
UNCERTAINTY QUANTIFICATION IN MULTI-PHYSICS MODEL FOR WIND TURBINE ASSET MANAGEMENT

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A thesis submitted in partial fulfilment of the requirements for the degree of

Doctor of Philosophy
(in Computer Science)

publicly defended on March 12, 2024 in front of the following jury:

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Contents

List of Figures	ix
List of Tables	xiii
Introduction	1
I Introduction to uncertainty quantification and wind energy	11
1 Uncertainty quantification in computer experiments	13
1.1 Introduction	14
1.2 Black-box model specification	14
1.3 Enumerating and modeling the uncertain inputs	15
1.3.1 Sources of the input uncertainties	15
1.3.2 Modeling uncertain inputs with the probabilistic framework	16
1.3.3 Joint input probability distribution	17
1.4 Uncertainty propagation for central tendency study	20
1.4.1 Numerical integration	20
1.4.2 Numerical design of experiments	26
1.4.3 Summary and discussion	29
1.5 Uncertainty propagation for rare event estimation	31
1.5.1 Problem statement	31
1.5.2 Rare event estimation methods	34
1.5.3 Summary and discussion	43
1.6 Global sensitivity analysis	45
1.6.1 Screening methods	46
1.6.2 Variance-based importance measures	47
1.6.3 Moment-independent importance measures	50
1.6.4 Summary and discussion	52
1.7 Surrogate modeling	52
1.7.1 Common framework	52

1.7.2	General purposes surrogate model	54
1.7.3	Goal-oriented active surrogate model	56
1.7.4	Summary and discussion	60
1.8	Conclusion	60
2	Introduction to wind turbine modeling and design	61
2.1	Introduction	62
2.2	Metocean conditions simulation	62
2.2.1	Turbulent wind generation	63
2.2.2	Wake modeling	66
2.2.3	Irregular wave generation	67
2.3	Wind turbine multi-physics modeling	68
2.3.1	Aerodynamics of horizontal axis wind turbines	69
2.3.2	Hydrodynamics	71
2.3.3	Control	72
2.3.4	Structural dynamics	72
2.3.5	Fatigue damage	73
2.4	Design and operation practices	76
2.4.1	Types of technologies and preliminary design	77
2.4.2	Further design considerations	78
2.5	Uncertain inputs	81
2.5.1	Environmental inputs	82
2.5.2	System inputs	82
2.5.3	Probabilistic fatigue assessment	83
2.6	Conclusion	85
II	Contributions to uncertainty quantification and propagation	87
3	Kernel-based uncertainty quantification	89
3.1	Introduction	90
3.2	Dependence modeling with nonparametric copula	91
3.2.1	Preliminary definitions and properties	92
3.2.2	Empirical and checkerboard copula	94
3.2.3	Empirical Bernstein and Beta copula	94
3.3	<i>Copulogram</i> : a tool for multivariate data visualization	99
3.3.1	From the pairwise plot to the copulogram	99
3.3.2	Implementation in a Python package	99
3.4	Semiparametric inference of the South Brittany metocean conditions	102
3.4.1	Inference of the marginals	102
3.4.2	Nonparametric inference of the dependence	104

3.4.3	Summary and discussion	106
3.5	Quantifying and clustering the wake-induced perturbations within a wind farm	106
3.5.1	Uncertainty propagation on a wake model	107
3.5.2	Statistical metric of wake-induced perturbations	109
3.5.3	Clustering the wake-induced perturbations	110
3.5.4	Summary and discussion	111
3.6	Conclusion	112
4	Kernel-based central tendency estimation	113
4.1	Introduction	114
4.2	Treatment of uncertainties on the Teesside wind farm	116
4.2.1	Numerical simulation model	117
4.2.2	Measured environmental data	117
4.2.3	Non parametric fit with empirical Bernstein copula	121
4.2.4	Fatigue assessment	121
4.3	Numerical integration procedures for mean damage estimation	123
4.3.1	Quadrature rules and quasi-Monte Carlo methods	124
4.3.2	Kernel herding sampling	125
4.3.3	Bayesian quadrature	128
4.4	Numerical experiments	132
4.4.1	Illustration on analytical toy-cases	132
4.4.2	Application to the Teesside wind turbine fatigue estimation	133
4.5	Conclusion	137
5	Kernel-based surrogate models validation	141
5.1	Introduction	142
5.2	Predictivity assessment criteria for an ML model	143
5.2.1	The predictivity coefficient	144
5.2.2	Weighting the test sample	145
5.3	Test-set construction	147
5.3.1	Fully-Sequential Space-Filling design	147
5.3.2	Support points	148
5.3.3	Kernel herding	150
5.3.4	Numerical illustration	151
5.4	Numerical results I: construction of a training set and a test set	151
5.4.1	Test-cases	154
5.4.2	Results and analysis	155
5.5	Numerical results II: splitting a dataset into a training set and a test set	161
5.5.1	Industrial test-case CATHARE	161
5.5.2	Benchmark results and analysis	162
5.6	Conclusion	165

III Contributions to rare event estimation	169
6 Nonparametric rare event estimation	171
6.1 Introduction	172
6.1.1 Background	173
6.2 Bernstein adaptive nonparametric conditional sampling	174
6.2.1 Empirical Bernstein copula	174
6.2.2 Bernstein adaptive nonparametric conditional sampling algorithm . .	176
6.3 Numerical experiments	177
6.3.1 Results analysis	179
6.4 Application to wind turbine fatigue reliability	180
6.5 Conclusion	180
7 Sequential reliability oriented sensitivity analysis	183
7.1 Introduction	184
7.2 PLI + application on the case	184
7.3 HSIC for GSA	184
7.4 HSIC for TSA & CSA	184
7.5 Sequential ROSA	184
7.6 Application to wind turbine fatigue reliability	184
7.7 Conclusion	184
Conclusion and perspectives	185
Bibliography	187
Appendix A Univariate distribution fitting	207
A.1 Main parametric methods	207
A.2 Main nonparametric methods	208
Appendix B Dissimilarity measures between probability distributions	211
B.1 Csizár f -divergences	211
B.2 Integral probability metrics	211
B.2.1 Kernel discrepancy	211
Appendix C Advanced rare event estimation algorithms	217
C.1 Subset simulation (SS)	217
C.2 Nonparametric adaptive importance sampling (NAIS)	217
Appendix D Uncertainty quantification practice with OpenTURNS	219

Appendix E Résumé étendu de la thèse	223
E.1 Introduction	223
E.2 Résumés des chapitres relatifs à l'état de l'art des méthodes et outils mis en œuvre dans la thèse	227
E.3 Résumés des chapitres relatifs aux contributions méthodologiques et apports vis-à-vis des applications	228
E.4 Conclusion	232
E.4.1 Communications et publications dans revues à comité de lecture . . .	233
E.4.2 Développements informatiques open source	234

List of Figures

1	General uncertainty quantification framework (de Rocquigny et al. (2008), adapted by Ajenjo (2023))	3
1.1	Samples of three joint distributions with identical marginals and different dependence structures	18
1.2	Ranked samples represented in the Fig. 1.1	18
1.3	Univariate quadratures nodes for increasing sizes ($1 \leq n \leq 15$)	22
1.4	Two identical univariate Gauss-Legendre quadratures combined as a tensor product (left) and a Smolyak sparse grid (right).	23
1.5	Nested Monte Carlo and quasi-Monte Carlo designs ($n = 256$)	26
1.6	Latin hypercube designs with poor and optimized space-filling properties ($n = 8$)	29
1.7	One-dimensional reliability analysis example	32
1.8	FORM and SORM approximation on a two-dimensional reliability problem	35
1.9	Multi-FORM approximation on an example with two MPFPs	36
1.10	Illustration of a rare event estimation.	39
1.11	Illustration of a rare event estimation by subset sampling ($n = 4 \cdot 10^4, p_0 = 0.1$).	44
1.12	Illustration of a k -fold cross-validation (with $k = 4$)	54
1.13	Illustration of an ordinary kriging model fitted on a limited set of observations ($n = 7$). The predictor is represented in and several trajectories of the conditioned Gaussian process are drawn represented in purple.	56
1.14	Illustration of the expected improvement learning criterion	58
1.15	Illustration of the deviation number learning criterion	59
2.1	Wind spectrum from Brookhaven, USA (source: Burton et al. 2021)	63
2.2	[Should we keep this representation? If so, it will be done properly.]	64
2.3	Example of a turbulent wind field generated by TurbSim (source: Jonkman 2009)	65
2.4	Illustration of the wake created downstream a wind farm (source: Veers et al. 2019)	66
2.5	Peirson-Moskowitz and JONSWAP spectra at significant wave height $H_s = 3$ m and peak period $T_p = 7$ s (source: Milano 2021)	67
2.6	Chained numerical model of offshore wind turbine.	68

2.7	Hierarchy of structural (a) and aerodynamic (b) wind energy systems models (source: Veers et al. 2019)	69
2.8	Actuator disk model of the energy extraction (source: Burton et al. 2021). Longitudinal evolution of the air pressure and wind speed along the wind stream.	70
2.9	Blade element forces. With the lift and drag forces L and D , the flow angle ϕ , the pitch angle β and the angle of attack α (source: Burton et al. 2021).	71
2.10	Illustration of a soft-stiff design strategy, placing the structure's natural frequency f_0 away from the wind and wave power spectra, and the rotor excitation frequencies f_{1P} and f_{3P} (source: Kallehave et al. 2015).	73
2.11	Illustration of the Haigh diagram representing the combination of stress mean and amplitude leading to the same fatigue endurance.	76
2.12	Main offshore wind turbine technologies (sources: Ahmed et al. 2015; Mei and Xiong 2021).	78
2.13	Diagrams of an offshore wind turbine structure (source: (Chen et al., 2018)) and nacelle (source: US ODE).	79
2.14	Illustration of a probabilistic S-N curve according to the model defined in Guédé et al. (2007).	84
3.1	Evolution of m_{IMSE} for different dimensions and sample sizes.	97
3.2	Bernstein approximations of the empirical copula C_n (with size $n = 10$) of a Clayton copula (with parameter $\theta = 2.5$). The polynomial orders are assumed equal in the two dimensions $m_1 = m_2 \in \{3, 10, 20\}$	98
3.3	Copulogram of the iris flower dataset with colors assigned by the iris species.	101
3.4	Copulogram of Monte Carlo sample (with size $n = 10^3$) of the inputs and outputs of the modified Ishigami problem.	101
3.5	Marginal inference results of the South Brittany metocean data.	103
3.6	Empirical distributions of the maximum mean discrepancy between the validation sample \mathbf{X}' and the sample $\widehat{\mathbf{X}}_n \stackrel{i.i.d.}{\sim} \widehat{F}_X$ (repeated for 100 samples $\widehat{\mathbf{X}}_n$).	104
3.7	Copulogram of the South Brittany metocean data.	105
3.8	South Brittany wind farm layout, the vertical direction does not represent the exact north (left). South Brittany wind rose from the ANEMOC data (right).	107
3.9	Joint distributions of the wake-perturbed wind conditions at WT 13, 19, and 25 (in color) compared with the ambient wind conditions (in black).	109
3.10	Ambient (in black) and wake-perturbed (in color) distributions of wind distributions.	110
3.11	South Brittany layout and wake effects measured by the squared MMD on wind conditions. Note that the vertical direction does not represent the north direction.	110
3.12	K-medoids clustering solution for five clusters. The representative elements of the clusters are tagged with the mention "r".	111
4.1	Diagram of the chained OWT simulation model.	116

4.2	Teesside wind farm layout (left). Monopile OWT diagram (Chen et al., 2018) (right)	118
4.3	Copulogram of the Teesside measured data ($N = 10^4$ in grey), kernel herding subsample ($n = 500$ in orange). Marginals are represented by univariate kernel density estimation plots (diagonal), the dependence structure with scatter plots in the rank space (upper triangle). Scatter plots on the bottom triangle are set in the physical space.	120
4.4	Angular distribution of the wind and waves with a horizontal cross-section of the OWT structure and the mudline. Red crosses represent the discretized azimuths for which the fatigue is computed	123
4.5	Histogram of the log-damage, at mudline, azimuth 45 deg. (Monte Carlo reference sample)	123
4.6	Greedy kernel herding algorithm	127
4.7	Kernel illustrations (left to right: energy-distance, squared exponential, and Matérn 5/2)	128
4.8	Sequential kernel herding for increasing design sizes ($n \in \{10, 20, 40\}$) built on a candidate set of $N = 8196$ points drawn from a complex Gaussian mixture π .	128
4.9	Bayesian quadrature on a one-dimensional case	130
4.10	Analytical benchmark results on the toy-case #1	134
4.11	Analytical benchmark results on the toy-case #2	135
4.12	Mean damage estimation workflows for the industrial use case. The orange parts represent optional alterations to the workflow: the first one is an alternative to input data subsampling where the underlying distribution is sampled from, the second one improves mean damage calculation by using optimal weights over the output data	136
4.13	Copulogram of the kernel herding design of experiments with corresponding outputs in color (log-scale) on the Teesside case ($n = 10^3$). The color scale ranges from blue for the lowest values to red for the largest. Marginals are represented by histograms (diagonal), the dependence structure with scatter plots in the ranked space (upper triangle). Scatter plots on the bottom triangle are set in the physical space.	137
4.14	Mean estimation convergence (at the mudline, azimuth $\theta = 45$ deg.) on the Teesside case. Monte Carlo confidence intervals are all computed by bootstrap	138
5.1	Additional points (ordered, green) complementing an initial design (red crosses), μ is uniform on $[0, 1]$, the candidate points are in gray.	152
5.2	Additional points (ordered, green) complementing an initial design (red crosses), μ normal, the candidate points are in gray. [Should we keep only two options in vertical or move the legends to the side?]	153
5.3	Left: $f_1(\mathbf{x})$ (test-case 1); right: $f_2(\mathbf{x})$ (test-case 2); $\mathbf{x} \in \mathcal{X} = [0, 1]^2$	155

5.4	Test-case 1: predictivity assessment of a poor (left), good (center) and very good (right) model with kernel herding, support points and FSSF test sets.	158
5.5	Test-case 2: predictivity assessment of a poor (left), good (center) and very good (right) model with kernel herding, support points and FSSF test sets.	159
5.6	Test-case 3: predictivity assessment of a poor (left), good (center) and very good (right) model with kernel herding, support points and FSSF test sets.	160
5.7	Test-case CATHARE: inputs output scatter plots ($N = 10^3$)	163
5.8	Test-case CATHARE: estimated Q^2 . The box-plots are for random cross-validation, the red diamond (left) is for Q_{LOO}^2	165
5.9	Test-case CATHARE: sum of the weights Eq. (5.7).	166
6.1	Evolution of m_{IMSE} for different dimensions and sample sizes.	176
6.2	BANCS on toy-case #1: illustration of conditional sampling and nonparametric fit at the first and second iterations.	176
6.3	QQ-plot for KDE of marginals of the conditional distribution from Fig. 6.2. . .	178
6.4	Kendall plot for EBC on the copula of a conditional distribution from Fig. 6.2. .	178
6.5	BANCS sampling steps on toy-case #1.	180
6.6	BANCS sampling steps on toy-case #2.	181
A.1	Adequation of two different Weibull models using their likelihood with a sample of observations (black crosses).	208
A.2	Fit of a bimodal density by KDE using different tuning parameters.	209
A.3	QQ-plot between the data from Example 2 and a KDE model.	210
B.1	Kernel mean embedding of a continuous and discrete probability distribution .	213
E.1	Schéma générique de la quantification des incertitudes (de Rocquigny et al. (2008), adapté par Ajenjo (2023))	226

List of Tables

2.1	S-N curve numerical values of welded tubular joints in different environmental conditions (source: DNV-RP-C203 2016)	75
2.2	Marginal distributions of the environmental random variables	83
2.3	Marginal distributions of the system random variables	83
3.1	Marginal inference results of the South Brittany metocean data.	103
4.1	Teesside Offshore Wind turbine datasheet	118
4.2	Description of the environmental data.	119
4.3	Kernels considered in the following numerical experiments.	127
4.4	Analytical toy-cases	133
6.1	Results of the numerical experiments (subset sample size $N = 10^4$, $p_0 = 0.1$). . .	180

Introduction

Industrial context and motivation

The current challenge of energy transition involves, among other things, reducing the share of fossil fuels in the global electricity mix. In this context, offshore wind energy offers several advantages ([Beauregard et al., 2022](#)). Offshore energy benefits from more consistent winds than onshore, mainly due to the absence of terrain roughness, it also makes possible the installation of larger and more powerful wind turbines. Since the construction of the first offshore wind farm in Vindeby, Denmark, in 1991, the industry has experienced rapid growth, with a total capacity of 56 GW in operation worldwide in 2021. Over time, offshore wind technology has matured, resulting in significant achievements such as securing projects in Europe through “zero-subsidy bids” where the electricity produced is directly sold on the wholesale market ([Beauregard et al., 2022](#)).

However, despite the progress of this sector, scaling limitations and numerous scientific challenges emerge. To meet ambitious national and regional development targets, the wind energy industry must address various scaling issues, including port logistics, the demand for critical natural resources, and sustainable end-of-life processes. Furthermore, the field presents several scientific challenges that often involve coupling data with numerical simulations of physical systems and their surrounding environment. The wind energy community is focused on different objectives, including enhancing the design of floating offshore wind turbines, refining wind resource estimation techniques, and optimizing maintenance operations. In general, several decisions are made throughout the lifespan of a wind turbine by its designer, installer, and operator, all while having only partial knowledge of certain physical phenomena. Therefore, modeling and controlling the various sources of uncertainties associated with offshore wind energy proved to be a key success factor in this highly competitive industry.

Overall, the offshore wind industry needs methods for uncertainty management regarding safety margins and industrial asset management (at the component, wind turbine, and overall wind farm levels) ([Van Kuik et al., 2016](#)). For wind project developers, the primary focus is on improving the wind potential assessment of candidate sites by combining various sources of information and modeling the multivariate distribution of environmental conditions. In the case of floating wind projects, the goal is to incorporate a probabilistic aspect from the design phase

(e.g., of the floaters) to define safer, more robust, and more cost-effective solutions. For wind farm owners, end-of-life management is another significant concern. An owner of a wind farm at the end of its life has three options: extend the operational life of assets, replace current wind turbines with newer models, or decommission and sell the wind farm. The first two options require evaluating the structural reliability and residual lifespan, with quantitative assessments reviewed by certification bodies and insurers to issue operating permits. To provide rigorous risk assessments, the generic methodology of *uncertainty quantification methodology* is a widely accepted approach in industrial sectors facing these types of issues ([de Rocquigny et al., 2008](#)).

Generic methodology for uncertainty quantification

Computer experiment is a discipline that emerged with the advent of informatics. This practice produces numerical models that allow the simulation of complex system behavior based on initial conditions defined by the analyst. Numerical models quickly became essential for the analysis, design, and certification of complex systems in cases where experiments or physical measurements are too costly or even unfeasible. However, such numerical models are mostly deterministic: the reproducible result of a simulation is associated with a fixed input set of parameters. The issue of managing uncertainties associated with these inputs arises when performing analysis with numerical models.

Uncertainty quantification aims at modeling and controlling uncertainties around a numerical model. To do so, a generic methodology has been proposed to quantify and analyze uncertainties between input and output variables of a numerical model ([de Rocquigny et al., 2008](#)). An overview of the mathematical tools used in this field is provided by [Sullivan \(2015\)](#). This approach improves the understanding of a system, ultimately contributing to more robust decision-making. Figure 1 illustrates the main step of the generic uncertainty quantification method, which are briefly summarized hereafter:

- **Step A – Problem specification:** This step involves identifying the system under study and constructing a numerical model capable of precisely simulating its behavior. Specifying the problem also involves the definition of a set of parameters inherent to the numerical model. These parameters include both the input variables and the output variables generated by the simulation. In this document, the numerical model is considered a black box, in contrast to approaches that are integrated within the numerical solution schemes for the system's behavioral equations (referred to as intrusive approaches ([Le Maître and Knio, 2010](#))). Generally, these numerical models are first calibrated against measured data and pass a process of validation and verification to reduce modeling errors ([Oberkampf and Roy, 2010](#)).
- **Step B – Uncertainty modeling:** The objective of the second step is to identify and model all the sources of uncertainty related to the input variables. Most of the time the uncertainty modeling is done in the probabilistic framework.

- **Step C – Uncertainty propagation:** This step consists in propagating the uncertain inputs through the computer model. Consequently, the output of the numerical model (commonly scalar) also becomes uncertain. The goal is to estimate a quantity of interest, which is a statistic related to the studied random output variable. The uncertainty propagation method may differ depending on the quantity of interest targeted (e.g., central tendency, a quantile, a rare event probability, etc.).
- **Step C' – Inverse analysis:** In this additional step, a sensitivity analysis can be performed to study the role allocated to each uncertain input leading to the uncertain output.
- **Metamodeling:** Considering the high computational cost associated with some simulations, statistical approaches emulate these expensive simulators with a limited number of simulations. Uncertainty quantification can then be carried out using a “surrogate model” (or metamodel) for a reduced computational cost. This optional step of statistical learning is not strictly a part of uncertainty quantification, but it often proves to be essential for enabling its practical implementation.

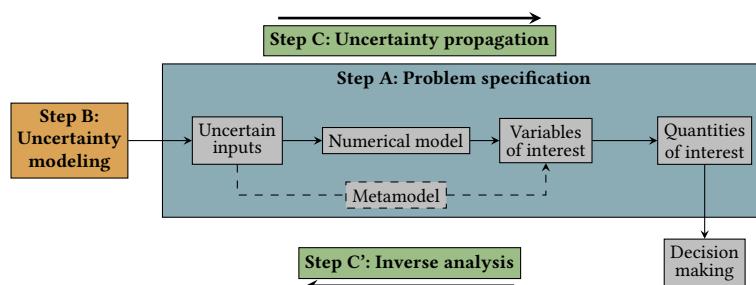


Figure 1 General uncertainty quantification framework ([de Rocquigny et al. \(2008\)](#), adapted by [Ajenjo \(2023\)](#))

Problem statement and outline of the thesis

Risk and uncertainty management in the field of wind energy is a significant concern for the electric utility Électricité de France (EDF). This thesis aims at adapting and applying the generic uncertainty quantification methodology to industrial offshore wind energy studies. As such, this use case raises scientific challenges related to its specific characteristics, described in the following:

- The numerical model exploited in the present work consists of a series of numerical models executed sequentially. This chain is divided into three parts: first, a temporal and stochastic generation of wind and wave velocity fields, followed by the simulation of the coupled hydro-aero-servo-elastic behavior of the wind turbine, and finally a post-processing phase to obtain scalar quantities of interest, aggregated over the temporal outputs.

- The complexity of this simulator, along with the high unit computational cost (about 40 minutes per simulation), requires the use of efficient sampling methods and high-performance computing systems. In addition to the complexity associated with the numerical model, modeling the input uncertainties also represents a challenge. Indeed, the joint distribution associated to environmental conditions present a complex dependence structure. The quality of the inference step is critical as it directly impacts the conclusions of uncertainty propagation.

In order to apply the generic methodology for uncertainty quantification to the offshore wind turbine case, this thesis aims at answering the following questions:

- Q1.** *How to accurately model the dependence structure associated with the joint environmental distribution?* (⇒ Step B)
- Q2.** *How to perform uncertainty propagation through a computationally expensive numerical chain uniquely based on an empirical description (measured data) of input uncertainties?* (⇒ Step C)
- Q3.** *How to estimate rare event probabilities related to the fatigue failure of offshore wind turbine structures?* (⇒ Step C)
- Q4.** *How to assess and analyse the sensitivity of uncertain inputs regarding quantities of interest resulting from structural reliability (i.e., reliability-oriented sensitivity analysis)?* (⇒ Step C')

To propose an answer to these questions, this manuscript is divided into three parts. The first part offers an introduction to uncertainty quantification methods and offshore wind turbine numerical modeling. The second part presents the contributions of this thesis to uncertainty quantification and propagation while the third part describes the contributions to rare event estimation. This manuscript is divided into seven chapters, which are summarized hereafter:

Chapter 1 – Introduction to uncertainty quantification. This chapter gives a brief overview of various topics in uncertainty quantification ([Sullivan, 2015](#)). After a reminder of some mathematical concepts, the model specification step is described, considering a black box and its input and output variables. The different types and sources of uncertainties are then presented, along with their modeling within a probabilistic framework. Uncertainty propagation depends on the estimated quantities of interest, therefore, one section addresses propagation methods for central tendency studies, and another focuses on rare event probability estimation (a statistic related to the tails of output distributions). The section dedicated to central tendency presents numerical integration, sampling, and design of experiments methods ([Fang et al., 2018](#)). The one about rare event probabilities introduces usual methods from the field of structural reliability ([Lemaire et al., 2009; Morio and Balesdent, 2015](#)).

This chapter also covers the main methods for global sensitivity analysis ([Da Veiga et al., 2021](#)). This field divides its methods into two major classes: screening methods and importance

measures. Screening techniques, typically applied in high-dimensional problems, aim to identify variables with low impact on the variability of the output of interest. Importance measures, on the other hand, quantitatively allocate, for each input variable, a share of the output variability, enabling the ranking of variables based on their influence.

Finally, this chapter presents an overview of the families of metamodels commonly used in uncertainty quantification ([Forrester et al., 2008](#)). Special attention is given to the Gaussian process regression, which involves conditioning a Gaussian process on a set of observations from the numerical model. Once conditioned, the Gaussian process provides richer information than other types of metamodels. This method simultaneously offers a surrogate model (mean of the Gaussian process, also called predictor) and an error function (variance of the process). Some iterative methods (called “active”) use this additional information to progressively enrich the metamodel and improve its predictability. These techniques were quite successful in the 1990s for solving optimization problems with expensive functions ([Jones et al., 1998](#)). Since then, their use has expanded to solve problems in structural reliability [Echard et al. \(2011\)](#).

Chapter 2 – Introduction to wind turbine modeling and design. The simulation of an offshore wind turbine involves modeling multiple physical aspects interacting with random environmental conditions. This chapter first introduces spectral methods used to generate wind and wave velocity fields by applying inverse Fourier transforms (e.g., as implemented in the TurbSim tool ([Jonkman, 2009](#))). These simulated wind velocity fields then become the inputs of a multi-physics wind turbines numerical model. Such simulation includes a simplified modeling of the interactions between fluids and structures (using the blade element momentum theory), dynamic modeling of the structure using flexible multibody methods, and modeling of wind turbine control systems [source]. The numerical code studied generates time series of several physical quantities describing the system’s behavior.

This thesis particularly focuses on the probabilistic evaluation of fatigue damage in wind turbine structures. Fatigue damage is a phenomenon that deteriorates the mechanical properties of a material as a result of exposure to many of cyclic, low-amplitude stresses. Currently, standards recommend the use of deterministic safety factors to address this failure mode ([IEC-61400-1, 2019](#); [DNV-ST-0437, 2016](#)). A probabilistic approach enhances the analysis and can sometimes reveal conservative of safety margins. Several recent studies have addressed this topic from different methodological perspectives ([Huchet, 2019](#); [Lataniotis, 2019](#); [Cousin, 2021](#); [Hirvoas, 2021](#); [Petrovska, 2022](#)).

In this context, this chapter enumerates the input parameters of the calculation chain that are considered uncertain. These random variables are grouped into two groups: the random vector related to the environment (e.g., average wind speed, wind speed standard deviation, wind direction, significant wave height, wave period, and wave direction), and the random vector related to the system (e.g., controller wind misalignment error, soil stiffness, fatigue calculation curve parameters).

Chapter 3 – Kernel-based uncertainty quantification. This chapter examines perturbations in environmental conditions within an offshore wind farm induced by wake effects [Larsen et al. \(2008\)](#). A theoretical offshore wind farm off the southern coast of Brittany is considered as a use case, and a simplified numerical model of wake in this wind farm is used. This model provides an analytical prediction of the wind speed deficit and turbulence created by the wake, taking into account the influence of the floaters' positions due to rigid body dynamics.

In a second phase, uncertainty propagation is carried out thought the wake model, considering the joint distribution of ambient environmental conditions as inputs. In the end, an environmental distribution perturbed by the wake is simulated for each wind turbine. A dissimilarity measure between distribution, based of kernels and named the *maximum mean discrepancy* (MMD), is used to compare the distributions perceived by each wind turbine. This measure allows the clustering of wind turbines exposed to similar environmental conditions, resulting in identical structural responses. Given the high computational cost of aero-servo-hydro-elastic simulations for offshore wind turbines, this preliminary study enables reliability analysis at the wind farm scale without repeating the analysis for each turbine. Ultimately, only four classes are selected to represent a wind farm of 25 turbines.

Chapter 4 – Kernel-based central tendency estimation. Chapter four presents the use of the kernel-based dissimilarity measure (MMD) in the context of probability distribution sampling, a method known as "kernel herding" introduced by [Chen et al. \(2010\)](#). This quadrature technique belongs to the family of "Bayesian quadratures" [Briol et al. \(2019\)](#), which can be viewed as a generalization of quasi-Monte Carlo methods [Li et al. \(2020\)](#).

The properties of this method are highlighted through an industrial application dedicated to estimating the mean fatigue damage of a wind turbine structure. Although this quantity is crucial in the design and certification of wind turbines, the methods used to estimated it are known to be suboptimal (i.e., regular grids). The study is conducted on a model of a fixed offshore wind turbine belonging to a farm in the North Sea. Uncertainties in input environmental conditions are inferred from in-situ measured data.

Finally, a numerical comparison with Monte Carlo and quasi-Monte Carlo sampling reveals the performance and practical advantages of kernel herding. This method allows for direct subsampling from a large environmental database without the need for inference (step B).

Chapter 5 – Kernel-based metamodel validation. This chapter proposes the use of kernel-based sampling methods in the context of model validation for machine learning (or surrogate models). Estimating the predictivity of supervised learning models requires an evaluation of the learned surrogate model on a set of test points that were not used during training. The quality of the validation naturally depends on the properties of the test set and the metric used to summarize the prediction error. This contribution first suggests using space-filling sampling methods to "optimally" select a test set, then, it introduces a new predictivity coefficient that weights the observed errors to improve the global error estimation. A numerical comparison between several sampling methods based on geometric approaches ([Shang and Apley, 2020](#)) or

kernel methods [Chen et al. \(2010\)](#); [Mak and Joseph \(2018\)](#) is carried out. Our results show that weighted versions of kernel methods offer superior performance. An application to simulated mechanical loads in an offshore wind turbine model is also presented. This experiment illustrates the practical relevance of this technique as an effective alternative to costly cross-validation techniques.

Chapter 6 – Nonparametric rare event estimation. Estimating rare events probabilities is a common issue in industrial risk management, especially in the field of structural reliability ([Chabridon, 2018](#)). To address this, several techniques have been proposed to overcome the known limitations of the Monte Carlo method. Among them, “subset sampling” ([Au and Beck, 2001](#)) is a technique based on the split of a rare probability into a product of less rare (and thus easier to estimate) conditional probabilities associated with nested failure events. However, this technique relies on conditional simulation using Markov chain Monte Carlo (MCMC) methods. These algorithms, while converging, often produce samples that are not independent and identically distributed (i.i.d.) due to the correlation between the Markov chains. In this chapter another conditional sampling method is proposed, with the advantage of preserving the i.i.d. property. Independent sampling is particularly relevant for reusing these samples in a posterior reliability-oriented sensitivity analysis. The algorithm introduced is based on the non-parametric inference of the conditional joint distribution using kernel density estimation of marginals combined with dependence inference using the empirical Bernstein copula ([Sancetta and Satchell, 2004](#)). The so-called “Bernstein adaptive nonparametric conditional sampling” (BANCS), is compared to the subset sampling method for several structural reliability problems. The initial results are promising, but further investigation is needed to control the estimator’s bias.

Chapter 7 – Sequential reliability oriented sensitivity analysis. This chapter deals with sensitivity analysis for risk measures (e.g., rare event probabilities). Global sensitivity analysis ([Da Veiga et al., 2021](#)) assigns a portion of the global output variability to each variable (or group of variables), often using a functional decomposition of the output variance. However, when studying risk measure (often located in the distributions’ tails), the global sensitivity might be very different to the sensitivity to the risk measure. “Reliability-oriented sensitivity analysis” (ROSA), studies the impact of the inputs in regard to a risk-measure such as a rare event probability (see e.g., [Chabridon \(2018\)](#)). Using the nested subsets obtained with the BANCS algorithm (presented in Chapter 6), the idea of this chapter is to study the ROSA evolution as the subsets get closer to the failure domain. For each subset, a ROSA is carried out with a kernel-based importance measure called the “Hilbert-Schmidt Independence Criterion” adapted to this context ([Marrel and Chabridon, 2021](#)).

Numerical developments

Several implementations developed in this thesis are available on different platforms, allowing the reader to reproduce some numerical results in an open-data approach:

- This Python package generates designs of experiments based on kernel methods such as Kernel Herding and Support Points. A tensorized implementation of the algorithms was proposed, significantly increasing their performances. Additionally, optimal weights for Bayesian quadrature are provided.

- This Python package, developed in collaboration with J.Muré, is available on the platform Pypi and fully documented.

-
- This Python package proposes an implementation of the “Bernstein Adaptive Non-parametric Conditional Sampling” method for rare event estimation.

- bancs²
- This Python package is available on the PyPI platform and is illustrated with examples and analytical benchmarks.

-
- This Python package presents a standardized process to benchmark different sampling methods for central tendency estimation.

- ctbenchmark³
- This Python package is available on a GitHub repository with analytical benchmarks.

-
- This Python package proposes an implementation of a synthetic visualization tool for multivariate distributions.

- copulogram⁴
- This Python package, developed in collaboration with V.Chabridon, is available on the Pypi platform.

