HACETTEPE ÜNİVERSİTESİ

SORULAR:

1) $A = \begin{bmatrix} -1 & 1 & 2 & 1 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & -1 & 1 \\ 1 & 1 & 2 & -1 \end{bmatrix}$ matrisi tersinir midir? Evet ise tersini bulunuz

$$\left[\begin{array}{cccc|cccc} -1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{3R_1+R_3\to R_3 \\ R_1+R_4\to R_4}]{R_1+R_2\to R_2} \left[\begin{array}{cccc|cccc} -1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 & 0 & 0 \\ 0 & 4 & 5 & 4 & 3 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{\frac{R_4}{2}\to R_4 \\ \frac{R_2}{3}\to R_2}]{-R_1\to R_1} \left[\begin{array}{cccc|cccc} 1 & -1 & -2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 1/3 & 1/3 & 0 & 0 \\ 0 & 4 & 5 & 4 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & \frac{1}{2} & 0 & 1/2 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cccc|cccc} 1 & -1 & -2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 4 & 5 & 4 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & 1/3 & 1/3 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[-4R_2+R_3\to R_3]{R_2+R_1\to R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 2 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & -3 & 4 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2/3 & 1/3 & 1/3 & 0 & 0 \end{array}\right] \xrightarrow[-2R_4+R_2\to R_2]{3R_4+R_3\to R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -4/3 & -1/3 & -2/3 & 1/2 & 0 \\ 0 & 0 & 0 & 6 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2/3 & 1/3 & 1/3 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3/6\to R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -4/3 & -1/3 & -2/3 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 1/3 & 1/6 & -1/6 & 0 \\ 0 & 0 & 1 & 2/3 & 1/3 & 1/3 & 0 & 0 \end{array}\right] \xrightarrow[\substack{\frac{4}{3}R_3+R_2\to R_2 \\ -\frac{2}{3}R_3+R_4\to R_4}]{R_3+R_1\to R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1/6 & 1/6 & 1/3 & 0 \\ 0 & 1 & 0 & 0 & 1/9 & -4/9 & 5/18 & 0 \\ 0 & 0 & 0 & 1 & 1/3 & 1/6 & -1/6 & 0 \\ 0 & 0 & 1 & 0 & 1/9 & 2/9 & 1/9 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I} \underbrace{\begin{bmatrix} -1/6 & 1/6 & 1/3 & 0 \\ 1/9 & -4/9 & 5/18 & 0 \\ 1/9 & 2/9 & 1/9 & 0 \\ 1/3 & 1/6 & -1/6 & 0 \end{bmatrix}}_{A^{-1}}$$

$\Rightarrow$ A tersinir. Gerçekten de $A \cdot A^{-1} = I$ dir. Kontrol edebilirsiniz!

2) $A=\begin{bmatrix} 1 & -1 & [illegible] & [illegible] \\ 2 & -1 & [illegible] & [illegible] \\ -1 & 1 & 1 & -3 \\ 1 & -1 & k & 4 \end{bmatrix}$ matrisinin tersinin olması için [illegible] ?

Ayrıca $k=-2$ için tersini bulunuz (varsa)

$$A \xrightarrow[\substack{R_1+R_3\to R_3 \\ -R_1+R_4\to R_4}]{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & -1 & k & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & k+1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_4/2\to R_4} \begin{bmatrix} 1 & -1 & k & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & k+1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[-2R_4+R_1\to R_1]{\substack{R_4+R_3\to R_3 \\ -R_4+R_2\to R_2}} \begin{bmatrix} 1 & -1 & k & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k+1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Bir matrisin tersinir olabilmesi için 0 satırı içermemesi gerekir. O halde $\boxed{k+1\neq 0}$ olmalıdır.

Tersinin olması için $k+1\neq 0$ yani $\boxed{k\neq -1}$ olması gerekir.

$\boxed{k=-2}$ için $A=\left[\begin{array}{cccc|cccc} 1 & -1 & -2 & 2 & 1 & 0 & 0 & 0 \\ 2 & -1 & -5 & 5 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & -3 & 0 & 0 & 1 & 0 \\ 1 & -1 & -2 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R_1+R_3\to R_3 \\ -R_1+R_4\to R_4}]{-2R_1+R_2\to R_2}$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & -2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{R_4}{2}\to R_4} \left[\begin{array}{cccc|cccc} 1 & -1 & -2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1/2 & 0 & 1/2 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{-R_4+R_2\to R_2 \\ -2R_4+R_1\to R_1}]{R_4+R_3\to R_3}\left[\begin{array}{cccc|cccc} 1 & -1 & -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -3/2 & 1 & -1/2 & 0 \\ 0 & 0 & -1 & 0 & 1/2 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 1 & -1/2 & 0 & 1/2 & 0 \end{array}\right]\xrightarrow{-R_3\to R_3}$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & -2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -3/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 0 & -3/2 & 0 \\ 0 & 0 & 0 & 1 & -1/2 & 0 & 1/2 & 0 \end{array}\right]\xrightarrow[2R_3+R_1\to R_1]{R_3+R_2\to R_2}\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & -1 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 0 & -3/2 & 0 \\ 0 & 0 & 0 & 1 & -1/2 & 0 & 1/2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_1\to R_1}\left[\underbrace{\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}}_{I}\right|\left.\underbrace{\begin{array}{cccc} -3 & 1 & -2 & 0 \\ -2 & 1 & 2 & 0 \\ -1/2 & 0 & -3/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \end{array}}_{A^{-1}}\right]$$

