MAT 122 → 5. HAFTA UYGULAMALARI (Has-olmayan integraller)

1) Aşağıdaki integrallerin yakınsak veya ıraksaklıklarını inceleyiniz.

a) $\int_0^\infty \frac{dx}{(x+2)^{3/2}}$, b) $\int_3^\infty \frac{\ln x}{x}\,dx$, c) $\int \frac{dx}{2^x}$

Çözüm a) $\int_0^\infty \frac{dx}{(x+2)^{3/2}} = \lim_{R\to\infty} \int_0^R \frac{dx}{(x+2)^{3/2}} = \lim_{R\to\infty} \left(-\frac{2}{\sqrt{x+2}} \Big|_0^R \right)$

$= \lim_{R\to\infty} \left(-\frac{2}{\sqrt{R+2}} + 1 \right) = +1$ (yakınsak)

b) $\int_3^\infty \frac{\ln x}{x}\,dx = \lim_{R\to\infty} \int_3^R \frac{\ln x}{x}\,dx = \lim_{R\to\infty} \left(\frac{\ln^2 x}{2} \Big|_3^R \right)$

$= \lim_{R\to\infty} \left(\frac{\ln^2 R}{2} - \frac{\ln^2 3}{2} \right) = \infty - \frac{\ln^2 3}{2} = \infty \Rightarrow$ ıraksak.

c) $\int_0^\infty \frac{dx}{2^x} = \lim_{R\to\infty} \int_0^R (1/2)^x dx = \lim_{R\to\infty} \left(\frac{(1/2)^x}{\ln(1/2)} \Big|_0^R \right)$

$= \lim_{R\to\infty} \left(-\frac{1}{2^R \cdot \ln 2} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}$ yakınsak.

2) a) $\int_0^3 \frac{dx}{x^2-6x+8}$, b) $\int_0^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$, c) $\int_0^1 \frac{x^5}{\sqrt{1-x^2}}\,dx$

Çözüm: a) $\int_0^3 \frac{dx}{x^2-6x+8} \overset{\text{II. tip}}{=} \int_0^2 \frac{dx}{x^2-6x+8} + \int_2^3 \frac{dx}{x^2-6x+8}$

$f(x) = \frac{1}{x^2-6x+8} \rightsquigarrow f(2) = \frac{1}{0} = \infty$

$= \lim_{b\to 2^-} \int_0^b \frac{dx}{x^2-6x+8} + \lim_{a\to 2^+} \int_a^3 \frac{dx}{x^2-6x+8}$

$= \lim_{b\to 2^-} \ln\sqrt{\left|\frac{x-4}{x-2}\right|}\Big|_0^b + \lim_{a\to 2^+} \ln\sqrt{\left|\frac{x-4}{x-2}\right|}\Big|_a^3$

$= \lim_{b\to 2^-} \left(\ln\sqrt{\left|\frac{b-4}{b-2}\right|} - \ln\sqrt{2} \right) + \lim_{a\to 2^+} \left[\ln\sqrt{|-1|} - \ln\sqrt{\left|\frac{a-4}{a-2}\right|} \right]$

$= \lim_{b\to 2^-} \frac{1}{2}\ln\left|\frac{b-4}{b-2}\right| - \frac{1}{2}\ln 2 - \lim_{a\to 2^+} \frac{1}{2}\ln\left|\frac{a-4}{a-2}\right| \to$ ıraksak.

b) $\int_0^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx \overset{\text{II. tip}}{=} \lim_{a\to 0^+} \int_a^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \left(2\sin\sqrt{x} \Big|_a^{\pi^2/4} \right)$

$$= \lim_{a\to 0^+} \left(2\sin(\pi/2) - 2\sin\sqrt{a}\right) = 2, \text{ yakınsak.}$$

$x=\sin t \to dx = \cos t\,dt$

c) $$\int_0^1 \frac{x^5}{\sqrt{1-x^2}}\,dx \overset{\text{II. tip}}{=} \lim_{b\to 1^-}\int_0^b \frac{x^5}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-}\int_0^{\sin^{-1}(b)} \frac{\sin^5 t\cdot \cos t}{\cos t}\,dt$$

$$= \lim_{b\to 1^-}\int_0^{\sin^{-1}(b)} (\sin^2 t)^2\cdot \sin t\,dt = \lim_{b\to 1^-}\int_0^{\sin^{-1}(b)} (1-\cos^2 t)^2\cdot \sin t\,dt$$