¹Documentation: <https://efekhari27.github.io/otkerneldesign/master/>

²Repository: <https://github.com/efekhari27/bancs>

³Repository: <https://github.com/efekhari27/ctbenchmark>

⁴Repository: <https://github.com/efekhari27/copulogram>

Publications and communications

The research contributions in this manuscript are based on the following publications:

Book Chap.	<u>E. Fekhari</u> , B. Iooss, J. Muré, L. Pronzato and M.J. Rendas (2023). "Model predictivity assessment: incremental test-set selection and accuracy evaluation". In: <i>Studies in Theoretical and Applied Statistics</i> , pages 315–347. Springer.
Jour. Pap.	<u>E. Fekhari</u> , V. Chabridon, J. Muré and B. Iooss (2023). "Given-data probabilistic fatigue assessment for offshore wind turbines using Bayesian quadrature". In: <i>Data-Centric Engineering</i> .
Int. Conf.	<u>E. Fekhari</u> , B. Iooss, V. Chabridon, J. Muré (2022). "Numerical Studies of Bayesian Quadrature Applied to Offshore Wind Turbine Load Estimation". In: <i>SIAM Conference on Uncertainty Quantification (SIAM UQ22)</i> , Atlanta, USA. (Talk)
	<u>E. Fekhari</u> , B. Iooss, V. Chabridon, J. Muré (2022). "Model predictivity assessment: incremental test-set selection and accuracy evaluation". In: <i>22nd Annual Conference of the European Network for Business and Industrial Statistics (ENBIS 2022)</i> , Trondheim, Norway. (Talk)
	<u>E. Fekhari</u> , B. Iooss, V. Chabridon, J. Muré (2022). "Efficient techniques for fast uncertainty propagation in an offshore wind turbine multi-physics simulation tool". In: <i>Proceedings of the 5th International Conference on Renewable Energies Offshore (RENEW 2022)</i> , Lisbon, Portugal. (Paper & Talk)
	<u>E. Fekhari</u> , V. Chabridon, J. Muré and B. Iooss (2023). "Bernstein adaptive nonparametric conditional sampling: a new method for rare event probability estimation" ⁵ . In: <i>Proceedings of the 13th International Conference on Applications of Statistics and Probability in Civil Engineering (ICASP 14)</i> , Dublin, Ireland. (Paper & Talk)
	<u>E. Vanem</u> , <u>E. Fekhari</u> , N. Dimitrov, M. Kelly, A. Cousin and M. Guiton (2023). "A joint probability distribution model for multivariate wind and wave conditions". In: <i>Proceedings of the ASME 2023 42th International Conference on Ocean, Offshore and Arctic Engineering (OMAE 2023)</i> , Melbourne, Australia. (Paper)
	<u>A. Lovera</u> , <u>E. Fekhari</u> , B. Jézéquel, M. Dupoirion, M. Guiton and E. Ardillon (2023). "Quantifying and clustering the wake-induced perturbations within a wind farm for load analysis". In: <i>Journal of Physics: Conference Series (WAKE 2023)</i> , Visby, Sweden (Paper)
Nat. Conf.	<u>E. Fekhari</u> , B. Iooss, V. Chabridon, J. Muré (2022). "Kernel-based quadrature applied to offshore wind turbine damage estimation". In: <i>Proceedings of the Mascot-Num 2022 Annual Conference (MASCOT NUM 2022)</i> , Clermont-Ferrand, France (Poster)
	<u>E. Fekhari</u> , B. Iooss, V. Chabridon, J. Muré (2023). "Rare event estimation using nonparametric Bernstein adaptive sampling". In: <i>Proceedings of the Mascot-Num 2023 Annual Conference (MASCOT-NUM 2023)</i> , Le Croisic, France (Talk)
Invited Lec.	Le Printemps de la Recherche 2022, Nantes, France. "Traitement des incertitudes pour la gestion d'actifs éoliens". (Talk)
	Journées Scientifiques de l'Eolien 2024, Saint-Malo, France. "Evaluation probabiliste de la fiabilité en fatigue des structures éoliennes en mer". (Talk)

⁵This contribution was rewarded by the "CERRA Student Recognition Award"

PART I:

INTRODUCTION TO UNCERTAINTY QUANTIFICATION AND WIND ENERGY

Toute pensée émet un coup de dé.

S. MALLARMÉ

Chapter **1**

Uncertainty quantification in computer experiments

1.1	Introduction	14
1.2	Black-box model specification	14
1.3	Enumerating and modeling the uncertain inputs	15
1.3.1	Sources of the input uncertainties	15
1.3.2	Modeling uncertain inputs with the probabilistic framework	16
1.3.3	Joint input probability distribution	17
1.4	Uncertainty propagation for central tendency study	20
1.4.1	Numerical integration	20
1.4.2	Numerical design of experiments	26
1.4.3	Summary and discussion	29
1.5	Uncertainty propagation for rare event estimation	31
1.5.1	Problem statement	31
1.5.2	Rare event estimation methods	34
1.5.3	Summary and discussion	43
1.6	Global sensitivity analysis	45
1.6.1	Screening methods	46
1.6.2	Variance-based importance measures	47
1.6.3	Moment-independent importance measures	50
1.6.4	Summary and discussion	52
1.7	Surrogate modeling	52
1.7.1	Common framework	52
1.7.2	General purposes surrogate model	54
1.7.3	Goal-oriented active surrogate model	56
1.7.4	Summary and discussion	60
1.8	Conclusion	60

1.1 Introduction

The progress of computer simulation gradually allows the virtual resolution of more complex problems in scientific fields such as physics, astrophysics, engineering, climatology, chemistry, or biology. This domain often provides a deterministic solution to complex problems depending on several inputs. Associating an uncertainty quantification (UQ) analysis with these numerical models is a key element to improving the understanding of the phenomena studied. A wide panel of UQ methods has been developed over the years to pursue these studies for a reasonable computational cost.

This chapter presents the essential tools and methods from the generic UQ framework, including elements partially inspired from [Sullivan \(2015\)](#) and [Chabridon \(2018\)](#). It is structured as follows: Section 1.2 describes the context of the model specification step; Section 1.3 presents a classification of the inputs uncertainties and the probabilistic framework to model them; Section 1.4 and 1.5 introduce various methods to propagate the input uncertainties through the numerical model for different purposes; Section 1.6 presents the main inverse methods to perform sensitivity analysis in our framework; Finally, 1.7 introduces the concept of surrogate models to emulate a model by realizing a statistical learning on a limited dataset.

OpenTURNS¹. Is a high-performance Python library dedicated to UQ ([Baudin et al., 2017](#)). OpenTURNS (“Open source initiative for the Treatment of Uncertainties, Risks’N Statistics”) is developed by industrial researchers from EDF R&D, Airbus Group, PHIMECA Engineering, IMACS and ONERA. It combines high-performance using C++ programming with high-accessibility through a Python API. Overall, this open source library provides tools for various steps of the UQ framework (e.g., uncertainty quantification, uncertainty propagation, surrogate modeling, reliability, sensitivity analysis and calibration). To guaranty software quality, the development follows robust processes such as exhaustive unit testing and multiplatform continuous integration. A dedicated forum hosts an active community, which is helping new users and discussing future developments. Finally, no-code users can benefit from OpenTURNS’s free-download Graphical User Interface software, named [Persalys](#)². In this chapter, the methodological concepts introduced are linked to minimal OpenTURNS implementations examples, available in the Appendix D.

1.2 Black-box model specification

In our computer experiments context, uncertainty quantification is performed around an input-output numerical simulation model. This numerical model, or code, is hereafter considered as *black-box* since the knowledge of the underlying physics doesn’t inform the UQ methods. Alternatively, one could consider *intrusive* UQ methods, introducing uncertainties within the

¹OpenTURNS installation guide and documentation are available at <https://openturns.github.io/www/>

²Persalys, a free-download graphical user interface available at <https://www.persalys.fr/obtenir.php>

resolution of the equations of the physics (see e.g., [Le Maître and Knio 2010](#)). In practice, numerical models might be a sequence of codes executed in series to obtain a variable of interest.

While simulation models are in most cases deterministic, they may also be qualified as intrinsically stochastic (i.e., two runs of the same model taking the same inputs return different outputs). Additionally, numerical simulation always presents modeling errors. In the following, it will be assumed that the numerical models passed a *validation & verification* phase, to quantify their confidence and predictive accuracy.

Formally, part of the problem specification is the definition of the set of d input variables $\mathbf{x} = (x_1, \dots, x_d)^\top$ considered as uncertain (e.g., wind speed, wave period, etc.). The outputs studied are also defined at this stage, which will only be of scalar type in the present work. UQ methods suited to other types of outputs exist (see e.g., for time series outputs [Lataniotis 2019](#), for functional outputs [Audet et al. 2012; Rollón de Pinedo et al. 2021](#)). Let us then define the following numerical model:

$$\mathcal{M} : \begin{cases} \mathcal{D}_x \subseteq \mathbb{R}^d & \longrightarrow \mathcal{D}_y \subseteq \mathbb{R} \\ \mathbf{x} & \longmapsto y. \end{cases} \quad (1.1)$$

Unlike the typical machine learning input-output dataset framework, the UQ analyst can simulate the output image of any inputs (in the input domain), using a numerical model. However, numerical simulations often come with an important computational cost. Therefore, UQ methods should be efficient and require as few simulations as possible. In this context, surrogate models (or metamodels) are statistical approximations of the costly numerical model, that can be used to perform tractable UQ. Surrogate models are built and validated on a limited number of simulations (in a *supervised learning* framework). In practice, note that the model specification step is often associated with the development of a *wrapper* of the code. It is an overlay of code allowing its execution in a parametric way, which is often associated with *high-performance computer* (HPC) deployment. Once the model is specified, a critical step in uncertainty quantification is enumerating the input uncertainties and building their associated mathematical model.

1.3 Enumerating and modeling the uncertain inputs

1.3.1 Sources of the input uncertainties

The analyst should construct a list of uncertain inputs as exhaustive as possible, to ensure a complete risk assessment (e.g., associated with the exploitation of a wind energy asset). Even if these uncertainties might have different origins, they should all be considered jointly in the UQ study. Authors proposed to classify them for practical purposes into two groups:

- **aleatory uncertainty** regroups the uncertainties arising from natural randomness (e.g., wind turbulence). From a risk management point of view, these uncertainties are qualified

as *irreducible* since the industrials facing them will not be able to acquire additional information to reduce them (e.g., additional measures).

- **epistemic uncertainty** gathers the uncertainties resulting from a lack of knowledge (e.g., material properties). Contrarily to the aleatory ones, epistemic uncertainties might be reduced by investigating their origin (often at a certain cost).

Der Kiureghian and Ditlevsen (2009) discuss the relevance of this classification. They affirm that this split is practical for decision-makers to identify possible ways to reduce their uncertainties. However, it should not affect the way of modeling or propagating uncertainties. In the following, the probabilistic framework is introduced to deal with uncertainties.

1.3.2 Modeling uncertain inputs with the probabilistic framework

Uncertainties are traditionally modeled with objects from the probability theory. In this thesis, the *probabilistic framework* is adopted. Alternative theories exist to mathematically model uncertainties. For example, imprecise probability theory allows more general modeling of the uncertainties (Beer et al., 2013; Schöbi and Sudret, 2017). It becomes useful when dealing with very limited and possibly contradictory information (e.g., expert elicitation). The core probabilistic tools and objects are introduced hereafter.

The *probability space* is a measure space with total measure summing to one, also called probability triple and denoted $(\Omega, \mathcal{A}, \mathbb{P})$. This mathematical concept first includes a sample space Ω , which contains a set of outcomes $\omega \in \Omega$. Note that an *event* is defined as a set of outcomes in the sample space. Then, a σ -algebra \mathcal{A} , also called event space, is a set of events. Finally, a probability function $\mathbb{P} : \mathcal{A} \rightarrow [0, 1]$, is a positive probability measure associated with an event. Most often, the choice of the probability space will not be specified. The main object will be functions defined over this probability space: random variables.

The *random vector* \mathbf{X} (i.e., multivariate random variable) is a measurable function defined as:

$$\mathbf{X} : \begin{cases} \Omega & \longrightarrow \mathcal{D}_{\mathbf{x}} \subseteq \mathbb{R}^d \\ \omega & \longmapsto \mathbf{X}(\omega) = \mathbf{x}. \end{cases} \quad (1.2)$$

In the following, the random vector \mathbf{X} will be considered to be a squared-integrable function against the measure \mathbb{P} (i.e., $\int_{\Omega} |\mathbf{X}(\omega)|^2 d\mathbb{P}(\omega) < \infty$). Moreover, the present thesis deals with continuous random variables.

The *probability distribution* of the random vector \mathbf{X} is the pushforward measure of \mathbb{P} by \mathbf{X} . Which is a probability measure on $(\mathcal{D}_{\mathbf{x}}, \mathcal{A})$, denoted $\mathbb{P}_{\mathbf{X}}$ and defined by:

$$\mathbb{P}_{\mathbf{X}}(B) = \mathbb{P}(\mathbf{X} \in B) = \mathbb{P}(\omega \in \Omega : \mathbf{X}(\omega) \in B), \quad \forall B \in \mathcal{A}. \quad (1.3)$$

The *cumulative distribution function* (CDF) is a common tool to manipulate random variables. It is a function $F_{\mathbf{X}} : \mathcal{D}_{\mathbf{x}} \rightarrow [0, 1]$ defined for all $\mathbf{x} \in \mathcal{D}_{\mathbf{x}}$ as:

$$F_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}(\mathbf{X} \leq \mathbf{x}) = \mathbb{P}(X_1 \leq x_1, \dots, X_d \leq x_d) = \mathbb{P}_{\mathbf{X}}([-\infty, x_1] \times \dots \times [-\infty, x_d]). \quad (1.4)$$

The CDF is a positive, increasing, right-continuous function, which tends to 0 as \mathbf{x} tends to $-\infty$ and to 1 as \mathbf{x} tends to $+\infty$. In the continuous case, one can also define a corresponding *probability density function* (PDF) $f_{\mathbf{X}} : \mathcal{D}_{\mathbf{x}} \rightarrow \mathbb{R}_+$ with $f_{\mathbf{X}}(\mathbf{x}) = \frac{\partial^d F_{\mathbf{X}}(\mathbf{x})}{\partial x_1 \dots \partial x_d}$.

The expected value of a random vector $\mathbb{E}[\mathbf{X}]$, also called first moment, is a vector defined as:

$$\mu_{\mathbf{X}} = \mathbb{E}[\mathbf{X}] = \int_{\Omega} \mathbf{X}(\omega) d\mathbb{P}(\omega) = \int_{\mathcal{D}_{\mathbf{x}}} \mathbf{x} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = (\mathbb{E}[X_1], \dots, \mathbb{E}[X_d])^\top. \quad (1.5)$$

In addition, considering two random variables X_i and X_j , with $i, j \in \{1, \dots, d\}$, one can write their respective variance:

$$\text{Var}(X_i) = \mathbb{E}[(X_i - \mathbb{E}[X_i])^2], \quad (1.6)$$

and a covariance describing their joint variability:

$$\text{Cov}(X_i, X_j) = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])]. \quad (1.7)$$

The *standard deviation* $\sigma_{X_j} = \sqrt{\text{Var}(X_j)}$ and *coefficient of variation* $\delta_{X_j} = \frac{\text{Var}(X_j)}{|\mathbb{E}[X_j]|}$ are two quantities directly associated to the two first moments.

1.3.3 Joint input probability distribution

This section introduces various techniques to model and infer a joint probability distribution (or multivariate distribution). It will first define the *copula*, a mathematical tool used to model the dependence structure of a joint distribution. Then, a few methods to fit a joint distribution over a dataset will be mentioned. Finally, a panel of tools to evaluate the goodness of fit between a probabilistic model and a dataset will be recalled.

In general, the single effects of multivariate distributions tend to be well modeled. However, modeling the dependence structure underlying in a joint distribution is often overlooked. To illustrate the importance of this step, Fig. 1.1 represents three i.i.d samples from three bivariate distributions sharing the same single effects (e.g., here two exponential distributions) but different dependence structures. Judging from this example, one can assume that the joint distribution results from the composition of the single effects, also called marginals, and an application governing the dependence between them.

An empirical way of isolating the dependence structures from this example is to transform the samples in the ranked space. Let us consider an n -sized sample $\mathbf{X}_n = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\} \in \mathcal{D}_{\mathbf{x}}^n$. The corresponding ranked sample is defined as: $\mathbf{R}_n = \{\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(n)}\}$, where³ $r_j^{(i)} = \sum_{l=1}^n \mathbb{1}_{\{x_j^{(l)} \leq x_j^{(i)}\}}$, $\forall j \in \{1, \dots, d\}, i \in \{1, \dots, n\}$. Ranking a multivariate dataset allows us to isolate the dependence

³The *indicator function* is defined such that $\mathbb{1}_{\{\mathcal{A}\}}(x) = 1$ if $x \in \mathcal{A}$ and is equal to zero otherwise.

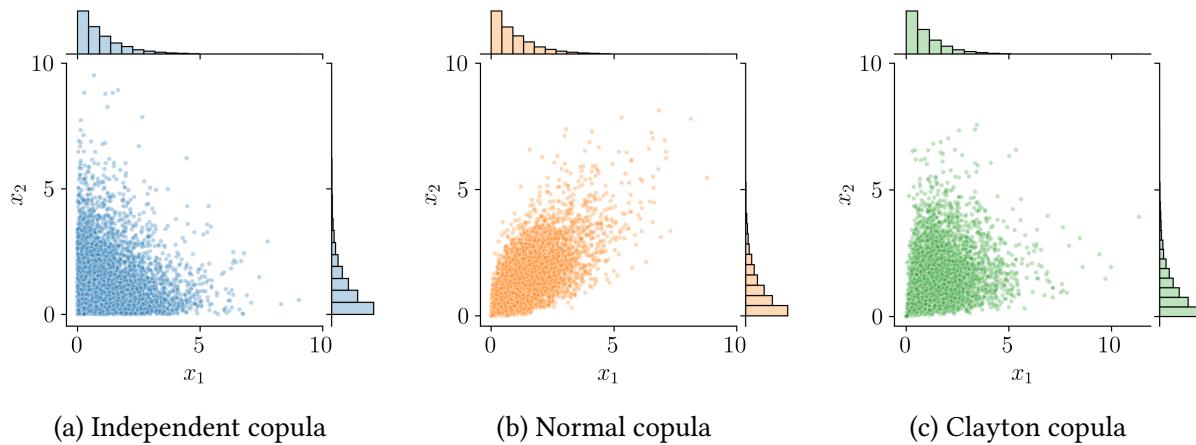


Figure 1.1 Samples of three joint distributions with identical marginals and different dependence structures

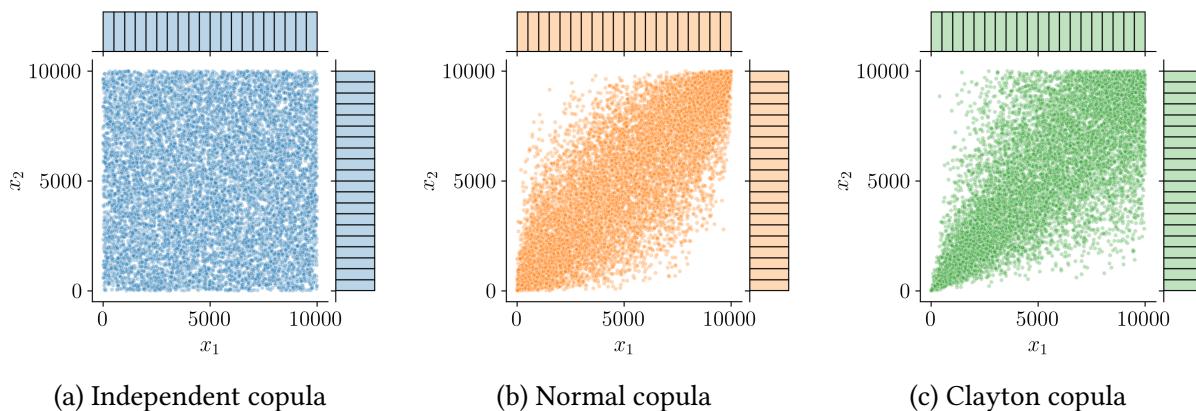


Figure 1.2 Ranked samples represented in the Fig. 1.1

structure witnessed empirically (Saporta, 2006). Fig. 1.2 shows the same three samples from Fig. 1.1 in the ranked space. One can first notice that the marginals are uniform since each rank is uniformly distributed. Then, the scatter plot from the distribution with independent copula (left plot) is uniform while the two others present different patterns.

A theorem states that the multivariate distribution of any random vector can be broken down into two objects (Joe, 1997). First, a set of univariate marginal distributions describing the behavior of the individual variables; Second, a function describing the dependence structure between all variables: a copula.

Theorem 1 (Sklar's theorem). *Let $\mathbf{X} \in \mathbb{R}^d$ be a random vector and its joint CDF $F_{\mathbf{X}}$ with marginals $\{F_{X_j}\}_{j=1}^d$, there exists a copula $C : [0, 1]^d \rightarrow [0, 1]$, such that:*

$$F_{\mathbf{X}}(x_1, \dots, x_d) = \mathbb{P}(X_1 \leq x_1, \dots, X_d \leq x_d) = C(F_{X_1}(x_1), \dots, F_{X_d}(x_d)). \quad (1.8)$$

If the marginals F_{X_i} are continuous, then this copula is unique. If the multivariate distribution has a PDF $f_{\mathbf{X}}$, it can also be expressed:

$$f_{\mathbf{X}}(x_1, \dots, x_d) = c(F_{X_1}(x_1), \dots, F_{X_d}(x_d)) \times f_{X_1}(x_1) \times \dots \times f_{X_d}(x_d), \quad (1.9)$$

where c is the density of the copula, sometimes also called copula by misuse of language. The reader might refer to [Durante and Sempi \(2015\)](#) for three different mathematical proofs.

Theorem 1 expresses the joint CDF by combining marginal CDFs and a copula, which is practical for sampling joint distributions. Conversely, the copula can be defined by using the joint CDF and the marginal CDFs:

$$C(u_1, \dots, u_d) = F_{\mathbf{X}}(F_{X_1}^{-1}(u_1), \dots, F_{X_d}^{-1}(u_d)) \quad (1.10)$$

This equation allows us to extract a copula from a joint distribution by knowing its marginals. Additionally, copulas are invariant under increasing transformations. This property is important to understand the use of rank transformation to display the copula without the marginal effects.

Identically to the univariate continuous distributions, a large catalog of families of copulas exists (e.g., independent, Normal, Clayton, Frank, Gumbel copula, etc.). Note that the independent copula Π implies that the distribution is defined as the product of its marginals $\Pi = \prod_{j=1}^d u_j$. In an inference context, this theorem divides the fitting problem into two independent problems: fitting the marginals and fitting the copula. Provided a dataset, this framework allows the potential combination of a parametric (or nonparametric) fit of marginals with a parametric (or nonparametric) fit of the copula.

To infer a joint distribution over a dataset, the analyst should determine a fitting strategy. Appropriate data visualization helps to choose the fitting methods susceptible to be relevant to the problem. In practice, the following points can be asked at this early stage:

- Is the distribution unimodal? If not, mixtures methods or nonparametric models might be required;
- Is the validity domain restrictive? If so, specific families of parametric distributions can be chosen or truncation can be applied;
- Is there a dependence structure? Does it concern all the variables together or only some groups of variables?
- Is the dependence structure complex? Transforming the dataset in the ranked space gives an empirical description of the dependence.

To ease the reading, a few techniques for estimating marginal distributions are available in Appendix A. In addition, two nonparametric methods are introduced in Chapter 3 to infer a copula: the “empirical Bernstein copula” and the “Beta copula”. The adequation between a fitted probabilistic model and a dataset should be validated, therefore, Appendices A recall visual and quantitative tools for univariate goodness-of-fit evaluation.

OpenTURNS 1 (Bivariate distribution). The Python code available in Appendix D proposes a minimalistic OpenTURNS example of a probabilistic uncertainty modeling. Figures illustrating the present section may be reproduced, using the OpenTURNS scripts available on GitHub⁴.

1.4 Uncertainty propagation for central tendency study

The previous section aimed at building a probabilistic model of the uncertainties considering the knowledge available. This one introduces diverse methods for forward propagation of the input uncertainties through a numerical model. In the present section, uncertainty propagation is dedicated to the “central tendency” as its goal is to study the mean and variance of the output distribution. This approach contrasts with the uncertainty propagation committed to rare event probability estimation, which will be introduced in Section 1.5 (e.g., used to assess reliability).

The difficulties related to any uncertainty propagation mostly arise from the practical properties of the numerical model. Its potential high dimension, irregularity and nonlinearity each represent a challenge. Such studies rely on a finite number of observations of the numerical model, depending on the computational budget affordable. Uncertainty propagation is at the end of generic UQ approach (step C), however, it is affected by the “garbage in, garbage out” concept. Meaning that its conclusions depend on the accuracy of the inputs’ uncertainty modeling.

This section introduces the main methods of global uncertainty propagation, outlining the links between numerical integration (i.e., Lebesgue integration or central tendency estimation) and numerical design of experiments.

1.4.1 Numerical integration

Forward uncertainty propagation aims at integrating a measurable function $g : \mathcal{D}_X \rightarrow \mathbb{R}$ with respect to a probability measure \mathbb{P}_X . Numerical integration provides algorithmic tools to help the resolution of this probabilistic integration (i.e., Lebesgue integration). Note that the measurable function g , in the context of computer experiments, becomes the numerical model \mathcal{M} introduced in Eq. (1.1).

In practice, this integral is approximated by summing a finite n -sized set of realizations $y_n = \{g(\mathbf{x}^{(1)}), \dots, g(\mathbf{x}^{(n)})\}$ from a set of input samples $X_n = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\}$. A *quadrature* establishes a rule selecting the input samples X_n (also called nodes), and an associated set of weights

⁴https://github.com/efekhari27/thesis/blob/main/numerical_experiments/chapter1/copulas.ipynb

$\mathbf{w}_n = \{w_1, \dots, w_n\} \in \mathbb{R}^n$. The approximation given by a quadrature rule is defined as a weighted arithmetic mean of the realizations:

$$I_{\mathbb{P}_X}(g) := \int_{\mathcal{D}_X} g(\mathbf{x}) d\mathbb{P}_X(\mathbf{x}) \approx \sum_{i=1}^n w_i g(\mathbf{x}^{(i)}). \quad (1.11)$$

For a given sample size n , the goal is to find a set of tuples $\{\mathbf{x}^{(i)}, w_i\}_{i=1}^n$ (i.e., quadrature rule), giving the best approximation of our quantity. Ideally, the approximation quality should be fulfilled for a wide class of integrands. Most quadrature rules only depend on the measure space $(\Omega, \mathcal{A}, \mathbb{P}_X)$, regardless of the integrand values. In the context of a costly numerical model, this property allows the analyst to massively distribute the calls to the numerical model.

This section aims at presenting the main multivariate integration techniques. These methods encompass different properties: some are deterministic and some are aleatory; some are sequential (i.e., nested) some are not; some are victims of the curse of dimensionality and some are not.

Classical multivariate deterministic quadrature

Historically, quadrature methods have been developed for univariate integrals. The Gaussian rule and the Fejér-Clebschaw-Curtis rule are two univariate deterministic quadratures that will be briefly introduced (see [Sullivan 2015](#) for further elements).

Gaussian quadrature is a powerful univariate quadrature building together a set of irregular nodes and a set of weights. The computed weights are positive, which ensures a numerically stable rule even for large sample sizes.

Different variants of Gaussian rules exist, the most common being the Gauss-Legendre quadrature. In this case, the function g to be integrated with respect to the uniform measure on $[-1, 1]$ is approximated by Legendre polynomials. Considering the Legendre polynomial of order n , denoted l_n , the quadrature nodes $x^{(i)}_{i=1}^n$ are given by the polynomial roots. The respective weights are given by the following formula:

$$w_i = \frac{2}{\left(1 - (x^{(i)})^2\right) (l'_n(x^{(i)}))^2}. \quad (1.12)$$

Gauss-Legendre quadrature guarantees a very precise approximation provided that the integrand is well-approximated by a polynomial of degree $2n - 1$ or less on $[-1, 1]$. This rule is deterministic but not sequential, meaning that two rules with sizes n_1 and n_2 , $n_1 < n_2$ will not be nested. However, a sequential extension is proposed by the Gauss-Kronrod rule ([Laurie, 1997](#)), at the expense of a slightly lower accuracy.

To overcome this practical drawback, Fejér then Clebschaw with Curtis proposed a nested rule with mostly equivalent accuracy as Gaussian quadratures. This method is usually presented to integrate a function with respect to the uniform measure on $[-1, 1]$ and starts with a change

of variables:

$$\int_{-1}^1 g(x) dx = \int_0^\pi g(\cos(\theta)) \sin(\theta) d\theta. \quad (1.13)$$

This expression can be written as an expansion of the integrand using cosine series. Therefore, knowing that cosine series are closely related to the Chebyshev polynomials of the first kind. Fejér's "first rule" (Trefethen, 2008) uses the Chebyshev polynomials roots as nodes $x^{(i)} = \cos(\theta^{(i+1/2)})$, associated with the following weights:

$$w_i = \frac{2}{n} \left(1 - 2 \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{1}{4j^2 - 1} \cos(j\theta^{(2i+1)}) \right). \quad (1.14)$$

These two univariate integration schemes are both very efficient on a wide panel of functions. Yet, Fejér-Clebschaw-Curtis is sequential and offers easy implementations, benefitting from powerful algorithms such as the *fast Fourier transform*. Fig. 1.3 illustrates the nested properties of Fejér-Clebschaw-Curtis quadrature by representing the nodes of quadrature rules with increasing size.

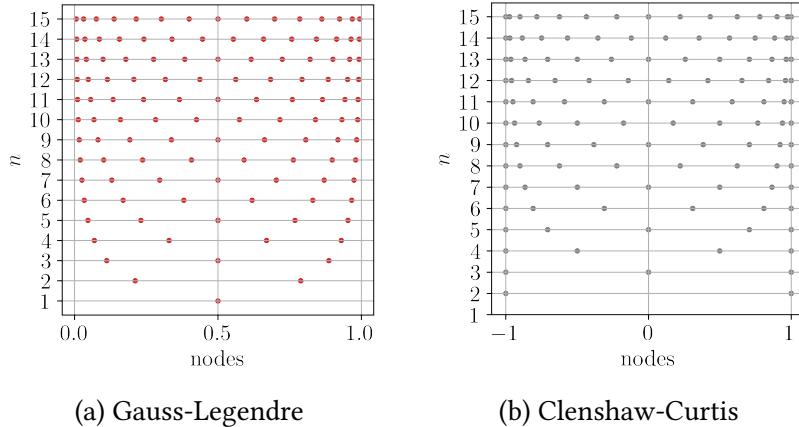


Figure 1.3 Univariate quadrature nodes for increasing sizes ($1 \leq n \leq 15$)

Uncertainty quantification problems are rarely unidimensional, but one can directly build a multivariate quadrature rule by defining the tensor product (also called full grids) of univariate rules. This exhaustive approach quickly shows its practical limits as the problem's dimension increases. In Fig. 1.4, the left plot represents a two-dimensional tensor product of identical Gauss-Legendre quadratures. Alternatively, sparse multivariate quadratures (i.e., Smolyak sparse grid) explore the joint domain more efficiently. Using the Smolyak recursive formula (see e.g., Sullivan 2015), two univariate quadratures can be combined as illustrated on the right of Fig. 1.4.

Monte Carlo methods

Monte Carlo methods were initially developed in the 1940s to solve problems in neutronics. Ever since, this frequentist technique has been applied to the resolution of the Lebesgue integral.

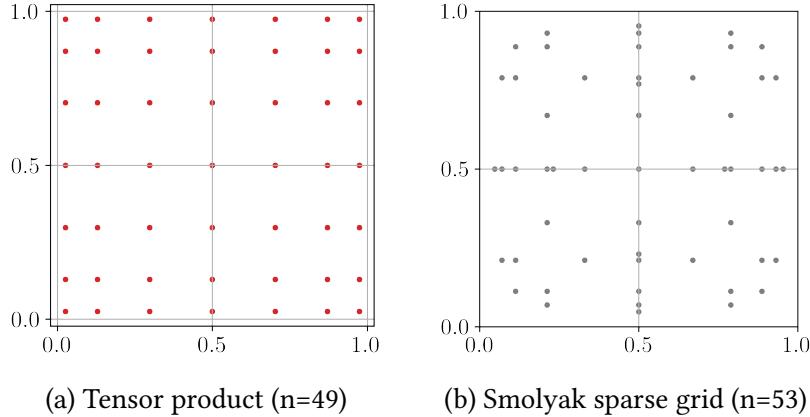


Figure 1.4 Two identical univariate Gauss-Legendre quadratures combined as a tensor product (left) and a Smolyak sparse grid (right).

To integrate a function g against a measure \mathbb{P}_X , it randomly generates points following the input measure. The integral is estimated by taking the uniform arithmetic mean of the nodes' images obtained by this random process.

This aleatory method requires to be able to generate points following a given distribution. To do so, the most common approach is to first uniformly generate a sequence of random points on $[0, 1]$. These sequences mimic actual randomness but are in fact generated by deterministic algorithms, also called pseudorandom number generators. Pseudorandom algorithms generate a sequence of numbers with a very large, but finite length. This sequence can be exactly repeated by fixing the same initial point, also called *pseudorandom seed*. Most programming languages use the Mersenne-Twister pseudorandom generator (Matsumoto and Nishimura, 1998), offering a very long period (around 4.3×10^{6001} iterations).

Formally, the “Vanilla” Monte Carlo (sometimes called “crude” Monte Carlo) method uses a set of i.i.d samples $\mathbf{X}_n = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\}$ following the joint distribution of \mathbb{P}_X . The Monte Carlo estimator of the integral is given by:

$$I_{\mathbb{P}_X}(g) \approx \bar{y}_n^{\text{MC}} = \frac{1}{n} \sum_{i=1}^n g(\mathbf{x}^{(i)}). \quad (1.15)$$

By construction, the law of large numbers makes this estimator unbiased, however, it converges relatively slowly. Considering the images of the sample \mathbf{X}_n , one can also estimate the variance of the output random variable $\hat{\sigma}_Y^2$. The variance of the Monte Carlo estimator results from a manipulation of the central limit theorem:

$$\text{Var}\left(\bar{y}_n^{\text{MC}}\right) = \frac{1}{\sqrt{n}} \text{Var}(g(\mathbf{X})). \quad (1.16)$$

This estimator also comes with theoretical confidence intervals at $\alpha\%$, regardless of the output distribution:

$$I_{\mathbb{P}_X}(g) \in \left[\bar{y}_n^{\text{MC}} - q_\alpha \frac{\text{Var}(g(\mathbf{X}))}{\sqrt{n}}, \bar{y}_n^{\text{MC}} + q_\alpha \frac{\text{Var}(g(\mathbf{X}))}{\sqrt{n}} \right], \quad (1.17)$$

where q_α is the α -quantile of the standard normal distribution. Monte Carlo presents the advantage of being a universal method, with no bias and strong convergence guarantees. Moreover, it is worth noting that its convergence properties do not depend on the dimension of the input domain. Unlike the previous multivariate deterministic quadrature, it doesn't suffer from the curse of dimensionality. The main limit of crude Monte Carlo is its convergence speed, making it intractable for most practical cases. More recent methods aim at keeping the interesting properties of this technique while making it more efficient. Among the *variance reduction* methods, let us mention importance sampling, stratified sampling (e.g., Latin hypercube sampling), control variates and multi-level Monte Carlo. For further details, the reader may refer to Chapters 8, 9 and 10 from [Owen \(2013\)](#) and [\(Giles, 2008\)](#).

Quasi-Monte Carlo and Koksma-Hlawka inequality

Among the methods presented so far, classical deterministic quadratures are subject to the curse of dimension while Monte Carlo methods deliver contrasted performances. Quasi-Monte Carlo is a deterministic family of numerical integration schemes with respect to the uniform measure on $[0, 1]$. It offers powerful performances with strong guarantees by choosing nodes according to *low discrepancy* sequences.

The discrepancy of a set of nodes (or a design) can be seen as a metric of its uniformity. The lowest the discrepancy of a design is, the “closest” it is to uniformity.

The Koksma-Hlawka theorem ([Morokoff and Caflisch, 1995](#); [Leobacher and Pillichshammer, 2014](#)) is a fundamental result for understanding the role of the discrepancy in numerical integration.

Theorem 2 (Koksma-Hlawka). *If $g : [0, 1]^d \rightarrow \mathbb{R}$ has a bounded variation (i.e., its total variation is finite), then for any design $\mathbf{X}_n = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\} \in [0, 1]^d$:*

$$\left| \int_{[0,1]^d} g(\mathbf{x}) d\mathbf{x} - \frac{1}{n} \sum_{i=1}^n g(\mathbf{x}^{(i)}) \right| \leq V(g) D^*(\mathbf{X}_n). \quad (1.18)$$

Where $D^*(\mathbf{X}_n)$ is the star discrepancy of the design \mathbf{X}_n , while $V(g)$ quantifies the complexity of the integrand, which is related to its total variation. The reader might refer to [Leobacher and Pillichshammer \(2014\)](#) Section 3.4 for further mathematical proof.

