Uygulama 1

1-) Aşağıdaki formülleri ters türevin tanımını kullanarak doğrulayın veya yanlışlayın.

a-) $\int \frac{1}{(1+x)^2}\,dx = \underbrace{\frac{-1}{1+x} + C}_{F}$ (Doğru)

herhangi bir f fonk için $\underbrace{F'(x) = f(x)}$ o. şekilde $F(x)$ varsa F'ye f'nin ters türevi denir.

$$\int F'(x)\,dx = \int f(x)\,dx$$

$$F(x) =$$

$$F(x) = -\frac{1}{1+x} + C \qquad (1+x)^{-1}$$ olsun.

$$F'(x) = +1 \cdot \frac{1}{(1+x)^2}$$

b-) $\int \sec^2(5x-1)\,dx = \underbrace{\frac{1}{5}\tan(5x-1) + C}_{F(x)}$ (Doğru).

$$F(x) = \frac{1}{5}\tan(5x-1) + C \Rightarrow F'(x) = \frac{1}{\cancel{5}} \cdot \sec^2(5x-1) \cdot \cancel{5}$$

c-) $\int \tan\theta \cdot \sec^2\theta \, d\theta = \underbrace{\frac{\sec^3\theta}{3} + C}_{F}$ (Yanlış)

$$\cos^{-1}\theta$$

$$\left(\frac{1}{\cos\theta}\right)' = -1 \cdot \cos^{-2}(-\sin\theta) = -\frac{1}{\cos^2\theta} \cdot (-\sin\theta)$$

$$F'(x) = \frac{1}{\cancel{3}} \cdot \cancel{3} \cdot \sec^2\theta \cdot (\sec\theta)'$$

$$= \sec^2\theta \cdot \frac{(\sin\theta)}{\cos^2\theta} = \sin\theta \cdot \sec^4\theta.$$

d-) $\int \sqrt{2x+1}\,dx = \underbrace{\frac{1}{3}\left(\sqrt{2x+1}\right)^3}_{F(x)} + C$ (Doğru).

$$F(x) = \frac{1}{3}\left((2x+1)^{1/2}\right)^3 + C = \frac{1}{3}(2x+1)^{3/2} + C$$

$$\Rightarrow F'(x) = \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(2x+1)^{1/2}\cdot\boxed{\cancel{2}} = (2x+1)^{1/2} = \sqrt{2x+1}.$$

2-) Bir roket dünya yüzeyinden 20 m/sn² 'lik bir sabit ivmeyle ayrılıyor. Buna göre 1 dakika sonra hızı ne olur?

Çözüm:

$$\text{ivme} = a(t) = \frac{dv(t)}{d(t)} \to \text{Hız} \quad \Rightarrow \quad v(t) = \int a(t)\,dt$$

$$\text{Hız} = v(t) = \frac{d\,r(t)}{dt} \to \text{Konum}$$

$$a(t) = 20 = \frac{dv(t)}{dt}$$

$$\Rightarrow \int 20\,dt = \int \frac{dv(t)}{dt}\,dt = v(t)$$

$\Rightarrow v(t) = 20t + C.$ (Roketin başlangıçtaki hızının 0 old. göz önüne alınırsa)

$v(0) = 0$ olduğundan $0 = 20.0 + C \Rightarrow C = 0$

$v(t) = 20t$

$v(1dk) = v(60) = 20.60 = 1200$

3-) Bir parçacık $a(t) = \sqrt{t} - \left(\frac{1}{\sqrt{t}}\right)$ ivmesi ile hareket ediyor. $\underline{t=0}$ anında hızının $v = 4/3$ ve konumunun $r = -4/15$ old. varsayarak parçacığın konumunu ve hızını veren fonk. elde ediniz.

$\underline{\text{Çözüm}}$: $a(t) = \frac{dv(t)}{dt} = \sqrt{t} - \frac{1}{\sqrt{t}}$

$$\Rightarrow v(t) = \int \sqrt{t} - \frac{1}{\sqrt{t}}\, dt = \int \underbrace{\sqrt{t}}_{t^{1/2}}\, dt - \int \underbrace{\left(\frac{1}{\sqrt{t}}\right)}_{t^{-1/2}} dt$$

$$= \frac{2}{3} \cdot t^{3/2} - 2\, t^{1/2} + C$$

$v(0) = 4/3 \quad \Rightarrow 4/3 = \frac{2}{3} \cdot 0^{3/2} - 2 \cdot 0^{1/2} + C$

