

1. X ve Y kesikli raslantı değişkeninin bileşik olasılık fonksiyonu aşağıda verilmiştir:

$$
\begin{aligned}
p_{X,Y}(x,y) &= \frac{x+y}{21} \qquad x = 1,2,3\ ve\ y = 1,2 \\
&= 0, \qquad ö.d.
\end{aligned}
$$



Marjinal olasılık fonksiyonları:

$$
\begin{aligned}
p_X(x) &= \frac{(2x+3)}{21} \quad , \quad x = 1,2,3 \\
&= 0 \quad , \quad diğer\ x\ değerleri\ için \\
p_Y(y) &= \frac{(y+2)}{7} \quad , \quad y = 1,2 \\
&= 0 \quad , \quad diğer\ y\ değerleri\ için
\end{aligned}
$$

Bileşik dağılım fonksiyonu bulunuz.

$$
\begin{aligned}
F_{X,Y}(x,y) &= \frac{1}{21}\sum_{t=1}^{x}\sum_{z=1}^{y}(t+z) \\
&= \frac{1}{21}\left[\sum_{t=1}^{x}\sum_{z=1}^{y} t + \sum_{t=1}^{x}\sum_{z=1}^{y} z\right] \\
&= \frac{1}{21}\left[\sum_{t=1}^{x} ty + \sum_{t=1}^{x}\frac{y(y+1)}{2}\right] \\
&= \frac{1}{21}\left[y\frac{x(x+1)}{2} + x\frac{y(y+1)}{2}\right] \\
&= \frac{1}{21}\left[\frac{xy(x+1)}{2} + \frac{xy(y+1)}{2}\right] = \frac{xy(x+y+2)}{42}
\end{aligned}
$$

$$
\begin{aligned}
F_{X,Y}(x,y) &= \frac{xy(x+y+2)}{42}, \quad x = 1,2,3, y = 1,2 \\
&= 0, \quad x < 1, y < 1 \\
&= 1, \quad x \geq 3, y \geq 2
\end{aligned}
$$

Sağlama: $F_{X,Y}(3,2) = 1$ olmalıdır.

Marjinal dağılım fonksiyonlarını bulunuz.

$$F_X(x) = \lim_{y\to 2} F_{X,Y}(x,y) = F_{X,Y}(x,2) = \frac{2x(x+2+2)}{42} = \frac{x(x+4)}{21}$$

$$\begin{aligned} F_X(x) &= \frac{x(x+4)}{21}, & x &= 1,2,3 \\ &= 0, & x &< 1, \\ &= 1, & x &\geq 3 \end{aligned}$$

$$F_Y(y) = \lim_{x\to 3} F_{X,Y}(x,y) = F_{X,Y}(3,y) = \frac{3y(3+y+2)}{42} = \frac{y(y+5)}{14}$$

$$\begin{aligned} F_Y(y) &= \frac{y(y+5)}{14}, & y &= 1,2 \\ &= 0, & y &< 1 \\ &= 1, & y &\geq 2 \end{aligned}$$

a) $p(x|y) =?$ $\quad E(3X+5|Y=1) =?$ $\quad F(x|y) =?$ $\quad P(X \leq 2|Y=1) =?$

$$p(x|y) = P(X=x|Y=y) = \frac{p(x,y)}{p_Y(y)} = \frac{\frac{x+y}{21}}{\frac{y+2}{7}} = \frac{(x+y)}{3(y+2)}$$

$$\begin{aligned} p(x|y) &= \frac{(x+y)}{3(y+2)} \quad , & & x = 1,2,3; \ (y=1,2) \\ &= 0 \quad , & & \textit{öteki } x,y \textit{ değerleri için} \end{aligned}$$

$$E(3X+5|Y=1) = 3E(X|Y=1) + 5$$

$$E(X|Y=1) = \sum_{Rx|y=1} xp(x|Y=1)$$

$$p(x|Y=1) = \frac{(x+1)}{3(1+2)} = \frac{(x+1)}{9}$$

$$\begin{aligned} p(x|Y=1) &= \frac{(x+1)}{9} \quad , & & x = 1,2,3; \ (y=1,2) \\ &= 0 \quad , & & \textit{öteki } x \textit{ değerleri için} \end{aligned}$$

$$E(X|Y=1)=\sum_{Rx|y=1} xp(x|Y=1)=\sum_{x=1}^{3} x\frac{(x+1)}{9}=\frac{1}{9}(1.2+2.3+3.4)=\frac{20}{9}$$

$$E(3X+5|Y=1)=3E(X|Y=1)+5=3\left(\frac{20}{9}\right)+5=\frac{35}{3}$$

$F(x|y)$ koşullu dağılım fonksiyonunu bulalım:

1.Yol:

$$F(x|y)=P(X\le x|Y=y)=\frac{P(X\le x,Y=y)}{P_Y(y)}=\frac{\sum_{t=-\infty}^{x} p(t,y)}{\frac{(y+2)}{7}}=\frac{\sum_{t=1}^{x}\frac{t+y}{21}}{\frac{(y+2)}{7}}$$
$$=\frac{2xy+x(x+1)}{6(y+2)}$$

$$F(x|y)=\begin{cases}\dfrac{2xy+x(x+1)}{6(y+2)}, & x=1,2,3\ (y=1,2)\\ =0, & x<0\\ =1, & x\ge 3\end{cases}$$