The function's variation $V(g)$ in Eq. (1.18) can be formally defined as the Hardy-Klause variation:

$$V(g) = \sum_{u \subseteq \{1, \dots, p\}} \int_{[0,1]^u} \left| \frac{\partial^u g}{\partial \mathbf{x}^u}(\mathbf{x}_u, 1) \right| d\mathbf{x}_u. \quad (1.19)$$

In which the L_p star discrepancy of a design \mathbf{X}_n is defined as the L_p -norm of the difference between the empirical CDF of the design $\widehat{F}_{\mathbf{X}_n}$ and the CDF of the uniform distribution F_U :

$$D_p^*(\mathbf{X}_n) = \|\widehat{F}_{\mathbf{X}_n} - F_U\|_p = \left(\int_{[0,1]^d} |\widehat{F}_{\mathbf{X}_n}(\mathbf{x}) - F_U(\mathbf{x})|^p d\mathbf{x} \right)^{1/p}. \quad (1.20)$$

Additionally, the L_∞ star discrepancy can be defined from a geometric point of view. Let us first consider the number of elements from a design \mathbf{X}_n , falling in a subdomain $[\mathbf{0}, \mathbf{x})$ as $\#(\mathbf{X}_n \cap [\mathbf{0}, \mathbf{x}))$. Then, if this empirical quantification is compared with the volume of the rectangle $[\mathbf{0}, \mathbf{x})$, denoted by $\text{vol}([\mathbf{0}, \mathbf{x}))$, the star discrepancy is expressed as:

$$D^*(\mathbf{X}_n) = \sup_{\mathbf{x} \in [0,1]^d} \left| \frac{\#(\mathbf{X}_n \cap [\mathbf{0}, \mathbf{x}))}{n} - \text{vol}([\mathbf{0}, \mathbf{x})) \right|. \quad (1.21)$$

Let us point out that this star discrepancy is equivalent to the Kolmogorov-Smirnov statistic, verifying whether the design follows a uniform distribution.

One can notice how the Koksma-Hlawka inequality dissociates the quadrature performance into a contribution from the function complexity and one from the repartition of the quadrature nodes. Knowing that the complexity of the studied integrand is fixed, this property explains the motivation to generate low-discrepancy quadratures in numerical integration.

Note that the design can also be considered as a discrete distribution (uniform sum of Dirac distributions). The discrepancy can then be expressed as a probabilistic distance between this discrete distribution and the uniform distribution. A generalized discrepancy between distributions called *maximum mean discrepancy* is introduced in the Appendix D and used for efficient sampling in Chapter 4 of this manuscript.

Some famous low-discrepancy sequences (e.g., van der Corput, Halton, Sobol', Faure, etc.) can offer a bounded star discrepancy $D^*(\mathbf{X}_n) \leq \frac{C \log(n)^d}{n}$, with the constant C depending on the sequence. Therefore, using these sequences as a quadrature rule with uniform weights provides the following absolute error upper bound:

$$\left| \int_{[0,1]^d} g(\mathbf{x}) d\mathbf{x} - \frac{1}{n} \sum_{i=1}^n g(\mathbf{x}^{(i)}) \right| \leq \frac{V(g) \log(n)^d}{n}. \quad (1.22)$$

The generation of such sequences does not necessarily require more effort than pseudo-random sampling. Chapter 15 in Owen (2013) offers an extended presentation of the ways to generate different low-discrepancy sequences. For example, the van der Corput and Halton sequences rely on congruential generators.

Halton sequences in medium dimension, unfortunately introduce pathological patterns when looking at their subprojections. To overcome these limits, digital nets such as the famous Sobol' or Faure sequences were developed. Sobol' sequences are in base two and have the advantage of being extensible in dimension. Note that by construction, these sequences offer significantly lower discrepancies for specific size values. Typically, designs with sizes equal to powers of two or power of prime numbers will be favorable. To illustrate the different repartition and properties of the methods, Fig. 1.5 represents the three Monte Carlo and quasi-Monte Carlo designs (with size $n = 256$). Each is split into the first 128 points (in red) and the following 128 points (in black) to show the nested properties of the QMC sequences.

Crude Monte Carlo estimators provide a confidence associated to the estimate. This complementary information is essential to deliver an end-to-end uncertainty quantification, which

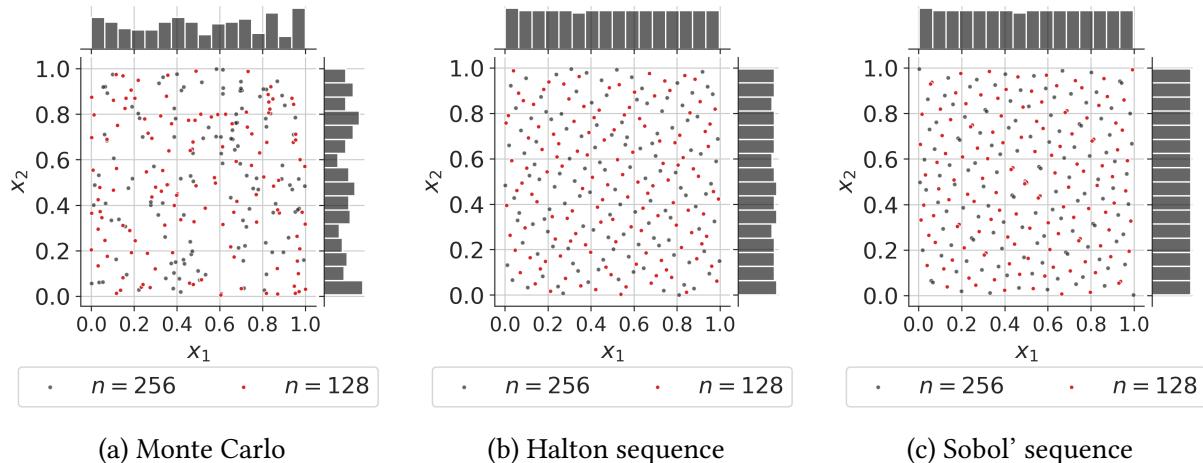


Figure 1.5 Nested Monte Carlo and quasi-Monte Carlo designs ($n = 256$)

misses in deterministic QMC methods. *Randomized quasi-Monte Carlo* (RQMC) is a method introducing some randomness in QMC in order to compute confidence intervals while benefiting from a low variance. A specific review of the randomized (also called “scrambled”) QMC is proposed by [L’Ecuyer \(2018\)](#). Various authors recommend the use of RQMC by default instead of QMC as a good practice. Recent works aim at exploring the use of these methods to estimate different quantities of interest, such as an expected value ([Gobet et al., 2022](#)) or a quantile ([Kaplan et al., 2019](#)).

Ultimately, quasi-Monte Carlo methods generate powerful integration schemes. The Koksma-Hlawka inequality associates an upper bound and a convergence rate to most integrals. A randomization overlay fades the deterministic property of these designs, allowing to compute confidence intervals. In the following, sampling techniques are presented from the numerical *design of experiments* point of view. Even if the goal might look different from numerical integration, these two topics share many methods and concepts.

OpenTURNS 2 (Numerical integration). The Python code available in Appendix D proposes a minimalistic OpenTURNS example to build multivariate quadrature rules. Figures illustrating the present section may be reproduced, using the OpenTURNS scripts available on GitHub⁵.

1.4.2 Numerical design of experiments

Numerical design of experiments aim at uniformly exploring the input domain, e.g., to build a learning set for a regression model, or to initialize a multi-start optimization strategy. A design of experiment (also simply called design) is qualified as *space-filling* when it uniformly covers a domain. As well as in integration, a design is used to propagate uncertainties through a numerical model (or a physical experiment). However, a difference comes from the fact that

⁵https://github.com/efekhari27/thesis/blob/main/numerical_experiments/chapter1/integration.ipynb

this community often works with designs of very limited size. Users of designs of experiments also consider various properties.

- Some might be interested in the sequentiality of a sampling method, to eventually add new points as they get their computational budget extended.
- Some might request a sampling method conserving its properties in any subdomains. This second property can be useful to reduce the problem's dimension by dropping a few unimportant variables (see the following Section 1.6 on global sensitivity analysis).

Different metrics are commonly used to quantify how space-filling a design of experiments is. The previously introduced of discrepancies are an example of space-filling metrics. Other types of space-filling metrics rely on purely geometrical considerations.

This section will first define a few space-filling metrics. Secondly, the *Latin hypercube sampling* (LHS) will be introduced as a variance-reduction method that became popular the UQ community. Finally, a general discussion on uncertainty propagation with respect to non-uniform measures will be presented.

Space-filling metrics and properties

Space-filling criteria are key to evaluating designs and are often used to optimize their performances. In the previous section, the star discrepancy was introduced as a distance of a finite design to uniformity. However, the L_∞ star discrepancy is hard to estimate, fortunately, Warnock (1972) elaborated an explicit expression specific to the L_2 star discrepancy:

$$[D_2^*(\mathbf{X}_n)]^2 = \frac{1}{9} - \frac{2}{n} \sum_{i=1}^n \prod_{l=1}^d \frac{(1-x_l^{(i)})}{2} + \frac{1}{n^2} \sum_{i,j=1}^n \prod_{l=1}^d \left[1 - \max(x_l^{(i)}, x_l^{(j)}) \right]. \quad (1.23)$$

One can notice that this expression is similar to the Cramér-von Mises test statistic. Even if this expression is tractable, Fang et al. (2018) detailed its limits: the star L_2 discrepancy generates designs that are not robust to projections in sub-spaces; it is not an invariant metric by rotation and reflection; and finally, by construction, L_p discrepancies give a disproportionate role to the point $\mathbf{0}$ by anchoring the box $[\mathbf{0}, \mathbf{x}]$.

Two improved criteria were proposed by Hickernell (1998) with the *centered L_2 discrepancy* and the *wrap-around L_2 discrepancy*. Those are widely used in practice since they solve the previous limits while satisfying the Koksma-Hlawka inequality with a modification of the total variation. Let us introduce the explicit formula of the centered L_2 discrepancy:

$$\begin{aligned} CD_2^*(\mathbf{X}_n) = & \left(\frac{13}{12} \right)^d - \frac{2}{n} \sum_{i=1}^n \prod_{l=1}^d \left(1 + \frac{1}{2} |x_l^{(i)} - 0.5| - \frac{1}{2} |x_l^{(i)} - 0.5|^2 \right) \\ & + \frac{1}{n^2} \sum_{i,j=1}^n \prod_{l=1}^d \left(1 + \frac{1}{2} |x_l^{(i)} - 0.5| + \frac{1}{2} |x_l^{(j)} - 0.5| - \frac{1}{2} |x_l^{(i)} - x_l^{(j)}| \right). \end{aligned} \quad (1.24)$$

As an alternative to discrepancies, many geometrical criteria exist to assess a space-filling design. The most common way to do so is to maximize the minimal distance among the pairs of Euclidian distances between the points of a design. The criterion to maximize is then simply called the *minimal distance* of a design (denoted ϕ_{\min}). For numerical reasons, the ϕ_p criterion is often used instead of the minimal distance. The following ϕ_p criterion converges towards the minimum distance as $p \geq 1$ tends to infinity:

$$\phi_{\min}(\mathbf{X}_n) = \min_{i \neq j} \|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\|_2, \quad \phi_p(\mathbf{X}_n) = \sum_{i=1}^j \sum_{j=1}^n \left(|x^{(i)} - x^{(j)}|^{-p} \right)^{\frac{1}{p}}. \quad (1.25)$$

More space-filling criteria are reviewed in [Abtini \(2018\)](#) and in the Appendix A from [Da Veiga et al. \(2021\)](#). Further relations between some mathematical objects related to space-filling are developed in [Pronzato and Müller \(2012\)](#). These space-filling metrics are widely used to optimize a different sampling technique.

Latin hypercube sampling

Latin hypercube sampling is a method introduced in 1979 ([Mckay et al., 1979](#)), initially for numerical integration. In a bounded domain, this stratified sampling technique forces the distribution of each sub-projection to be as uniform as possible. To do so, for an n -sized design, each marginal domain is divided into n identical segments. This creates a regular grid of n^d squared cells over the domain.

Then, a Latin hypercube design (LHD) does not allow more than one point within a segment. That way, new LHDs can be built as a permutation of the marginals of an existing LHD. Inside each selected cell from the grid, the point can be placed at the center or randomly.

Various contributions proposed a variance, and a central limit theorem to LHS ([Koehler and Owen, 1996](#)). Similarly to the Monte Carlo variance in Eq. (1.16), LHS variance can be expressed as:

$$\text{Var}\left(\bar{y}_n^{\text{LHS}}\right) = \frac{1}{\sqrt{n}} \text{Var}(g(\mathbf{X})) - \frac{C}{n} + o\left(\frac{1}{n}\right). \quad (1.26)$$

Where C is a positive constant, showing that the LHS usually reduces the variance for numerical integration. Because of its stratified structure, LHS can generate poor designs from a space-filling point of view (see e.g., the illustration in Fig. 1.6a). The following section presents various methods aiming at optimizing LHDs.

Optimized Latin hypercube sampling

To improve the space-filling property of LHD, it is common to add an optimization step. The goal of this optimization is to improve a space-filling criterion by generating LHD from permutations of an initial LHD. [Damblin et al. \(2013\)](#) reviews LHS optimization using different discrepancy criteria and subprojection properties. This optimization can be performed by different algorithms, such as the stochastic evolutionary algorithm or simulated annealing. The results from

this work show that LHD optimized by L_2 centered or wrap-around discrepancies offer strong robustness to two-dimensional projections. It also shows that these designs keep this property for dimensions larger than 10, while scrambled Sobol' sequences lose it. Fig. 1.6 illustrates two LHD, optimized by the L_2 centered discrepancy and the geometrical ϕ_p . The space-filling difference is not obvious in two dimensions, and they both spread uniformly.

More recent work developed different ways to optimize LHD. Among them, let us mention the maximum projection designs from Joseph et al. (2015) which relies on the optimization of a geometrical criterion and delivers interesting performances. In the same vein, the uniform projection designs from Sun et al. (2019) is also a method to optimize LHS, this time based on a criterion averaging discrepancies between each pairs of marginals.

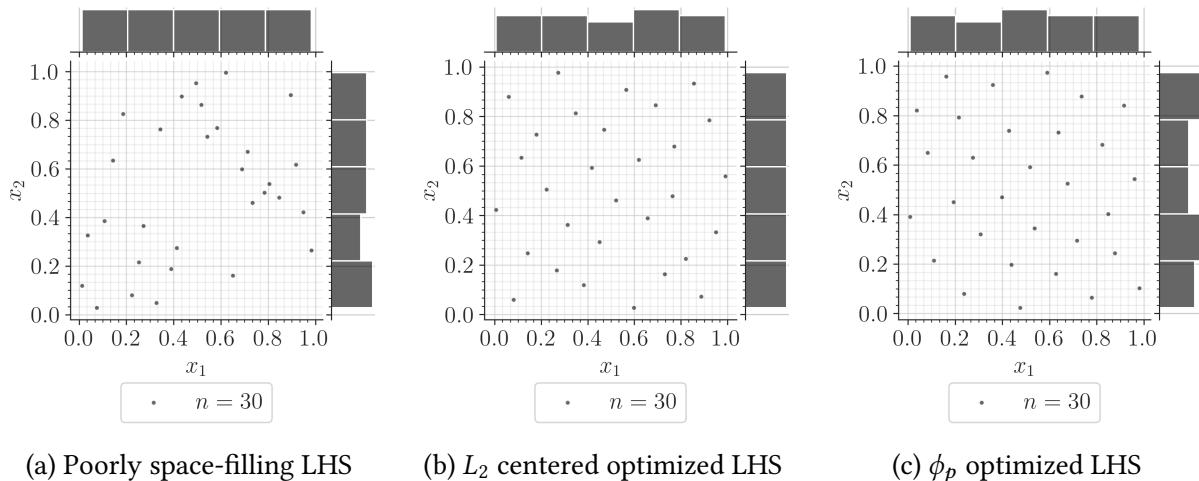


Figure 1.6 Latin hypercube designs with poor and optimized space-filling properties ($n = 8$)

OpenTURNS 3 (Design of experiments). The Python code available in Appendix D proposes a minimalistic OpenTURNS example to build an LHS and an LHS optimized w.r.t. to a space-filling metric (here the L2-centered discrepancy) using the simulated annealing algorithm. Figures illustrating the present section may be reproduced, using the OpenTURNS scripts available on GitHub⁶.

1.4.3 Summary and discussion

A wide panel of sampling techniques exists for numerical integration or design of experiments purposes. In both cases, the studied domain is bounded and the targeted measure is uniform. However, uncertainty propagation is often performed on complex input distributions, with possibly unbounded domains. In uncertainty quantification, this step might be referred to as the estimation of the output random variable's central tendency (i.e., its mean and variance). Central tendency estimation is a numerical integration with respect to any input distribution,

⁶https://github.com/efekhari27/thesis/blob/main/numerical_experiments/chapter1/designofexperiments.ipynb

sometimes called *probabilistic integration* (Briol et al., 2019) as part of *probabilistic numerics* (Oates and Sullivan, 2019).

To generate i.i.d samples following any distribution (i.e., non-uniform), one may use *inverse transform* sampling. After generating samples in the unit hypercube, the inverse CDF function (i.e., quantile function) is applied on marginals. Finally, possible dependence effects may be added using the Sklar theorem Eq. (1).

One may wonder if the properties from the uniform design are conserved after this nonlinear transformation. Li et al. (2020) explores this question from a discrepancy point of view. The authors find correspondences between discrepancies with respect to uniformity and discrepancy with respect to the target distribution. However, this result show practical limits, sometimes making the interpretation of the last discrepancy easier. This question will be further discussed in the [Chapter 4], using a more general framework.

Let us also remark that, depending on the distribution, defining the inverse CDF is not always possible. For example, samples following truncated distributions or mixture distributions might sometime be generated with a different technique. The *acceptance-rejection* method offers a versatile generation only based on the PDF f_x . Assuming that a well-known proposal PDF f_x^* exists such that $f_x \leq c \times f_x^*, c \in [1, +\infty]$. One may generate a sample according to $c \times f_x^*$ and only retain from this sample the points under the PDF f_x . Note that some sampling methods, such as QMC, are not well suited with acceptance-rejection since their structure gets perturbed.

In this section, many methods were presented to propagate input uncertainties against a deterministic function. The propagation with the three following goals and contexts were introduced:

- building a quadrature rule for numerical integration against a uniform distribution,
- creating a space-filling design of experiments to uniformly explore the space, often in a small data context (e.g., to build the learning set of a surrogate model),
- generating a design for central-tendency estimation, which is simply a numerical integration against a nonuniform density.

These three objectives have been explored in different communities, but they actually share similar methods. They all have in common the general analysis (i.e., global behavior) of the output random variable. However, some studies require to shift the focus towards specific areas of the output random variables. When using uncertainty propagation to perform risk analysis, the events studied are often contained in the tails of the output distribution. In this case, dedicated uncertainty propagation methods will significantly improve the estimation of the associated statistical quantities.

1.5 Uncertainty propagation for rare event estimation

This section aims at presenting another type of uncertainty propagation. In the context of a risk analysis applied to the engineering field, the reliability of a system needs to be assessed. Most often, a risk measure associated with a failure mode of the studied system is estimated.

Since most systems studied in risk analysis must be highly reliable, the occurrence of such event is qualified as rare. Only a small amount of extreme input conditions or an unlikely unfavorable combinations of inputs lead to the failure of the system. Hence, the usage of the equivalent terms *reliability analysis* and *rare event estimation*. The notion of risk associated with an event is often decomposed as a product of its likelihood and its consequences. The failure of a system might be very rare, but its consequences can be severe (e.g., civil engineering structures, nuclear infrastructure, telecommunication networks, electrical grid, railway signalling, etc.).

Different risk measures (i.e., quantities of interest related to the tail of the distributions) can be studied depending on the type of risk analysis. Quantiles are a first conservative measure, widely used for risk analysis. The α -quantile q_α of the output random variable Y is defined as:

$$q_\alpha = \inf_{y \in \mathbb{R}} \{F_Y(y) \geq \alpha\}, \quad \alpha \in [0, 1]. \quad (1.27)$$

As an alternative, one can define a scalar safety threshold y_{th} that should not be exceeded to keep the system safe. Then, a second risk measure is the probability of exceeding this safety threshold, also called *failure probability*:

$$p_f = \mathbb{P}(Y \geq y_{\text{th}}), \quad y_{\text{th}} \in \mathbb{R}. \quad (1.28)$$

To illustrate this quantity, Fig. 1.7 shows the one-dimensional propagation of a normal distribution (represented by the PDF on the left), through a function $g(\cdot)$. The probability of exceeding a given threshold y_{th} is represented by the area in red under the output PDF on top. An interesting reflection on the use and the interpretation of risk measures including measures from the finance domain such as the *conditional value-at-risk* (also called superquantile) is presented in [Rockafellar and Royset \(2015\)](#).

In the following section, the formalism for reliability analysis problems will be first presented, then the main methods solving this specific problem will be introduced. Note however that the present work will not address the problems of time-dependent reliability analysis tackled in [Hawchar et al. \(2017\)](#).

1.5.1 Problem statement

Following to the UQ methodology, the behavior of the system is modeled by $\mathcal{M}(\cdot)$. Considering the problem of exceeding a safety threshold in Eq. (1.28), the system's performance is commonly defined as the difference between the model's output and the safety threshold $y_{\text{th}} \in \mathbb{R}$. Formally,

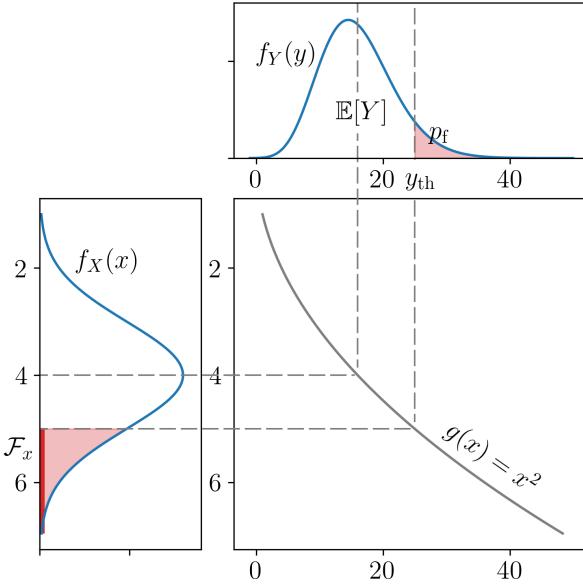


Figure 1.7 One-dimensional reliability analysis example

the *limit-state function* (LSF) is a deterministic function $g : \mathbb{R} \rightarrow \mathbb{R}$ quantifying this performance:

$$g(\mathbf{x}) = y_{\text{th}} - \mathcal{M}(\mathbf{x}). \quad (1.29)$$

Depending on the sign of its images, this function splits the inputs space into two disjoint and complementary domains called the *failure domain* \mathcal{F}_x , and the *safe domain* \mathcal{S}_x which are defined as:

$$\mathcal{F}_x = \{\mathbf{x} \in \mathcal{D}_x \mid g(\mathbf{x}) \leq y_{\text{th}}\}, \quad \mathcal{S}_x := \{\mathbf{x} \in \mathcal{D}_x \mid g(\mathbf{x}) > y_{\text{th}}\}. \quad (1.30)$$

The border between these two domains is a hypersurface called *limit-state surface* (LLS), defined by $\mathcal{F}_x^0 := \{\mathbf{x} \in \mathcal{D}_x \mid g(\mathbf{x}) = 0\}$. Similarly to any UQ study using a numerical model, this problem may require to be resolved using a limited number of calls to a black-box simulator. The difficulties in a reliability problem come from the properties of the LSF: nonlinear, costly to evaluate or with a multimodal failure domain. Additionally, note that the reliability problem can be the composition of multiple reliability problems, often modeled as system of problems in series and parallel.

A rare event estimation results from a particular uncertainty propagation through the LSF. Considering the output variable of interest $g(\mathbf{X})$, its probability of being negative (i.e., in the failure domain) is a common risk measure. The so-called *failure probability*, denoted by p_f , is the quantity of interest for reliability analysis considered in this work. This quantity is formally written⁷:

$$p_f = \mathbb{P}(Y \geq y_{\text{th}}) = \mathbb{P}(g(\mathbf{X}) \leq 0) = \int_{\mathcal{F}_x} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = \int_{\mathcal{D}_x} \mathbb{1}_{\mathcal{F}_x}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \quad (1.31)$$

⁷Note that this probabilistic integration is usually written using the PDF $f_{\mathbf{X}}(\cdot)$, but it could identically be expressed in terms of probability measure by taking $f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = d\mathbb{P}_{\mathbf{X}}(\mathbf{x})$, $\forall \mathbf{x} \in \mathcal{D}_x$.

were the indicator function applied to the failure domain returns $\mathbb{1}_{\{\mathcal{F}_X\}}(x) = 1$ if $x \in \mathcal{F}_X$ and $\mathbb{1}_{\{\mathcal{F}_X\}}(x) = 0$ otherwise. Rare event estimation implies both contour finding (i.e., characterizing the LSF) and an estimation strategy targeting the failure domain (often with a limited number of simulations). Note that failure events are qualified as rare when their failure probability has an order of magnitude between $10^{-2} \leq p_f \leq 10^{-9}$ (see e.g., [Lemaire et al. 2009](#)).

Instead of directly performing a reliability analysis in the physical space (i.e., \mathbf{x} -space), these problems are usually solved in the *standard normal space* (i.e., \mathbf{u} -space). Working in the standard space reduces numerical issues caused by potentially unscaled or asymmetric marginals. Moreover, a larger panel of methods can be applied in the standard space since the random inputs become independent. The bijective mapping between these two spaces is called an “iso-probabilistic transformation”, denoted by $T : \mathcal{D}_X \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\mathbf{x} \mapsto T(\mathbf{X}) = \mathbf{u} = (u_1, \dots, u_d)^\top$. When considering any random vector $\mathbf{X} = (X_1, \dots, X_d)^\top$ and the independent standard Gaussian vector $\mathbf{U} = (U_1, \dots, U_d)^\top$, the following equalities hold:

$$\mathbf{U} = T(\mathbf{X}) \Leftrightarrow \mathbf{X} = T^{-1}(\mathbf{U}). \quad (1.32)$$

A reliability problem can be expressed in the standard normal space. Let us first consider the transformed limit-state function \check{g} defined as:

$$\check{g} : \begin{array}{ccc} \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ \mathbf{u} & \longmapsto & \check{g}(\mathbf{u}) = (g \circ T^{-1})(\mathbf{u}). \end{array} \quad (1.33)$$

Since this transformation is a diffeomorphism⁸, one can apply the change of variable $\mathbf{x} = T(\mathbf{u})$ to express the reliability problem from Eq. (1.31) in the standard space:

$$p_f = \mathbb{P}(\check{g}(\mathbf{U}) \leq 0) = \int_{\mathcal{F}_u} \varphi_d(\mathbf{u}) d\mathbf{u} = \int_{\mathbb{R}^d} \mathbb{1}_{\mathcal{F}_u}(\mathbf{u}) \varphi_d(\mathbf{u}) d\mathbf{u}, \quad (1.34)$$

with the transformed failure domain denoted by $\mathcal{F}_u = \{\mathbf{u} \in \mathbb{R}^d \mid \check{g}(\mathbf{u}) \leq 0\}$, and the d -dimensional standard Gaussian PDF $\varphi_d(\mathbf{u}) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{\|\mathbf{u}\|_2^2}{2}\right)$. The fact that the failure probability is invariant by this transformation allows the analyst to estimate this quantity in both spaces.

Different types of transformations exist, such as the Rosenblatt or the generalized Nataf transformation introduced by [Lebrun \(2013\)](#). In practice, the transformation choice depends on the properties of the input distribution studied. For example in OpenTURNS, depending on the three following cases, different types of transformations are applied:

- for elliptical distributions, a linear Nataf transformation is applied;
- for distributions with an elliptical copula, the generalized Nataf transformation is used;
- otherwise, the Rosenblatt transformation is used.

⁸Considering two manifolds A and B , a transformation $T : A \rightarrow B$ is called a diffeomorphism if it is a differentiable bijection with a differentiable inverse $T^{-1} : B \rightarrow A$.

1.5.2 Rare event estimation methods

The main risk measure chosen for rare event estimation in this work is the previously introduced failure probability. Therefore, let us recall that the goal is to build an efficient estimation (or approximation) of the following d -dimensional integral:

$$p_f = \int_{\mathcal{D}_x} \mathbb{1}_{\mathcal{F}_x}(\mathbf{x}) f_x(\mathbf{x}) d\mathbf{x} \quad (1.35)$$

In the context of rare event estimation using costly to evaluate numerical models, the simulation budget is often limited to n runs with $p_f \ll \frac{1}{n}$. Which explains the need for specific methods offering approximations or simulations targeting the unknown failure domain. Two types of rare event estimation methods are classically presented: first, using approximation approaches, second, using sampling techniques. This section introduced the commonly used rare event methods, see [Morio and Balesdent \(2015\)](#) for a more exhaustive review.

First and second order reliability methods (FORM/SORM)

The well-known first and second order reliability methods (FORM and SORM) both rely on a geometric approximation to estimate a failure probability ([Lemaire et al., 2009](#)). They extrapolate a local approximation of the LSF built in the vicinity of a *most-probable-failure-point* (MPFP), also called *design point*.

Working in the standard space, the methods first look for this MPFP, denoted P^* , with coordinates \mathbf{u}^* . To find it, one can solve the following quadratic optimization problem:

$$\mathbf{u}^* = \arg \max_{\mathbf{u} \in \mathbb{R}^d} (\mathbb{1}_{\mathcal{F}_u}(\mathbf{u}) \varphi_d(\mathbf{u})). \quad (1.36)$$

Using the properties of the standard space allows us to rewrite it as:

$$\mathbf{u}^* = \arg \max_{\mathbf{u} \in \mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{\mathbf{u}^\top \mathbf{u}}{2}\right) \quad \text{s.t. } \mathbf{u} \in \mathcal{F}_u \quad (1.37)$$

$$= \arg \min_{\mathbf{u} \in \mathbb{R}^d} \mathbf{u}^\top \mathbf{u} \quad \text{s.t. } \check{g}(\mathbf{u}) \leq 0. \quad (1.38)$$

This problem then becomes a quadratic optimization under nonlinear constraint. It is classically solved by gradient decent algorithms (e.g., Abdo-Rackwitz algorithm [Abdo and Rackwitz, 1991](#)) but can also use gradient-free techniques (e.g., Cobyla algorithm [Powell, 1994](#)). This point defines the smallest Euclidian distance between the LSS and the origin of the standard space. To understand its role in the reliability problem, let us recall that the density of the standard normal present an exponential decay in its radial and tangential direction. Then, P^* is the point with the biggest contribution to the failure probability (see the illustration in Fig. 1.8).

This distance between the origin and P^* is another risk measure, defined as the *Hasofer-Lind reliability index* (Lemaire et al., 2009), $\beta \in \mathbb{R}$ such that:

$$\beta = \|\mathbf{u}^*\|_2 = \boldsymbol{\alpha}^\top \mathbf{u}^*, \quad \text{s.t.} \quad \boldsymbol{\alpha} = \frac{\nabla_{\mathbf{u}} \check{g}(\mathbf{u})}{\|\nabla_{\mathbf{u}} \check{g}(\mathbf{u})\|_2}. \quad (1.39)$$

The vector $\boldsymbol{\alpha}$ is the unit vector pointing at P^* from the origin point.

Then, FORM aims at approximating the limit-state function $\check{g}(\cdot)$ by its first-order Taylor expansion around the MPFP, denoted $\check{g}_1(\mathbf{u}^*)$:

$$\begin{aligned} \check{g}(\mathbf{u}) &= \check{g}_1(\mathbf{u}^*) + o\left(\|\mathbf{u} - \mathbf{u}^*\|_2^2\right) \\ &= \check{g}(\mathbf{u}^*) + \nabla_{\mathbf{u}} \check{g}(\mathbf{u}^*)^\top (\mathbf{u} - \mathbf{u}^*) + o\left(\|\mathbf{u} - \mathbf{u}^*\|_2^2\right) \\ &= \|\nabla_{\mathbf{u}} \check{g}(\mathbf{u})\|_2 (\boldsymbol{\alpha}^\top \mathbf{u}^* - \boldsymbol{\alpha}^\top \mathbf{u}) + o\left(\|\mathbf{u} - \mathbf{u}^*\|_2^2\right) \end{aligned} \quad (1.40)$$

Using $\check{g}_1(\cdot)$ as an approximation of the LSF, the failure probability can be approximated as:

$$p_f \approx p_f^{\text{FORM}} = \mathbb{P}(-\boldsymbol{\alpha}^\top \mathbf{u} \leq -\beta) = \Phi(-\beta), \quad (1.41)$$

with $\Phi(\cdot)$ the CDF of the standard Gaussian. Depending on the properties of the LFS, this approximation will be more or less accurate. Note that for a purely linear LFS, $p_f = p_f^{\text{FORM}}$. When the function is nonlinear, adding a quadratic term to the Taylor expansion can help the approximation. The approximation method is then called SORM for *second order reliability method*. However, this added complexity implies the computation of Hessian matrices, which can be complicated (see the Chapter 1 from Bourinet 2018 for their estimation).

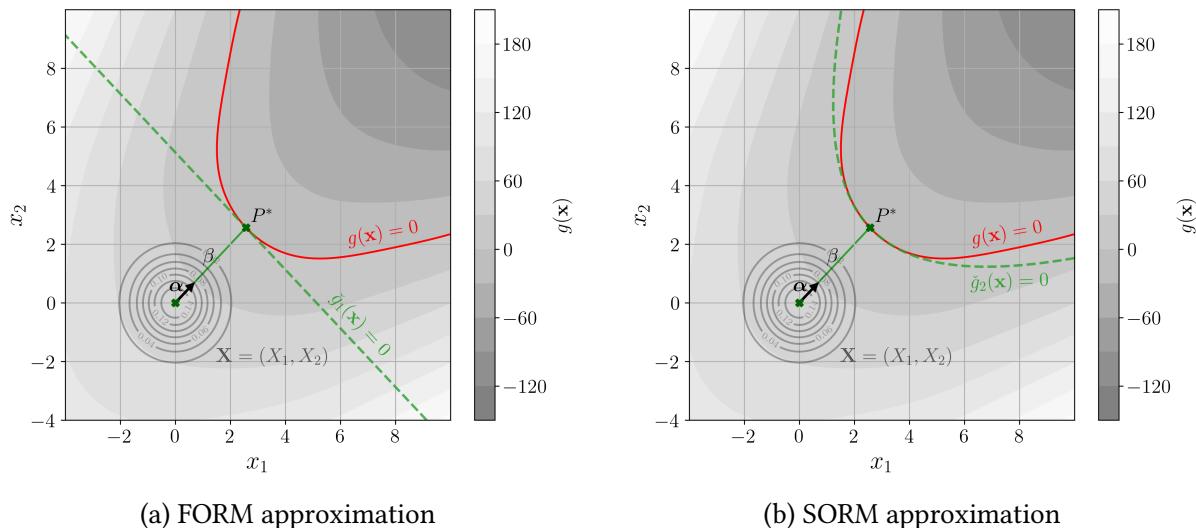


Figure 1.8 FORM and SORM approximation on a two-dimensional reliability problem

When the MPFP is not unique, the application of these methods might lead to important errors. From a geometrical point of view, having more than one MPFP means that more than one failure zones are at the same euclidean distance of the origin. Applying a FORM or SORM resolution in this particular case leads to the estimation of only one of the failure areas. The

muti-FORM algorithm (see [Der Kiureghian and Dakessian \(1998\)](#)) prevents this situation by applying successive FORM. Once the first MPFP $P^{*(1)}$ found, the LSS is modified by removing a nudge to find to following MPFP $P^{*(2)}$, positioned at a similar distance but in a different direction.