$y=\cos t \to dy = -\sin t\,dt$

$$= \lim_{b\to 1^-}\left(-\cos t + \frac{2}{3}\cos^3 t - \frac{1}{5}\cos^5 t\right)\Big|_0^{\sin^{-1}(b)}$$

$$= \lim_{b\to 1^-}\left(-\cos(\sin^{-1} b) + \frac{2}{3}\cos^3(\sin^{-1}(b)) - \frac{1}{5}\cos^5(\sin^{-1}(b))\right) + \frac{8}{15}$$

$$= \left(-\cos(\pi/2) + \frac{2}{3}\cos^3(\pi/2) - \frac{1}{5}\cos^5(\pi/2)\right) + 8/15 = 8/15 \text{ yak.}$$

3) $$\int_{-\infty}^{2} \frac{4x^2-3}{5\cdot\sqrt{x^2}}\,dx \left(\longrightarrow \text{Hem I. hem de II. tip}\right) = \int_{-\infty}^{-1}\frac{4x^2-3}{5\cdot\sqrt{x^2}}\,dx + \int_{-1}^{0}\frac{4x^2-3}{5\cdot|x|}\,dx + \int_{0}^{2}\frac{4x^2-3}{5\cdot|x|}\,dx$$

$$= \lim_{B\to-\infty}\int_B^{-1}\frac{4x^2-3}{5\cdot|x|}\,dx + \lim_{b\to 0^-}\int_{-1}^{b}\frac{4x^2-3}{5\,|x|}\,dx + \lim_{a\to 0^+}\int_a^2\frac{4x^2-3}{5\,|x|}\,dx$$

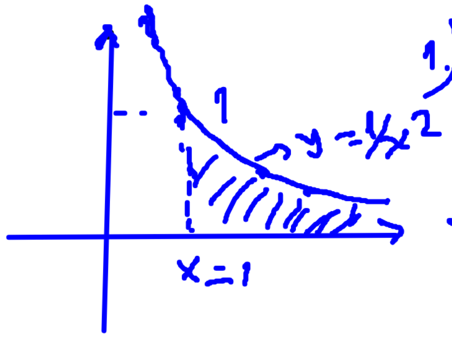
$$= \lim_{B\to-\infty}\frac{1}{5}\int_B^{-1}\frac{4x^2-3}{\underbrace{-x}_{=|x|}}\,dx + \lim_{b\to 0^-}\frac{1}{5}\int_{-1}^{b}\frac{4x^2-3}{-x}\,dx + \lim_{a\to 0^+}\frac{1}{5}\int_a^2\frac{4x^2-3}{x}\,dx$$

$$= \frac{1}{5}\left[\lim_{B\to-\infty}\int_B^{-1}\left(-4x+\frac{3}{x}\right)dx + \lim_{b\to 0^-}\int_{-1}^{b}\left(-4x+3/x\right)dx + \lim_{a\to 0^+}\int_a^2\left(4x-3/x\right)dx\right]$$

$$= \frac{1}{5}\left[\lim_{B\to-\infty}\left(-2x^2+3\ln|x|\Big|_B^{-1}\right) + \lim_{b\to 0^-}\left(-2x^2+3\ln|x|\Big|_{-1}^{b}\right) + \lim_{a\to 0^+}\left(2x^2-3\ln x\Big|_a^2\right)\right]$$

$= - - - .$

4) $y=1/x^2$ eğrisi, $x=1$ doğrusu ve x-ekseniyle belirlenen bölgenin alanını (varsa) bulunuz.

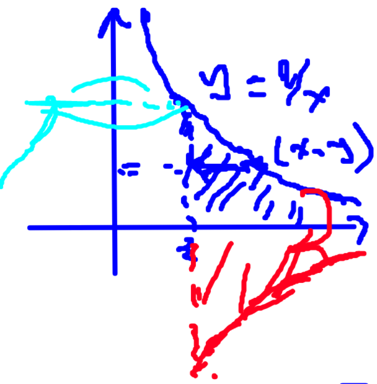


1) $$\text{Alan} = \int_1^{\infty}\frac{dx}{x^2} = \lim_{R\to\infty}\left(-\frac{1}{x}\Big|_1^R\right) = \lim_{R\to\infty}\left(-\frac{1}{R}+1\right) = 1.$$

b) $$= \int_0^1\left(\frac{1}{\sqrt{y}}-1\right)dy = \lim_{a\to 0^+}\int_a^1\left(\frac{1}{\sqrt{y}}-1\right)dy = - - -$$

5) $y=1/x$, $x=1$ ve x-ekseniyle belirlenen bölgenin
a) alanını, b) x-ekseni etrafında döndü... cismin hacmi
c) y-ekseni etrafında döndü...