2.yol:

$$F(x|y)=\sum_{t=-\infty}^{x} p(t|y)=\sum_{t=1}^{x}\frac{t+y}{3(y+2)}=\frac{1}{3(y+2)}\sum_{t=1}^{x}(t+y)=\frac{\frac{x(x+1)}{2}+xy}{3(y+2)}$$

$$F(x|y)=\begin{cases}\frac{2xy+x(x+1)}{6(y+2)}, & x=1,2,3\ (y=1,2)\\ =0, & x<0\\ =1, & x\ge 3\end{cases}$$

$$F(x|Y=1)=\begin{cases}\dfrac{2x+x(x+1)}{18}, & x=1,2,3\\ =0, & x<0\\ =1, & x\ge 3\end{cases}$$

$$P(X\le 2|Y=1)=P(X=1|Y=1)+P(X=2|Y=2)=\frac{1+1}{9}+\frac{2+1}{9}=\frac{5}{9}$$

ya da

$$P(X\le 2|Y=1)=F(2|Y=1)=\frac{2\times 2+2(2+1)}{18}=\frac{5}{9}$$

b) $p(y|x) = ?$ $V(2Y|X=3) = ?$ $F(y|x) = ?$ $P(Y \leq 1|X=2) = ?$

$$p(y|x) = P(Y=y|X=x) = \frac{p(x,y)}{p_X(x)} = \frac{\frac{x+y}{21}}{\frac{2x+3}{21}} = \frac{(x+y)}{2x+3}$$

$$\begin{aligned} p(y|x) &= \frac{(x+y)}{2x+3} \quad , \quad y = 1,2 \ \ (x = 1,2,3) \\ &= 0 \quad , \quad \textit{öteki x ve y değerleri için} \end{aligned}$$

$$V(2Y|X=3) = 4V(Y|X=3) = 4\{E(Y^2|X=3) - [E(Y|X=3)]^2\}$$

$$p(y|X=3) = \frac{(3+y)}{2*3+3} = \frac{(y+3)}{9}$$

$$\begin{aligned} p(y|X=3) &= \frac{(y+3)}{9} \quad , \quad y = 1,2 \\ &= 0 \quad , \quad \textit{öteki x değerleri için} \end{aligned}$$

$E(Y|X=3)$ koşullu beklenen değeri bulalım:

1.Yol:

$$E(Y|X=3) = \sum_{y=1}^{2} yp(y|X=3) = \sum_{y=1}^{2} y\frac{(y+3)}{9} = \frac{1}{9}(1+3+4+6) = \frac{14}{9}$$

$$E(Y^2|X=3) = \sum_{y=1}^{2} y^2p(y|X=3) = \sum_{y=1}^{2} y^2\frac{(y+3)}{9} = \frac{1}{9}(1+3+8+12) = \frac{24}{9}$$

$$V(2Y|X=3) = 4V(Y|X=3) = 4\left[\frac{24}{9} - \left(\frac{14}{9}\right)^2\right] = 4\left(\frac{216-196}{81}\right) = \frac{80}{81}$$

2.yol:

$$E(Y|X) = \sum_{y=1}^{2} yp(y|x) = \sum_{y=1}^{2} y\frac{(x+y)}{2x+3} = \frac{x+1+2(x+2)}{2x+3} = \frac{3x+5}{2x+3}$$

$$E(Y|X=3) = \frac{3.3+5}{2.3+3} = \frac{14}{9}$$

$$E(Y^2|X) = \sum_{y=1}^{2} y^2 p(y|x) = \sum_{y=1}^{2} \frac{y^2(x+y)}{2x+3} = \frac{x+1+4(x+2)}{2x+3} = \frac{5x+9}{2x+3}$$

$$E(Y^2|X=3) = \frac{5.3+9}{2.3+3} = \frac{24}{9}$$

$$V(2Y|X=3) = 4V(Y|X=3) = 4\left[\frac{24}{9} - \left(\frac{14}{9}\right)^2\right] = 4\left(\frac{216-196}{81}\right) = \frac{80}{81}$$

$F(y|x)$ koşullu dağılım fonksiyonunu bulalım:

1.Yol:

$$F(y|x) = P(Y \le y|X=x) = \frac{P(X=x, Y \le y)}{p_X(x)} = \frac{\sum_{z=-\infty}^{y} p(x,z)}{\frac{(2x+3)}{21}} = \frac{\sum_{z=1}^{y} \frac{x+z}{21}}{\frac{(2x+3)}{21}}$$

$$= \frac{2xy + y(y+1)}{2(2x+3)}$$

2.yol:

$$F(y|x) = \sum_{z=-\infty}^{y} p(z|x) = \sum_{z=1}^{y} \frac{x+z}{2x+3} = \frac{1}{2x+3}\left(xy + \frac{y(y+1)}{2}\right) = \frac{2xy + y(y+1)}{2(2x+3)}$$

$$F(y|x) = \begin{cases} \frac{2xy+y(y+1)}{2(2x+3)} & , \quad y = 1,2 \ (x = 1,2,3) \\ 0 & , \quad y < 0 \\ 1 & , \quad y \ge 2 \end{cases}$$

$$P(Y \le 1|X=2) = F(1|X=2) = \frac{2.2.1 + 1(1+1)}{2(2.2+3)} = \frac{6}{14} = \frac{3}{7}$$