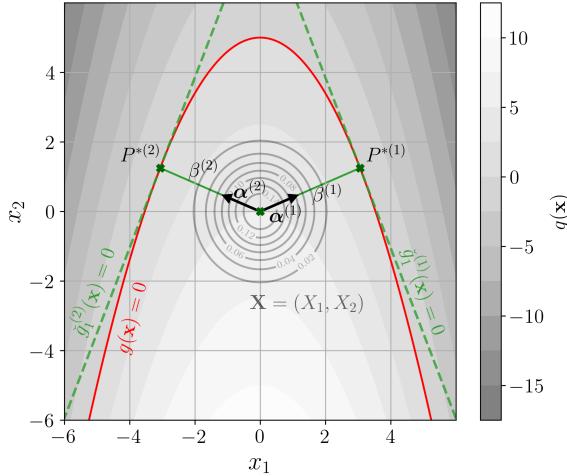


Figure 1.9 Multi-FORM approximation on an example with two MPFPs

Overall, FORM and SORM methods deliver a very efficient approximation of small probabilities for relatively simple problems (in terms of linearity and dimension). For this reason, they have been widely used in the practical context of limited simulation budget. [Straub \(2014\)](#) illustrates the efficiency of FORM approaches on industrial cases such as probabilistic fatigue damage. However, these methods present serious limits as the dimension increases (see the discussion in the Chapter 1 from [Chabridon 2018](#)). Additionally, their main drawback is the lack of complementary information concerning the confidence of the results. The textbook example illustrated in Fig. 1.9 shows that the method might miss some important areas of the failure domain, leading to poor estimations. As an alternative to approximation methods, simulation-based methods often provide to the analyst an assessment of the estimation's confidence.

Monte Carlo

Crude Monte Carlo sampling is a universal and empirical method for uncertainty propagation. As introduced earlier, it relies on the pseudo-random generation of i.i.d. samples $\{\mathbf{x}^{(i)}\}_{i=1}^n \stackrel{\text{i.i.d.}}{\sim} f_{\mathbf{X}}$. Only the estimator is now written using the indicator function applied to the LSF:

$$p_f \approx \hat{p}_f^{\text{MC}} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\mathcal{F}_x}(\mathbf{x}^{(i)}). \quad (1.42)$$

Provided that the failure probability is bounded, this estimator converges towards it almost surely according to the LLN. Once again, Monte Carlo offers an unbiased estimator, regardless of the problem's dimension or the regularity of the function $g(\cdot)$. Additionally, the variance of

this estimator is fully known:

$$\text{Var}(\widehat{p}_f^{\text{MC}}) = \frac{1}{n} p_f(1 - p_f). \quad (1.43)$$

The variance of this estimator can be used to build its confidence interval according to the central limit theorem (similarly to the ones from Eq. (1.17)). Because of the small scale of the quantities manipulated in rare event estimation, the estimator's coefficient of variation is also widely used:

$$\delta_{\widehat{p}_f^{\text{MC}}} = \frac{\sqrt{\text{Var}(\widehat{p}_f^{\text{MC}})}}{\mathbb{E}[\widehat{p}_f^{\text{MC}}]} = \sqrt{\frac{1 - p_f}{np_f}}. \quad (1.44)$$

In theory, Monte Carlo estimation presents multiple advantages for rare event estimation. First, this method can be applied directly in the physical space, without transformation (which is practical for complex input distributions). Second, it does not suffer from the curse of dimensionality. Third, it is qualified as embarrassingly parallel method since each of the numerical simulations are independent. Finally, it offers strong convergence guarantees and complementary information on the estimation confidence. These properties often make Monte Carlo the reference method in rare event estimation benchmarks.

However, the advantages of this estimator are shadowed by its slow convergence. To estimate a target failure probability $p_f = 10^{-\alpha}$, a Monte Carlo estimation with a convergence level $\delta_{\widehat{p}_f^{\text{MC}}} = 0.1$ famously requires $n = 10^{\alpha+2}$ simulations.

In the context of rare event estimation, Monte Carlo needs a number of simulation that is often prohibitive in practice. This excessive simulation budget comes from the fact that the vast majority of the samples drawn from the input distribution are not in the failure domain.

Importance sampling

Importance sampling (IS) is a variance reduction method, aiming at improving the performances of crude Monte Carlo sampling. In the context of rare event estimation, the main idea is to deliberately introduce a bias in the sampled density, shifting it towards the failure domain. If this shift actually goes towards the failure domain, it allows drawing more points in it, leading to a better estimate of our quantity.

The challenge in importance sampling is to pick a relevant *instrumental* distribution h_X (also called *auxiliary* distribution) to replace the distribution f_X . Then, by introducing the fully known *likelihood ratio* $w_X(x) = \frac{f_X(x)}{h_X(x)}$, one can rewrite $f_X(x) = w_X(x)h_X(x)$ and inject it in the failure probability expression:

$$p_f = \int_{\mathcal{D}_X} \mathbb{1}_{\mathcal{F}_X}(x) f_X(x) dx = \int_{\mathcal{D}_X} \mathbb{1}_{\mathcal{F}_X}(x) w_X(x) h_X(x) dx. \quad (1.45)$$

This simple writing trick allows us to integrate against the auxiliary distribution. With a Monte Carlo method, this task should be easier than integrating directly against the initial distribution.

The importance sampling estimator of the failure probability is defined for a sample drawn on the auxiliary distribution $\{\mathbf{x}^{(i)}\}_{i=1}^n \stackrel{\text{i.i.d.}}{\sim} h_{\mathbf{X}}$:

$$\hat{p}_f^{\text{IS}} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}^{(i)}) w_{\mathbf{X}}(\mathbf{x}^{(i)}). \quad (1.46)$$

Similarly to Monte Carlo, this estimator is unbiased, however, its variance is defined as:

$$\text{Var}(\hat{p}_f^{\text{IS}}) = \frac{1}{n} \left(\mathbb{E}_h \left[\left(\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{X}) \frac{f_{\mathbf{X}}(\mathbf{X})}{h_{\mathbf{X}}(\mathbf{X})} \right)^2 \right] - p_f^2 \right). \quad (1.47)$$

The quality of the variance reduction associated to this technique fully depends on the choice of the instrumental distribution. In fact, importance sampling can lead to higher variance than crude Monte Carlo when the instrumental distribution is poorly chosen (Owen and Zhou, 2000). However, an optimal instrumental distribution h_{opt} theoretically gives the smallest variance by setting it equal to zero in Eq. (1.47):

$$h_{\text{opt}}(\mathbf{x}) = \frac{\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})}{p_f}. \quad (1.48)$$

The optimal expression above is unfortunately not usable in practice since it includes the targeted quantity p_f . Considering this framework, various techniques intend to define instrumental distributions as close as possible to this theoretical result. An important review of the use of importance sampling in the context of reliability analysis was proposed by Tabandeh et al. (2022).

The most immediate solution is to combine the information provided by the results of FORM with importance sampling, simply called FORM-IS. In practice, the instrumental distribution is defined as the initial distribution centered on the design point resulting from FORM. Fig. 1.10 illustrates on the same two-dimensional case, the estimation by Monte Carlo and importance sampling centered on the design point. The points in red reached the failure domain and their number seem insufficient for Monte Carlo. Note that comparing the results from FORM and FORM-IS allows to assess the nonlinearity of the LSF in the vicinity of the design point. This strategy is simple to implement, but it inherits the main drawbacks of FORM, such as the limits related to multiple failure areas (see the example illustrated in Fig. 1.9). Finally, other importance sampling schemes integrate adaptive mechanisms, progressively leading the sampling towards the failure domain (Bugallo et al., 2017).

Adaptive importance sampling by cross-entropy

The *cross-entropy-based adaptive importance sampling* (CE-AIS) is an adaptive strategy, optimizing the IS variance reduction by searching the best instrumental distribution within a parametric

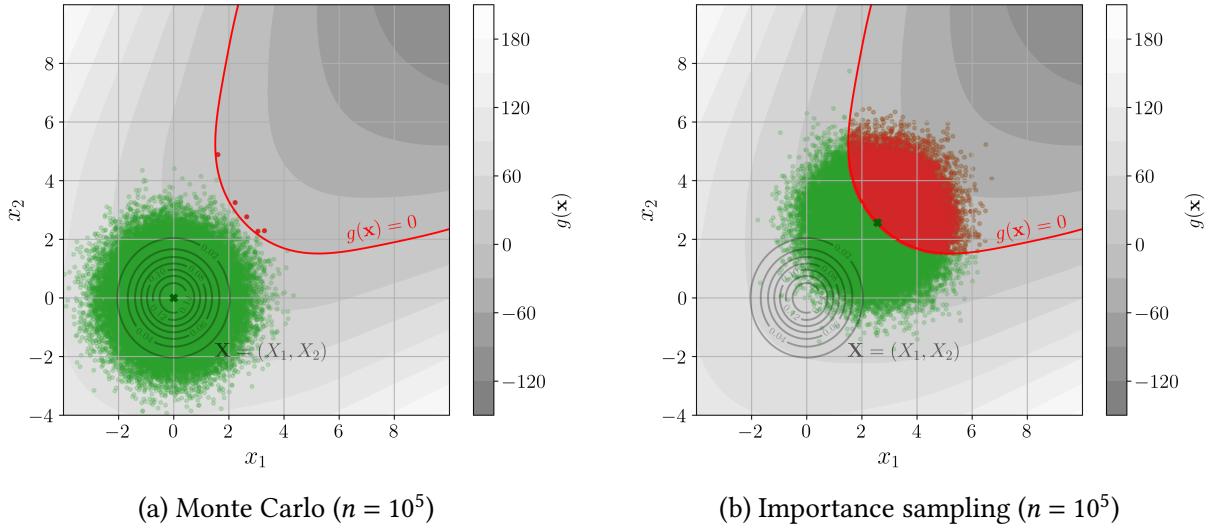


Figure 1.10 Illustration of a rare event estimation.

family. Let us consider the distribution h_λ , belonging to the parametric family \mathcal{H}_λ , defined as:

$$\mathcal{H}_\lambda = \left\{ \mathbf{x} \mapsto h_X(\mathbf{x}|\boldsymbol{\lambda}) = h_\lambda(\mathbf{x}), \quad \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_p) \in \mathcal{D}_\lambda \subseteq \mathbb{R}^p \right\}. \quad (1.49)$$

The early work of [Bucher \(1988\)](#) only included normal distributions to minimize the IS variance w.r.t. the parameter $\boldsymbol{\lambda}$, using Eq. (1.47) the optimization simplifies as:

$$\boldsymbol{\lambda}^* = \arg \min_{\boldsymbol{\lambda} \in \mathcal{D}_\lambda} \mathbb{E}_{h_\lambda} [\mathbb{1}_{\mathcal{F}_x}(X) w_X(X)^2]. \quad (1.50)$$

However, this optimization strategy requires sampling with respect to the instrumental distribution at each optimization iteration, which was overcome by a different approach.

The “cross-entropy” (CE) method uses Kullback-Leibler (KL) divergence to optimize importance sampling. KL divergence is a dissimilarity measure between distributions, expressed between the parametric instrumental distribution h_λ and the optimal one h_{opt} :

$$D_{\text{KL}}(h_{\text{opt}} || h_\lambda) = \int_{\mathcal{D}_x} \log \left(\frac{h_{\text{opt}}(\mathbf{x})}{h_\lambda(\mathbf{x})} \right) h_{\text{opt}}(\mathbf{x}) d\mathbf{x} \quad (1.51a)$$

$$= \int_{\mathcal{D}_x} \log(h_{\text{opt}}(\mathbf{x})) h_{\text{opt}}(\mathbf{x}) d\mathbf{x} - \int_{\mathcal{D}_x} \log(h_\lambda(\mathbf{x})) h_{\text{opt}}(\mathbf{x}) d\mathbf{x}. \quad (1.51b)$$

[Rubinstein and Kroese \(2004\)](#) simplify the expression of the optimization problem minimizing the KL divergence, which is most often convex and differentiable w.r.t. $\boldsymbol{\lambda}$:

$$\boldsymbol{\lambda}^* = \arg \min_{\boldsymbol{\lambda} \in \mathcal{D}_\lambda} D_{\text{KL}}(h_{\text{opt}} || h_\lambda). \quad (1.52)$$

By injecting the expression in Eq. (1.51b), the optimization problem simply becomes function of an expected value over the initial density f_X :

$$\lambda^* = \arg \max_{\lambda \in \mathcal{D}_\lambda} \int_{\mathcal{D}_x} \log(h_\lambda(\mathbf{x})) h_{\text{opt}}(\mathbf{x}) d\mathbf{x} = \arg \max_{\lambda \in \mathcal{D}_\lambda} \mathbb{E}_{f_X} [\mathbb{1}_{\mathcal{F}_X}(\mathbf{X}) \log(h_\lambda(\mathbf{X}))]. \quad (1.53)$$

To directly estimate this expected value, the failure probability p_f should not be too rare, which allows to use an empirical estimator of the expected value:

$$\lambda^* = \arg \max_{\lambda \in \mathcal{D}_\lambda} \sum_{i=1}^n \mathbb{1}_{\mathcal{F}_X}(\mathbf{x}^{(i)}) \log(h_\lambda(\mathbf{x}^{(i)})), \quad \{\mathbf{x}^{(i)}\}_{i=1}^n \stackrel{\text{i.i.d.}}{\sim} f_X. \quad (1.54)$$

Eventually, this optimization can be solved by cancelling the gradient:

$$\sum_{i=1}^n \mathbb{1}_{\mathcal{F}_X}(\mathbf{x}^{(i)}) [\nabla \log(h_{\lambda^*})](\mathbf{x}^{(i)}) = \mathbf{0}. \quad (1.55)$$

According to Rubinstein and Kroese (2004), this system of equations has a unique analytical solution when assuming that the instrumental distribution belongs to the “natural exponential family”.

However, when dealing with rare probabilities, the empirical estimation does not draw enough points in the failure domain to get an accurate estimate. The adaptive version of this technique, called *multilevel cross-entropy*, gradually builds a set of intermediate levels, decreasing towards the failure level (equal to zero). By working on a set of individually less rare events, the empirical estimation in Eq. (1.53) is made possible.

The algorithm starts by generating and evaluating an initial sample $\{g(\mathbf{X}_{[1]}^{(i)})\}_{i=1}^n$, on which a threshold level $q_{[1]}^{p_0}$ is computed as the empirical p_0 -quantile. Using the samples below the first threshold $q_{[1]}^{p_0}$, a first instrumental distribution $h_{\lambda_{[1]}^*}$ is optimized. At the next steps $k \in \{1, \dots, k_{\#}\}$, the sample $\{\mathbf{X}_{[k]}^{(i)}\}_{i=1}^n$ is generated from the density $h_{\lambda_{[k-1]}^*}$ and the rest of the process repeats until the estimated threshold level becomes negative, $q_{[k_{\#}]}^{p_0} \leq 0$.

The final instrumental density $h_{\lambda_{[k_{\#}]}^*}$ is then considered for importance sampling as defined in Eq. (1.46):

$$\hat{p}_f^{\text{CE-AIS}} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{g(\mathbf{x}) \leq 0\}} \frac{f_X(\mathbf{x}_{[k_{\#}]}^{(i)})}{h_{\lambda_{[k_{\#}]}^*}(\mathbf{x}_{[k_{\#}]}^{(i)})}, \quad \{\mathbf{X}_{[k_{\#}]}^{(i)}\}_{i=1}^n \stackrel{\text{i.i.d.}}{\sim} h_{\lambda_{[k_{\#}]}^*}. \quad (1.56)$$

CE-AIS is widely used in rare event estimation, as it develops an adaptive technique while conserving the explicit IS variance given in Eq. (1.47). According to Rubinstein and Kroese (2004), the successive instrumental distributions $h_{\lambda_{[k]}^*}$ converge towards h_{opt} under a few hypotheses. The most important one is that the optimal density must belong to the parametric family considered, which should offer enough flexibility to describe a wide range of distributions.

When failure domain is composed of multiple regions, different improvements of the CE-AIS were proposed. [Kurtz and Song \(2013\)](#) proposed to optimize h_{λ^*} among a mixture of Gaussian distributions. This method was further studied by [Wang and Song \(2016\)](#) and [Papaioannou et al. \(2019\)](#) using advanced mixtures in the standard space. However, when using mixtures, the optimization problem does not have an analytical expression anymore ([Geyer et al., 2019](#)).

In the parametric framework, the family choice leads to a complicated trade-off between optimization complexity and flexibility allowed by the family. A similar mechanism is used by other importance sampling methods, inferring the optimal instrumental density by applying kernel density estimation to the points in the failure domain.

Nonparametric adaptive importance sampling

The use of multivariate kernel density estimation (KDE) to approximate the importance sampling optimal density h_{opt} was introduced in the context of structural reliability by [Ang et al. \(1992\)](#), latter followed by [Zhang \(1996\)](#). Let us first present the nonparametric importance sampling from [Zhang \(1996\)](#), considering the instrumental density h_0 (for now, $h_0 \neq f_X$), on which a sample $\{\mathbf{X}_{[1]}^{(i)}\}_{i=1}^n$ is generated. A first failure probability can be roughly estimated, assuming that enough samples lead to the failure domain:

$$\widehat{p}_{f[1]} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{g(\mathbf{x}) \leq 0\}} \left(\mathbf{x}_{[1]}^{(i)} \right) \frac{f_X \left(\mathbf{x}_{[1]}^{(i)} \right)}{h_0 \left(\mathbf{x}_{[1]}^{(i)} \right)} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{g(\mathbf{x}) \leq 0\}} \left(\mathbf{x}_{[1]}^{(i)} \right) w_{[1]}^{(i)}. \quad (1.57)$$

On this biased sample, another density can be fitted using KDE, using the previously defined $\widehat{p}_{f[1]}$ as a normalization term:

$$\widehat{h}_{[1]}(\mathbf{x}) = \frac{\det(\mathbf{H}_{[1]})^{-1/2}}{n \widehat{p}_{f[1]}} \sum_{i=1}^n \mathbb{1}_{\{g(\mathbf{x}) \leq 0\}} \left(\mathbf{x}_{[1]}^{(i)} \right) w_{[1]}^{(i)} K \left(\mathbf{H}_{[1]}^{-1/2} (\mathbf{x} - \mathbf{x}_{[1]}^{(i)}) \right). \quad (1.58)$$

Where the kernel K is commonly taken as the multivariate Gaussian centered density with the covariance matrix \mathbf{H} . The tuning of \mathbf{H} is usually done by minimizing an asymptotic mean integrated squared error (AMISE) criterion ([Glad et al., 2007](#)). In the previous expression, the normalization constant insures to build a probability density while the weights $w_{[1]}^{(i)}$, defined above, reflect the contribution of each point to $\widehat{p}_{f[1]}$. After performing this KDE, the estimated density can be used as instrumental density in Eq. (1.46).

As for the CE-IS methods, the risk is that barely any points sampled from the instrumental density h_0 hit the failure domain, leading to poor estimates. [Zhang \(1996\)](#) proposed to couple an adaptive mechanism with a nonparametric inference of the optimal density. This method is further referred to as NAIS for *nonparametric adaptive importance sampling*. Later, the NAIS method was adapted by [Morio \(2011\)](#) to the reliability analysis problem, using a similar mechanism to the CE-AIS method.

In this framework, a series of intermediate threshold are computed as empirical p_0 -quantiles $q_{[1]}^{p_0} > \dots > q_{[k_\#]}^{p_0}$ of the successive importance sampling steps. This algorithm is initiated by setting $h_0 = f_x$ and stops at the step $k_\#$, when $q_{[k_\#]}^{p_0} < 0$.

At the step k , the intermediate normalization constant is written as:

$$\widehat{p}_f[k] = \frac{1}{kn} \sum_{j=1}^k \sum_{i=1}^n \mathbb{1}_{\{g(\mathbf{x}) \leq q_{[j]}^{p_0}\}} \left(\mathbf{x}_{[j]}^{(i)} \right) \frac{f_x \left(\mathbf{x}_{[j]}^{(i)} \right)}{\widehat{h}_{[j-1]} \left(\mathbf{x}_{[j]}^{(i)} \right)} = \frac{1}{kn} \sum_{j=1}^k \sum_{i=1}^n \mathbb{1}_{\{g(\mathbf{x}) \leq q_{[j]}^{p_0}\}} \left(\mathbf{x}_{[j]}^{(i)} \right) w_{[j]}^{(i)}, \quad (1.59)$$

with $\left\{ \mathbf{X}_{[j]}^{(i)} \right\}_{i=1}^n \stackrel{\text{i.i.d.}}{\sim} h_{[j-1]}$. Then, an intermediate instrumental density is inferred by KDE on the samples exceeding the threshold $q_{[k]}^{p_0}$ such that:

$$\widehat{h}_{[k+1]}(\mathbf{x}) = \frac{\det(\mathbf{H}_{[k]})^{-1/2}}{kn \widehat{p}_f[k]} \sum_{j=1}^k \sum_{i=1}^n \mathbb{1}_{\{g(\mathbf{x}) \leq q_{[j]}^{p_0}\}} \left(\mathbf{x}_{[j]}^{(i)} \right) w_{[j]}^{(i)} K \left(\mathbf{H}_{[k]}^{-1/2} (\mathbf{x} - \mathbf{x}_{[j]}^{(i)}) \right). \quad (1.60)$$

The last instrumental density $\widehat{h}_{[k_\#]}$ is finally considered as an approximation of the optimal density for importance sampling introduced in Eq. (1.46):

$$\widehat{p}_f^{\text{NAIS}} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{g(\mathbf{x}) \leq 0\}} \frac{f_x \left(\mathbf{x}_{[k_\#]}^{(i)} \right)}{\widehat{h}_{[k_\#]} \left(\mathbf{x}_{[k_\#]}^{(i)} \right)}, \quad \left\{ \mathbf{X}_{[k_\#]}^{(i)} \right\}_{i=1}^n \stackrel{\text{i.i.d.}}{\sim} h_{k_\#}. \quad (1.61)$$

Overall, the NAIS offers more flexibility to infer the optimal importance sampling density. This property might suit problems presenting highly nonlinear limit state function. Then, relying on importance sampling still provides an expression of the estimator's variance, by adapting Eq. (1.47) to the recurrent mechanism in NAIS. Because this approach depends on KDE, it inherits its drawbacks. As discussed in Morio (2011), tuning the KDE can create numerical issues and KDE famously suffers from the curse of dimension. In practice, the performances of NAIS seriously decrease for problems in dimension larger than ten.

Subset sampling

Although the concept of splitting already existed, the name of *subset sampling* (SS) was first introduced by Au and Beck (2001) in the structural reliability community. This concept was generalized as a sequential Monte Carlo method under the name of “adaptive multilevel splitting”, as reviewed by Cérou et al. (2019).

Subset sampling splits the failure event \mathcal{F}_x into an intersection of $k_\#$ intermediary events $\mathcal{F}_x = \cap_{k=1}^{k_\#} \mathcal{F}_{[k]}$. Each are nested such that $\mathcal{F}_{[1]} \supset \dots \supset \mathcal{F}_{[k_\#]} = \mathcal{F}_x$. The failure probability is then expressed as a product of conditional probabilities:

$$p_f = \mathbb{P}(\mathcal{F}_x) = \mathbb{P}(\cap_{k=1}^{k_\#} \mathcal{F}_{[k]}) = \prod_{k=1}^{k_\#} \mathbb{P}(\mathcal{F}_{[k]} | \mathcal{F}_{[k-1]}). \quad (1.62)$$

From a practical point of view, the analyst tunes the algorithm⁹ by setting the intermediary probabilities $\mathbb{P}(\mathcal{F}_{[k]} | \mathcal{F}_{[k-1]}) = p_0, \forall k \in \{1, \dots, k_{\#}\}$. Then, the corresponding quantiles $q_{[1]}^{p_0} > \dots > q_{[k_{\#}]}^{p_0}$ are estimated for each conditional subset samples $\mathbf{X}_{[k],N}$ of size N . Note that the initial quantile is estimated by crude Monte Carlo sampling on the input PDF f_X . Following conditional subset samples are generated by *Monte Carlo Markov Chain* (MCMC) sampling of $f_X(\mathbf{x} | \mathcal{F}_{[k-1]})$, using as seeds initialization points the $n = Np_0$ samples given by $\mathbf{A}_{[k],n} = \{\mathbf{X}_{[k-1]}^{(i)} \subset \mathbf{X}_{[k-1],N} | g(\mathbf{X}_{[k-1]}^{(i)}) > \hat{q}_{[k-1]}^{p_0}\}_{i=1}^n$. This process is repeated until an intermediary quantile becomes negative: $\hat{q}_{[k_{\#}]}^{p_0} < 0$. Finally, the failure probability is estimated by:

$$\hat{p}_f^{\text{SS}} = p_0^{k_{\#}-1} \frac{1}{N} \sum_{i=1}^n \mathbb{1}_{\{g(\mathbf{x}) \leq 0\}}(\mathbf{X}_{[k_{\#}],N}^{(i)}). \quad (1.63)$$

In practice, the subset sample size should be large enough to properly estimate intermediary quantiles, leading to the usual recommendation of $p_0 = 0.1$. Fig. 1.11 illustrates the consecutive subset samples moving towards the failure domain. At each step of the algorithm (corresponding to a color), a subset is generated and an intermediate quantile estimated.

Au and Beck (2001) also provide bounds to the coefficient of variation of \hat{p}_f^{SS} . The first one results from a first order Taylor expansion of Eq. (1.63) and is often considered as an upper bound. While the second assumes the estimations of the conditional probabilities to be independent, and tends to underestimate the coefficient of variation.

As discussed in (Papaioannou et al., 2015), the efficiency of the SS method depends on the choice and tuning of the MCMC algorithm. The Metropolis–Hastings (MH) algorithm is widely used as MCMC algorithm for subset sampling, however, it quickly becomes inefficient as the dimension increases. Different improvements of the MH are made possible by working in the standard space, such as the “component-wise” (or “modified M–H”). More recently, alternative MCMC methods including physical system dynamics (e.g., Hamiltonian MCMC) showed promising results in high-dimension reliability problems (Papakonstantinou et al., 2023).

The subset sampling is a versatile method, presenting consistent performances even for rare probabilities. Its flexibility allows dealing with highly nonlinear LSF, but its drawbacks arise from the use of MCMC sampling. The convergence of MCMC is complex to control and depends on its tuning, in addition, the MCMC samples are dependent. Unlike the methods derived from importance sampling, the variance of \hat{p}_f^{SS} is only approximated.

1.5.3 Summary and discussion

This section introduced the generic formulation and the main methods for rare event estimation. Even if the problem is generic rare event estimation requires tailored solutions. Depending on the properties of the problem tackled, some methods might outperform others. Beyond the one introduced previously, many more methods are worth mentioning in the field of reliability analysis, such as the *directional sampling* Bjerager (1988), or *line sampling* (Koutsourelakis,

⁹An algorithmic presentation of the generic subset sampling method is given in Appendix C.

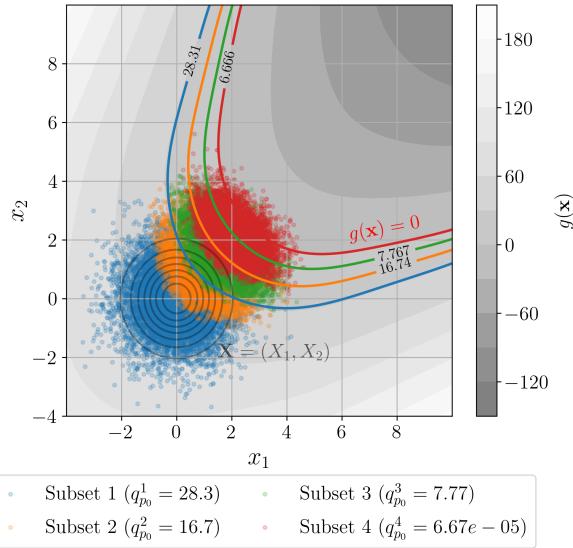


Figure 1.11 Illustration of a rare event estimation by subset sampling ($n = 4 \cdot 10^4, p_0 = 0.1$).

2004), moving particles (Walter, 2015). Morio and Balesdent (2015) compares the advantages and drawbacks of the most commonly used one, with corresponding algorithmic descriptions and numerical benchmarks.

Overall, the main properties increasing the complexity of reliability problems are related to:

- the computational cost of the limit-state function evaluation;
- the strong nonlinearity of the limit-state function;
- the rareness of the failure event.

In regard to the methods, the estimation is made easier by algorithms with simple tuning or allowing to work in the physical space (avoiding a possibly complex iso-probabilistic transform). Considering all these elements the analyst may set up a sampling strategy, possibly coupled with the use of a surrogate model (further discussed in Section 1.7).

Nevertheless, the unified formulation of reliability analysis problems (see 1.31) is an opportunity for the community to share standardized benchmark problems. Following the well-accepted benchmark platform for optimization “Comparing Continuous Optimizers” (COCO) (Hansen et al., 2021), an equivalent initiative was proposed for structural reliability. In 2019, the “black-box reliability challenge”, was organized as a hackathon by the Dutch organization for applied scientific research (TNO) (Rozsas and Slobbe, 2019). This platform proposed a large catalog of reliability problems with their respective solutions. Most of them were encapsulated as a Python package called `otbenchmark`¹⁰ (Fekhari et al., 2021), based on core OpenTURNS objects.

When working with computationally expensive numerical models, the direct use of rare event estimation methods is most often intractable. Many contributions were dedicated to the coupling of surrogate models with sampling methods for rare event estimation. Moustapha et al.

¹⁰<https://github.com/mbaudin47/otbenchmark/>

(2022) presented the results of a wide benchmark on the challenge from TNO, obtained by using surrogate models for reliability developed in the UQLab software (Marelli and Sudret, 2014).

In any case, risk assessment analysts should favor the methods offering convergence guarantees over punctual performance demonstrations. Finally, the robustness of the failure probability to the input uncertainty model is a major question, which was studied from probabilistic (Lemaître et al., 2015) and extra-probabilistic (Ajenjo et al., 2022) frameworks.

OpenTURNS 4 (Rare event estimation). The Python code available in Appendix D proposes a minimalistic OpenTURNS example to estimate rare event probabilities. Figures illustrating the present section may be reproduced, using the OpenTURNS scripts available on GitHub¹¹.

1.6 Global sensitivity analysis

The aim of sensitivity analysis (SA) is to determine the impact of a single (or a group of random inputs) on a random output(s). As described earlier, this step is qualified as an inverse analysis in the general UQ framework (illustrated in Fig. 1), in opposition with the forward uncertainty propagation step. In fact, the analyst studies the effect of the inputs at different scales, hence the distinction between “local” and “global” SA. Local SA focuses on the impact of small perturbations around nominal values of the inputs (i.e., derivative-based approaches), while global sensitivity analysis (GSA), typically studies the general variability (e.g., the variance) of the output. Two types of GSA methods exist in the literature, either proposing qualitative and quantitative approaches:

- *screening methods*: determines the non-influential variables in a UQ study (qualitative);
- *importance measures*: assess the contribution of inputs in the global variability of the output (quantitative).

Screening methods are typically used in a statistical learning process, to drop the irrelevant variables to the learning. In this context, *feature selection* serves the same purpose with a slight difference. Screening methods usually assume the inputs to be independent while feature selection does not. Moreover, feature selection not only looks for the irrelevant features to the learning but also the redundant features (Fan and Lv, 2010).

The global sensitivity of an output can be explained by different elements: the single variability of the inputs, their dependence, and their interactions. Two variables present interactions when their simultaneous effect on an output is not additive. Note that SA on dependent inputs is an active field of research and the inputs will mostly be considered as independent in the following.

¹¹https://github.com/efekhari27/thesis/blob/main/numerical_experiments/chapter1/reliability.ipynb

1.6.1 Screening methods

Many UQ methods suffer from the curse of dimensionality, thankfully, high-dimensional problems often only depend on a few variables. This observation was formalized with the concept of *effective dimension* introduced by [Owen \(2003\)](#). Screening methods allow to discriminate the non-influential variables, which can be considered afterwards as determinist to simplify the problem.

Morris method

The Morris method ([Morris, 1991](#)) is a screening method historically used in engineering applications. It starts by mapping the input domain \mathcal{D}_X into a unit hypercube $[0, 1]^d$, which is discretized as a regular grid with step $\Delta \in \mathbb{R}$. The algorithm computes local elementary sensitivity by building “one at a time” (OAT) local trajectories over the regular grid. Each OAT design starts at a random node $\mathbf{x}^{(t)} = (x_1^{(t)}, \dots, x_j^{(t)}, \dots, x_d^{(t)})$ of the grid, and moves only in one direction by an increment equal to the elementary step such that: $\mathbf{x}^{(t)} + \Delta_j = (x_1^{(t)}, \dots, x_j^{(t)} + \Delta, \dots, x_d^{(t)})$. The elementary effect in the direction of the variable i from a OAT trajectory t is expressed as a finite difference:

$$\text{EE}_j^{(t)} = \frac{g(\mathbf{x}^{(t)}) - g(\mathbf{x}^{(t)} + \Delta_j)}{\Delta}. \quad (1.64)$$

The Morris method generates $T \in \mathbb{N}$ OAT trajectories and computes theirs respective elementary effects in each direction i . To assess the global sensitivity of the function, the mean $\overline{\text{EE}}_j$ and variance $\widehat{\text{Var}}(\text{EE}_j)$ of the elementary effects are computed:

$$\overline{\text{EE}}_j = \frac{1}{n} \sum_{t=1}^T |\text{EE}_j^{(t)}|, \quad \widehat{\text{Var}}(\text{EE}_j) = \frac{1}{n-1} \sum_{t=1}^T \left(\text{EE}_j^{(t)} - \overline{\text{EE}}_j \right)^2. \quad (1.65)$$

It allows to divide the variables into three categories, regardless of any regularity hypothesis on the function: (i) negligible effects; (ii) linear effects without interaction; and (iii) nonlinear effects with possible interactions. This method is very intuitive but quickly shows its limits as the dimension increases since it relies on a discretization of the space by a regular grid. Another disadvantage of this method is that it does not distinguish interactions and nonlinear effects of inputs.

Derivative-based global sensitivity measures

The Derivative-based global sensitivity measures (DGSM) are a GSA method introduced in [Sobol and Gresham \(1995\)](#) and further studied in [Kucherenko et al. \(2009\)](#). As the Morris method, they study the mean value of local derivatives of the model output with regard to the inputs:

$$v_j = \int_{\mathcal{D}_X} \left(\frac{\partial g(\mathbf{x})}{\partial x_j} \right)^2 f_X(\mathbf{x}) d\mathbf{x} = \mathbb{E} \left[\left(\frac{\partial g(\mathbf{X})}{\partial X_j} \right)^2 \right]. \quad (1.66)$$

This continuous formulation does not require using OAT designs, which was proven to be more efficient when exploiting sampling methods such as quasi-Monte Carlo. The efficiency of the DGSMs for screening purposes was outlined in many papers (e.g., [Kucherenko and Iooss \(2017\)](#)). Since their value depends on the probability distribution of the input, a normalized version was developed. The connections between DGSM and variance-based GSA measures (i.e., Sobol' indices introduced hereafter), revealed bounding properties between DGSMs and Sobol' total indices ([Lamboni et al., 2013](#)).