Çözüm: a)



$$\text{Alan}=\int_1^\infty \frac{dx}{x}=\lim_{R\to\infty}\left(\ln x\Big|_1^R\right)=\lim_{R\to\infty}(\ln R-0)$$

$$=\lim_{R\to\infty}\ln R=+\infty \text{ ıraksak, yok}$$

b) $$V_{ox}\overset{\text{Disk}}{=}\pi\int_1^\infty y^2dx=\pi\int_1^\infty\frac{1}{x^2}dx$$

$$=\pi.\lim_{R\to\infty}\int_1^R\frac{1}{x^2}dx=\pi\lim_{R\to\infty}\left(-\frac{1}{x}\Big|_1^R\right)=\pi \text{ (yakınsak)}$$

c) $$V_{oy}\overset{\text{Kabuk}}{=}2\pi\int_1^\infty x.y\,dx=2\pi\int_1^\infty(x-1)\frac{1}{x}dx=2\pi\lim_{R\to\infty}\int_1^R\left(1-\frac{1}{x}\right)dx$$

$$=2\pi\lim_{R\to\infty}\left(x-\ln x\Big|_1^R\right)=2\pi\lim_{R\to\infty}(R-\ln R-1)=\text{ıraksak}$$

Disk: $V_{oy}=\pi\int_0^1\left(\frac{1}{y^2}-1\right)dy=\dots$

6) $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}\Rightarrow\int_0^\infty x^4.e^{-x^2}dx=?$

$$\int_0^\infty x^4.e^{-x^2}dx=\lim_{R\to\infty}\left[\int_0^R x^4e^{-x^2}dx\right]=\lim_{R\to\infty}\left[-\frac{x^3}{2}e^{-x^2}+\frac{3}{2}\int_0^R x^2e^{-x^2}dx\right]$$

$u=x^3\to du=3x^2dx$
$dv=x.e^{-x^2}dx\Rightarrow v=-\frac{1}{2}e^{-x^2}$

$u=x\to du=dx$
$dv=xe^{-x^2}dx\to v=-\frac{1}{2}e^{-x^2}$

$$=\lim_{R\to\infty}\left[-\frac{x^3}{2}e^{-x^2}-\frac{3}{4}xe^{-x^2}\Big|_0^R+\frac{3}{4}\int_0^R e^{-x^2}dx\right]$$

$$=\lim_{R\to\infty}\left[-\frac{R^3}{2}e^{-R^2}-\frac{3}{4}Re^{-R^2}+0\right]+\frac{3}{4}\lim_{R\to\infty}\int_0^R e^{-x^2}dx$$

$$=-\frac{1}{2}\lim_{R\to\infty}\frac{R^3}{e^{R^2}}-\frac{3}{4}\lim_{R\to\infty}\frac{R}{e^{R^2}}+\frac{3}{4}\int_0^\infty e^{-x^2}dx$$

$$=0+0+\frac{3}{4}\int_0^\infty e^{-x^2}dx=\frac{3}{4}.\frac{\sqrt{\pi}}{2}=\frac{3}{8}\sqrt{\pi}\text{ bulunur.}$$

7) $\int_0^\infty\frac{dx}{\sqrt{x+x^3}}$ integralinin yakınsaklık veya ıraksaklığını inceleyin

Çözüm: $$\int_0^\infty\frac{dx}{\sqrt{x+x^3}}\overset{I_1}{=}\boxed{\int_0^1\frac{dx}{\sqrt{x+x^3}}}+\boxed{\int_1^\infty\frac{dx}{\sqrt{x+x^3}}}\to I_2$$

II. tip   I. tip

Has olmayan integraller.

I_1 için; $\forall x \in (0,1]$ olmak üzere $0 \leq f(x) = \frac{1}{\sqrt{x+x^3}} \leq \frac{1}{\sqrt{x}}$ ve

$\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 \frac{dx}{x^{1/2}}$ olup, $\int_0^1 \frac{dx}{x^{1/2}}$, II. tip hasolmayan integrali p-testi

gereği ($p = 1/2 < 1 \Rightarrow$ yakınsak) yakınsaktır $\xrightarrow[\text{testi}]{\text{karşılaşt.}}$ $\int_0^1 \frac{dx}{\sqrt{x+x^3}}$ int. de yakınsak olur.

