SORULAR:

1) $A=\begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & 7 \end{bmatrix} \Rightarrow \operatorname{adj}(A) = ?$

$a_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 1 & -1 \\ 1 & 7 \end{vmatrix} = 8$, $a_{21} = (-1)^3 \cdot \begin{vmatrix} -3 & 1 \\ 1 & 7 \end{vmatrix} = 22$, $a_{31} = (-1)^4 \cdot \begin{vmatrix} -3 & 1 \\ 1 & -1 \end{vmatrix} = 2$

$a_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & -1 \\ 3 & 7 \end{vmatrix} = -3$, $a_{22} = (-1)^4 \begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix} = 11$, $a_{32} = (-1)^5 \cdot \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = 2$

$a_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = -3$, $a_{23} = (-1)^5 \cdot \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} = -11$, $a_{33} = (-1)^6 \cdot \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} = 2$

Kofaktör matrisi $= \begin{bmatrix} 8 & -3 & -3 \\ 22 & 11 & -11 \\ 2 & 2 & 2 \end{bmatrix} \Rightarrow \operatorname{adj}(A) = \begin{bmatrix} 8 & 22 & 2 \\ -3 & 11 & 2 \\ -3 & -11 & 2 \end{bmatrix}$ //

2) $\boxed{\operatorname{adj}(A^T) = (\operatorname{adj}(A))^T}$

$B = A^T$ diyelim :

B'nin kofaktörü

B den j satır i sütunu atılarak elde edilen matris.

$(\operatorname{adj}(A^T))_{ij} = (\operatorname{adj}(B))_{ij} = b_{ji} = (-1)^{j+i} \cdot \det \underbrace{B[j|i]}$

A'nın kofaktörü

$= (-1)^{j+i} \det \underline{A^T[j|i]} = (-1)^{j+i} \cdot \det \underline{A[i|j]} = a_{ij}$

$= (\operatorname{adj}(A))_{ji} = (\operatorname{adj}(A))^T_{ij}$

(3) $\boxed{adj(c.A) = c^{n-1} . adj(A)}$, c bir skaler

$cA = B$ diyelim :

$$(adj(cA))_{ij} = (adj(B))_{ij} = \underset{\text{kofaktör}}{b_{ji}} = (-1)^{j+i} . \det B[j|i]$$

$$= (-1)^{i+j} . \det(c \underbrace{A[j|i]}_{\substack{(n-1)\times(n-1)\\ \text{tipinde}}}) = (-1)^{i+j} . c^{n-1} . \det A[j|i]$$

$$= c^{n-1} . \underbrace{(-1)^{i+j} . \det A[j|i]}_{} = c^{n-1} . a_{ji}$$

$$= c^{n-1} . (adj A)_{ij}$$

(4) $\boxed{\det(adj(A)) = (\det(A))^{n-1}}$, $A_{n\times n}$, $n>1$

<u>1.durum</u>: A tersinir değilse $\Rightarrow \det(A) = 0 \Rightarrow \det(A)^{n-1} = 0$ dır.

$\det(adj(A)) = 0$ mı? Bunu görelim : Kabul edelim ki: $\det(adj A) \neq 0$ olsun.

$\det(adj(A)) \neq 0 \Rightarrow adj(A)$ tersinirdir.

$$adj(A).A = (\underset{0}{\det A}).I = 0 \underset{\substack{(adj A)\\ \text{tersinir}}}{\Rightarrow} \underbrace{adj(A)^{-1}.adj(A)}_{I}.A = adj(A)^{-1}.0$$

$$\Rightarrow A = 0 \Rightarrow \det(adj(A)) = 0 \ \#$$

O halde $\det(Adj(A)) = 0$ olmalıdır. $\boxed{\det(adj A) = (\det A)^{n-1} = 0}$ olur

2. durum: A tersinir $\Rightarrow \det(A) \neq 0$

$$A \cdot adj\,A = \det A \cdot I \xRightarrow[\text{al}]{\text{det'ini}} \det(A \cdot adj(A)) = \det(\underbrace{\det A}_{\text{sayı}} \cdot \overset{n\times n}{I})$$

$$\Rightarrow (\det A) \cdot \det(adj(A)) = (\det A)^n \cdot \det I$$

$$\Rightarrow \boxed{\det(adj(A)) = (\det(A))^{n-1}} \text{ olur.}$$

5) $B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ matrisinin hiçbir reel bileşenli matrisin adjointi olamayacağını gösteriniz.