1.6.2 Variance-based importance measures

Screening methods determine the non-influential variables in a UQ problem. Beyond this information, importance measures quantify the influence of inputs, allowing to rank the inputs according to their contribution to the output variability.

Functional variance decomposition and Sobol' indices

Sobol' indices are the most popular importance measure in GSA. Their universality comes from the functional decomposition of the output's variance, attributing variance share to the inputs. Considering a squared-integrable and measurable function $g(\cdot)$ and an independent random vector \mathbf{X} . The output random variable $Y = g(\mathbf{X})$ can be decomposed, according to [Hoeffding \(1948\)](#), as:

$$Y = g(\mathbf{X}) = g_0 + \sum_{j=1}^d g_j(X_j) + \sum_{j < l}^d g_{jl}(X_j, X_l) + \dots + g_{1\dots d}(\mathbf{X}), \quad (1.67)$$

with the previous terms defined according this recurrence:

$$g_0 = \mathbb{E}[g(\mathbf{X})] \quad (1.68a)$$

$$g_j(X_j) = \mathbb{E}[g(\mathbf{X})|X_j] - g_0 \quad (1.68b)$$

$$g_{jl}(X_j, X_l) = \mathbb{E}[g(\mathbf{X})|X_j, X_l] - g_j(X_j) - g_l(X_l) - g_0 \quad (1.68c)$$

$$\dots \quad (1.68d)$$

Sobol in [Sobol' \(1993\)](#) proved that this decomposition is unique by exploiting the orthogonality of the terms of the decomposition. Therefore, this decomposition can be transposed in terms of functional decomposition of variance (also called functional analysis of variance or FANOVA):

$$\text{Var}(Y) = \sum_{j=1}^d V_j(Y) + \sum_{j < l}^d V_{jl}(Y) + \dots + V_{1\dots d}(Y), \quad (1.69)$$

where the previous terms are defined in a recurrent way, in the same fashion as Eq. (1.68): $V_j(Y) = \text{Var}(\mathbb{E}[Y|X_j])$, $V_{jl}(Y) = \text{Var}(\mathbb{E}[Y|X_j, X_l]) - V_j(Y) - V_l(Y)$, and so on for higher order interaction

terms. The Sobol' indices of different order are defined as normalized shares of variance. The *first-order Sobol' index* S_j quantifies the share of variance of the output only explained by the marginal X_j (also called main effect). Second order S_{jl} (or higher order) Sobol' indices quantify the effect of the interactions between a group of marginals.

$$S_j = \frac{V_j(Y)}{\text{Var}(Y)} = \frac{\text{Var}(\mathbb{E}[Y|X_j])}{\text{Var}(Y)} \quad (1.70a)$$

$$S_{jl} = \frac{V_{jl}(Y)}{\text{Var}(Y)} = \frac{\text{Var}(\mathbb{E}[Y|X_j, X_l]) - V_j(Y) - V_l(Y)}{\text{Var}(Y)} \quad (1.70b)$$

$$\dots \quad (1.70c)$$

The generic definition of the Sobol' sensitivity indices associated to a subset of inputs $A \in \mathcal{P}_d$, with \mathcal{P}_d the set of all possible subsets of $\{1, \dots, d\}$, is given by:

$$S_A = \frac{V_A(Y)}{\text{Var}(Y)} = \frac{\sum_{B \subset A} (-1)^{|A|-|B|} \text{Var}(\mathbb{E}[Y|X_B])}{\text{Var}(Y)}. \quad (1.71)$$

By using the functional decomposition of variance in Eq. (1.69), one can show that the Sobol' indices add up to one:

$$\sum_{A \in \mathcal{P}_d} S_A = 1. \quad (1.72)$$

The so-called *closed Sobol' index* associated to a subset of inputs $A \in \mathcal{P}_d$ (equivalent to the first-order Sobol' index of A) is defined as:

$$S_A^{\text{clos}} = \sum_{A' \subset A} S_{A'} = \frac{\text{Var}(\mathbb{E}[Y|X_A])}{\text{Var}(Y)}. \quad (1.73)$$

Assessing Sobol' indices for every order becomes complex in medium to high dimension. The *total Sobol' index* S_j^T associated with the variable j , see Saltelli et al. (2008), quantifies the share of output variance which is explained by all the interactions of the variable X_j :

$$S_j^T = 1 - \frac{\text{Var}(\mathbb{E}[Y|X_{-j}])}{\text{Var}(Y)} = \frac{\mathbb{E}[\text{Var}(Y|X_{-j})]}{\text{Var}(Y)}, \quad (1.74)$$

where X_{-j} represents all the marginals from X but X_j . This definition can also be generalized for a subset of inputs $A \in \mathcal{P}_d$, such that:

$$S_A^T = 1 - S_{A^C}^{\text{clos}} = 1 - \frac{\text{Var}(\mathbb{E}[Y|X_{A^C}])}{\text{Var}(Y)}, \quad A^C = \mathcal{P}_d \setminus A \quad (1.75)$$

By analyzing jointly the first and total Sobol' indices, one can get an indication on the decomposition between the marginal and interaction effects. Note that the total indexes are only equal to the first indexes when the model does not present interactions (i.e., purely additive model).

Estimating Sobol' indices can be achieved in various ways, even if historically the *pick-freeze* scheme was the most popular. This method is based on two samples, but it often requires a prohibitive number of evaluations of the function. Many estimators using the pick-freeze generic scheme were developed to estimate Sobol' indices (e.g., Saltelli's, Jansen's, Martinez's etc.), see further details in the Chapter 3 of Da Veiga et al. (2021). Alternatively, the surrogate models were exploited to estimate such sensitivity measure. Using an input-output dataset, the analyst may build a *polynomial chaos expansion* (PCE) surrogate model, which gives an explicit expression of the Sobol' indices (Sudret, 2008). Authors such as Marrel et al. (2009) also studied the use of Gaussian processes for this purpose.

In the case of independent inputs, the first and total Sobol' indices is a complete tool for GSA. The main advantage of this approach is the quantitative nature of its results, allowing to objectively compare the effect of input variables. When the inputs present a dependence structure, it becomes complicated to distinguish its effects from possible interactions. However, many authors tried to adapt the Sobol's indices to this context. Chapter 5 of Da Veiga et al. (2021) reviews four of these approaches. For example, Mara and Tarantola (2012) proposed two extra Sobol' indices, called "full indices", detecting the contributions associated with the inputs' dependence. Note that the interpretation and estimation of this solution becomes complicated. Moreover, unlike the independent case, the four Sobol' indices do not divide the output variance between the inputs. Beyond Sobol' indices, another important GSA method was adapted from the theory of Shapley values by Owen (2014), allowing to work with dependent inputs.

OpenTURNS 5 (Sobol' indices). The Python code available in Appendix D gives a minimalist OpenTURNS implementation of the Sobol' indices to assess global sensitivity analysis on the Ishigami analytical problem. Further scripts are also available on GitHub¹².

Shapley effects

Shapley effects are an adaptation to GSA by Owen (2014) of the Shapley values from the cooperative games' theory (Shapley et al., 1953). This method is an alternative to the Sobol' indices in the case of dependent inputs, for which the natural interpretation of single interaction effects no longer holds. In the game theory, Shapley values act as a rule on how to share the value created by a team between its members (players). The Shapley value allocated to the player X_j is given considering the indices $\{1, \dots, d\} \setminus \{j\}$:

$$\varsigma_j = \sum_{A \subset \{1, \dots, d\} \setminus \{j\}} \binom{d-1}{\text{card}(A)}^{-1} (\text{val}(A \cup \{j\}) - \text{val}(A)), \quad (1.76)$$

where the value (or cost) function is denoted by $\text{val}(A)$, and A is a subset of $\{1, \dots, d\}$ with cardinality $\text{card}(A)$. The Shapley effects adapted this concept to perform a GSA by considering

¹²https://github.com/efekhari27/thesis/blob/main/numerical_experiments/chapter1/sensitivity_analysis.ipynb

the variables as players and the closed Sobol' indices for the value function:

$$Sh_j = \sum_{A \subset -\{j\}} \binom{d-1}{|A|}^{-1} \left(S_{A \cup \{j\}}^{\text{clos}} - S_A^{\text{clos}} \right). \quad (1.77)$$

Conceptually, this expression compares a performance defined by a cost function with or without the variable X_j , and averages it over all the possible combinations of inputs. This importance measure offers the following decomposition:

$$\sum_{j=1}^d Sh_j = 1. \quad (1.78)$$

In the case of independent inputs, the Shapley effects present properties related to the Sobol' indices. The following equation (see proof in [Owen \(2014\)](#)) reveal that the Shapley effects equally divide the interaction effects between the implicated variable:

$$S_j \leq Sh_j \leq S_j^T, \quad Sh_j = \sum_{A \in \mathcal{P}_d, j \in A} \frac{S_A}{\text{card}(A)}. \quad (1.79)$$

Unlike the Sobol' indices, Shapley effects are a nonnegative allocation of output variance with equitable division of the interaction effects. This method presents an interesting alternative in the dependent case, however, estimating Shapley effects creates computational difficulties. The reader may refer to the permutation-based algorithm from [Song et al. \(2016\)](#). Surrogate models were also coupled to estimate Shapley effects, using Gaussian processes in [Benoumechiara and Elie-Dit-Cosaque \(2019\)](#) and random forests in [Bénard et al. \(2022\)](#).

Shapley effects are a promising importance measure based on variance allocation. However, in some cases the variance of the output distribution does not represent well its variability (e.g., multimodal distribution). The following section introduces another family of GSA methods based on distances between distributions.

1.6.3 Moment-independent importance measures

Beyond variance-based GSA, many types of distances between distributions have been used to evaluate the dependence between the input and output distributions. Comparing the entire distributions instead of their moments might be more robust in some cases (e.g., when the variance is a poor indicator of the variability). The tools used to do so are generally called *dissimilarity measures* between distributions. Appendix D briefly introduces two families of dissimilarity measures: the class of f -Csiszár divergences (e.g., the Kullback-Leibler divergence, total variation distance) and the class of integral probability metrics (IPM) (e.g., Wasserstein distance, total variation distance, maximum mean discrepancy).

Considering the probability measures \mathbb{P}_{X_j} and \mathbb{P}_Y (associated with the random variables X_j and Y) and a dissimilarity measure $\Delta(\cdot, cdot)$, one can define two formulations for GSA:

- directly using a dissimilarity measure to assess $\Delta(\mathbb{P}_Y, \mathbb{P}_{Y|X_j})$;
- building a *dependence measures* evaluating $\Delta(\mathbb{P}_{(X_j, Y)}, \mathbb{P}_{X_j} \otimes \mathbb{P}_Y)$.

The first approach was studied in association with f -divergences in Da Veiga (2015); Rahman (2016). However, some f -divergences introduce estimation issues, and the resulting importance measures do not propose a functional decomposition of variance (also called FANOVA). Using kernel-based IPMs such as the maximum mean discrepancy (MMD), an alternative importance measure was proposed. The following section presents the *Hilbert-Schmidt Independence Criterion* (HSIC), which was initially introduced by Gretton et al. (2006) for dependence testing, and later adapted as a dependence measure in GSA by Da Veiga (2015).

Hilbert-Schmidt independence criterion

Let us first recall the definition of the maximum mean discrepancy (further discussed in Appendix D). This distance between two probability distributions π and ζ can be defined as the worst-case error for any function within a unit ball of a function space \mathcal{H} :

$$\text{MMD}(\pi, \zeta) := \sup_{\|g\|_{\mathcal{H}(k)} \leq 1} \left| \int_{\mathcal{D}_X} g(\mathbf{x}) d\pi(\mathbf{x}) - \int_{\mathcal{D}_X} g(\mathbf{x}) d\zeta(\mathbf{x}) \right| \quad (1.80)$$

This quantity is a distance in the RKHS by taking a characteristic kernel (e.g., the Gaussian or Matérn kernel). After a calculation developed in Appendix D, an unbiased one-sample estimator of the squared-MMD was proposed by Gretton et al. (2006), with a convergence rate of $O(n^{-1/2})$ in probability. Considering the two-samples $\{\pi^{(i)}\}_{i=1}^n \sim \pi$ and $\{\zeta^{(j)}\}_{j=1}^n \sim \zeta$:

$$\widehat{\text{MMD}}^2(\pi, \zeta) = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i}^n k(\pi^{(i)}, \pi^{(j)}) - k(\pi^{(i)}, \zeta^{(j)}) - k(\zeta^{(i)}, \pi^{(j)}) + k(\zeta^{(i)}, \zeta^{(j)}), \quad (1.81)$$

In the context of GSA, a first option is to directly use this dissimilarity measure to define the unnormalized index:

$$S_j^{\text{MMD}} = \text{MMD}(\mathbb{P}_Y, \mathbb{P}_{Y|X_j}). \quad (1.82)$$

da Veiga (2021) remarked that the unnormalized first order Sobol' indices are recovered by taking the linear kernel on the output $k_Y(y, y') = yy'$. Using this non-characteristic kernel (see the definition in Appendix D) brings us back to a moment-dependent importance measure.

Alternatively, the second option considers a couple of random variables (X_j, Y) , with probability distributions \mathbb{P}_{X_j} and \mathbb{P}_Y , and assumes the RKHS \mathcal{H} induced by the tensor product kernel $k((x_j, y), (x'_j, y')) = k_{X_j}(x_j, x'_j)k_Y(y_j, y'_j)$. The *Hilbert-Schmidt independence criterion* (HSIC) measures the dependence between \mathbb{P}_{X_j} and \mathbb{P}_Y by expressing the MMD between $\mathbb{P}_{(X_j, Y)}$ and $\mathbb{P}_{X_j} \otimes \mathbb{P}_Y$:

$$\text{HSIC}(X_j, Y) = \text{MMD}^2(\mathbb{P}_{(X_j, Y)}, \mathbb{P}_{X_j} \otimes \mathbb{P}_Y). \quad (1.83)$$

This technique showed very good results for screening, and corresponding independence tests were studied for screening in [De Lozzo and Marrel \(2016\)](#).

[da Veiga \(2021\)](#) proposed the functional decomposition of the two indices defined in Eq. (1.82) and Eq. (1.83), allowing to develop their respective normalized versions. Note that the HSIC decomposition requires a specific hypothesis on the structure of the kernel associated to the inputs.

1.6.4 Summary and discussion

This section introduced the GSA methods commonly used in uncertainty quantification. Either to reduce the dimension of a problem (screening) or to quantify the influence of inputs (with importance measures), GSA improves the understanding of an uncertainty quantification study. As for other steps of the generic UQ methodology, SA is made more complicated for computationally costly simulation models, hence the use of surrogates models. Additionally, the dependence between inputs still represents an important limit to interpret GSA results.

Alongside rare event estimation, a literature is dedicated to the influence of the random inputs on such tail statistics. The sensitivity is no more qualified as “global” but becomes “goal-oriented”. In the field of structural reliability, an overview of the reliability-oriented sensitivity analysis methods is presented in [Chabridon \(2018\)](#). Several techniques derive from rare event estimation (e.g., the FORM importance factors [Papaioannou and Straub \(2021\)](#)), or were adapted from GSA, like Sobol’ indices ([Ehre et al., 2020](#)), Target-HSIC ([Marrel and Chabridon, 2021](#)), or Shapley effects ([Demange-Chryst et al., 2023](#)).

Finally, sensitivity analysis may describe the effects of random inputs on the variation of the output, however, this study is done considering by assuming a model on the input uncertainties. The role of a regulatory agency auditing an uncertainty quantification approach for certification (i.e., a nuclear safety authority), might be to challenge the way to model the uncertainties on the inputs. In this case, various tools for *robustness analysis* exist to quantify the impact of misspecifying the random inputs on the quantity of interest studied. Among the methods to perturbate uncertainty models, some remain in the probabilistic framework, such as the “perturbed-law based indices” (PLI) ([Lemaître et al., 2015](#)), or on extra-probabilistic methods ([Ajenjo et al., 2022](#)).

1.7 Surrogate modeling

1.7.1 Common framework

The aim of *surrogate modeling* (or metamodeling) is to build a cheap-to-call statistical model, denoted by $\widehat{g}_n(\cdot)$, replacing a costly numerical model $g(\cdot)$ over the input domain \mathcal{D}_X . To do so, a statistical learning is performed on a finite number of observations of the costly function g . When manipulating computationally expensive simulations, its size can be limited (i.e.,

small-data context). This n -sized set is usually called *learning set* written:

$$\{\mathbf{X}_n, \mathbf{y}_n\} = \left\{ \mathbf{x}^{(i)}, y^{(i)} \right\}_{i=1}^n = \left\{ \mathbf{x}^{(i)}, g(\mathbf{x}^{(i)}) \right\}_{i=1}^n. \quad (1.84)$$

A very large catalog of regression methods exist, here is a list of the most encountered ones in the field of UQ: generalized linear regression, polynomial chaos expansion (PCE) (Soize and Ghanem, 2004; Blatman and Sudret, 2011), support vector machine (SVM) (Cortes and Vapnik, 1995), Gaussian processes (GP) (Rasmussen and Williams, 2006), low-rank tensor approximations (Grasedyck et al., 2013), and artificial neural network (ANN) (Hastie et al., 2009). The following section will provide a short focus on Gaussian process regression.

Validating the accuracy and precision of a surrogate model is an important step to guaranty its fidelity with regard to the numerical model. When an m -sized input-output set is dedicated to validating the surrogate model, independently of the learning set, it is called *test set* and denoted by $\{\mathbf{X}_m, \mathbf{y}_m\} = \left\{ \mathbf{x}^{(i)}, g(\mathbf{x}^{(i)}) \right\}_{i=1}^m$. Note that the analyst may work in two different frameworks, affecting the regression and validation method's choice:

- Given-data context: only using a fixed input-output dataset to build and validate the surrogate model.
- Computer experiment context: allowing to generate simulated data points (often at a certain cost).

Validating surrogate models in small-data context appears to be an important challenge. Different validation criteria and techniques exist. The *coefficient of validation*, denoted by R^2 , is a first validation metric that can be directly computed on the learning set:

$$R^2(\hat{g}_n) = 1 - \frac{\sum_{i=1}^n (y(\mathbf{x}^{(i)}) - \hat{g}(\mathbf{x}^{(i)}))^2}{\sum_{i=1}^n (y(\mathbf{x}^{(i)}) - \bar{y}_n)^2}, \quad (1.85)$$

where $\bar{y}_n = (1/n) \sum_{i=1}^n y^{(i)}$ denotes the empirical mean of the observations in the test sample. However, such metrics are not relevant for every regression method (typically, interpolant method have an $R^2 = 1$). The *predictivity coefficient* is an alternative defined as a normalized *integrated square error* (ISE):

$$Q^2(\hat{g}_n) = 1 - \frac{\text{ISE}(\hat{g}_n)}{\text{Var}(\hat{g}_n)}, \quad (1.86)$$

where

$$\text{ISE}(\hat{g}_n) = \int_{\mathcal{D}_X} (g(\mathbf{x}) - \hat{g}(\mathbf{x}))^2 d\mathbf{x}, \quad \text{Var}(\hat{g}_n) = \int_{\mathcal{D}_X} (g(\mathbf{x}) - \hat{g}(\mathbf{x}))^2 d\mathbf{x}. \quad (1.87)$$

This quantity can be estimated on a test set $\{\mathbf{X}_m, \mathbf{y}_m\}$:

$$\widehat{Q}^2(\hat{g}_n) = 1 - \frac{\sum_{i=1}^m (y(\mathbf{x}^{(i)}) - \hat{g}(\mathbf{x}^{(i)}))^2}{\sum_{i=1}^m (\bar{y}_m - y(\mathbf{x}^{(i)}))^2}. \quad (1.88)$$

Note that for either criteria, the higher the value, the better the quality of the fit.

Validating a surrogate model with an independent test-set is sometimes called *holdout* validation. In a small-data context, dedicating an independent test set to validation might be impossible. Then, *cross-validation* is a generic estimation strategy allowing to use only one sample for learning and testing. The most common cross-validation method is the *k-fold* validation, illustrated in Fig. 1.12. The idea is first to split the n -sized dataset in several equal parts, called folds. A first surrogate can be fitted on all the dataset but the first fold, on which a validation criterion is estimated (i.e., performance metric). The operation is repeated for each fold, providing a virtual validation on the entire dataset. Leave-One-Out validation (LOO) is an extreme case of *k*-fold cross-validation, for which $k = n - 1$. Note that multiple variations of these methods exist, for example by adding a permutation or shuffling step. The “bagging” validation method (for “bootstrap aggregating”) consists of a shuffled cross-validation repeated many times (Breiman, 1996).



Figure 1.12 Illustration of a k -fold cross-validation (with $k = 4$)

1.7.2 General purposes surrogate model

In this section, a particular focus is dedicated to Gaussian process (GP) regression (also called kriging after the geostatistician D.G. Krige). Gaussian processes are a widely used regression method in UQ for their performance, flexibility and their associated confidence model. In a small-data context, the way of placing the few points forming the surrogate’s learning set is critical. Intuitively, to build a versatile surrogate model, the learning set should collect information over the entire domain uniformly. Which is why space-filling designs of experiments are commonly used to build learning sets. In practice, QMC and optimized LHS design introduced in Section 1.4 are widely used.

Gaussian process regression

Considering a learning set \mathbf{X}_n , the goal is to approximate the function $g(\cdot)$ by a scalar Gaussian process conditioned on a set of observations $\mathbf{y}_n = \left\{ g\left(\mathbf{x}^{(i)}\right) \right\}_{i=1}^n$. Let us first define a prior structure G on the function approximated $g(\cdot)$, taken as a Gaussian process with a mean function $m(\cdot)$ and covariance function $k(\cdot, \cdot)$:

$$G \sim \text{GP}(m(\cdot), k(\cdot, \cdot)), \quad (1.89)$$

with a:

- *trend model*: $m(\mathbf{x}) = \mathbf{f}(\mathbf{x})^\top \boldsymbol{\beta}$, composed of a functional basis $\mathbf{f} = (f_1, \dots, f_d)^\top$ and a vector of coefficients $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d)^\top$,
- *covariance model*: $k(\mathbf{x}, \mathbf{x}')$, usually taken stationary, such that $k(\mathbf{x}, \mathbf{x}') = \sigma^2 k_s(\mathbf{x} - \mathbf{x}', \boldsymbol{\theta})$ with $\sigma^2 > 0$ and $\boldsymbol{\theta} \in \mathcal{D}_X$.

The trend model of a GP defines its general tendency, while the covariance model influences its regularity. Gaussian process regression takes different names depending on the knowledge of the trend model. It is called “simple kriging” when the trend is fully known, “ordinary kriging”, when the trend is unknown but supposed constant and “universal kriging” otherwise. Note that [Schobi et al. \(2015\)](#) introduced a hybrid method named PC-Kriging setting a PCE as the trend of a kriging model.

To ease the presentation, let us first consider the hyperparameters $\sigma, \boldsymbol{\theta}$ fully known and a zero trend $\boldsymbol{\beta} = \mathbf{0}$. At a given point $\mathbf{x} \in \mathcal{D}_b X$ the realization of the GP is a Gaussian random variable $G(\mathbf{x}) \sim \mathcal{N}(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x}))$. Working with Gaussian variables allows to easily write conditioning formulas between $G(\mathbf{x})$ and the observations \mathbf{y}_n . This Gaussian variable $G(\mathbf{x})$ conditioned on the observations \mathbf{y}_n is sometimes called conditional posterior $G_n(\mathbf{x}) := (G(\mathbf{x}) | \mathbf{y}_n) \sim \mathcal{N}(\eta_n(\mathbf{x}), s_n^2(\mathbf{x}))$. The well-known “Kriging equations” (see e.g., [Rasmussen and Williams 2006](#)) offer its explicit expression:

$$\begin{cases} \eta_n(\mathbf{x}) &:= \mathbf{k}^\top(\mathbf{x}) \mathbf{K}^{-1} \mathbf{y}_n \\ s_n^2(\mathbf{x}) &:= k(\mathbf{x}, \mathbf{x}) - \mathbf{k}^\top(\mathbf{x}) \mathbf{K}^{-1} \mathbf{k}(\mathbf{x}) \end{cases} \quad (1.90)$$

where $\mathbf{k}(\mathbf{x})$ is the column vector of the covariance kernel evaluations $[k(\mathbf{x}, \mathbf{x}^{(1)}), \dots, k(\mathbf{x}, \mathbf{x}^{(n)})]$ and \mathbf{K} is the $(n \times n)$ variance-covariance matrix such that the (i, j) -element is $\{\mathbf{K}\}_{i,j} = k(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$.

In practice the surrogate model is defined by the *predictor* function $\eta_n(\cdot)$. This regression model provides an important complementary information with the *kriging variance* $s_n^2(\mathbf{x})$, reaching zero at the learning points. Let us remark that the kriging variance fully depends on the covariance model (defined by its parametric structure and hyperparameters). In practice, the hyperparameters are unknown, therefore, their estimation is a key step in the construction of a kriging model. This estimation can be done using different approaches, most commonly using maximum likelihood estimation or a cross-validation.

The illustration in Fig. 1.13 is a typical one-dimensional representation of an ordinary kriging model. The mean of the conditioned process is plotted in red while its variability is represented by the many trajectories drawn on the process. In the simplest framework, the kriging model exactly interpolate the observations (black crosses).

Associated with kriging models, another validation criterion is relevant to evaluate the kriging variance $s_n^2(\mathbf{x})$. The predictive variance adequation (PVA) has been introduced by [Bachoc \(2013\)](#) to confirm that the kriging variance is reliable. For a validation performed by holdout,

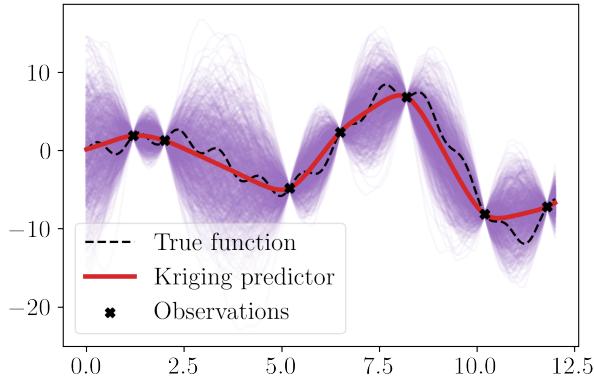


Figure 1.13 Illustration of an ordinary kriging model fitted on a limited set of observations ($n = 7$). The predictor is represented in and several trajectories of the conditioned Gaussian process are drawn represented in purple.

and using an independent m -sized test set, the PVA is defined as:

$$\text{PVA} = \left| \log \left(\frac{1}{m} \sum_{i=1}^n \frac{(y(\mathbf{x}^{(i)}) - \hat{g}(\mathbf{x}^{(i)}))^2}{s_n^2(\mathbf{x}^{(i)})} \right) \right|. \quad (1.91)$$

The smaller this quantity get, the better the quality of the kriging variance.

Gaussian process regression is an elegant solution, offering a lot of flexibility and an associated error model (i.e., the kriging variance). However, well known numerical issues appear during the estimation of the hyperparameters, especially as the learning size increases. More specifically, the computation and memory allocation for the variance-covariance matrix is a recurrent issue. Multiple techniques solve this issue by applying compression schemes on this matrix, e.g., based on sparse approximations (e.g., Hierarchical Matrices Geoga et al. 2020).

This section introduced a general purpose surrogate model, uniformly approximating a function on a domain, however surrogates are often used for specific purposes (e.g., contour finding for reliability analysis).

OpenTURNS 6 (Gaussian process regression). The Python code available in Appendix D proposes a minimalistic OpenTURNS example to fit an ordinary kriging model and active learning models. Figures illustrating the present section may be reproduced, using the OpenTURNS scripts available on GitHub^{13,14}.

1.7.3 Goal-oriented active surrogate model

Surrogates are often fitted for a specific purpose, requiring an accurate approximation over a limited subdomain only. In these cases, a more efficient approach might be to circumscribe

¹³https://github.com/efekhari27/thesis/blob/main/numerical_experiments/chapter1/surrogates.ipynb

¹⁴https://github.com/efekhari27/thesis/blob/main/numerical_experiments/chapter1/active_learning.ipynb

the learning to this subdomain (i.e., *goal-oriented learning*), rather than uniformly over the entire domain. For example, to fit a surrogate model for contour finding in reliability analysis, one should concentrate the learning set around the limit-state function. Similarly, to build a surrogate for a global optimization problem, one should focus the learning set around the optimum(s). Unfortunately, the area(s) of interest is usually unknown before evaluating the true function. *Active learning* is a general concept, aiming at iteratively increasing the learning set w.r.t. a *learning criterion* (also called “acquisition function”) depending on the surrogate’s goal to enhance the surrogate in the area(s) of interest. An exploration–exploitation trade-off arises in active learning, mostly sorted by the learning criterion.

Remark 1. This section introduces active learning methods in the computer experiment context, where the true function can be evaluated anywhere for a given computational cost. However, the “active learning” term is also used to handle big data frameworks in the machine learning community (Qiu et al., 2016). When datasets become so large that learning methods do not scale in practice, the analyst needs to select a relevant subset on which the learning is performed.

Active kriging for optimization

In the field of black-box optimization, many methods rely on approximating the function by a surrogate. The use of Gaussian processes as probabilistic surrogates for optimization was popularized by the *efficient global optimization* (EGO) algorithm (Jones et al., 1998). Ever since, many related methods were developed under the generic name of *Bayesian optimization*. The main idea is to exploit the uncertainty model from the GP to direct the point selection. Factually, the learning criterion depends on the Gaussian process variance model. Numerous reviews of this field were proposed by Shahriari et al. (2015); Gramacy (2020) and numerical benchmarks presented in Le Riche and Picheny (2021).

The generic black-box optimization problem tackled is defined as:

$$\mathbf{x}^* = \arg \min_{\mathbf{x} \in \mathcal{D}_X} g(\mathbf{x}) \approx \arg \min_{\mathbf{x} \in \mathcal{D}_X} \hat{g}(\mathbf{x}). \quad (1.92)$$

To illustrate Bayesian optimization, let us present the EGO algorithm, defined by its specific learning criterion: the “expected improvement”. Considering an initial learning set $\{\mathbf{X}_n, \mathbf{y}_n\}$ built on a space-filling input design \mathbf{X}_n to explore the domain. A first surrogate $G_n(\mathbf{x}) \sim \mathcal{N}(\eta_n(\mathbf{x}), s_n^2(\mathbf{x}))$ is fitted using Eq. (4.11). The expected improvement, to be maximized, is then written as:

$$\mathcal{A}^{EI}(\mathbf{x}; \mathbf{y}_n) = \mathbb{E}[\max(g_{\min} - G_n(\mathbf{x}))] \quad (1.93)$$

$$= (g_{\min} - \eta_n(\mathbf{x})) \Phi\left(\frac{g_{\min} - \eta_n(\mathbf{x})}{s_n(\mathbf{x})}\right) + s_n(\mathbf{x}) \phi\left(\frac{g_{\min} - \eta_n(\mathbf{x})}{s_n(\mathbf{x})}\right), \quad (1.94)$$

where $g_{\min} = \min(\mathbf{y}_n)$, ϕ and Φ respectively stand for the PDF and the CDF of the standard Gaussian distribution. This learning criterion is relatively inexpensive and allows to progressively enhance the Gaussian process to solve the optimization problem with a limited number of calls to the true function.

Three iterations of the EGO algorithm are represented in Fig. 1.14 to minimize a function (dashed line), knowing a few observations (black crosses). After fitting an initial kriging model (in red), the corresponding expected improvement function is represented underneath it (green line). This learning criterion determines the location of the observation to be added to the learning set to enhance the surrogate w.r.t. to the optimization problem.

Bayesian optimization is an active research field, with different open problems such as constrained Bayesian optimization (Petit, 2022), or Bayesian optimization on stochastic functions (Gramacy, 2020). Similarly, active learning was also adapted for structural reliability problems.

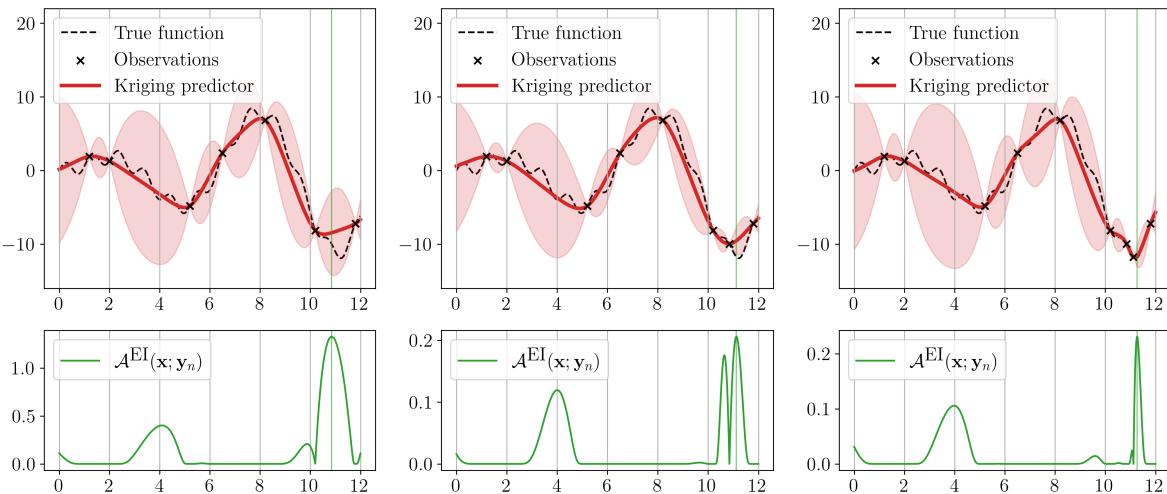


Figure 1.14 Illustration of the expected improvement learning criterion

Active kriging for reliability analysis

Rare event estimation often requires large amounts of evaluations of the limit-state function (becoming intractable for costly numerical models). Emulating this function by a surrogate model can drastically limit the number of calls to the LSF. This surrogate approximates the contour (i.e., border) of the failure domain. However, in most cases, the failure domain represents a very restricted area of the input domain. Active learning methods were proposed to iteratively concentrate the learning set around this border.

For rare event estimation, the surrogate only needs to be accurate near the limit state function. In other words, it should accurately discriminate the points leading to the safe domain from those leading to the failure domain. In fact, this problem can be seen as a binary classification. For example, active learning procedure using SVM classifiers were adapted to this specific goal (Bourinet, 2018).

The following paragraph introduces the most popular kriging-based learning criterion: the “deviation number” U (Echard et al., 2011). The reader may refer to Morio and Balesdent (2015) for further active learning techniques dedicated to rare event estimation. More recently, Teixeira et al. (2021) and Moustapha et al. (2022) reviewed this topic with the presentation of wide numerical benchmarks.