I_2 için: $\forall x \in [1,\infty)$ olmak üzere $f(x) = \frac{1}{\sqrt{x+x^3}} \leq \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$ dir,

ve de $\int_1^\infty \frac{dx}{x^{3/2}}$ I. tip Hasolmayan integrali, yine p-testi gereği ($p = 3/2 > 1 \Rightarrow$ yak.) yakınsak

olduğundan $\xrightarrow[\text{testi}]{\text{karş.}}$ $\int_1^\infty \frac{dx}{\sqrt{x+x^3}}$ de yakınsak olur. Bu durumda

hem I_1 ve hem de I_2 integralleri yakınsak olduğundan

$I_1 + I_2 = \int_0^\infty \frac{dx}{\sqrt{x+x^3}}$ int. de yakınsak olur.

8) $\int_0^1 \frac{dx}{x-\sin x}$ integralinin yakınsak veya ıraksaklığını inceleyiniz.

Çözüm. $x \in (0,1]$ için $f(x) = \frac{1}{x-\sin x}$, $f(0) = \frac{1}{0} = \infty$ olduğundan II. tip bir Has olmayan integraldir.

Her $x \in (0,1]$ için $\sin x \leq x$ olduğundan, $0 \leq x - \sin x$

$\Rightarrow 0 \leq \frac{1}{x-\sin x} = \frac{1}{x.\left(1-\frac{\sin x}{x}\right)}$ ve $0<x<1$ için $0 < 1 - \frac{\sin x}{x} < 1$ olduğundan, $\frac{1}{x} < \frac{1}{x\left(1-\frac{\sin x}{x}\right)} = \frac{1}{x-\sin x}$

elde edilir. Diğer yandan; $\int_0^1 \frac{1}{x}dx$, p testi ($p = 1 \leq 1$)

gereğince ıraksak olur $\xrightarrow[\text{karş. testi}]{}$ $\int_0^1 \frac{dx}{x-\sin x}$ de ıraksak olur.

II-yol: $\forall x \in (0,1]$ için $f(x) = \frac{1}{x-\sin x}$, $g(x) = \frac{1}{x^3} \geq 0$ ve

$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{x^3}{x-\sin x} = [0/0] \overset{\text{L'Hosp.}}{=} \lim_{x\to 0^+} \frac{3x^2}{1-\cos x} = [0/0] \overset{\text{L'Hosp.}}{=}$

$= \lim_{x\to 0^+} \frac{6x}{\sin x} = 6 \in (0,\infty)$ ve $\int_0^1 \frac{dx}{x^3}$ II. tip Hasolmayan int. (p-testinden) ıraksaktır. $\xrightarrow[\text{Karş. Testi}]{\text{Limit}}$ $\int_0^1 f(x)dx = \int_0^1 \frac{dx}{x-\sin x}$ de ıraksak olur.

9) $\int_2^\infty \frac{dx}{\ln x}$ in yakınsak veya ıraksak olup olmadığını belirleyiniz.

Çözüm: $\forall x \geq 2$ için $\ln x \leq x \Rightarrow \frac{1}{x} \leq \frac{1}{\ln x}$ dır.

Ayrıca $\int_2^\infty \frac{dx}{x} = \lim_{R\to\infty}\int_2^R \frac{dx}{x} = \lim_{R\to\infty}(\ln R - \ln 2) = \infty - 2$ ıraksak.

Karşılast. $\xrightarrow{}$ Testi $\int_2^\infty \frac{dx}{\ln x}$ int. de ıraksak olur.

10) $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$ in yakınsaklık veya ıraksaklığını inceleyiniz, yakınsaksa değerini bulunuz.

Çözüm: $\forall x\in(0,1]$ için $f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}}$, $g(x) = \frac{1}{\sqrt{x}} > 0$ dır ve

$\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}\frac{e^{-\sqrt{x}}/\sqrt{x}}{1/\sqrt{x}} = \lim_{x\to 0^+} e^{-\sqrt{x}} = e^0 = 1 \in (0,\infty)$ ve ak

$\int_0^1 \frac{1}{\sqrt{x}} = \int_0^1 \frac{dx}{x^{1/2}}$ has olmayan (II. tip) integrali p-testi ($p = 1/2 < 1$) gereğince yakınsaktır

Lim Karş. $\xrightarrow{}$ Testi $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$ int. de yakınsak olur. O zaman;

$$\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{a\to 0^+}\int_a^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$$

$x=a \Rightarrow \sqrt{a} = u$, $x=1 \Rightarrow u = 1$

$u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx$

$$= \lim_{a\to 0^+}\int_{\sqrt{a}}^1 2e^{-u}du = \lim_{a\to 0^+}\left(-2e^{-u}\Big|_{\sqrt{a}}^1\right) = \lim_{a\to 0^+}\left(-2e^{-1} + 2e^{-\sqrt{a}}\right)$$

$= -\frac{2}{e} + 2$ bulunur.