$$P(Y \le 1|X=2) = P(Y=1|X=2) = \frac{2+1}{2.2+3} = \frac{3}{7}$$

c) $p(y|X>1) = ?$ $\quad E(4Y|X>1) = ?$ $\quad F(y|X>1) = ?$

$$p(y|X>1) = P(Y=y|X>1) = \frac{P(X>1, Y=y)}{P(X>1)} = \frac{\sum_{x=2}^{3} \frac{x+y}{21}}{\sum_{x=2}^{3} \frac{2x+3}{21}}$$

$$= \frac{2+y+3+y}{4+3+6+3} = \frac{5+2y}{16}$$

$$
\begin{aligned}
p(y|X>1) &= \tfrac{5+2y}{16} \quad , \quad y=1,2 \\
&= 0 \quad , \quad \text{ö}.d.
\end{aligned}
$$

$$E(4Y|X>1)=4E(Y|X>1)=4\sum_{y=1}^{2} yp(y|X>1)=4\sum_{y=1}^{2} y\left(\frac{5+2y}{16}\right)$$

$$=\frac{1}{4}(7+18)=\frac{25}{4}$$

$F(y|X>1)$ koşullu dağılım fonksiyonunu bulalım:

1.Yol:

$$F(y|X>1)=P(Y\le y|X>1)=\frac{P(X>1,Y\le y)}{P(X>1)}=\frac{\sum_{x=2}^{3}\sum_{z=1}^{y}p(x,z)}{\sum_{x=2}^{3}p_X(x)}$$

$$=\frac{\sum_{x=2}^{3}\sum_{z=1}^{y}\left(\frac{x+z}{21}\right)}{\sum_{x=2}^{3}\left(\frac{2x+3}{21}\right)}=\frac{\sum_{x=2}^{3}\left[xy+\frac{y(y+1)}{2}\right]}{16}=\frac{5y+y(y+1)}{16}$$

2.yol:

$$F(y|X>1)=\sum_{z=-\infty}^{y}p(z|x>1)=\sum_{z=1}^{y}\left(\frac{5+2z}{16}\right)=\frac{1}{16}\left[5y+\frac{2y(y+1)}{2}\right]=\frac{y^2+6y}{16}$$

$$
\begin{aligned}
F(y|X>1) &= \tfrac{y^2+6y}{16} \quad , \quad y=1,2 \\
&= 0 \quad , \quad y<1 \\
&= 1 \quad , \quad y\ge 2
\end{aligned}
$$

d) $p(y|X\le 2)=?$ $F(y|X\le 2)=?$ $P(Y>1|X\le 2)=?$ $P(Y<4|X\le 2)=?$ $P(Y<1|X\le 2)=?$

$$p(y|X\le 2)=P(Y=y|X\le 2)=\frac{P(X\le 2,Y=y)}{P(X\le 2)}=\frac{\sum_{x=1}^{2}p(x,y)}{\sum_{x=1}^{2}p_X(x)}=\frac{\sum_{x=1}^{2}\frac{x+y}{21}}{\sum_{x=1}^{2}\frac{2x+3}{21}}$$

$$=\frac{1+y+2+y}{2+3+4+3}=\frac{2y+3}{12}$$

$$
\begin{aligned}
p(y|X\le 2) &= \frac{2y+3}{12} \quad , \quad y=1,2 \\
&= 0 \quad , \quad \text{ö}.d.
\end{aligned}
$$

$F(y|X\leq 2)$ koşullu dağılım fonksiyonunu bulalım:

1.Yol:

$$F(y|X\leq 2)=P(Y\leq y|X\leq 2)=\frac{\sum_{x=1}^{2}\sum_{z=1}^{y}p(x,z)}{\sum_{x=1}^{2}p_X(x)}=\frac{\sum_{x=1}^{2}\sum_{z=1}^{y}\left(\frac{x+y}{21}\right)}{\sum_{x=1}^{2}\left(\frac{2x+3}{21}\right)}$$

$$=\frac{\sum_{x=1}^{2}\left[xy+\frac{y(y+1)}{2}\right]}{12}=\frac{3y+y(y+1)}{12}=\frac{y^2+4y}{12}$$

2.yol:

$$F(y|X\leq 2)=\sum_{z=-\infty}^{y}p(z|X\leq 2)=\sum_{z=1}^{y}\frac{2z+3}{12}=\frac{1}{12}\left(2\frac{y(y+1)}{2}+3y\right)=\frac{y^2+4y}{12}$$

$$\begin{aligned}F(y|X\leq 2)&=\frac{y^2+4y}{12}, &&y=1,2\\ &=0, &&y<1\\ &=1, &&y\geq 2\end{aligned}$$

$P(Y>1|X\leq 2)=p(2|X\leq 2)=\frac{7}{12}$

$P(Y<4|X\leq 2)$=1

$P(Y<1|X\leq 2)=0$

2. X ve Y sürekli raslantı değişkeninin bileşik olasılık yoğunluk fonksiyonu aşağıda verilmiştir:

$$\begin{aligned}f_{X,Y}(x,y)&=\frac{x+y}{3}, &&0<x<1\,,0<y<2\\ &=0, &&\text{ö}.d.\end{aligned}$$

a) $f(x|y)=?$ $f(y|x)=?$

$$f_X(x)=\int_{Ry}f(x,y)dy=\int_0^2\left(\frac{x+y}{3}\right)dy=\frac{1}{3}\left(xy+\frac{y^2}{2}\bigg|_0^2\right)=\frac{(2x+2)}{3}$$

$$\begin{aligned}f_X(x)&=\frac{(2x+2)}{3}, &&0<x<1\\ &=0, &&\text{ö}.d\end{aligned}$$