Considering an initial learning set $\{\mathbf{X}_n, \mathbf{y}_n\}$ built on a space-filling input design \mathbf{X}_n to explore the domain. A first Gaussian process $G_n(\mathbf{x}) \sim \mathcal{N}(\eta_n(\mathbf{x}), s_n^2(\mathbf{x}))$ is fitted using Eq. (4.11). The deviation number U is looking for points close to the limit-state function while presenting a high kriging variance. This criterion to be minimized, is defined as:

$$\mathcal{A}^U(\mathbf{x}; \mathbf{y}_n) = \frac{|y_{\text{th}} - \eta_n(\mathbf{x})|}{s_n^2(\mathbf{x})}, \quad (1.95)$$

where $y_{\text{th}} \in R$ is a threshold defining the failure domain.

Fig. 1.15 reuses the same one-dimensional function as in Fig. 1.14 to create a rare event problem. In this case, the failure domain is defined for output values below the threshold y_{th} . Once again, three iterations of the AK algorithm are illustrated, with the corresponding learning criterion U (to minimize). In this simple case, the LSF is defined by the two intersections of the function with the threshold. Therefore, the AK method selects points near these intersections.

Unlike optimization problems, the surrogate is used for uncertainty propagation, meaning that the rare event estimation is the result of the approximation of the LSF (i.e., contour finding) and a sampling techniques. AK methods were coupled with most sampling techniques introduced in Section 1.5 (e.g., AK-MCS, AK-IS, AK-SS, etc.). As an agnostic strategy, [Moustapha et al. \(2022\)](#) recommend to start by applying an AK method (using the learning function U) paired with a subset sampling (taking an intermediary probability $p_0 = 0.2$).

The AK methods present the advantages to be easily implemented and interpreted, however, their learning criterion rely on a local approach. Alternatively, the *stepwise uncertainty reduction* (SUR) chooses iterative points by reducing the future expected uncertainty related to the quantity of interest ([Bect et al., 2012](#)). If this method was proven to be theoretically more consistent ([Bect et al., 2019](#)), its scaling ability is still a bottleneck.

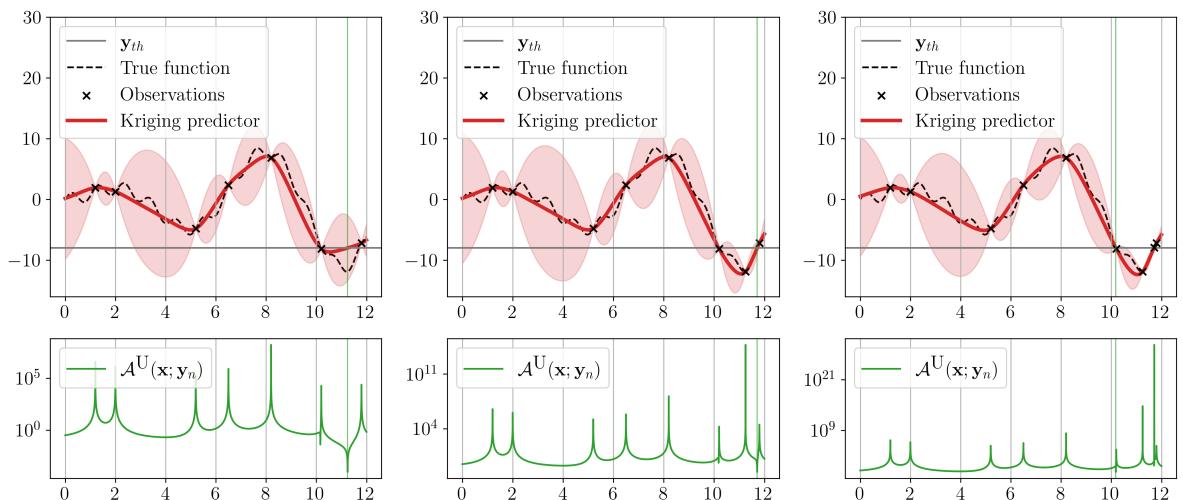


Figure 1.15 Illustration of the deviation number learning criterion

1.7.4 Summary and discussion

This section brought the attention on surrogate modeling in the context of computer experiments. Statistical learning in this framework is made specific by the capacity of the analyst to choose the repartition of the learning set and the small data constraint (mostly due to the costly numerical models manipulated). In this context, many methods are used, however Gaussian processes became popular in UQ as they consider a prior structure of uncertainty that is conditioned by observations (at the edge between a Bayesian and a frequentist approach). To enhance the learning for specific purposes (e.g., optimization or contour finding), active learning methods iteratively add learning points in the subdomain of interest. For some applications, the system studied might be modeled for different fidelities (each presenting different computational costs). Multi-fidelity surrogate modeling is an active field of research, associating observations from different fidelities to improve the learning (Fernández-Godino et al., 2016). Such methods are relevant for models with a very high computational cost (typically in computer fluid dynamics).

In UQ, surrogate models are used for uncertainty propagation (step C) and inverse analysis (step C'). Surrogate modeling is made difficult when the functions present discontinuities (or strong nonlinearities), high dimension, stochasticity, or nonscalar inputs or outputs. To deal with high dimensional problems, unimportant inputs can be screened using sensitivity analysis (see Section 1.6.1), otherwise, model order reduction methods might be used [ref ?]. When the function is stochastic, different approaches allow to fit the function and its intrinsic variability (Binois et al., 2019; Baker et al., 2020; X., 2022).

Provided a strict validation process, surrogate models are a great opportunity for uncertainty quantification. However, many regulatory authorities are still reluctant to the use of surrogates, although their error is often much smaller than the modeling error (i.e., the error between the actual physical behavior and its numerical modeling).

1.8 Conclusion

This section gives a literature overview of the main steps in uncertainty quantification. From uncertainty modeling, uncertainty propagation, sensitivity analysis to surrogate modeling. To ease the methodological presentation, all the illustrations from this section are reproducible using the Python/OpenTURNS scripts available on the GitHub repository mentioned earlier.

Finally this work, the numerical models exploited are supposed to be accurate, but they obviously carry some modeling uncertainty (Oberkampf and Roy, 2010). In fact, prior to uncertainty quantification the model should be calibrated to make it match some physical information (e.g., measurements). Numerical model calibration is also called data assimilation when a stream of measured data is available.

The aim of this work is to apply the tools presented in this chapter to offshore wind turbine models, therefore the next chapter introduces the numerical models manipulated in this thesis.

Chapter **2**

Introduction to wind turbine modeling and design

2.1	Introduction	62
2.2	Metocean conditions simulation	62
2.2.1	Turbulent wind generation	63
2.2.2	Wake modeling	66
2.2.3	Irregular wave generation	67
2.3	Wind turbine multi-physics modeling	68
2.3.1	Aerodynamics of horizontal axis wind turbines	69
2.3.2	Hydrodynamics	71
2.3.3	Control	72
2.3.4	Structural dynamics	72
2.3.5	Fatigue damage	73
2.4	Design and operation practices	76
2.4.1	Types of technologies and preliminary design	77
2.4.2	Further design considerations	78
2.5	Uncertain inputs	81
2.5.1	Environmental inputs	82
2.5.2	System inputs	82
2.5.3	Probabilistic fatigue assessment	83
2.6	Conclusion	85

2.1 Introduction

Wind energy is a highly competitive industry with increasing regulations regarding its impact on ecosystems, land and sea-use, landscapes, or air traffic management (Beauregard et al., 2022). During the long process to win calls for tenders, obtain construction permits, or through the wind farm exploitation, an advanced technical understanding of such systems might offer a competitive advantage.

The operation of offshore wind turbines (OWTs) is driven by multiple physics coupled. This behavior results from different external solicitations which are highly turbulent and uncertain. Among them, the *metocean* (abbreviation of “meteorology” and “oceanography”) environmental conditions play a primary role. Yet, many other types of solicitations affect the exploitation of offshore wind turbines e.g.: the corrosion of the structure, global scour, marine growth, stress concentration factor induced by the manufacturing quality, etc.

In this context, numerical models have been developed to certify the structural integrity of OWTs with respect to their solicitations. A wind farm project planned at given location should pass different validation procedures established by international standards such as the International Electrotechnical Commission ([IEC-61400-1, 2019](#)). As wind turbine structures face a large amount of stress cycles in their lifetime (up to 10^8 for 20 years of operation), this chapter will particularly focus on fatigue damage assessment.

The present thesis studied different steps of uncertainty quantification on two wind farm projects. First the Teesside wind farm, operating since 2014 in the North Sea, second the theoretical wind farm of south Brittany, currently at the stage of calls for tenders.

Considering the standard uncertainty quantification diagram presented in Fig. 1, the material of this chapter is related to the step A (problem specification) and the step B (uncertainty quantification). It briefly introduces wind turbine modeling and design, in the following layout: Section 2.2 presents the methods used for wind and wave generation and wake simulation at a farm scale; Section 2.3 recalls elements of theory associated with wind turbine modeling; Section 2.4 introduces recommended practices regarding design and operation; finally, Section 2.5 gives a description of the various sources of uncertainties considered in this thesis.

2.2 Metocean conditions simulation

In the atmosphere, the wind represents the air movements caused by the heterogeneous solar heating of Earth’s surface. Winds usually move from high-pressure to low-pressure regions. Earth’s rotation also impacts large-scale climate patterns, including winds by the intermediate of the well-known Coriolis effect. The wind is a highly variable resource, making its exploitation for energy production uncertain. This variability is both expressed in space and time with different behaviors depending on the scales studied.

Regarding large timescales, yearly seasonal fluctuations of wind conditions are well-defined using probability distributions (typically Weibull distributions). However, predictions at a

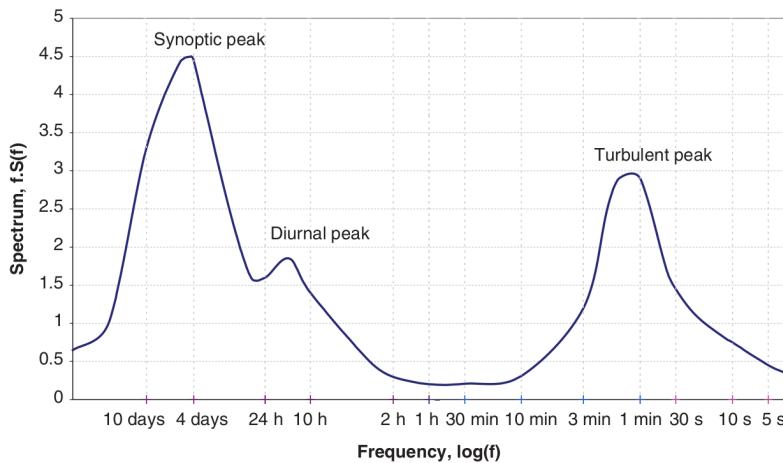


Figure 2.1 Wind spectrum from Brookhaven, USA (source: Burton et al. 2021)

shorter timescale are usually unreliable beyond a few days ahead. Under a few days, the spectral wind energy distribution per time unit is represented by its power spectral density (PSD). Historically, the spectral study of horizontal wind by [Van der Hoven \(1957\)](#) for timescales between a few seconds and ten days revealed distinct ranges of behaviors. The PSD, such as the one illustrated in Fig. 2.1, presents three main separated peaks, explaining how the wind energy is split. The two first peaks are named “synoptic” and “diurnal” peak, which respectively correspond to return periods around four days and one day. While these two peaks are relatively close together, the third peak is completely separated. This third peak describes the energy related to wind turbulence, which evolves in a range below ten minutes. Considering this typical energy distribution, wind behaviors are often referred to as “short-term” (for turbulent wind) and “long-term” (otherwise). In wind turbine simulation, ten minutes simulations became a common practice to fully consider turbulent winds.

Remark that the spectrum presented in the research paper of [Van der Hoven \(1957\)](#) (represented in Fig. 2.1) was build from wind measures near New York, USA. The same pattern between the three peaks is rather constant between sites, however, the geography (including the surface roughness, the topology, the proximity to the coast, etc.) may affect this distribution.

At a larger timescale than one year, assessing trends becomes more complicated. Additionally, wind resource assessment over decades is made more uncertain by climate change [Nagababu et al. \(2023\)](#), disrupting large weather trends and increasing the occurrence of extreme events.

2.2.1 Turbulent wind generation

Wind turbulence is a complex and aleatory process, often described as chaotic, since a small perturbation of its initial conditions might have an important impact on the response. However, the wind over short-term periods (i.e., ten minutes periods) is usually assumed to be a Gaussian process with constant mean \bar{U} and standard deviation σ_U ([Burton et al., 2021](#)). Its mean is modeled by the long-term wind conditions (i.e., mean wind speed), often described by a probabilistic model such as a Weibull distribution. [This short-term / long-term modeling

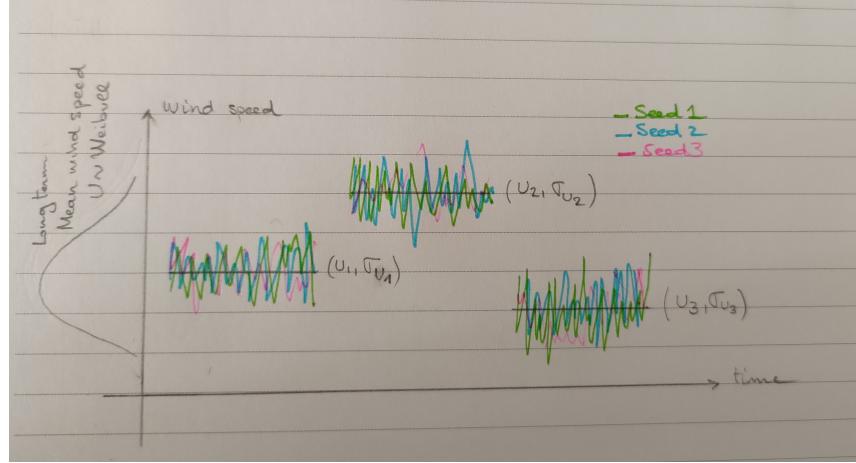


Figure 2.2 [Should we keep this representation? If so, it will be done properly.]

hypothesis is represented in Fig. ??.] Note that this assumption is based on the bimodal wind energy distribution observed in Fig. 2.1, which might vary at some specific locations.

The *turbulence intensity* is a commonly used normalized statistic of the wind variability:

$$I = \frac{\sigma_U}{\bar{U}}. \quad (2.1)$$

As the wind depends on differences between pressure, humidity, air density, different models exist to represent vertical wind profiles. The vertical change in wind conditions is referred to as *vertical wind shear*. Assuming a constant standard deviation over the altitude, the power law is a widely used model to approximate vertical shear ([IEC-61400-1, 2019](#)):

$$\bar{U}(z) = \bar{U}_0 \left(\frac{z}{z_0} \right)^\alpha, \quad (2.2)$$

with \bar{U}_0 a well-defined mean wind speed at the height z_0 (typically corresponding to a measurement height), z the studied height (e.g., the turbine's hub-height), and α the vertical shear coefficient (defined according to measures or standards' recommendations).

To generate a turbulent wind field on a mesh around the turbine, the general mechanism is to apply inverse Fourier transforms on a turbulent wind spectrum. Two types of parametric spectrums are commonly used in wind energy: the *Kaimal model* ([Kaimal et al., 1972](#)) and the *Mann model* ([Mann, 1998](#)). In this thesis, the Kaimal spectrum as defined in [IEC-61400-1 \(2019\)](#) is used for turbulent wind generation over the Cartesian component $k \in \{u, v, w\}$:

$$S_k(f) = \frac{4\sigma_k^2 \frac{L_k}{\bar{U}}}{\left(1 + 6f \frac{L_k}{\bar{U}}\right)^{5/3}}, \quad (2.3)$$

such that f is the frequency, \bar{U} is the longitudinal mean speed at hub-height, L_k are the Kaimal length scales, and σ_k standard deviations. Along with the Kaimal wind speed spectrum, a spacial

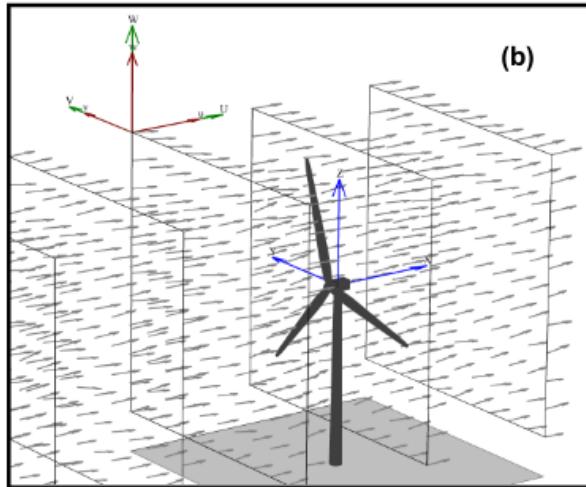


Figure 2.3 Example of a turbulent wind field generated by TurbSim (source: [Jonkman 2009](#))

coherence model is usually defined in the frequency domain. Each couples of nodes in the mesh are correlated, for example, using an exponential coherence model (see the complete definition in Annex C of [IEC-61400-1 2019](#)).

In this thesis, the full-field turbulent wind fields (i.e., over a regular mesh) are generated using TurbSim, a software developed by the National Renewable Energy Laboratory (NREL) ([Jonkman, 2009](#)). TurbSim generates time realizations by adapting the spectral method proposed in [Veers \(1988\)](#) (relying on the inverse Fourier transforms of each axial component). Considering a wind spectrum (e.g., Kaimal model) and a vertical shear model (e.g., power law), TurbSim takes as inputs a mean wind speed, a turbulence standard deviation and a mean wind orientation. Fig. 2.3 illustrates the corresponding wind field generated by a ten-minutes TurbSim simulation, considering a set of input long-term conditions.

In their recent review of the challenges in wind energy, ([Veers et al., 2019](#)) list some limits of the two spectral turbulence models recommended by the standards. First, their parameters were fitted using a restricted amount of data ([Dimitrov et al., 2017](#)). Second, the spacial coherence models associated with Kaimal models showed differences with turbulence measured on site ([Saranyasoontorn et al., 2004](#)). Finally, recent studies showed that the choice of spectral model impacts the resulting wind turbines loads ([Doubrawa et al., 2019](#)). These approximations generally tend to overestimate wind flows, leading to conservative designs.

To ensure more realistic turbulent wind field generation, two research perspectives are actively explored. Authors recently developed hybrid methods, including measurement data to enhance spectral models ([Dimitrov and Natarajan, 2017](#)). Alternatively, higher fidelity models were studied in this domain, see for example the use of vortex methods ([Branlard, 2017](#)) and large eddy simulations (LES) ([Doubrawa et al., 2019; Bui and Bakhoday-Paskyabi, 2022](#)). Such complex models allow the simulation of mesoscale conditions (e.g., at the farm scale), and extreme transient events (e.g., gusts and storms). However, their computational cost is often prohibitive for uncertainty quantification studies. When studying the wind resources at a wind farm scale, modeling wind energy losses induced by the turbines' wake becomes essential.

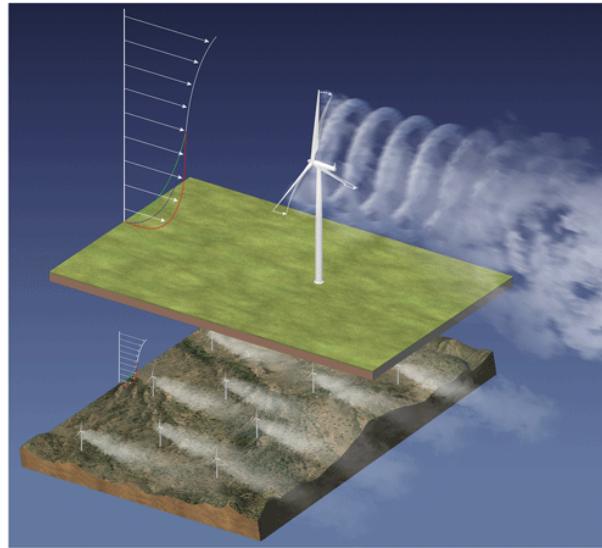


Figure 2.4 Illustration of the wake created downstream a wind farm (source: [Veers et al. 2019](#))

2.2.2 Wake modeling

The wake is caused by the extraction of the wind kinetic energy, reducing the wind speed and increasing the turbulence downstream of the turbines (see the illustration in Fig. 2.4). In a wind farm, this effect depends on the spacing between turbines, as well as the ambient wind speed and turbulence intensity. The turbines positioned at the center of the farm are indeed the most impacted by the wake. As a wind farm owner, the consequence of the wake is twofold: a loss of energy production (in the range of 10 to 20 percents depending on the farm), and an increase of fatigue loads (due to the asymmetric loading from the created turbulences).

The initiation of the wake is a complex physical mechanism, however, the wake almost becomes axisymmetric after two turbine diameters downstream. At this stage, the wind speed deficit often presents a Gaussian profile centered on the hub ([Burton et al., 2021](#)). Numerical models of different fidelities aim at simulating the wake. For example, computational fluid dynamics (CFD) models give a detailed description of the wake (including near the turbine) but require high computational efforts. In practice, simple analytical models (often called “engineering models”) are widely used and recommended by standards (see e.g., Annex E in [IEC-61400-1 2019](#)). These models mostly rely on the equivalence between the thrust load and the turbine wind energy deficit. Since the seminal engineering model proposed by [Jensen \(1983\)](#), multiple enhancements were proposed. A wide benchmark of the wake modeling solutions for different fidelities was performed in [Doubrava et al. \(2020\)](#) and [Ardillon et al. \(2023\)](#). The optimal tuning of these engineering models was studied using measurements from a Doppler wind lidar in [Zhan et al. \(2020\)](#). Different software programs propose wake engineering models, such as: FLORIS (developed by the NREL [Fleming et al. \(2020\)](#)), FarmShadow (developed by IFPEN).

To take into account the wake effect, control strategies increasingly move from the turbine scale to the farm scale. This concept, called “active wake control”, introduces small yaw

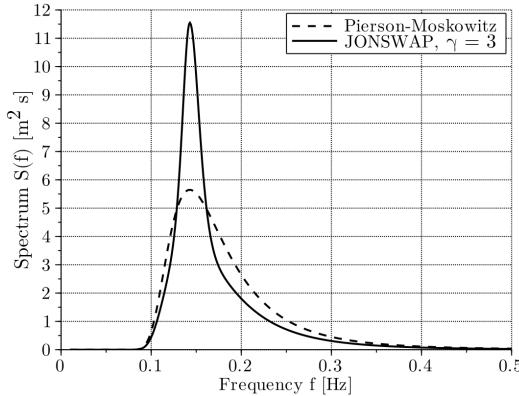


Figure 2.5 Peirson-Moskowitz and JONSWAP spectra at significant wave height $H_s = 3$ m and peak period $T_p = 7$ s (source: [Milano 2021](#))

misalignments (making the control of turbines individually suboptimal) to optimize the global wake inside the farm ([Rott et al., 2018](#); [Simley et al., 2020](#); [Meyers et al., 2022](#)).

2.2.3 Irregular wave generation

The propagation of wind generated waves has long been studied in hydrodynamics under the prism of different theories such as Airy's, and Stokes'. Airy's wave theory (also referred to as the linear wave theory) models sea states under the hypothesis of small waves relatively to the water depth. This spectral approach superposes many regular waves, following the same wave spectrum, to model irregular waves. Two standard statistics are used in oceanography to represent sea states and their corresponding wave spectra: the wave period T_p (with the corresponding frequency f_p), and the significant wave height H_s (average over the highest third of the waves measured).

The most commonly used parametric wave spectrum is named JONSWAP, after the “Joint North Sea Wave Project” ([Hasselmann et al., 1973](#)):

$$S(f) = \delta \frac{H_s^2}{f} \left(\frac{f_p}{f} \right)^4 \exp \left[-\frac{5}{4} \left(\frac{f_p}{f} \right)^4 \right] \gamma^\alpha. \quad (2.4)$$

The JONSWAP spectrum is a corrected version of the Pierson-Moskowitz spectrum (developed in 1964), adding a peak enhancement factor γ^α . Further details regarding the numerical values to choose in Eq. (2.4) are given in [Burton et al. \(2021\)](#). An illustration of the two spectra is presented in Fig. 2.5, revealing the enhancement factor proposed in the JONSWAP model to better fit sea states measurements.

Swell waves are the result of weather conditions occurring far away from the location studied. Such waves usually present long wavelength, allowing them to propagate over long distances with little dissipation. To take them into account, the unimodal wave spectra introduced in Eq. (2.4) was improved. Different methods allow to build a parametric bimodal distribution,

with a mode in the low frequencies corresponding to the swell. [Ewans et al. \(2004\)](#) reviews different bimodal wave spectra, and compares their adequacy with measured sea states.

2.3 Wind turbine multi-physics modeling

Offshore wind turbine models are coupling multiple physics such as aerodynamics, hydrodynamics, mechanical elasticity, control and mooring dynamics for floating OWT. Similarly to the usual practices from the offshore oil & gas industry, OWT have been first modeled in the frequency domain. At an early design stage, a study in the frequency domain gives a rough idea of the system's feasibility by computing its natural frequencies. An OWT should not have its natural frequencies in the same range as the main frequencies of the wave energy spectra. Otherwise, such systems can be subject to critical dynamic resonance, leading to their failure.

Beyond this preliminary check, frequency-domain approaches present limits for OWT modeling. As they rely on linear assumptions, they are unable to model the nonlinearities and transient loading phases ([Matha et al., 2011](#)). These aspects happen to be essential in the design of OWTs ([Cordle and Jonkman, 2011](#)). As an alternative, the behavior of OWT systems are also simulated in the time domain.

In the time domain, such systems may be models for different fidelities. The diagram in Fig. 2.7 illustrates the increasing complexities for two physics involved in OWT modeling (aerodynamics and structural dynamics). Generally, the computational cost increases with the model fidelity, making uncertainty quantification intractable at some point. In the present work, the numerical model of an OWT studied is actually a chain of three models executed sequentially (as illustrated in Fig. 2.6):

- **TurbSim:** a turbulent wind generator (see Section 2.2.1);
- **DIEGO:** a multiphysics wind turbine model in the time domain (see Section 2.3);
- **Fatigue assessment:** a post-processing computing fatigue damage (see Section 2.3.5).

DIEGO is a numerical model developed by EDF R&D to simulate the aero-hydro-servo-elastic behavior of OWTs in the time domain. Different extensive code-to-code comparisons between DIEGO and other aero-hydro-servo-elastic models showed close results. Considering bottom-fixed OWT ([Popko et al., 2021](#)) or floating OWT ([Robertson et al., 2020; Kim et al., 2022](#)), DIEGO was compared to “FAST” (developed by NREL), “HAWC2” (developed by DTU), “BLADED” (developed by DNV), and “DeepLines Wind” (developed by IFPEN).

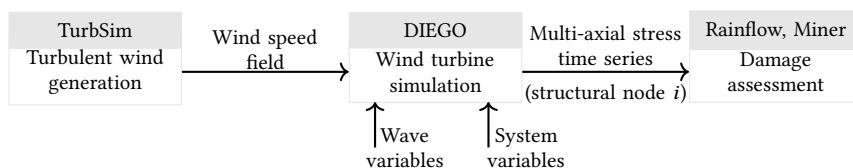


Figure 2.6 Chained numerical model of offshore wind turbine.

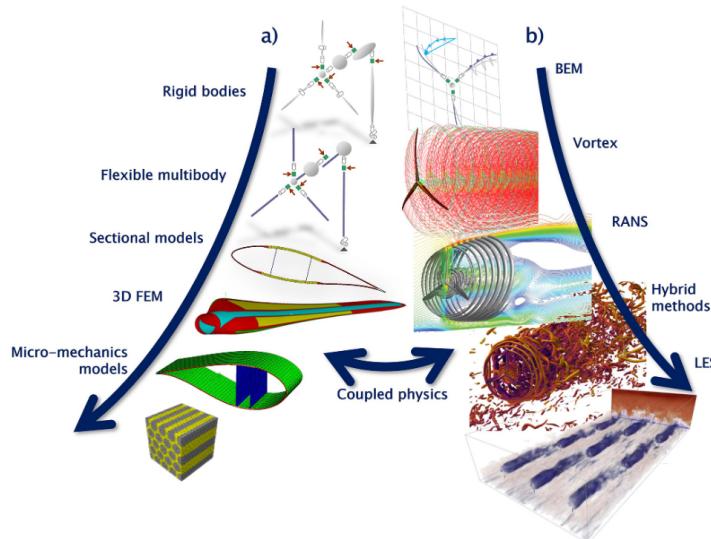


Figure 2.7 Hierarchy of structural (a) and aerodynamic (b) wind energy systems models (source: Veers et al. 2019)

2.3.1 Aerodynamics of horizontal axis wind turbines

The blade element momentum theory mixes different concepts to compute the aerodynamic forces on the rotating blades of the wind turbine. In this coupled physics models, the aerodynamics affects the structural response and vice-versa. To solve this problem, algorithms used in DIEGO first assess displacement of elementary blades, to recover the lift and drag coefficients. The elementary loads are then integrated over each blade and communicated to the structural model.

Momentum theory. At the core of wind turbine's aerodynamics, the concept of *momentum theory*, also called *actuator disk theory* assumes that the air stream passing thought the rotor disk is bounded by a stream tube of circular surface (not mixing with the ambient air). Fig. 2.8 is a longitudinal representation of the actuator disk and the way it affects the air upstream and downstream the rotor. The associated momentum theory assumes the conservation of airflow at any cross-section (of area A) during a time period. Passing through the actuator disk, the wind speed slows down and a drop in static pressure is at the origin of the wake. This pressure drop generates an axial force (called *axial thrust force*) and a torque on the actuator disk.

Considering the upstream flow, the flow at the rotor disk and the airflow in the wake, respectively denoted by the subscripts $\{\infty, d, \text{wake}\}$, the following equality comes:

$$\rho A_\infty U_\infty = \rho A_d U_d = \rho A_{\text{wake}} U_{\text{wake}}, \quad (2.5)$$

where U is the wind speed, A the stream-tube area, and ρ the air density. The wind speed in at the rotor disk can be expressed using the induction factor a in the following expression:

$$U_d = U_\infty(1 - a), \quad 0 \leq a \leq 1. \quad (2.6)$$

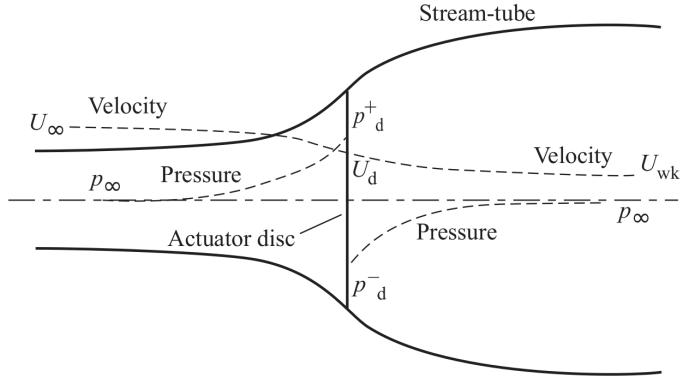


Figure 2.8 Actuator disk model of the energy extraction (source: Burton et al. 2021). Longitudinal evolution of the air pressure and wind speed along the wind stream.

Using the momentum theory and Bernoulli's incompressible flow equation, one can express the aerodynamic thrust T and power P (see Milano 2021):

$$T = (p_d^+ - p_d^-)A_d = 2\rho A_d U_\infty^2 a(1-a) \quad (2.7a)$$

$$P = TU_d = 2\rho A_d U_\infty^3 a(1-a)^2 \quad (2.7b)$$

The widely used power coefficient (respectively thrust coefficient) is the ratio of the power captured by the turbine against to the total kinetic wind power available in the stream tube:

$$C_P = \frac{P}{\frac{1}{2}\rho A_d U_\infty^3} = 4a(1-a)^2, \quad (2.8a)$$

$$C_T = \frac{T}{\frac{1}{2}\rho A_d U_\infty^2} = 4a(1-a). \quad (2.8b)$$

Betz's law is a theoretical limit value of the power coefficient, obtained by cancelling the power coefficient gradient. To this day, no wind turbine has exceeded this limit value: $C_P^{\text{Betz}} = 0.593$ (Burton et al., 2021).

Blade element theory. Assuming a purely two-dimensional flow (meaning that the forces are only determined by the lift and drag coefficients), the blade element theory expresses the thrust dT and torque dQ applied on a blade element.

Let us consider a wind turbine with B blades, a pitch length R and a pitch angle β . Assuming the blade element represented in Fig. 2.9 at the blade length r , with airfoil chord c , angle of attack α , lift C_L and drag C_D coefficients, lift L and drag D forces, and the axial and tangential induction factors a and a' . Under these assumptions, the axial thrust and torque exerted on a blade element are:

$$dT = \frac{1}{2}\rho W^2 B c (C_L \cos(\varphi) + C_D \cos(\varphi)) dr, \quad (2.9a)$$

$$dQ = \frac{1}{2}\rho W^2 B c (C_L \sin(\varphi) + C_D \sin(\varphi)) dr. \quad (2.9b)$$

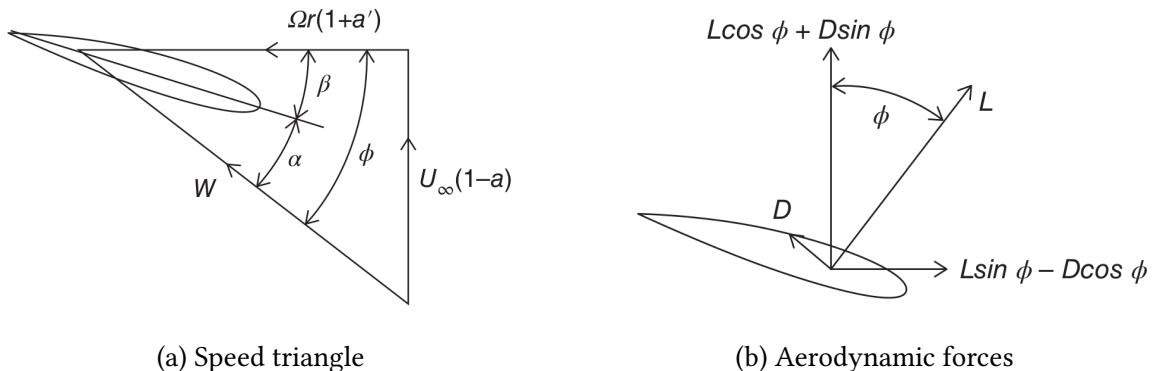


Figure 2.9 Blade element forces. With the lift and drag forces L and D , the flow angle ϕ , the pitch angle β and the angle of attack α (source: [Burton et al. 2021](#)).

Blade element momentum theory (BEMT) combines the results from blade element theory in Eq. (2.9) with the results from momentum theory in Eq. (2.7) to obtain the induction factors a and a' . The resolution of this system of equations is often solved by iterative approaches (e.g., Dai et al. 2011). Global axial thrust over the blade are then computed by integrating the elementary loads over all the elements. Note that various corrections are applied to the BEMT model, for example to take into account the non-homogeneous loss of momentum over the rotor disk. The BEMT also fails to model non-linear aerodynamic effects, occurring with sudden change of angle of attack. Such effects are sometimes called “dynamic stall” and are represented in DIEGO by the Beddoes-Leishman model (see Burton et al. 2021 for further details).