3) $A=\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ ve $B=\begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ verilsin. $A.X=B$ denklemini çözünüz.

$A_{3\times 3}.X_{?\times ?}=B_{3\times 4} \Rightarrow X$ matrisi 3×4 tipinde olmalıdır.

X matrisini nasıl bulmalıyız:

Eğer A tersinir matris ise; $A.X=B$ eşitliğinin her iki tarafını soldan A^{-1} ile çarparsam; A tersinir olduğundan $A^{-1}.A=I$,

$$\underbrace{A^{-1}A}_{I}.X=A^{-1}.B \Rightarrow \underbrace{I.X}=A^{-1}.B$$

$$\Rightarrow \boxed{X=A^{-1}.B} \text{ olur.}$$

A matrisi tersinir mi bulalım:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\quad \cdots\cdots \quad} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\underbrace{\quad}_{I} \qquad \underbrace{\qquad\qquad}_{A^{-1}}$$

A matrisi tersinirdir. O halde

$$X = A^{-1} \cdot B = \begin{bmatrix} -1/2 & 1/2 & 1 \\ 1/2 & 1/2 & -1 \\ 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 5/2 & -1 & 1 \\ 1/2 & -1/2 & 0 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix} = X \text{ elde edilir.}$$

4) A ve B birbirine satır denk matrisler olsun.

g.s.5 a) A tersinirdir $\Longleftrightarrow$ B tersinirdir.

b) A O matrisine satır denktir $\Longleftrightarrow$ $A = O$'dır. } ifadelerini ispatlayınız.

Çözüm: a) ($\Longrightarrow$:) A tersinir olsun. O zaman A birim matrise (I'ya) satır denktir. $\boxed{A \approx I}$

Aynı zamanda A, B'ye satır denk ise B'nin de A'ya satır denk olduğunu biliyorum. $A \approx B \Rightarrow \boxed{B \approx A}$.

$$B \approx A \text{ ve } A \approx I \Longrightarrow \boxed{B \approx I}\text{'dır.}$$

$\Longrightarrow$ B tersinirdir.

($\Leftarrow$:) B tersinir olsun. O halde $B \approx I$. Aynı zamanda $A \approx B$ idi.

$$A \approx B \text{ ve } B \approx I \Rightarrow A \approx I$$

$\Rightarrow$ A tersinirdir.

b) ($\Rightarrow$:) $A \approx O$ olsun. O zaman $O = P.A$ olacak şekilde P tersinir matrisi vardır.

$$\underbrace{P^{-1}.O}_{O} = \underbrace{P^{-1}.P}_{I}.A = I.A = A.$$

$\Rightarrow \boxed{A = O}$ olur.

($\Leftarrow$:) $A = O$ ise her matris kendisine satır denk olduğundan;

$A \approx O$'dır. ✓

5) $A = \begin{bmatrix} -1 & k & 3 & -2 \\ 2 & 1 & 1 & k \\ 1 & k+1 & 4 & k-2 \\ 2 & 1 & 1 & k+1 \end{bmatrix}$ matrisinin tersinir olması için k ne olmalıdır?

$$A \xrightarrow[2R_1+R_4 \to R_4]{\substack{2R_1+R_2 \to R_2 \\ R_1+R_3 \to R_3}} \begin{bmatrix} -1 & k & 3 & -2 \\ 0 & 1+2k & 7 & k-4 \\ 0 & 1+2k & 7 & k-4 \\ 0 & 1+2k & 7 & k-3 \end{bmatrix} \xrightarrow{-R_2+R_3 \to R_3} \begin{bmatrix} -1 & k & 3 & -2 \\ 0 & 1+2k & 7 & k-4 \\ 0 & 0 & 0 & 0 \\ 0 & 1+2k & 7 & k-3 \end{bmatrix}$$

Matris 0 satırı içerdiği için hiçbir zaman tersinir olamaz. Böyle k'lar yoktur!