$$f_Y(y) = \int_{Rx} f(x,y)dx = \int_0^1 \left(\frac{x+y}{3}\right)dx = \frac{1}{3}\left(\frac{x^2}{2} + xy\Big|_0^1\right) = \frac{(2y+1)}{6}$$

$$\begin{aligned} f_Y(y) &= \tfrac{(2y+1)}{6} \quad , \quad 0 < y < 2 \\ &= 0 \quad , \quad \text{ö}.d \end{aligned}$$

$$f(x|y) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{x+y}{3}}{\frac{(2y+1)}{6}} = \frac{2(x+y)}{(2y+1)}$$

$$\begin{aligned} f(x|y) &= \frac{2(x+y)}{(2y+1)} \quad , \quad 0 < x < 1, \quad (0 < y < 2) \\ &= 0 \quad , \quad \text{ö}.d \end{aligned}$$

$$f(y|x) = \frac{f(x,y)}{f_Y(x)} = \frac{\frac{x+y}{3}}{\frac{(2x+2)}{3}} = \frac{(x+y)}{(2x+2)}$$

$$\begin{aligned} f(y|x) &= \frac{(x+y)}{(2x+2)} \quad , \quad 0 < y < 2, \quad (0 < x < 1) \\ &= 0 \quad , \quad \text{ö}.d \end{aligned}$$

b) $E(2X|Y=1) =?$ $E(3Y|X=0.5) =?$

$$E(2X|Y=1) = \int_0^1 2xf(x|Y=1)dx = \int_0^1 2x\frac{2(x+1)}{(2+1)}dx = \frac{4}{3}\int_0^1 x(x+1)dx$$
$$= \frac{4}{3}\left(\frac{x^3}{3} + \frac{x^2}{2}\Big|_0^1\right) = \frac{4}{3}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{10}{9}$$

$$E(3Y|X=0.5) = \int_0^2 3yf(y|X=0.5)dy = \int_0^2 3y\frac{(0.5+y)}{(1+2)}dy = \int_0^2 y(0.5+y)dy$$
$$= \left(\frac{y^2}{4} + \frac{y^3}{3}\Big|_0^2\right) = \left(\frac{4}{4} + \frac{8}{3}\right) = \frac{44}{12} = \frac{11}{3}$$

c) $F(x|y) =?$ $F(y|x) =?$

$$F(x|y) = \int_0^x f(t|y)dt = \int_0^x \frac{2(t+y)}{(2y+1)}dt = \frac{2}{2y+1}\int_0^x (t+y)dt$$

$$= \frac{2}{2y+1}\left(\frac{t^2}{2} + yt\Big|_0^x\right) = \frac{2}{2y+1}\left(\frac{x^2}{2} + xy\right) = \frac{x^2+2xy}{1+2y}$$

$$\begin{aligned} F(x|y) \quad &= \frac{x^2+2xy}{1+2y}, \quad 0 < x < 1, \ \ (0 < y < 2) \\ &= 0, \qquad\qquad x \le 0 \\ &= 1, \qquad\qquad x \ge 1 \end{aligned}$$

$$F(y|x) = \int_0^y f(z|x)dz = \int_0^y \frac{x+z}{(2x+2)}dz = \frac{1}{2x+2}\int_0^y (x+z)dz$$

$$= = \frac{1}{2x+2}\left(xz + \frac{z^2}{2}\Big|_0^y\right) = \frac{1}{2x+2}\left(xy + \frac{y^2}{2}\right) = \frac{y^2+2xy}{4x+4}$$

$$\begin{aligned} F(y|x) \quad &= \frac{y^2+2xy}{4x+4}, \quad 0 < y < 2, \ \ (0 < x < 1) \\ &= 0, \qquad\qquad y \le 0 \\ &= 1, \qquad\qquad y \ge 2 \end{aligned}$$

d) $P(X \le 0.5|Y=1) = ?$ $\qquad P(0.6 < Y < 0.5|X=0.2) = ?$

$$P(X \le 0.5|Y=1) = F(0.5|Y=1) = \frac{0.5^2 + 2\times 0.5\times 1}{1+2\times 1} = \frac{125}{300} = \frac{5}{12}$$

ya da

$$P(X \le 0.5|Y=1) = \int_0^{0.5} f(x|Y=1)\,dx = \int_0^{0.5}\frac{2(x+y)}{(2+1)}dx = \frac{5}{12}$$

$$P(0.6 < Y < 1.5|X=0.2) = F(1.5|X=0.2) - F(0.6|X=0.2)$$

$$= \frac{1.5^2 + 2\times 0.2\times 1.5}{4\times 0.2+4} - \frac{0.6^2+2\times 0.2\times 0.6}{4\times 0.2+4} = 0.46875$$

$$P(0.6 < Y < 1.5|X = 0.2) = \int_{0.6}^{1.5} f(y|X = 0.2)\,dy = \int_{0.6}^{1.5} \frac{(0.2 + y)}{(2 * 0.2 + 2)}\,dy$$
$$= 0.46875$$

e) $f(y|X \leq 0.5) =?$ $\quad\quad$ $F(y|X \leq 0.5) =?$

$$\begin{aligned} F(x,y) &= \frac{x^2y + xy^2}{6}, \quad 0 < x < 1, 0 < y < 2 \\ &= 0, \quad x \leq 0, y \leq 0 \\ &= 1, \quad x \geq 1, y \geq 2 \end{aligned}$$