$\Rightarrow 4/3 = C$

$v(t) = \frac{2}{3} t^{3/2} - 2t^{1/2} + 4/3 \qquad \left(v(t) = \frac{d\, r(t)}{dt} \right)$ → konum

$$\Rightarrow r(t) = \int \left(\frac{2}{3} t^{3/2} - 2 t^{1/2} + 4/3\right) dt$$

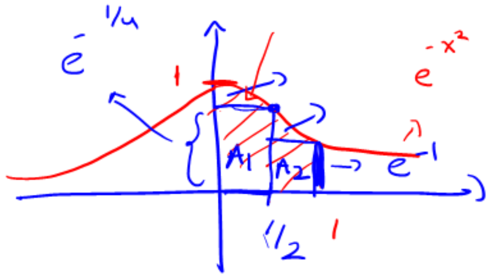
ödev

$$= \frac{4}{15} \cdot t^{5/2} - \frac{4}{3} t^{3/2} + \frac{4}{3} t + \tilde{C} \to -4/15$$

$r(0) = -4/15 \quad \Rightarrow \tilde{C} = -4/15$

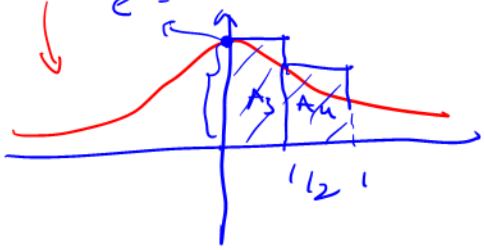
4-) $\frac{1}{2}\left[e^{-1/4} + e^{-1}\right] \le \underbrace{\int_0^1 e^{-x^2} dx}_{} \le \frac{1}{2}\left[1 + e^{-1/4}\right]$ old. ispatlayınız.

$\underline{\text{Çözüm}}$: $\int e^{-x^2} dx$ integralinin sonucunu elementer yollardan elde etmek mümkün değildir.



$$\boxed{A_1+A_2 \le \int_0^1 e^{-x^2}\,dx \le A_3+A_4}$$

$$A_1 = \frac{1}{2}\cdot e^{-1/4}, \quad A_2 = \frac{1}{2}\, e^{-1}$$



$$A_3 = \frac{1}{2}\cdot 1, \quad A_4 = \frac{1}{2}\cdot e^{-1/4}$$

$$A_1+A_2 = \frac{1}{2}\left[e^{-1/4}+e^{-1}\right]$$

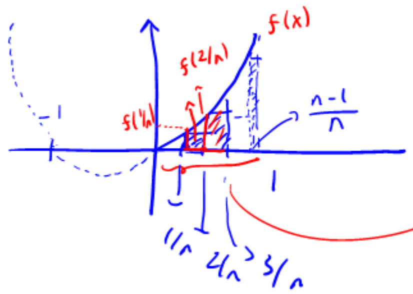
$$A_3+A_4 = \frac{1}{2}\left[1+e^{-1/4}\right]$$ bulunur ve

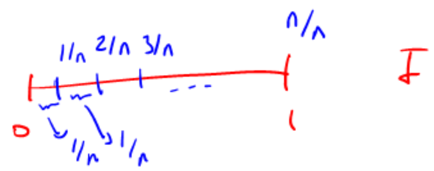
böylece ispat tamamlanır.

5-) Belirli integralin tanımını kullanarak

$\int_0^1 (x^2+x)\,dx$ int. hesaplayınız.

Çözüm:





I aralığını n eşit parçaya bölelim.

$$A = \frac{1}{n}\cdot f(1/n) + \frac{1}{n} f(2/n) + \dots + \frac{1}{n} f\left(\frac{n-1}{n}\right)$$

$$= \frac{1}{n}\left[\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right)\right]$$

$$\Rightarrow \int_0^1 f(x)\,dx = \int_0^1 (x^2+x)\,dx = \lim_{n\to\infty} \frac{1}{n}\left[\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\sum_{k=1}^{n-1} \frac{k^2}{n^2}+\frac{k}{n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left\{ \sum_{k=1}^{n-1} \frac{k^2}{n^2} + \sum_{k=1}^{n-1} \frac{k}{n} \right\}$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n-1} k^2 + \frac{1}{n^2} \sum_{k=1}^{n-1} k$$

?

$$\sum_{k=1}^{n-1} k = \frac{(n-1).n}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\dots+n^2 = \frac{n.(n+1)(2n+1)}{6}$$

[Ödev: Bu eşitliğin ispatını araştırın.]