2.3.2 Hydrodynamics

Morison's equations are a widely-used semi-empirical model to assess the hydrodynamic forces on thin fixed structures such as offshore oil platforms and wind turbines. Considering a slender cylindrical structure of diameter D , a flow velocity $u(t)$, the drag and inertial coefficients C_m and C_D , the axial force (parallel to the flow direction) is given by:

$$F = C_m \rho \frac{\pi}{4} D^2 \frac{du}{dt} + C_d \frac{1}{2} \rho D u |u|. \quad (2.10)$$

Standard values for the drag and inertial coefficients are often considered (DNV-OS-J103, 2013). DIEGO uses Morison's equation together with first order potential solution to perform hydrodynamical simulations in the time domain. An extended introduction to hydrodynamics of fixed slender structures, as well as large floating structures is given in the Chapter 1 from (Milano, 2021).

To design floating structures, more complex wave loading modeling should be considered. For this purpose, Rongé et al. (2023) reviews nonlinear theories applied to the fluid-structure interactions of FOWT and compares them to CFD results.

2.3.3 Control

To maximize their energy production under turbulent wind conditions, wind turbines rely on their control systems. This aspect of wind turbines is usually kept confidential by manufacturers, as it gives them a competitive advantage. Nevertheless, the general control mode of a wind turbine depends on the wind speed. Two main ranges of operation are usually defined: first between the cut-in and rated wind speed, second between the rated and cut-off wind speed. These characteristic wind speed values are given by the turbine manufacturer, for example a turbine may present a cut-in at 4 m s^{-1} , a rated at 13 m s^{-1} and a cut-off at 25 m s^{-1} . Let us then recall the wind turbine power derived from the momentum theory:

$$P = \frac{1}{2} \rho A_d U_\infty^3 C_p(\lambda, \beta), \quad (2.11)$$

with the power coefficient C_p , function of the pitch angle β and the blade tip speed ratio λ , defined between the tangential speed on top of the blade and the wind speed: $\lambda = \frac{\Omega R}{U_\infty}$, for the rotation speed Ω and a rotor radius R .

Below the rated wind speed. The goal of the control system is to extract as much power as available. A control strategy among the family of the *maximum power point tracking* can be deployed (Abdullah et al., 2012). For example, the “power signal feedback” uses the electromagnetic torque to control the power. This method first computes the maxima of the extracted power as a function of the rotation speed (using Eq. (2.11)), for different speed values. Then, for a measured wind rotation speed, the system can determine the reference maximal power. Considering this reference power, a controller (such as a proportional integral controller) intends to match the generated power with the reference by acting on the electromagnetic torque.

Above the rated wind speed. The control system switches to a *power limiting* mode by increasing the blades’ pitch angles. By operating on the pitch, the rotation speed and the power produced are kept at their nominal values. This control is also often realized by a proportional integral system (Bossanyi, 2003).

A more exhaustive description of wind turbines control systems is available in Chapter 8 from Burton et al. (2021). More recent strategies often consider the control at the farm scale. As explained earlier, the operation of one turbine affects the others via the effect of its wake. Moreover, since the wind energy production becomes important in the electric mix, its production might be constrained to respect the stability of the grid (e.g., quality of the utility frequency). The work of Gionfra (2018) studied the optimal control of wind farms considering the effects of the wake and the grid restrictions.

2.3.4 Structural dynamics

The structural elements of modern wind turbines, such as the tower and the blades, compose a dynamic system subject to important elastic deformations. Modeling an operating wind

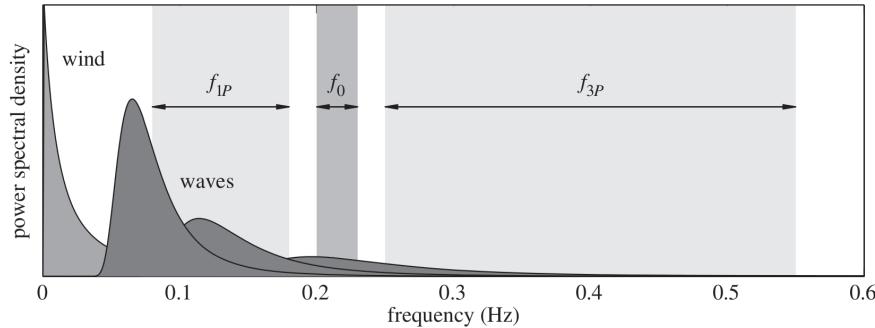


Figure 2.10 Illustration of a soft-stiff design strategy, placing the structure's natural frequency f_0 away from the wind and wave power spectra, and the rotor excitation frequencies f_{1P} and f_{3P} (source: Kallehave et al. 2015).

turbine therefore requires rigid body dynamics and nonlinear elastic deformations. All together, various approaches were developed to model the structural dynamics of wind turbines: modal analysis, multibody methods and finite element methods (FEM). At the stage of preliminary designs, modal approaches can be used to represent the dynamics under linear assumptions (Hegseth and Bachynski, 2019). Then, the tower's natural frequencies assessed by a modal analysis can be compared with the wind, waves, and the rotor's frequencies. As illustrated in Fig. 2.10, the structure's natural frequency (denoted by f_0) should not coincide with the main excitation frequencies to avoid critical dynamic resonance. In the case of a wind turbine, the rotor imbalance creates a first dynamic load of frequency f_{1P} , while the blades passing in front of the tower generate a second excitation of frequency f_{3P} . The *soft-stiff* design strategy places the structure's natural frequency between the two rotor frequencies (i.e., $f_{1P} < f_0 < f_{3P}$ as described in Fig. 2.10) and avoids main frequencies of the wind and waves.

However, modal analysis does not model transient loading phases and their corresponding non-linearities, which is crucial beyond early design. For a higher fidelity, simulations in the time domain using flexible multibody approaches are commonly used to describe the nonlinear dynamics (Holm-Jørgensen, 2009; Al-Solihat and Nahon, 2018). DIEGO implements such an approach by combining rigid multibody dynamics with a deflection model based on Lagrangian equations (Milano, 2021). Note that for floating wind turbines modeling, a preliminary step of rigid body dynamics is added to define the coordinate system of the floater. Otter et al. (2022) reviews the state-of-the-art of numerical and experimental modelling techniques for multi-physics OWT systems.

2.3.5 Fatigue damage

Mechanical fatigue damage is an important phenomenon to consider when designing wind turbines. It refers to the progressive weakening of a material when subjected to cyclic or repeated loading, which may be significantly lower than the material's ultimate strength. Understanding the mechanisms behind mechanical fatigue damage is essential for designing durable and reliable structures. To quantify the fatigue damage on offshore wind turbine structures, standards (DNV-

RP-C203, 2016) recommend simulating the stresses in the time domain and identifying a series of stress cycles. Then, the *stress-number of cycles curve* of a specific material (S-N curve) gives the number of cycles before failure at a given constant stress amplitude. As the stress cycles identified on the results of the OWT simulation are not constant, a linear aggregation method called *Miner's rule* gathers the elementary damages over the stress time series studied.

Stress cycles identification. Offshore wind turbine simulators as DIEGO, deliver a time-dependent stress tensor. To ease the manipulation of this tensor, the equivalent Von Mises stress is computed, turning a multiaxial stress into an equivalent uniaxial stress. One can also consider a “plane strain” hypothesis on the Cauchy stress tensor $\underline{\sigma}$, which is expressed as:

$$\underline{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}. \quad (2.12)$$

This assumption simplifies the expression of the equivalent Von Mises stress:

$$\sigma_{VM} = \sqrt{\frac{1}{2} [(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2] + 3\sigma_{12}^2}. \quad (2.13)$$

Stress cycles can now be identified on the equivalent Von Mises stress time series. The usual method to identify fatigue stress cycles is called *rainflow counting* (Dowling, 1972). In this approach, fatigue stress cycles are only defined by their amplitude (also called “range”) and mean value, regardless of their chronology. Rainflow counting returns a list of stress ranges identified denoted by s in the following.

S-N curve. The S-N curve is also called the “Wöhler curve” after the pioneer work of August Wöhler, who demonstrated that fatigue damage was at the origin of railway accidents in the mid-19th century (Schütz, 1996). As a result of repeated fatigue experiments, this tool determines the number of similar stress cycles necessary to reach a fatigue ruin for a defined stress cycle amplitude. Its values depend on the material studied and on external conditions (i.e., in the offshore industry, the S-N curves distinguish the fatigue in the air vs. underwater).

A well admitted simplification of the S-N curve is to consider it as log-linear¹ on two segments:

$$\log(N_c(s)) = \begin{cases} \log(a_1) - m_1 \log(s), & \text{for } s \in [s_{\min}, s_e] \\ \log(a_2) - m_2 \log(s), & \text{for } s \in [s_e, s_{\max}] \end{cases} \quad (2.14)$$

Where N_c is the predicted number of cycles to failure for stress range s , m is the negative inverse slope of the S-N curve, $\log(a)$ is the intercept of log N-axis by the S-N curve, s_{\min} is the minimal (resp. maximal) stress range identified by the rainflow counting, and s_e is the stress range axis of the intersection of the two log-lines formed by the S-N curve.

¹The logarithms related to the S-N curves in this document are logarithms in base 10.

The expression of this curve in two linear segments arise from the concept of endurance limit of a material, s_e , under which the effect of fatigue on a material should be considerably smaller. According to DNV-RP-C203 (2016), the S-N curve is altered for welded tubular joints by taking into account the tube's thickness:

$$N_c(s) = \begin{cases} a_1 \left(s \left(\frac{t}{t_{\text{ref}}} \right)^h \right)^{-m_1}, & \text{for } s \in [s_{\min}, s_e] \\ a_2 \left(s \left(\frac{t}{t_{\text{ref}}} \right)^h \right)^{-m_2}, & \text{for } s \in [s_e, s_{\max}] \end{cases} \quad (2.15)$$

With t_{ref} the reference thickness (for tubular welded joints $t_{\text{ref}} = 25$ mm); t the plate thickness, and h the thickness exponent. The numerical values considered in the present work derive from the Section 2.4.6 of DNV-RP-C203 (2016), reproduced in Table 2.1.

Table 2.1 S-N curve numerical values of welded tubular joints in different environmental conditions (source: DNV-RP-C203 2016)

Environment	m_1	$\log(a_1)$	m_2	$\log(a_2)$	h
Air	3.0	12.48	5.0	16.13	0.25
Seawater with cathodic protection	3.0	12.18	5.0	16.13	0.25
Seawater free corrosion	3.0	12.03	3.0	12.03	0.25

Non-zero mean correction. Most S-N curves are built over zero mean stress cycles, however, different empirical models were developed to consider different stress mean s_m (Suresh, 1998). The S-N curve becomes a three-dimensional envelope depending on the number of cycles N_c , the stress amplitude s , and the mean stress s_m . The “Goodman line” and the “Gerber parabola” are two models relating the stress amplitude s to the mean stress s_m :

$$\text{Goodman : } \frac{s}{s_e} + \frac{s_m}{R_m} = 1 \quad (2.16)$$

$$\text{Gerber : } \frac{s}{s_e} + \left(\frac{s_m}{R_m} \right)^2 = 1 \quad (2.17)$$

Where the material’s yield stress is denoted by R_m and the endurance limit by s_e . The Haigh diagram represented in the Fig. 2.11 is a slice of the three-dimensional envelope for fixed values of fatigue endurance (i.e., number of cycles). By comparing the two models visually, the Goodman line is more conservative and is mostly used in the literature. Further discussion in the field of wind turbines was proposed in the early 2000s with a focus on the fatigue endurance of glass fiber materials (Sutherland, 2000). In the present work, the non-zero correction presented above are not considered as the values of mean stress were found to be negligible compared to the yield stress of the steel material studied.

Cumulative damage theory. A popular approach to assess the damage cumulated on a stress time series is to consider the fatigue contribution of each stress cycle according to the S-N curve. Palmgren-Miner’s rule defines the *cumulative damage* d_c by summing the fatigue

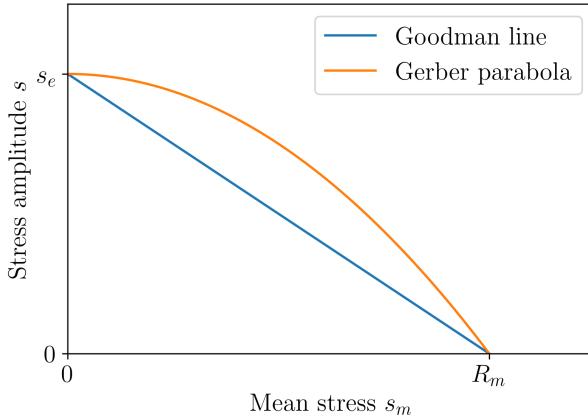


Figure 2.11 Illustration of the Haigh diagram representing the combination of stress mean and amplitude leading to the same fatigue endurance.

contributions of each stress cycle k , regardless of their order of appearance:

$$d_c = \sum_{j=1}^k \frac{1}{N_c(s^{(j)})}. \quad (2.18)$$

In this theory, the material reaches fatigue ruin when the cumulative damage exceeds one. A common practice when using Palmgren-Miner's rule is to gather the stress cycles in a set of bins. This practice induces an integration error, which becomes significant as the number of bins is reduced. In the following, the cumulative damage is computed without binning, as defined in Eq. (2.18).

Spectral methods were also introduced to quantify fatigue damage in the late 80s. The main idea is to infer a PDF over the amplitudes of the stress cycles identified by rainflow counting, typically using a mixture of parametric distributions. From this PDF, one can derive the fatigue endurance and therefore a cumulated damage (see further details in the review of [Dirlik and Benasciutti 2021](#)). In the context of wind turbine fatigue assessment, spectral approaches showed to be unsuited in some cases (e.g., for blades' fatigue [Ragan and Manuel 2007](#)). Overall, fatigue estimation in the time-domain does not represent important computational effort compared to the simulation of the wind turbine's physics.

Nonlinear fatigue models were developed in the 90s ([Fatemi and Yang, 1998](#)) to take into account the order in which the loading cycles were applied to the structure. For offshore wind turbine applications, the recent work of [Rocher et al. \(2020\)](#) studied a probabilistic version of a nonlinear fatigue models. Unfortunately, this refined approach requires a larger computational effort and the calibration over experimental tests of various parameters ([Freyssinet et al., 2023](#)).

2.4 Design and operation practices

The design and operation of offshore wind turbines is at the intersection of various engineering, environmental and social considerations. Regardless of the different bottom-fixed or floating

technologies, OWTs are dynamically excited structures evolving in a harsh offshore environment. To operate such assets over up to 25 years of lifespan, multiple aspects should be assessed, from soil modeling, studies of environmental impact, grid integration, manufacturing quality, port logistics, to marine growth management, and maintenance. This section resumes the main types of OWT technologies, as well as the main design and operation practices.

2.4.1 Types of technologies and preliminary design

The multiple OWTs technologies developed over the last two decades can be gathered into two groups: bottom-fixed or floating technologies. Fig. 2.12a and 2.12b respectively illustrates the different types of bottom-fixed or floating technologies. At this stage, the bottom-fixed solutions present more maturity while floating technologies are still transitioning from the phase of large demonstrators to industrial wind farms. In France, the current development of offshore wind energy lead to the construction of the two first industrial projects (both managed by EDF Renewables). On the coast of Saint-Nazaire, 80 bottom-fixed wind turbines were built on monopile foundations, altogether producing up to 480 MW. On the Mediterranean coast, the first French industrial floating project was recently installed 20 km offshore the coast of Marseille. This pilot project, called “*Provence grand large*”, is composed of three turbines operating on so called “tension-leg platforms”, delivering 25 MW of nominal power.

In order to lift water depth limitations associated with bottom fixed technologies (technical limit around 60 meters), floating pilot projects emerge across the world. However, the wind energy industry still tests different floating technologies in terms of cost efficiency and durability (as listed in [MacKinnon et al. 2022](#)). An example of some farm projects with different types of technologies is described hereafter:

- **Semi-submersible:** a pilot project of three 10MW turbines called “*les éoliennes du Golfe du Lion*” in the south of France relies on a semi-submersible technology developed by the company Principle Power ([Cermelli et al., 2018](#)).
- **Tension-leg:** a pilot project of three 8MW turbines called “*Provence grand large*” exploits tension-leg platforms co-developed between the IFPEN national laboratory and the company SBM ([Caillé et al., 2017](#)).
- **Barge:** a pilot project of three 10MW turbines called “EOLMED” uses the floater developed by the company Ideol ([Guignier et al., 2016](#)).
- **Spar:** the Norwegian oil and gas company Equinor chose the spar technology ([Driscoll et al., 2016](#)) to equip its floating wind farm of 88MW, named “Hywind Tampen”.

The turbines installed offshore over bottom-fixed foundations or floating structures present the same properties and components. As described in Fig. 2.13, the structure of a wind turbine is composed of blades made in composite materials, while the tower, the transition piece and foundation (e.g., monopile) are made out of steel. The steels used for the foundation and the tower are typical structural steels (i.e., steels with low carbon concentration such as the S355).

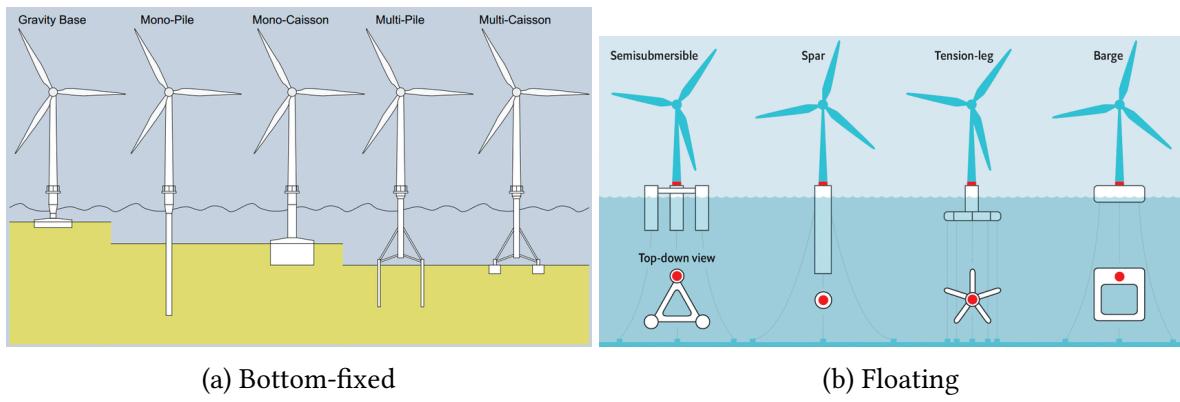


Figure 2.12 Main offshore wind turbine technologies (sources: Ahmed et al. 2015; Mei and Xiong 2021).

Inside the nacelle, the gearbox adapts the rotation speed to suit the energy conversion system (i.e., generator). To improve the reliability of the components, manufacturers offer without gearboxes, called “direct-drive”. This technology is relevant offshore, as the maintenance constraints are higher. However, the corresponding generators used in this situation operate at lower rotation speed. Adapting the generators significantly increase their weight and requires the use of larger permanent magnets increasing their price.

The construction of an offshore wind farm requires several years of project planning, administrative procedures, consultation of the public opinion, and design. Internationals standards define the recommended practices and requirements related to the design and operation of OWTs. Among them, the IEC 61400 is subdivided in many parts, including the general one ([IEC-61400-1, 2019](#)) and other parts detailing specific topics. To validate the structural integrity of a wind turbine design, the standards recommend simulating the behavior of the OWT (using the methods described in Section 2.3) for many environmental conditions, called “design load cases” (DLC). As the environmental conditions depend on the site studied, the standards provide generic DLCs depending on a rough classification of the environmental conditions. Following the terminology in civil engineering, the structure is designed for “fatigue limit states” (FLS) and “ultimate limit states” (ULS) with respect to the environmental conditions. For fatigue, advanced sampling methods relying on environmental data measured on site will be introduced in Chapter 4. Beyond the main solicitations resulting from environmental loading, various aspects should be considered around offshore wind turbines.

2.4.2 Further design considerations

The present section focuses on different topics to be addressed ahead of, or during the design and operation of offshore wind turbines.

Soil modeling. The accurate geotechnical description of an offshore site plays an important role in the design and stability of bottom-fixed offshore wind turbines. The seabed soil properties are far from uniform in a wind farm, forcing the designer to adapt the foundations within a

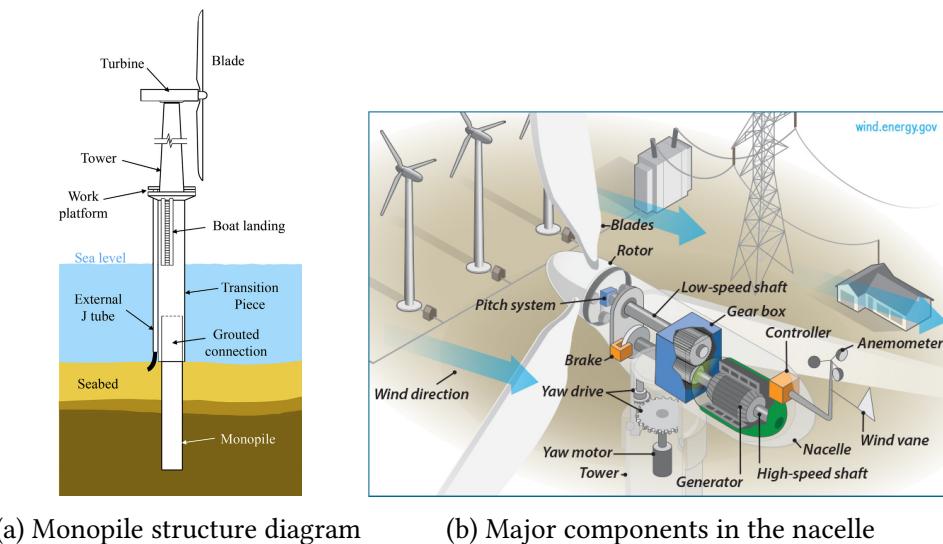


Figure 2.13 Diagrams of an offshore wind turbine structure (source: (Chen et al., 2018)) and nacelle (source: US ODE).

farm. Prior to the installation, geotechnical surveys and soil testing are conducted to assess parameters such as soil composition, density, strength, and seabed stability.

To model the dynamic behavior of foundations, certification companies adapted their methods from the oil & gas industry to the offshore wind energy (DNV-ST-0126, 2018). For monopile foundations, the “ $p - y$ ” method is often used to model soil-structure interactions. Assuming that these interactions are purely lateral, this method defines a set of non-linear lateral springs along the foundation’s height. Together, the springs model the relation between the soil resistance “ p ” and the lateral displacement “ y ”. Generally speaking, monopile foundations for OWT tend to be more rigid than for oil & gas platforms, as the cyclic loading on wind turbines induces more fatigue (see the case-study presented in Le et al. (2014)). However, various contributions in wind energy extended the use of $p - y$ curves to the case of multidirectional and irreversible displacements (Lovera, 2019). In summary, geotechnical considerations are essential for offshore wind turbine design, and the variability of soil properties within a wind farm necessitates a tailored approach to foundation design. Finally, the consideration of uncertainties in this field is still an open research topic (Reale et al., 2021).

Marine growth. The bio-colonization of offshore structures and submarine cables is a significant concern in the maintenance and operation of OWTs. Elements exposed to the colonization of marine organisms, such as mussels, can cause several adverse effects. Firstly, the added weight increases the mass of the turbine and its foundation, potentially changing the dynamics of the systems and its structural integrity (Ameryoun et al., 2019; Schoefs and Tran, 2022). Secondly, marine growth changes the surface’s roughness of the submerged components, which can create fluctuating hydrodynamic loads and vibrations (Marty et al., 2021). To limit its impact on the reliability of OWTs, this phenomenon is addressed with regular preventive cleaning measures as part of the maintenance planning.

Global scour. The large-scale erosion of seabed sediment around bottom-fixed offshore wind turbine foundations, also called “global scour”, poses different problems. The stability of the foundation is first reduced, potentially leading to tilting. Moreover, the load distributions changes, causing uneven stresses and increased fatigue. Finally, submarine cable exposure increases the risk of damage and electrical faults. As global scour is a critical element of the long-term OWT reliability, various mitigation measures are reviewed in [Fazeres-Ferradosa et al. \(2021\)](#), including scour protection, and scour-robust foundation design.

Port logistics. In the installation and maintenance of such large scale systems, port logistics plays an important role considering the international supply chain involved. The coordination, transportation and assembly of massive wind turbine components, foundations, and supporting infrastructure requires meticulous planning and execution. In accordance, the costs of handling operations and maintenance represent an important share of the *levelized cost of energy* (LCOE) ([Shields et al., 2021](#)).

In his review of OWT installation techniques, [Jiang \(2021\)](#) describes the foundations’ and components’ installation processes depending on the OWT technology. Because of their large scale, most structural assembly (e.g., blades, or floater) are done on dedicated port docks, making the port choice critical. The assembled turbines are then transferred offshore with specialized vessels, such as installation jack-ups. Timing and synchronization are critical, as weather windows for handling operations can be limited.

Grid integration. Unlike traditional centralized energy production plants (i.e., nuclear and fossil), wind energy has considerable impact on the grid management. The intermittency of offshore wind generation is driven by variable wind conditions, which disrupts the electricity supply ([Heier, 2014](#)). Then, grid balancing becomes more complex as variable and distributed production sources are introduced. Wind turbine integration often require more flexibility from the grid, resulting in grid infrastructure upgrades (e.g., energy storage) and advanced grid management.

Environmental impact and social acceptance. The fast development of offshore wind turbines in Europe raises questions regarding environmental and social impact. In their review, [Galparsoro et al. \(2022\)](#) showed that the installation and operation was shown to disrupt marine ecosystems. Further studies should be realized to better understand the reliance of the ecosystems to this change. This industry also affects other marine activities (e.g., fishing or tourism), and coastal landscapes, which need to be discussed during the regional marine spatial planning. Finally, social acceptance of offshore wind projects varies across Europe, often split between local disturbances and the regional economic activity generated.

Manufacturing quality. The manufacturing of structural wind turbine components is subject to several uncertainties that can affect the overall quality and performance of OWTs. For example, the manufacturing process of composite blades can lead to inconsistencies in the final

product. Imperfections in the composite material, like air pockets or delamination, can weaken the blades and reduce their lifespan. Additionally, variations in manufacturing processes can result in differences in blade weight, which impacts the turbine's performance. Regarding steel components, OWTs are mostly assembled by bolted and soldered joint. Inconsistent soldering, variations in material properties, and potential flaws in the joints can compromise the structural integrity of OWTs (Veers et al., 2019). These uncertainties in manufacturing quality can pose significant challenges in ensuring the reliability and longevity of the structures. Note that at the design phase, *stress concentration factors* are defined by standards to take into account the local change in material properties created by soldering. Rigorous quality control, material testing, and manufacturing standards are essential to maintain the safety and efficiency of wind energy installations.

Maintenance and end-of-life management. To ensure the continued performance and availability of wind turbines, advanced maintenance planning is essential. Maintenance activities involve inspections, repairs, component replacements, and addressing issues as corrosion, or electrical faults. Preventive maintenance strategies (reviewed by Ren et al. (2021)) minimize the asset's unavailability and extends its lifespan.

Once the wind farms reach their planned lifetime (typically between 20–25 years), the operator has the choice between decommissioning, “repowering”, or “revamping” the assets. Usually, revamping implies an intermediate renovation of the WT. In most cases, the underperforming major components are replaced while the structural components are kept. Alternatively, repowering is a strategy reusing the foundations of a wind farm to install brand new turbines. This approach is often an opportunity to increase the scale and performances of the old turbines.

As the first generation of wind farms currently reach their end-of-life, an important problematic raises from recycling large amounts of blades made out of composite materials. Different processes for recycling composite material are reviewed in Jensen and Skelton (2018), including mechanical, pyrolysis and chemical techniques. However, recycling composites is a complex and energy-consuming operation, that needs to be further studied. The recent lifecycle study of floating OWT in the Mediterranean region by Pulselli et al. (2022) showed that effective maintenance and proper decommissioning planning are essential for ensuring cost-effective yet durable lifecycle management.

2.5 Uncertain inputs

Following the general diagram of uncertainty quantification in Fig. 1, this section focuses on the definition of the uncertain inputs and their corresponding probabilistic model (step B). In our case, the generic term of “inputs” refers to the inputs to the wind turbine numerical model illustrated in Fig. 2.6, which will be considered as random afterwards.

The random variables studied in this work are split into two groups (assumed independent) which are respectively called *environmental variables* and *system variables*. First, the

environmental variables are a collection of variables characterizing the long-term metocean conditions near a wind farm. Even if the associated random vector presents a complex dependence structure, this source of variability is well-defined after the wind potential measurement campaigns.

The second group of uncertain inputs is related to the wind turbine system. A wide range of uncertainties can be taken into account in such systems, such as material properties, manufacturing quality, soil conditions, control error, corrosion, marine growth, aerodynamic damping, etc. Among them, a restricted list of four variables are kept according to sensitivity analysis results from the literature and expert knowledge.

2.5.1 Environmental inputs

During the planning and operation of a wind farm project, the metocean conditions are studied using different sources of information. At the early stage, datasets generated by fine mesoscale numerical simulation can be used to assess the wind potential. This was typically the case during the call for tenders² issued by the French government regarding the construction of two floating offshore wind farm in the south coast of Brittany (of respectively 250 and 500 MW of nominal power). Open-access environmental data of the sea-states in this region were available, as a result of mesoscale simulations (Raoult et al., 2018) realized by EDF R&D. In a second time, the local conditions are measured using a meteorological mast with wind speed cup anemometers at different heights and wave bores are generally installed in the vicinity of the future farm. As a cheaper alternative to met masts, new measurement technologies such as floating LIDARs (standing for “light detection and ranging”) were studied by Gottschall et al. (2017). Then, different adequation methods between the local measures and the data obtained by mesoscale simulations were reviewed in Sempreviva et al. (2008). Finally, after the installation of the turbines, the acquisition system (usually called SCADA, for “supervisory control and data acquisition”) measures wind conditions with a sampling period of ten minutes.

In the present work, two wind farms projects are partially studied: the Teesside wind farm, operating in the North Sea, and the south Brittany floating project, at the stage of tenders call. Table 2.2 summarizes the variables considered as random hereafter. The inference of such data will be discussed in Chapter 3 of this manuscript. Note that the environmental data resulting from the SCADA system of the Teesside wind farm is confidential, and will be represented as anonymized data in the following.

2.5.2 System inputs

As mentioned earlier, multiple parameters in a wind turbine system can be considered as uncertain. Our study focuses on the effects of uncertainties on the fatigue damage over the structure. Therefore, the literature review of the sensitivity analysis on offshore wind turbine fatigue helped us narrow down a few system variables. Petrovska (2022) explored the sensitivity

²<https://eolbretsud.debatpublic.fr/>

Name	Notation	Description
Mean wind speed	U	10-min. average horizontal at 10m
Turbulence	σ_s	10-min. standard deviation
Wind direction	θ_{wind}	Wind directions
Significant wave height	H_s	Significant wave height per hour
Peak wave period	T_p	Peak 1-hour spectral wave period
Wave direction	θ_{wave}	Wave directions

Table 2.2 Marginal distributions of the environmental random variables

analysis of many variables on the fatigue of a wind turbine in Teesside. Even if the use of the Morris method is questionable, the results allowed to screen out some variables. For example, the uncertainties related to the corrosion, the wind shear exponent, or the nacelle mass showed a limited impact on the fatigue. By crossing the conclusions of various research with the expert knowledge among partners from the HIPERWIND European project, the system variables considered uncertain in the following are summarized in Table 2.3. Each of them is assumed independent, with a marginal probabilistic model arising from the literature.

Name	Notation	Marginal model	Description
Soil coefficient	S	Normal ($\mu = 1., \sigma = 0.3$)	Applied to the soil stiffness matrix
Yaw misalignment	θ_m	Normal ($\mu = 0., \sigma = 0.3$)	Error in wind alignment
SN curve coefficient	a	Log-normal ($\mu = 1, \sigma = 0.3$)	See Guédé et al. (2007)
Critical damage	D_{cr}	Log-normal ($\mu = 1, \sigma = 0.3$)	See Drexler and Muskulus (2021)

Table 2.3 Marginal distributions of the system random variables

2.5.3 Probabilistic fatigue assessment

The definition of a fatigue endurance model has a main impact on fatigue damage assessment. However, the S-N curves usually describing the endurance of a material are built on repeated laboratory experiments. Even if the need for random S-N curves has long been expressed in the field of fatigue experiments (Lieudarde, 1982), their probabilistic description was better formalized in (Guédé et al., 2007; Sudret, 2013).

The models proposed in Guédé et al. (2007) are based on the experimental procedure used to build the S-N curves. For identical steel specimens, a cyclic loading with fixed amplitude is repeated until fatigue ruin. Because of variations in the material's microstructure, the fatigue endurance for the same cyclic solicitation is random. This variation is commonly assumed to follow a log-normal distribution in the literature. A probabilistic model of the S-N curve naturally comes:

$$\log(N_c(s, \omega)) = \log(a) - m \log(s) + \log(\varepsilon(\omega)) \quad (2.19a)$$

$$\Rightarrow N_c(s, \omega) = a s^{-m} \varepsilon(\omega), \quad (2.19b)$$

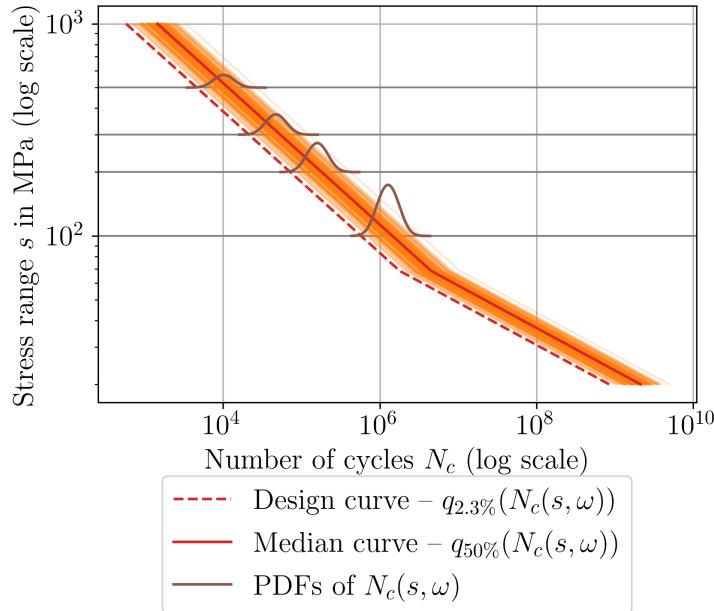


Figure 2.14 Illustration of a probabilistic S-N curve according to the model defined in Guédé et al. (2007).

where $\log(\varepsilon(\omega)) \sim \mathcal{N}(0, \sigma_{N_c} = 0.2)$ is assumed when no measurement is available (according to Appendix F.5 from DNV-RP-C203 2016).

This uncertainty can be injected in the Miner-Palmgren rule defined in Eq. (2.18):

$$d_c(\omega) = \sum_{j=1}^k \frac{1}{N_c(s^{(j)}, \omega)} = \sum_{j=1}^k \frac{1}{a (s^{(j)})^{-m} \varepsilon(\omega)} = \frac{1}{\varepsilon(\omega)} \sum_{j=1}^k \frac{1}{a (s^{(j)})^{-m}} \quad (2.20)$$

Then, this uncertainty can be assessed as a pure post-processing of fatigue damage results computed with a single deterministic S-N curve.