$$\begin{aligned} F_X(x) &= \tfrac{x^2+2x}{3}, \quad 0 < x < 1 \\ &= 0, \quad x \leq 0 \\ &= 1, \quad x \geq 1 \end{aligned}$$

$$F(y|X \leq 0.5) = P(Y \leq y|X \leq 0.5) = \frac{P(X \leq 0.5, Y \leq y)}{P(X \leq 0.5)} = \frac{F(0.5, y)}{F_X(0.5)}$$
$$= \frac{\frac{0.5^2y + 0.5y^2}{6}}{\frac{0.5^2 + 2 \times 0.5}{3}} = \frac{y + 2y^2}{10}$$

$$\begin{aligned} F(y|X \leq 0.5) &= \tfrac{y+2y^2}{10} \quad , \quad 0 < y < 2 \\ &= 0 \quad , \quad y \leq 0 \\ &= 1 \quad , \quad y \geq 2 \end{aligned}$$

$f(y|X \leq 0.5)$ koşullu olasılık yoğunluk fonksiyonunu bulalım:

1.yol:

$$f(y|X \leq 0.5) = \frac{d}{dy}F(y|X \leq 0.5) = \frac{d}{dy}\left(\frac{y + 2y^2}{10}\right) = \frac{1 + 4y}{10}$$

$$\begin{aligned} f(y|X \leq 0.5) &= \frac{1 + 4y}{10} \quad , \quad 0 < y < 2 \\ &= 0 \quad , \quad \text{ö.}d. \end{aligned}$$

2.yol:

$$f(y|X \leq 0.5) = \frac{P(X \leq 0.5, Y = y)}{P(X \leq 0.5)} == \frac{\int_0^{0.5} f(x,y)dx}{\int_0^{0.5} f_X(x)} = \frac{\int_0^{0.5} \left(\frac{x+y}{3}\right) dx}{\int_0^{0.5} \frac{(2x+2)}{3} dx} =$$

$$= \frac{\frac{1}{3}\left(\frac{x^2}{2} + xy\Big|_0^{0.5}\right)}{\frac{2}{3}\left(\frac{x^2}{2} + x\Big|_0^{0.5}\right)} = \frac{\frac{1}{8} + \frac{y}{2}}{2\left(\frac{1}{8} + \frac{1}{2}\right)} = \frac{1+4y}{10}$$

$$\begin{aligned} f(y|X \leq 0.5) &= \frac{1+4y}{10} \quad , \quad 0 < y < 2 \\ &= 0 \quad , \quad ö.d. \end{aligned}$$

f) $E(7Y + 8|X \leq 0.5) =?$

$$\begin{aligned} E(7Y + 8|X \leq 0.5) &= 7E(Y|X \leq 0.5) + 8 \\ &= 7\int_0^2 yf(y|X \leq 0.5)dy + 8 \\ &= 7\int_0^2 y\left(\frac{1+4y}{10}\right)dy + 8 \\ &= \frac{7}{10}\left(\frac{y^2}{2} + \frac{4y^3}{3}\Big|_0^2\right) + 8 \\ &= \frac{7}{10}\left(2 + \frac{32}{3}\right) + 8 = \frac{253}{15} \end{aligned}$$

g) $f(x|Y \geq 1) =?$ $\quad\quad F(x|Y \geq 1) =?$

$$\begin{aligned} F_Y(y) &= \frac{y+y^2}{6}, \quad 0 < y < 2 \\ &= 0, \quad y \leq 0 \\ &= 1, \quad x \geq 2 \end{aligned}$$

$$F(x|Y \geq 1) = P(X \leq x|Y \geq 1) = \frac{P(X \leq x, Y \geq 1)}{P(Y \geq 1)} = \frac{F(x,2) - F(x,1)}{1 - F_Y(1)}$$

$$= \frac{\frac{x^2.2 + x.2^2}{6} - \frac{x^2.1 + x.1^2}{6}}{1 - \frac{1+1^2}{6}} = \frac{x^2 + 3x}{4}$$

$$\begin{aligned} F(x|Y \geq 1) &= \frac{x^2+3x}{4} \quad , \quad 0 < x < 1 \\ &= 0 \quad , \quad x \leq 0 \\ &= 1 \quad , \quad x \geq 1 \end{aligned}$$

$f(x|Y \geq 1)$ koşullu olasılık yoğunluk fonksiyonunu bulalım:

1.Yol:

$$f(x|Y \geq 1) = \frac{d}{dx}F(x|Y \geq 1) = \frac{d}{dx}\left(\frac{x^2+3x}{4}\right) = \frac{2x+3}{4}$$

$$\begin{aligned} f(x|Y \geq 1) &= \frac{2x+3}{4} \quad , \quad 0 < x < 1 \\ &= 0 \quad , \quad \text{ö}.d. \end{aligned}$$