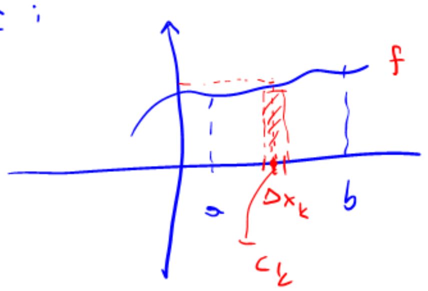
$$= \lim_{n\to\infty} \frac{1}{n^3}.\left[\frac{\overbrace{(n-1).(n).(2n-1)}^{2n^3+\dots}}{6}\right] + \frac{1}{n^2}.\frac{\overbrace{(n-1).n}^{n^2}}{2}$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

6-) Aşağıdaki limiti belirli integral ile ifade ediniz.

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k-1+2n}{k-1+n}.\left(\frac{1}{n}\right)$$

Çözüm:



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k).\Delta x_k$$

$$\frac{k-1+2n}{k-1+n} = \frac{n\left\{\frac{k-1}{n}+2\right\}}{n\left\{\frac{k-1}{n}+1\right\}} = \frac{\frac{k-1}{n}+2}{\frac{k-1}{n}+1}, \qquad \frac{k-1}{n} = c_k$$

$$\frac{1}{n} = \Delta x_k$$

$$= \frac{c_k+2}{c_k+1}$$

$f(x) := \dfrac{x+2}{x+1}$ fonk. tanımlayalım.

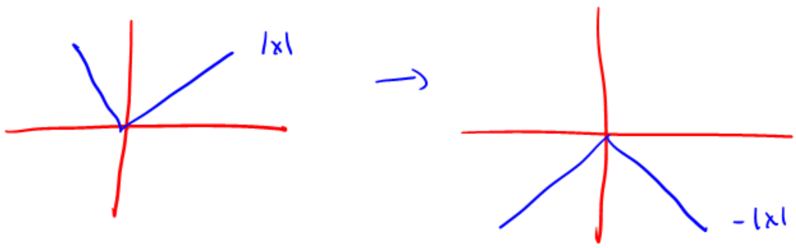
$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \int_a^b f(x)\,dx$$

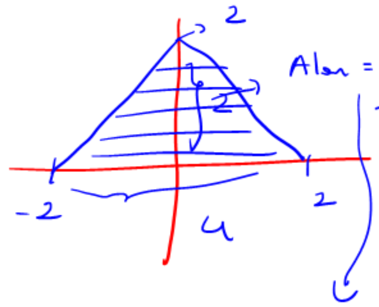
7-) İntegralin geometrik tanımını kullanarak

a-) $\displaystyle\int_{-2}^{2} (2-|x|)\,dx$ ve $\displaystyle\underbrace{\int_{-1}^{1} \left(1+\sqrt{1-x^2}\right)dx}_{\text{ödev}}$ hesaplayınız.

Çözüm: a-)

1. Yol; $f(x) := 2-|x|$ fonk. tanımlayalım.

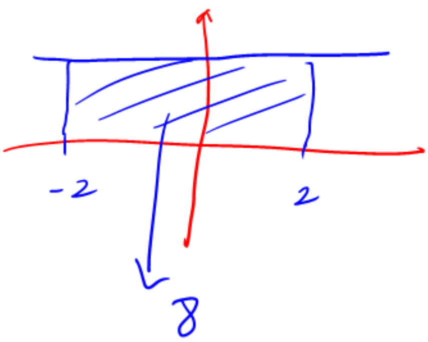




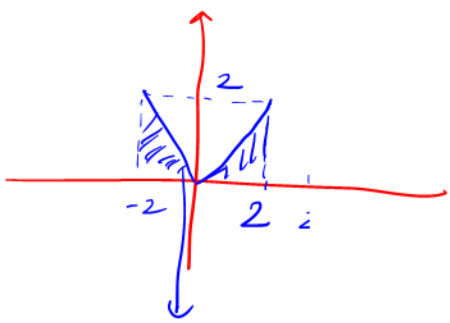
$$\text{Alan} = \int_{-2}^{2} (2-|x|)\,dx = \frac{4 \cdot 2}{2} = 4.$$

2. Yol; İntegralin toplamsallık özelliğini kullanarak

$$\int_{-2}^{2} 2-|x| = \int_{-2}^{2} 2 - \int_{-2}^{2} |x|$$



$f(x) = 2$



$$\frac{2 \cdot 2}{2} + \frac{2 \cdot 2}{2} = 4$$

$\Rightarrow 8 - 4 = 4.$