11) $\int_{-\infty}^{\infty} e^{-|x|}dx$ integralinin yak. veya ıraks. ... inceleyiniz

Çözüm: $\int_{-\infty}^{\infty} e^{-|x|}dx = \int_{-\infty}^{0} e^{-|x|}dx + \int_0^\infty e^{-|x|}dx$

$$= \lim_{A\to-\infty}\int_A^0 e^{-|x|}dx + \lim_{B\to\infty}\int_0^B e^{-|x|}dx = \lim_{A\to-\infty}\int_A^0 e^{x}dx + \lim_{B\to\infty}\int_0^B e^{-x}dx$$

$$= \lim_{A\to-\infty}(e^0 - e^A) + \lim_{B\to\infty}(-e^{-B} + e^{-0}) = 1 + 1 = 2 \text{ birim}$$

$\Rightarrow$ İntegral yakınsak ve değeri 2 dir.

12) $\int_1^\infty \frac{1-e^{-x}}{x}dx$ int. nin yakınsaklık veya ıraksaklığını inceleyeniz.

Çözüm: $\forall x\in[1,\infty)$ için $f(x)=\frac{1-e^{-x}}{x}$, $g(x)=1/x\geq 0$
ve $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\left(\frac{1-e^{-x}}{x}\right)\Big/\frac{1}{x}=\lim_{x\to\infty}(1-e^{-x})=1$
dır. Ayrıca $\int_1^\infty g(x)dx=\int_1^\infty\frac{1}{x}dx$, I. tip Has olmayan int.
p-testi ($p=1\leq 1$) gereği ıraksak olur. $\xRightarrow{\text{Lim Karş. Testi}}$
$\int_1^\infty\frac{1-e^{-x}}{x}dx$ integrali de ıraksak olur.

13) $\int_1^\infty\frac{x+3}{(x-1)(x^2+1)}dx=\underbrace{\int_1^2\frac{x+3}{(x-1)(x^2+1)}dx}_{I_1\to \text{II. tip}}+\underbrace{\int_2^\infty\frac{x+3}{(x-1)(x^2+1)}dx}_{I_2\to\text{I. tip dir.}}$

Öncelikle, $\int\frac{x+3}{(x-1)(x^2+1)}dx=\int\left(\frac{A}{x-1}+\frac{Bx+C}{x^2+1}\right)dx$

$x+3=A(x^2+1)+(Bx+C)(x-1)$
$=(A+B)x^2+(B-C)x+A-C$

$\Leftrightarrow \left.\begin{array}{l}A+B=0\\B-C=1\\A-C=3\end{array}\right\}\Rightarrow \begin{array}{l}A=2\\B=-2\\C=-1\end{array}$

$=2\ln|x-1|-\int\frac{2x}{x^2+1}dx-\int\frac{dx}{x^2+1}$
$=\ln\left|\frac{(x-1)^2}{x^2+1}\right|-\text{Arctan}x$ dır.

O zaman;

$I_1=\int_1^2\frac{x+3}{(x-1)(x^2+1)}dx=\lim_{b\to1^+}\int_b^2\frac{x+3}{(x-1)(x^2+1)}dx=\lim_{b\to1^+}\left[\ln\left(\frac{(x-1)^2}{x^2+1}\right)-\text{Arctan}x\right]_b^2$
$=\lim_{b\to1^+}\left[\left(\ln\frac{1}{5}-\text{Arctan}2\right)-\ln\left(\frac{(b-1)^2}{b^2+1}\right)+\text{Arctan}b\right]$
$=-\ln5-\text{Arctan}2-\underbrace{\lim_{b\to1}\ln\left(\frac{(b-1)^2}{b^2+1}\right)}_{+\infty}+\pi/4\Rightarrow$ ıraksak.

$\Rightarrow I_1$ ıraksak olduğundan, $I_1+I_2=\int_1^\infty\frac{x+3}{(x-1)(x^2+1)}dx$ ıraks. olur.

Buradaki $I_2=\int_2^\infty\frac{x+3}{(x-1)(x^2+1)}dx=\lim_{R\to\infty}\left[\ln\left(\frac{(x-1)^2}{x^2+1}\right)-\text{Arctan}x\right]_2^R$
$=\lim_{R\to\infty}\left[\left(\ln\left(\frac{(R-1)^2}{R^2+1}\right)-\text{Arctan}R\right)-\ln(1/5)+\text{Arctan}2\right]=\ln1-\pi/2-\ln5+\text{Arctan}2$
$=\text{Arctan}2-\ln5-\pi/2$ olup yakınsaktır.