2.yol:

$$f(x|Y \geq 1) = \frac{P(X = x, Y \geq 1)}{P(\,Y \geq 1)} = \frac{\int_1^2 f(x,y)dy}{\int_1^2 f_Y(y)} = \frac{\int_1^2 \left(\frac{x+y}{3}\right)dy}{\int_1^2 \frac{(2y+1)}{6}dy} = \frac{\frac{1}{3}\left(xy + \frac{y^2}{2}\Big|_1^2\right)}{\frac{1}{6}(y^2 + y|_1^2)}$$

$$= \frac{(2x + 2 - x - 0.5)}{\frac{1}{2}(4 + 2 - 1 - 1)} = \frac{2x+3}{4}$$

$$\begin{aligned} f(x|Y \geq 1) &= \frac{2x+3}{4} \quad , \quad 0 < x < 1 \\ &= 0 \quad , \quad \text{ö}.d. \end{aligned}$$

h) $P(0.4 < X < 0.9|Y \geq 1) =?$ $P(X < 2|Y \geq 1) =?$ $P(X > 4|Y \geq 1) =?$

$$\begin{aligned} P(0.4 < X < 0.9|Y \geq 1) &= F(0.9|Y \geq 1) - F(0.4|Y \geq 1) \\ &= \frac{0.9^2 + 3 \times 0.9}{4} - \frac{0.4^2 + 3 \times 0.4}{4} \\ &= \frac{3.51 - 1.36}{4} = \frac{1.88}{4} = 0.47 \end{aligned}$$

$P(X < 2|Y \geq 1) = 1$

$P(X > 4|Y \geq 1) = 0$

i) $V(4X + 3|Y \geq 1) =?$

$$V(4X + 3|Y \geq 1) = 16V(X|Y \geq 1) = 16\left[E(X^2|Y \geq 1) - \left(E(X|Y \geq 1)\right)^2\right]$$

$$E(X|Y \geq 1) = \int_0^1 x f(x|Y \geq 1)dx = \int_0^1 x\left(\frac{2x+3}{4}\right)dx$$
$$= \frac{1}{4}\left(\frac{2x^3}{3} + \frac{3x^2}{2}\Big|_0^1\right) = \frac{1}{4}\left(\frac{2}{3} + \frac{3}{2}\right) = \frac{13}{24}$$

$$E(X^2|Y \geq 1) = \int_0^1 x^2 f(x|Y \geq 1)dx = \int_0^1 x^2\left(\frac{2x+3}{4}\right)dx$$
$$= \frac{1}{4}\left(\frac{2x^4}{4} + \frac{3x^3}{3}\Big|_0^1\right) = \frac{1}{4}\left(\frac{2}{4} + \frac{3}{3}\right) = \frac{3}{8}$$

3. X ve Y sürekli raslantı değişkeninin bileşik olasılık yoğunluk fonksiyonu aşağıda verilmiştir:

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{xy}{9}, \quad 0 \leq x \leq 3\,, 0 \leq y \leq 2 \\ &= 0, \quad \text{ö}.d. \end{aligned}$$

a) X ve Y raslantı değişkenleri bağımsız mıdır?

$$f(x,y) = f_X(x)f_Y(y) \qquad F(x,y) = F_X(x)F_Y(y)$$

$$f_X(x) = \int_0^2 f(x,y)dy = \int_0^2 \frac{xy}{9}dy = \frac{1}{9}\left(\frac{xy^2}{2}\Big|_0^2\right) = \frac{1}{18}(4x) = \frac{2x}{9}$$

$$\begin{aligned} f_X(x) &= \frac{2x}{9} \quad , \quad 0 < x < 3 \\ &= 0 \quad , \quad \text{ö}.d. \end{aligned}$$

$$f_Y(y) = \int_0^3 f(x,y)dx = \int_0^3 \frac{xy}{9}dx = \frac{1}{9}\left(\frac{yx^2}{2}\Big|_0^3\right) = \frac{1}{18}(9y) = \frac{y}{2}$$

$$\begin{aligned} f_Y(y) &= \frac{y}{2} \quad , \quad 0 < y < 2 \\ &= 0 \quad , \quad \text{ö}.d. \end{aligned}$$

$$\begin{aligned} f(x,y) &= f_X(x)f_Y(y) \\ \frac{xy}{9} &= \left(\frac{2x}{9}\right)\left(\frac{y}{2}\right) \end{aligned}$$

Eşitlik sağlandığı için X ve Y bağımsız raslantı değişkenlerdir.

b) $f(x|Y \leq 1) = ?$

X ve Y bağımsız raslantı değişkenleri olduğu için, $f(x|Y \leq 1) = f_X(x)$ olur.

c) $E(7X|Y \leq 1.5) = ?$

X ve Y bağımsız raslantı değişkenleri olduğu için, koşul kalkar:

$$E(7X|Y \leq 1.5) = 7E(X) = 7\int_0^3 x f_X(x)dx = 7\int_0^3 x\frac{2x}{9}dx = 7\left(\frac{2x^3}{27}\bigg|_0^3\right) = 14$$

4. X ve Y kesikli raslantı değişkeninin bileşik olasılık fonksiyon aşağıda verilmiştir:

$$\begin{aligned} p(x,y) &= \frac{1}{25}, \quad x,y = 1,2,\dots,5 \\ &= 0, \quad\quad ö.d. \end{aligned}$$

a) X ve Y bağımsız raslantı değişkenleri midir?

$$p(x,y) = p_X(x)p_Y(y) \qquad\qquad F(x,y) = F_X(x)F_Y(y)$$

$$p_X(x) = \sum_{y=1}^{5} p(x,y) = \sum_{y=1}^{5} \frac{1}{25} = \frac{5}{25} = \frac{1}{5}$$

$$\begin{aligned} p_X(x) &= \frac{1}{5}, \quad x = 1,2,\dots,5 \\ &= 0, \quad\quad ö.d. \end{aligned}$$

$$p_Y(y) = \sum_{x=1}^{5} p(x,y) = \sum_{x=1}^{5} \frac{1}{25} = \frac{5}{25} = \frac{1}{5}$$

$$\begin{aligned} p_Y(y) &= \frac{1}{5}, \quad y = 1,2,\dots,5 \\ &= 0, \quad\quad ö.d. \end{aligned}$$

$$p(x,y) = p_X(x)p_Y(y)$$
$$\frac{1}{25} = \frac{1}{5} \times \frac{1}{5}$$

Eşitlik sağlandığı için X ve Y bağımsız raslantı değişkenlerdir.