Kabul edelim ki: B matrisi bir reel bileşenli $A \in \mathbb{R}^{3\times 3}$ matrisinin adjointi olsun.

$B = adj(A)$ olsun. Determinantlara geçelim:

$$\Rightarrow \underbrace{\det(adj(A))}_{\|} = \underbrace{\det(B)}_{\|} \qquad |B| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{vmatrix} = -10$$

4. soru

$$\Rightarrow (\det A)^{\overset{n-1}{\underset{3}{\|}}} = -10$$

$$\Rightarrow (\det A)^2 = -10 \Rightarrow \text{Reel bir sayının karesi negatif olamaz}$$

#

$\Rightarrow$ B böyle bir A'nın adjointi olamaz!

6 $A^{4\times4}$ ve $\det A = \frac{1}{2}$ olsun. $\det(\mathrm{adj}\,(\mathrm{adj}\,(A^{-1}))) = ?$

$\mathrm{adj}\,(A^{-1}) = B$ diyelim:

$$\Rightarrow \det(\mathrm{adj}\,(\underbrace{\mathrm{adj}\,(A^{-1})}))= \det(\mathrm{adj}\,B) = (\det B)^{4-1} = (\det B)^3$$

$$= (\det(\mathrm{adj}\,(A^{-1})))^3$$

$A^{-1} = C$ diyelim;

$$(\det(\mathrm{adj}\,(A^{-1})))^3 = (\underbrace{\det(\mathrm{adj}\,(C))})^3 = ((\det(C))^{4-1})^3 = (\det C)^9$$

$$= (\det(A^{-1}))^9 = 2^9$$

$\det A = \frac{1}{2} \Rightarrow \boxed{\det(A^{-1}) = 2}$

7 $A = \begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 3 & -6 & 0 & 0 & 0 \\ 2 & -4 & 2 & 4 & 0 \\ -2 & 3 & 5 & 1 & -5 \\ 0 & 9 & 3 & 4 & 7 \end{bmatrix} \Rightarrow (\mathrm{adj}\,A)^{-1} = ?$

$\det(\mathrm{adj}\,(A)) = ?$

$\det A = 146$

$\det(\text{adj} A) = (\det A)^{5-1} = (146)^4 \neq 0 \Rightarrow \text{Adj}(A)$ tersinir.

$A . \text{adj}(A) = \det A . I \Rightarrow \underbrace{A . \text{adj}(A) . \text{adj}(A)^{-1}}_{} = \det(A) . I . \text{adj}(A)^{-1}$

$$\Rightarrow A . \frac{1}{\det A} = \text{adj}(A)^{-1}$$

$$= \frac{1}{146} . \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \text{adj}(A)^{-1} \text{ dır.}$$

8) $A^{-1} = \begin{bmatrix} a & 2 & -3 \\ b & 4 & 1 \\ 5 & 0 & 0 \end{bmatrix} \Rightarrow \text{adj}(A^T) = ?$

$$\det(A^{-1}) = \begin{vmatrix} a & 2 & -3 \\ b & 4 & 1 \\ 5 & 0 & 0 \end{vmatrix} = 5 . \begin{vmatrix} 2 & -3 \\ 4 & 1 \end{vmatrix} = 5 . (2 + 12) = \underline{\underline{90}}$$

$\Rightarrow \boxed{\det A = \frac{1}{90}} \neq 0 \Rightarrow A$ ve $\text{adj}(A)$ tersinir.

$A . \text{adj}(A) = \det A . I \Rightarrow (\text{adj} A) = A^{-1} . \det A .$

$$\Rightarrow \text{adj}(A^T) = (\text{adj} A)^T = (A^{-1} . \det A)^T$$

$$= \text{adj}(A^T) = \frac{1}{90} . \begin{bmatrix} a & b & 5 \\ 2 & 4 & 0 \\ -3 & 1 & 0 \end{bmatrix} //$$