b) $P(X < 3|Y \geq 2) =?$

X ve Y bağımsız raslantı değişkenleri oldukları için, aşağıdaki gibi hesaplanır:

$$P(X < 3|Y \geq 2) = P(X < 3) = P(X = 1) + P(X = 2) = \frac{2}{5}$$

c) $E(Y|X < 4) =?$

X ve Y bağımsız raslantı değişkenleri oldukları için, $E(Y|X < 4) = E(Y)$ olur.

$$E(Y|X < 4) = E(Y) = \sum_{y=1}^{5} y\frac{1}{5} = \frac{1}{5}\left(\frac{5.6}{2}\right) = 3$$

5. X ve Y sürekli raslantı değişkeninin bileşik olasılık yoğunluk fonksiyonu aşağıda verilmiştir:

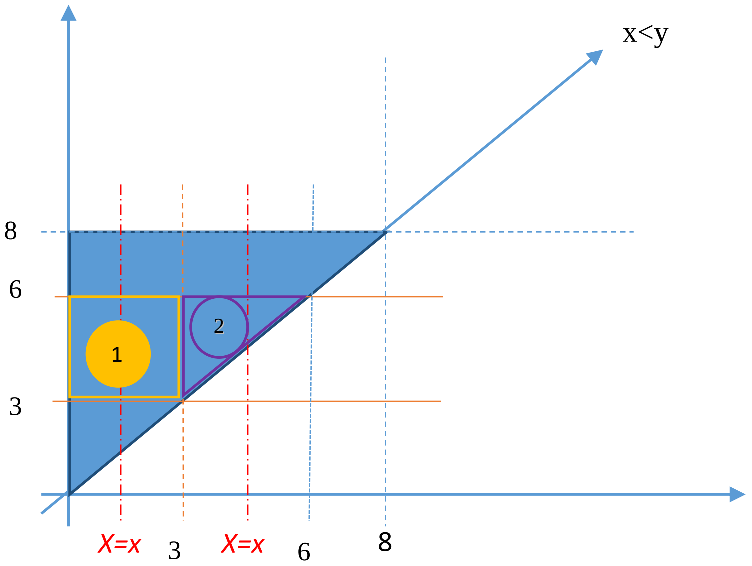
$$\begin{aligned} f_{X,Y}(x,y) &= \frac{xy}{512}, \quad 0 < x < y < 8 \\ &= 0, \quad \text{ö}.d. \end{aligned}$$

$f(x|3 \leq Y \leq 6) =?$

$$f_Y(y) = \int_{Rx} f(x,y)dx = \int_0^y f(x,y)dx = \int_0^y \frac{xy}{512}dy = \frac{x^2y}{1024}\Bigg|_0^y = \frac{y^3}{1024}$$

$$\begin{aligned} f_Y(y) &= \frac{y^3}{1024} \quad , \quad 0 < y < 8 \\ &= 0 \quad , \quad \text{ö}.d. \end{aligned}$$

$$f(x|3 \leq Y \leq 6) = \frac{P(X = x, 3 \leq Y \leq 6)}{P(3 \leq Y \leq 6)}$$



$$0 < x < 3 \; için \qquad P(X = x, 3 \leq Y \leq 6) = \int_3^6 \frac{xy}{512} dy = \frac{xy^2}{1024}\Big|_3^6 = \frac{27x}{1024}$$

$$3 < x < 6 \; için \qquad P(X = x, 3 \leq Y \leq 6) = \int_x^6 \frac{xy}{512} dy = \frac{xy^2}{1024}\Big|_x^6 = \frac{36x - x^3}{1024}$$

$$P(3 \leq Y \leq 6) = \int_3^6 f_Y(y) dy = \int_3^6 \frac{y^3}{1024} dy = \frac{y^4}{4096}\Big|_3^6 = \frac{6^4 - 3^4}{4096} = \frac{1296 - 81}{4096} = \frac{1215}{4096}$$

$$0 < x < 3 \; için \qquad f(x|3 \leq Y \leq 6) = \frac{P(X=x, 3\leq Y \leq 6)}{P(3 \leq Y \leq 6)} = \frac{\frac{27x}{1024}}{\frac{1215}{4096}} = \frac{108x}{1215}$$

$$3 < x < 6 \; için \qquad f(x|3 \leq Y \leq 6) = \frac{P(X=x, 3\leq Y \leq 6)}{P(3 \leq Y \leq 6)} = \frac{\frac{36x - x^3}{1024}}{\frac{1215}{4096}} = \frac{4(36x - x^3)}{1215}$$

$$\begin{aligned} f(x|3 \leq Y \leq 6) &= \frac{108x}{1215} \quad , \quad 0 < x < 3 \\ &= \frac{4(36x - x^3)}{1215} \quad , \quad 3 < x < 6 \\ &= 0 \quad , \quad ö.d. \end{aligned}$$

6. X ve Y sürekli raslantı değişkeninin bileşik olasılık yoğunluk fonksiyonu aşağıda verilmiştir:

$$\begin{aligned} f_{X,Y}(x,y) &= 2(x + y - 2xy), \quad 0 \leq x, y \leq 1 \\ &= 0, \quad ö.d. \end{aligned}$$

a) $f(x|0.2 \leq Y \leq 0.6) = ?$ $\qquad F(x|0.2 \leq Y \leq 0.6) = ?$

$$
\begin{aligned}
f_Y(y) &= \int_0^1 f(x,y)dx = \int_0^1 2(x+y-2xy)dx \\
&= 2\left(\frac{x^2}{2} + yx - 2y\frac{x^2}{2}\Big|_0^1\right) \\
&= 2\left(\frac{1}{2} + y - y\right) = 1
\end{aligned}
$$

$$
\begin{aligned}
f(x|0.2 \le Y \le 0.6) &= \frac{P(X=x, 0.2 \le Y \le 0.6)}{P(0.2 \le Y \le 0.6)} = \frac{\int_{0.2}^{0.6} f(x,y)dy}{\int_{0.2}^{0.6} f_Y(y)dy} \\
&= \frac{\int_{0.2}^{0.6} 2(x+y-2xy)dy}{\int_{0.2}^{0.6} 1dy} \\
&= \frac{2\left(xy + \frac{y^2}{2} - 2x\frac{y^2}{2}\Big|_{0.2}^{0.6}\right)}{y|_{0.2}^{0.6}} \\
&= \frac{(2xy + y^2 - 2xy^2)|_{0.2}^{0.6}}{0.6 - 0.2} \\
&= \frac{1.2x + 0.36 - 0.72x - 0.4x - 0.04 + 0.08x}{0.4} \\
&= \frac{0.16x + 0.32}{0.4} = \frac{4x+8}{10} = \frac{2x+4}{5}
\end{aligned}
$$

$$
\begin{aligned}
f(x|0.2 \le Y \le 0.6) &= \frac{2x+4}{5}, \quad 0 \le x \le 1 \\
&= 0, \qquad\quad \text{ö}.d.
\end{aligned}
$$

$$
\begin{aligned}
F(x|0.2 \le Y \le 0.6) &= \int_0^x f(t|0.2 \le Y \le 0.6)dt \\
&= \int_0^x \frac{2t+4}{5}dt = \frac{2}{5}\left(\frac{t^2}{2} + 2t\Big|_0^x\right) \\
&= \frac{2}{5}\left(\frac{x^2}{2} + 2x\right) = \frac{x^2+4x}{5}
\end{aligned}
$$

$$
\begin{aligned}
F(x|0.2 \le Y \le 0.6) &= \frac{x^2+4x}{5}, \quad 0 \le x \le 1 \\
&= 0, \qquad\quad x < 0 \\
&= 1, \qquad\quad x \ge 1
\end{aligned}
$$

b) $P(X \geq 0.5|0.2 \leq Y \leq 0.6) = ?$

$$\begin{aligned} P(X \geq 0.5|0.2 \leq Y \leq 0.6) &= 1 - P(X < 0.5|0.2 \leq Y \leq 0.6) \\ &= 1 - P(X \leq 0.5|0.2 \leq Y \leq 0.6) \\ &= 1 - F(0.5|0.2 \leq Y \leq 0.6) \\ &= 1 - \frac{0.5^2 + 4 \times 0.5}{5} \\ &= 1 - \frac{0.25 + 2}{5} = \frac{2.75}{5} = \frac{11}{20} \end{aligned}$$

c) $V(5X + 9|0.2 \leq Y \leq 0.6) = ?$

$$V(X|0.2 \leq Y \leq 0.6) = E(X^2|0.2 \leq Y \leq 0.6) - [E(X|0.2 \leq Y \leq 0.6)]^2$$

$$\begin{aligned} E(X^2|0.2 \leq Y \leq 0.6) &= \int_0^1 x^2 f(x|0.2 \leq Y \leq 0.6)dx \\ &= \int_0^1 x^2 \left(\frac{2x + 4}{5}\right) dx \\ &= \frac{2}{5}\left(\frac{x^4}{4} + \frac{2x^3}{3}\bigg|_0^1\right) \\ &= \frac{2}{5}\left(\frac{1}{4} + \frac{2}{3}\bigg|_0^1\right) = \frac{11}{30} \end{aligned}$$

$$\begin{aligned} E(X|0.2 \leq Y \leq 0.6) &= \int_0^1 x f(x|0.2 \leq Y \leq 0.6)dx \\ &= \int_0^1 x \left(\frac{2x + 4}{5}\right) dx \\ &= \frac{2}{5}\left(\frac{x^3}{3} + \frac{2x^2}{2}\bigg|_0^1\right) \\ &= \frac{2}{5}\left(\frac{1}{3} + \frac{2}{2}\bigg|_0^1\right) = \frac{8}{15} \end{aligned}$$

$$\begin{aligned} V(X|0.2 \leq Y \leq 0.6) &= E(X^2|0.2 \leq Y \leq 0.6) - [E(X|0.2 \leq Y \leq 0.6)]^2 \\ &= \frac{11}{30} - \left(\frac{8}{15}\right)^2 = \frac{11}{30} - \frac{64}{225} = \frac{165 - 128}{450} = \frac{37}{450} \end{aligned}$$

$$V(5X+9|0.2 \leq Y \leq 0.6) = 25V(X|0.2 \leq Y \leq 0.6) = 25 \times \frac{37}{450} = \frac{37}{18}$$