In wind energy standards, S-N curves for design are actually a conservative envelope of the measured fatigue endurance. Annex F.7 from DNV-RP-C203 (2016) describes how to define a design S-N curve from fatigue measures. Assuming that the fatigue endurance follows a Gaussian distribution (on logarithmic scale), the design S-N curve $N_c^{\text{design}}(s)$ is the curve at two standard deviations σ_{N_c} below the median curve.

Using the design S-N curve given in Section 2.4.6 DNV-RP-C203 (2016) and the normality assumption, one can reconstruct the median S-N curve by taking:

$$q_{50\%}[\log(N_c(s, \omega))] = \log(N_c^{\text{design}}(s)) + 2 \sigma_{N_c}. \quad (2.21)$$

Fig. 2.14 illustrates the design curve defined by DNV for tubular joints (in Section 2.4.6) and the reconstructed probabilistic model according to the previous assumptions.

2.6 Conclusion

This chapter proposed an overview on offshore wind turbine modeling and design. It introduced concepts related to the description and simulation of metocean conditions. The impact of the wake on the performance and on unsymmetrical fatigue loading was also explained. Then, the different theories considered in OWT modeling were introduced such as aerodynamics, hydrodynamics, structural dynamics and control. A variety of softwares implementations exist for this purpose, but a special attention was brought on DIEGO a numerical model developed by EDF R&D. As a perspective, uncertainty quantification could benefit from the different fidelities proposed to model OWT systems presenting very nonlinear transient phases.

To understand the design of OWTs, the most common technologies of bottom-fixed and floating turbines were presented. Then, a focus on a few critical topics to be considered during design and exploitation was proposed. In the light of the previous elements, the variables considered as random in this work were listed with a particular focus on probabilistic fatigue.

This growing industry faces various challenges, for example related to the important use of primary commodities, the cohabitation of offshore dynamic structures with an ecosystem, composite materials recycling, etc. Uncertainty quantification is a tool to understanding some of these problematics, however, many uncertainties are hard to characterize and quantify. For example, manufacturing quality issues were revealed by Siemens Gamesa³ regarding wrinkles on the surface of some blades.

³L. Pitel and R.Millard. (August 7 2023). Siemens Energy warns of €4.5bn loss from ailing wind turbine division. *Financial Times*. <https://www.ft.com/content/df8947cd-4bab-46ff-804e-b28de4b5a0f0>

PART II:

CONTRIBUTIONS TO UNCERTAINTY QUANTIFICATION AND PROPAGATION

*Le doute est un état mental désagréable,
mais la certitude est ridicule.*

VOLTAIRE

Chapter **3**

Kernel-based uncertainty quantification

3.1	Introduction	90
3.2	Dependence modeling with nonparametric copula	91
3.2.1	Preliminary definitions and properties	92
3.2.2	Empirical and checkerboard copula	94
3.2.3	Empirical Bernstein and Beta copula	94
3.3	<i>Copulogram</i> : a tool for multivariate data visualization	99
3.3.1	From the pairwise plot to the copulogram	99
3.3.2	Implementation in a Python package	99
3.4	Semiparametric inference of the South Brittany metocean conditions	102
3.4.1	Inference of the marginals	102
3.4.2	Nonparametric inference of the dependence	104
3.4.3	Summary and discussion	106
3.5	Quantifying and clustering the wake-induced perturbations within a wind farm	106
3.5.1	Uncertainty propagation on a wake model	107
3.5.2	Statistical metric of wake-induced perturbations	109
3.5.3	Clustering the wake-induced perturbations	110
3.5.4	Summary and discussion	111
3.6	Conclusion	112

Parts of this chapter are adapted from the following references:

A. Lovera, E. Fekhari, B. Jézéquel, M. Dupoiron, M. Guiton and E. Ardillon (2023). “Quantifying and clustering the wake-induced perturbations within a wind farm for load analysis”. In: *Journal of Physics: Conference Series (WAKE 2023)*.

E. Vanem, E. Fekhari, N. Dimitrov, M. Kelly, A. Cousin and M. Guiton (2023). “A joint probability distribution model for multivariate wind and wave conditions”. In: *Proceedings of the ASME 2023 42th International Conference on Ocean, Offshore and Arctic Engineering (OMAE 2023)*.

3.1 Introduction

The main sources of solicitation in offshore design reside in the metocean conditions. To accurately verify a structural design against the joint wind and wave conditions, these random excitations must be carefully modelled. Offshore structures are usually certified against ultimate limit states (related to the occurrence of extreme metocean conditions) and fatigue limit states (related to the average fatigue over the metocean conditions). In this context, the probabilistic framework is typically used to model the joint distribution of random variables describing the metocean conditions (listed in Section 2.5).

Note that a given probabilistic model might describe well the central behavior of the environmental distribution but not its tail behavior (and vice-versa). Extreme value theory develops specific methods to model the far tails of distributions (Beirlant et al., 2006). Modeling the tails is not the priority in the present work since the focus is on mean fatigue estimation.

The environmental random variables studied present different particularities. First, an offshore wind turbine project leads to the collection of an important amount of metocean data (possibly merged with data from mesoscale simulations). Second, their dependence structure is complex, making the probabilistic modeling more complicated.

This chapter explores different aspects of the environmental conditions’ uncertainty quantification. The theory of some nonparametric copulas is introduced before their use in metocean conditions inference. A semiparametric approach is applied on the South Brittany data, mixing parametric modeling of the marginals with nonparametric modeling of the copula. To visually analyze multivariate distributions, the *copulogram* is a new tool that decomposes the marginal effects and the dependence structure of a joint distribution.

At the scale of a wind farm, each turbine perceives different metocean conditions as the wake of other turbines creates wind perturbations. To study this perturbation, an engineering wake model (see Section 2.2.2) was used to obtain one perturbed environmental distribution per turbine. This work applies a kernel-based discrepancy (the maximum mean discrepancy) to compare wake induced perturbations. In a second time, this discrepancy is used to gather wind

turbines perceiving similar perturbations. This clustering can be used to perform uncertainty propagation at the farm scale by considering a few turbines with are representative of a cluster.

3.2 Dependence modeling with nonparametric copula

In uncertainty quantification, the lack of knowledge can lead to rough assumptions regarding the dependence modeling. However, an accurate representation of the uncertain inputs is of prime importance. For example, the work of [Torre et al. \(2019\)](#) demonstrates the influence of the dependence model on the estimation of rare event probabilities by studying the same problem with different copula models.

When inferring a probabilistic model over a multivariate dataset, one can decompose the problem into the fit of a set of marginals and the fit of a copula (see the Sklar Theorem 1). In the case of metocean conditions, the fit of the marginals is not problematic considering the amounts of data available. However, the complex dependence structure appears to be more challenging. Different strategies to model the dependence for multivariate distributions are shortly summarized hereafter:

- **Vine copulas** (also known as pair copula) decompose the joint distribution as a product of conditioned bivariate copulas organized in a tree-like structure called a vine. This approach proved to be very efficient, but it requires the definition of the vine and the bivariate parametric copulas ([Joe and Kurowicka, 2011](#)).
- **Conditional modeling** defines the joint distribution as a product of univariate conditional distributions. In practice, the parameter of a marginal are defined as a function of other marginals (see e.g., [Vanem et al. 2023](#)).
- **Multivariate KDE** is another way to capture the dependence together with marginal effects. As the dimension and the size of the dataset increase, this method become less tractable ([Wand and Jones, 1994](#)).
- **Nonparametric copulas** are methods uniformly approximating an empirical copula without any assumption on a dependence structure. They will be further described and used for metocean conditions inference in the present chapter.

Remark 2. The strategy referred as “conditional modeling” can in fact be expressed as a copula ([Vanem, 2016](#)) for continuous variables. For example, in the bivariate case of a continuous random vector $\mathbf{X} = (X_1, X_2)$ with PDF $f_{\mathbf{X}}(\mathbf{x})$, CDF $F_{\mathbf{X}}(\mathbf{x})$ and density copula c :

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1) = f_{X_1}(x_1) f_{X_2}(x_2) c(F_{X_1}(x_1), F_{X_2}(x_2)) \quad (3.1a)$$

$$\Leftrightarrow c(F_{X_1}(x_1), F_{X_2}(x_2)) = \frac{f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1)}{f_{X_1}(x_1) f_{X_2}(x_2)} \quad (3.1b)$$

The notions related to the copula theory are further introduced in the monographs of [Nelsen \(2006\)](#); [Joe \(2014\)](#); [Durante and Sempi \(2015\)](#) while the key properties are introduced hereafter.

3.2.1 Preliminary definitions and properties

Let us consider a random vector $\mathbf{X} \in \mathcal{D}_x \subseteq \mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Its probability distribution $\mathbb{P}_{\mathbf{X}}$ can be represented by a CDF $F_{\mathbf{X}}$ and PDF $f_{\mathbf{X}}$. The functional definition of a *d-dimensional copula* (or simply “d-copula”) is a density function $C : [0, 1]^d \mapsto [0, 1]$ whose marginals are uniformly distributed on $[0, 1]$.

Theorem 3 (Copula). *A function $C : [0, 1]^d \mapsto [0, 1]$ is a d-copula if, and only if, it presents the following properties:*

- *The function C is “grounded” (also called “anchored”):*

$$C(u_1, \dots, u_d) = 0 \text{ if } u_j = 0, \forall j \in \{1, \dots, d\};$$
- *The marginals of C are uniform, then:* $C(1, \dots, u_j, \dots, 1) = u_j, \forall j \in \{1, \dots, d\}$;
- *The function C is “d-increasing”, meaning that for any hyperrectangle $A \subset [0, 1]^d$, the corresponding volume induced by C is positive (see [Durante and Sempi \(2015\)](#) p.7).*

A copula is bounded by two functions according to the Fréchet-Hoeffding bounds.

Theorem 4 (Fréchet-Hoeffding bounds). *If a function $C : [0, 1]^d \mapsto [0, 1]$ is a d-copula, then it respects the following bounds for all $\mathbf{u} \in [0, 1]^d$:*

$$W(\mathbf{u}) = \max(1 - d + u_1 + \dots + u_d, 0) \leq C(\mathbf{u}) \leq M(\mathbf{u}) = \min(u_1, \dots, u_d). \quad (3.2)$$

Where the upper bound M is still a copula while the lower bound W is only one for $d = 2$.

The rank transform plays an essential role to understand copulas. Considering a continuous random vector $\mathbf{X} \in \mathcal{D}_x$ and the sample $\mathbf{X}_n = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\} \sim \mathbf{X}$, its *ranks* $\mathbf{R}_n = \{\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(n)}\} \in \mathbb{N}^n$ correspond to the indexes of its order statistics:

$$r_j^{(i)} = n \widehat{F}_{X_j}(x_j^{(i)}) = \sum_{l=1}^n \mathbb{1}_{\{x_j^{(l)} \leq x_j^{(i)}\}}, \quad \forall j \in \{1, \dots, d\}, i \in \{1, \dots, n\}, \quad (3.3)$$

where \widehat{F}_{X_j} stands for the marginal empirical CDF associated to the random variable X_j .

Theorem 5 (Rank-invariance). *Considering a random vector $\mathbf{X} = (X_1, \dots, X_d)$, a set of mappings $\{r_j(\cdot)\}_{j=1}^d$, and the image random vector $\mathbf{R} = (r_1 \circ X_1, \dots, r_d \circ X_d)$. If the mappings are strictly increasing (which is the case for the rank transform introduced in Eq. (3.3)), then, the copula associated to \mathbf{R} is invariant by transformation: $C_{\mathbf{X}} = C_{\mathbf{R}}$. A proof is presented in [Durante and Sempi \(2015\)](#) p. 57.*

Transforming in the ranks generally reduces the effect of outliers and ensures more robust estimates. The invariance by rank transform of copulas allows the estimation of different *dependence measures* in the ranked space.

Spearman's rho. Is a well-known dependence measure, also called the “Spearman's rank correlation coefficient”, which is defined for two random variables X_i, X_j as:

$$\rho^S(X_i, X_j) = \frac{\text{Cov}(r_i(X_i), r_j(X_j))}{\sigma_{r_i(X_i)} \sigma_{r_j(X_j)}}, \quad (3.4)$$

an equivalent definition exists, using the copula C between the joint distribution of X_i and X_j and the independent copula $\Pi(u_i, u_j) = u_i u_j$:

$$\rho^S(X_i, X_j) = 12 \int_{[0,1]^2} C(u_i, u_j) du_i du_j - 3 = 12 \int_{[0,1]^2} (C(u_i, u_j) - \Pi(u_i, u_j)) du_i du_j. \quad (3.5)$$

Kendall's tau. Also referred to as the “Kendall's rank correlation coefficient”, is defined for a pair of random variables (X_i, X_j) and their respective independent copies (X'_i, X'_j) as:

$$\tau(X_i, X_j) = \mathbb{P}\left((X_i - X'_i)(X_j - X'_j) > 0\right) - \mathbb{P}\left((X_i - X'_i)(X_j - X'_j) < 0\right), \quad (3.6)$$

and can also be defined using the copula C between the joint distribution of the two random variables:

$$\tau(X_i, X_j) = 4 \int_{[0,1]^2} C(u_i, u_j) dC(u_i, u_j) - 1 = 1 - 4 \int_{[0,1]^2} \frac{\partial C(u_i, u_j)}{\partial u_i} \frac{\partial C(u_i, u_j)}{\partial u_j} du_i du_j \quad (3.7)$$

These dependence measures fully rely on the copula and are both bounded between -1 and 1. Further properties and estimators of Spearman's rho and Kendall's tau are presented in [Durante and Sempi \(2015\)](#) Section 2.4.

Upper/lower tail dependence. Considering the random vector $\mathbf{X} = (X_i, X_j)$ and the copula C underlying their joint distribution. The *upper/lower tail dependence* coefficients are defined as:

$$\lambda_U(X_i, X_j) = \lim_{\substack{u \rightarrow 1 \\ u < 1}} \mathbb{P}\left(X_i > F_{X_i}^{-1}(u) | X_j > F_{X_j}^{-1}(u)\right) = \lim_{\substack{u \rightarrow 1 \\ u < 1}} \left(2 - \frac{1 - C(u, u)}{1 - u}\right) \quad (3.8a)$$

$$\lambda_L(X_i, X_j) = \lim_{\substack{u \rightarrow 0 \\ u > 0}} \mathbb{P}\left(X_i \leq F_{X_i}^{-1}(u) | X_j \leq F_{X_j}^{-1}(u)\right) = \lim_{\substack{u \rightarrow 0 \\ u > 0}} \left(\frac{C(u, u)}{1 - u}\right) \quad (3.8b)$$

[Joe \(2014\)](#) further discusses asymptotic limit and outlines the particular case of the bivariate Gaussian copula, for which the tail dependence measures are null, $\lambda_U = \lambda_L = 0$. Note that Kendall's tau and the tail dependence coefficients both have their associated plots, allowing to compare the dependence of two distributions [\[add ref\]](#).

3.2.2 Empirical and checkerboard copula

The *empirical copula* was introduced by [Deheuvels \(1979\)](#), as an estimator of the copula C associated with the random vector \mathbf{X} . Since the normalized ranks are a creasing mapping that present uniform marginals by construction, they are a natural empirical representation of the density copula. Considering a sample $\mathbf{X}_n = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\} \sim \mathbf{X}$ with the respective ranks $\mathbf{R}_n = \{\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(n)}\}$, a definition of the empirical copula is:

$$C_n(u_1, \dots, u_d) = \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^d \mathbb{1} \left\{ \frac{r_j^{(i)}}{n} \leq u_j \right\}, \quad \mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d \quad (3.9)$$

Even if this function converges uniformly towards the copula C (according to the Glivenko-Cantelli theorem), it does not fulfill the conditions to be a copula (see e.g., [González-Barrios and Hoyos-Argüelles 2021](#)).

In this context, different methods may be applied to smooth the empirical copula into a genuine copula. This problem can be perceived as a functional approximation of the underlying copula C , which is unique for continuous variables (according to Sklar's Theorem 1). Let us consider a discretization of the unit hypercube as a grid:

$$G = \left\{ \frac{0}{m_1}, \dots, \frac{m_1}{m_1} \right\} \times \cdots \times \left\{ \frac{0}{m_d}, \dots, \frac{m_d}{m_d} \right\}, \quad \mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d. \quad (3.10)$$

The *checkerboard copula* is a simple approximation of the empirical copula using the discretization G . This method is comparable to a multivariate histogram of the empirical density copula c_n (see the formal multivariate definition proposed by [Cottin and Pfeifer 2014](#)). In the particular case for which $m_j = m, \forall j \in \{1, \dots, d\}$, the checkerboard copula is called the “rook” copula, and expressed by [Segers et al. \(2017\)](#) as:

$$C_n^{\#m}(u_1, \dots, u_d) = \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^d \min \left(\max(nu_j - r_j^{(i)} + 1, 0), 1 \right). \quad (3.11)$$

This empirical copula has a low complexity (see [Rose 2015](#)) and efficient results for large samples ([González-Barrios and Hoyos-Argüelles, 2021](#)), however, its variance is comparable to the empirical copula for small-sized samples ([Segers et al., 2017](#)). It is proven to be a genuine copula and its asymptotic behavior was studied by various authors such as [Li et al. \(1998\)](#); [Genest et al. \(2017\)](#). In the following, an approximation of the empirical copula with Bernstein polynomials is presented.

3.2.3 Empirical Bernstein and Beta copula

A few elements on Bernstein polynomials and their corresponding approximation is reminded before introducing the empirical Bernstein copula.

Bernstein polynomials and approximation

Let us first define the *Bernstein basis polynomial* of order $m \in \mathbb{N}$ as:

$$b_{m,t}(u) = \binom{m}{t} u^t (1-u)^{m-t}, \quad t \in \{0, \dots, m\}. \quad (3.12)$$

These polynomials present various interesting properties, such as their nonnegativity over $[0, 1]$, being bounded by one, and offering a partition of unity on $[0, 1]$ ([Lasserre, 2023](#)):

$$1 = \sum_{t=0}^n b_{m,t}(u)(x), \quad \forall x \in \mathbb{R}, \quad \forall n \in \mathbb{N}. \quad (3.13)$$

Bernstein's polynomials allow us to uniformly approximate any continuous and real-valued function defined on a compact set $f : [0, 1]^d \mapsto \mathbb{R}$ (as they were used to demonstrate the Weierstrass approximation theorem). In the multivariate case, the *Bernstein approximation* of the function f can be written on a grid over the unit hypercube $G = \left\{ \frac{0}{m_1}, \dots, \frac{m_1}{m_1} \right\} \times \dots \times \left\{ \frac{0}{m_d}, \dots, \frac{m_d}{m_d} \right\}$, $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$, as:

$$B_{\mathbf{m}}(f)(\mathbf{u}) = \sum_{t_1=0}^{m_1} \dots \sum_{t_d=0}^{m_d} f\left(\frac{t_1}{m_1}, \dots, \frac{t_d}{m_d}\right) \prod_{j=1}^d b_{m_j, t_j}(u_j), \quad \mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d. \quad (3.14)$$

The Bernstein polynomials approximate f such that $\lim_{m \rightarrow \infty} B_m(f) = f$ uniformly on $[0, 1]$.

Bernstein polynomials for copula approximation

Copulas are continuous and bounded functions defined on a compact set (the unit hypercube). Therefore, they are good candidates to be approximated by Bernstein polynomials. The Bernstein approximation applied on an empirical copula C_n was introduced as *empirical Bernstein copula* (EBC) by [Sancetta and Satchell \(2004\)](#) for applications in economics and risk management:

$$B_{\mathbf{m}}(C_n)(\mathbf{u}) = \sum_{t_1=0}^{m_1} \dots \sum_{t_d=0}^{m_d} C_n\left(\frac{t_1}{m_1}, \dots, \frac{t_d}{m_d}\right) \prod_{j=1}^d b_{m_j, t_j}(u_j), \quad \mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d. \quad (3.15)$$

In this expression, the evaluations of the empirical copula on the vertices of the grid are smoothed by the product of Bernstein polynomials. A respective approximation of the copula density can be directly expressed by deriving the previous formula. The EBC delivers a genuine copula, if and only if all the polynomial degrees $\{m_j\}_{j=1}^d$ are divisors of n (see [Segers et al. 2017](#), Proposition 2.5).

In the particular case of regular grids, $\{m_j = m\}_{j=1}^d$, the EBC can be expressed as a mixture of beta distributions ([Segers et al., 2017](#)). Let us consider an n -sized rank sample, $\mathbf{R} = (\mathbf{r}_1, \dots, \mathbf{r}_d) \in \mathbb{N}^n$, and the degree m taken as divisor of n . Note that the r^{th} order statistic of an n -sized sample following a uniform $[0, 1]$ is distributed according to the beta distribution $\mathcal{B}(r, n - r + 1)$.

Considering these hypotheses, the EBC can be written as:

$$B_m(C_n)(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^d F_{m,r_j^{(i)}}, \quad \mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d, \quad (3.16)$$

where $F_{m,r}$ is the CDF of the beta distribution $\mathcal{B}(r, m-r+1)$ (also called the “regularized incomplete beta function”):

$$F_{m,r} = \sum_{t=r}^m \binom{m}{t} u^t (1-u)^{m-t}, \quad u \in [0, 1], \quad r \in \{1, \dots, m\}. \quad (3.17)$$

Overall, the EBC is a very versatile tool which able to approximate complex dependence patterns. Moreover, Monte Carlo sampling on an EBC is straightforward and licit since it is a genuine copula. As a drawback, the estimation accuracy of this nonparametric method heavily relies on the polynomial order tuning.

Asymptotic behavior of the empirical Bernstein copula

In practice, the choice of polynomial degree for an EBC leads to a challenging bias-variance tradeoff. For example, the particular case of $\{m = n\}$, introduced as the *empirical Beta copula* by Segers et al. (2017), tends to reduce the bias while increasing the variance. In this paper, the beta copula presents interesting results compared to the Bernstein or the checkerboard copula for small sample sizes (i.e., $n < 100$). Theoretically, the tuning of the degree was first optimized to minimize an “Asymptotic Mean Integrated Squared Error” (AMISE) of $B_m(C_n)$:

$$\text{AMISE}(B_m(C_n)) = \mathbb{E}[\|B_m(C_n) - C\|_2^2] = \mathbb{E}\left[\int_{\mathbb{R}^d} (B_m(C_n)(\mathbf{u}) - C(\mathbf{u})) d\mathbf{u}\right]^2. \quad (3.18)$$

The seminal work of Sancetta and Satchell (2004) proves in Theorem 3 that:

- $B_m(C_n)(\mathbf{u}) \rightarrow C(\mathbf{u})$ for any $u_j \in]0, 1[$ if $\frac{m^{d/2}}{n} \rightarrow 0$, when $m, n \rightarrow \infty$.
- The optimal polynomial order in terms of AMISE is¹: $m \lesssim m_{\text{AIMSE}} = n^{2/(d+4)}$, $\forall u_j \in]0, 1[$.

To illustrate the previous theorem, Fig. 6.1 represents the evolution of the m_{AMISE} for different dimensions and sample sizes (adapted from Lasserre 2022). In medium dimension, the values of m_{IMSE} tend towards one, which is equivalent to the independent copula. Therefore, high-dimensional problems should rather be divided into a product of smaller problems on which the EBC is tractable.

The polynomial order for EBC estimation is still a bottleneck which was studied over the years by different authors (see e.g., Janssen et al. 2012; Bouezmarni et al. 2013; Rose 2015; Segers et al. 2017). Meanwhile, other nonparametric approaches such as the “penalized Bernstein” and the “penalized B-spline” estimators were compared to the EBC and vine copulas in a

¹The sign \lesssim stands for “less than or approximately”.

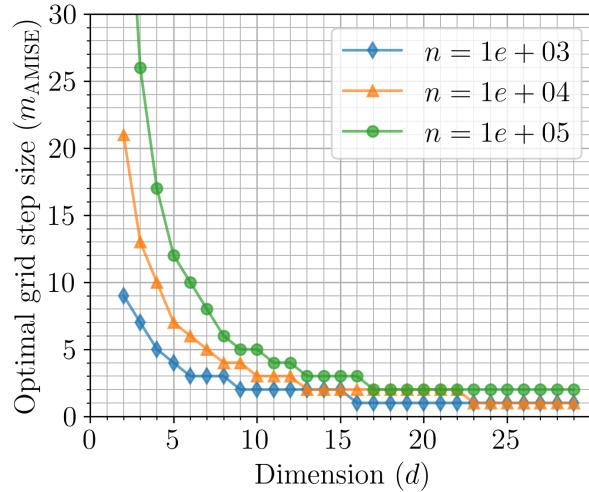


Figure 3.1 Evolution of m_{IMSE} for different dimensions and sample sizes.

benchmark realized by Nagler et al. (2017). The results showed that the most performant methods vary depending on the problem studied (for different dimension, sample sizes, strength of dependence). Regarding tail dependence modeling, nonparametric approaches are generally limited, but recent contributions introduced Bootstrap procedures to better this aspect (Kiriliouk et al., 2021).

Illustrative example on a Clayton copula

Let us consider a bivariate Clayton copula C with parameter $\theta = 2.5$ (see Nelsen 2006) A Monte Carlo sample with size $n = 10$ is generated on it, which is then used to build an empirical copula C_n as defined in Eq. (3.9). Fig. 3.2 (a) illustrates the empirical copula corresponding to the sample, with the shade of grey matching the CDF values. Then, the Bernstein approximation of the empirical copula (i.e., the EBC) is represented in Fig. 3.2 (b), (c), (d) according to the Eq. (6.4). The three versions of the EBC correspond to different polynomial orders, assuming that $(m_1 = m_2)$.

As the order increases, the bias between the EBC and the copula C tends to be reduced. Note that the second EBC in Fig. 3.2 (c) where $(m_1 = m_2 = n)$ is equivalent to the Beta copula. Moreover, increasing the order beyond the sample size definitely overfits of the copula.

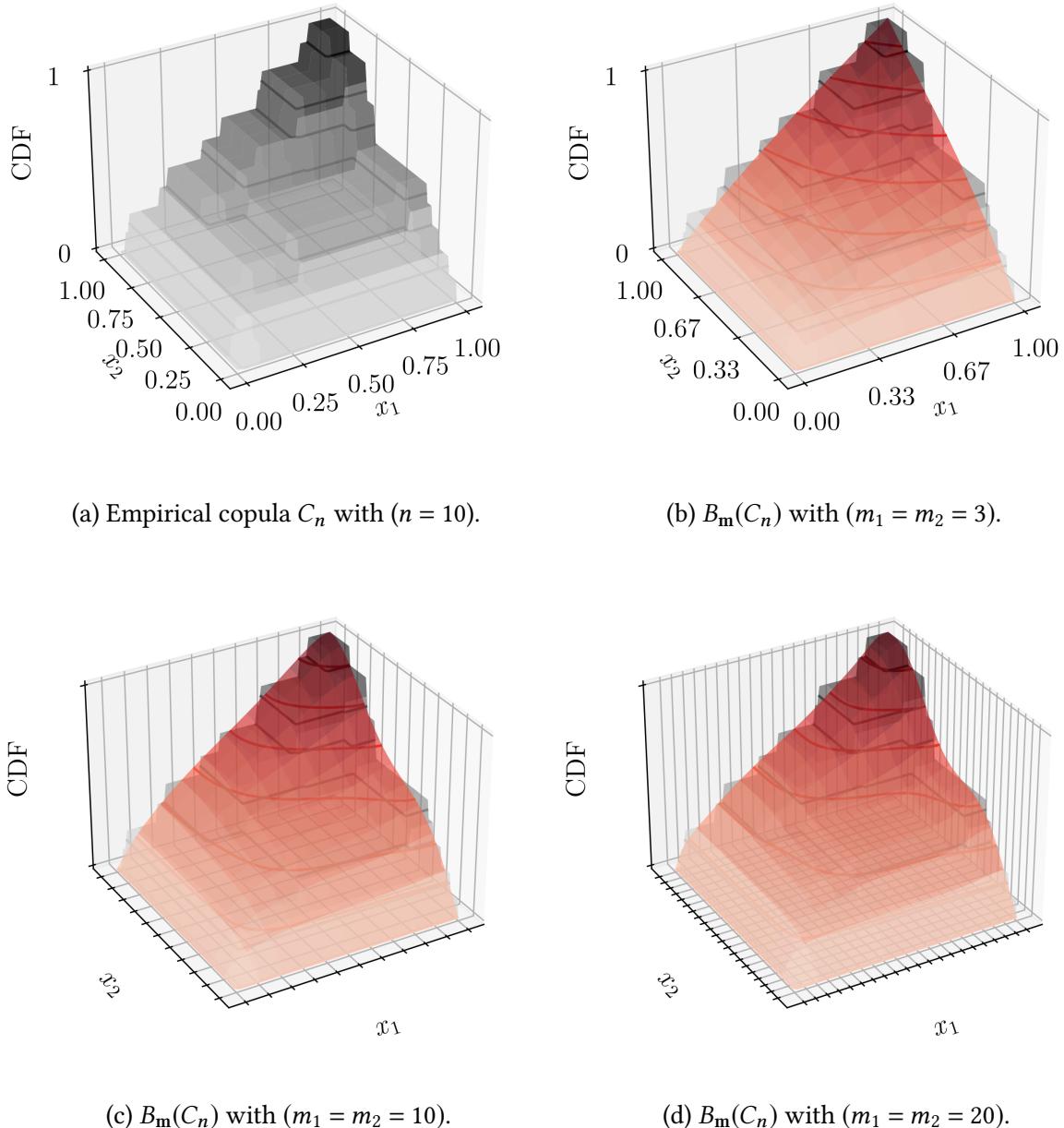


Figure 3.2 Bernstein approximations of the empirical copula C_n (with size $n = 10$) of a Clayton copula (with parameter $\theta = 2.5$). The polynomial orders are assumed equal in the two dimensions $m_1 = m_2 \in \{3, 10, 20\}$.

3.3 *Copulogram*: a tool for multivariate data visualization

In statistics, data visualization offers a wide set of tools to analyze data. Multivariate data visualization is of great help to apprehend problems with dimension higher than two. In the context of continuous variables, let us consider the n -sized sample $\mathbf{X}_n = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\} \sim \mathbf{X} \in \mathcal{D}_{\mathbf{x}} \subseteq \mathbb{R}^d$. The marginal samples of \mathbf{X}_n are denoted by $X_{n,j} = \{x_j^{(1)}, \dots, x_j^{(n)}\}, j \in \{1, \dots, d\}$.

Various techniques exist to represent multivariate data, such as the “parallel coordinate plot”, also called “cobweb plot” (see e.g., [Heinrich and Weiskopf, 2013](#)). For each sample $\mathbf{x}^{(i)} \in \mathbf{X}_n$, this plot draws a line passing by the values of $\mathbf{x}^{(i)} = [x_1^{(i)}, \dots, x_d^{(i)}]$. This representation was used in sensitivity analysis to illustrate the connections between a set of inputs and an output, however, it does not provide a good representation of the dependence structure between the inputs.

3.3.1 From the pairwise plot to the copulogram

Alternatively, the “pairwise plot”, also named “generalized draftsman plot”, was initially introduced by [Hartigan \(1975\)](#) to draw a matrix of scatter-plots between all the pairs of marginal samples $\{X_{n,i}, X_{n,j}\}, i \neq j \in \{1, \dots, d\}$ ². Because of the symmetry, the pairwise plot is usually represented on the lower triangle of the matrix. Later on, statisticians improved the pairwise plot by adding a histogram (or KDE) of the marginal samples $X_{n,j}, j \in \{1, \dots, d\}$ on the diagonal. Additionally, the upper triangle was completed with the values of linear correlation for each pairs of marginal sample $\{X_{n,i}, X_{n,j}\}, i \neq j$. This matrix of correlation coefficients is also known as “correlogram”. Altogether, this matrix plot became known as the “scatter plot of matrices” (SPLOM).

However, the linear correlation coefficient is known to give a poor description of the dependence in nonlinear cases. When analyzing a continuous sample $\mathbf{X}_n \sim \mathbf{X}$, the Sklar theorem states that the dependence structure within the random vector \mathbf{X} has a unique expression with its d -copula C . As mentioned in Section 3.2.1, the component wise normalized ranks of the original sample \mathbf{X}_n define the empirical copula density c_n (converging towards C as n increases).

To the best of our knowledge, the *copulogram* is a new multivariate data visualization tool improving the SPLOM by representing the empirical copula density c_n on the upper triangle of the matrix plot. This plot is an empirical decomposition of a multivariate sample in the vein of the Sklar theorem between marginals on the diagonal and copula on the upper triangle.

3.3.2 Implementation in a Python package

An open-source implementation is proposed in the python package `copulogram`. This code mostly relies on the Python package for data visualization `seaborn` ([Waskom, 2021](#)). The developments are tracked and archived in a GitHub repository² and the package can be installed from the package-management system “PyPI”.

Multiple visual options are offered by the `copulogram` package, as illustrated in the GitHub repository. For example, the user can represent the univariate samples on the diagonal,

or the bivariate samples in the triangles with kernel density estimation. Categorical variables can be used to assign different colors depending on the data class. The colors can also vary depending on a continuous variable after defining a mapping between the values of this variable and a set of color (also called colorbar).

Example #1: Iris flower dataset

The first example illustrates the copulogram on a widely used dataset in the machine learning community. The iris flower dataset was first introduced by Fisher and became a reference dataset for classification techniques. In the following lines of Python code, the dataset is loaded and the copulogram package is used to draw the new plot. The resulting copulogram applied to the iris flower data is represented in Fig. 3.3.

```

1  #!/usr/bin/python3
2  import seaborn as sns
3  import copulogram as cp
4  data = sns.load_dataset("iris")
5  copulogram = cp.Copulogram(data)
6  copulogram.draw(hue="species")
```

Since this data mostly presents linear dependencies, the copulogram is not very instructive. In other cases, the role of the dependence in the joint distribution is more important.

Example #2: Ishigami function

The Ishigami function is commonly used as a benchmark problem for global sensitivity analysis (GSA):

$$y = g(x_1, x_2, x_3) = \sin(x_1) + 7 \sin(x_2)^2 + \frac{x_3^4 \sin(x_1)}{10}. \quad (3.19)$$

This uncertainty quantification problem considers an independent random input vector $\mathbf{X} = \prod_{j=1}^3 X_j$. While the marginals in GSA benchmarks are usually assumed to be uniform, they will be considered Gaussian hereafter to distinguish the different elements of the joint distribution. Therefore, let us define $X_j \sim \mathcal{N}(0, 1) \forall j \in \{1, 2, 3\}$. In this setup, the random inputs are independent but they each present interesting dependencies with the random output $Y = g(\mathbf{X})$.

A Monte Carlo sample with size $n = 10^3$ is generated, $\mathbf{X}_n \stackrel{\text{i.i.d.}}{\sim} \mathbf{X}$, and evaluated such that $Y_n = g(\mathbf{x}_n)$. The copulogram of the input-output sample (\mathbf{X}_n, Y_n) is represented in Fig. 3.4. As expected, the scatter-plots between the inputs in the upper triangle are uniform (representing an independent density copula).

In GSA, the work of Póczos et al. (2012) studied the discrepancy between the empirical density copula $c_n(X_j, Y), j \in \{1, \dots, d\}$ and the independent copula to qualitatively assess the importance of X_j . In the same vein, the paper of Plischke and Borgonovo (2019) attempted to formalize a link between different GSA approaches based on copulas and to quantitative approaches as the Sobol' indices.

²GitHub repository: <https://github.com/efekhari27/copulogram>

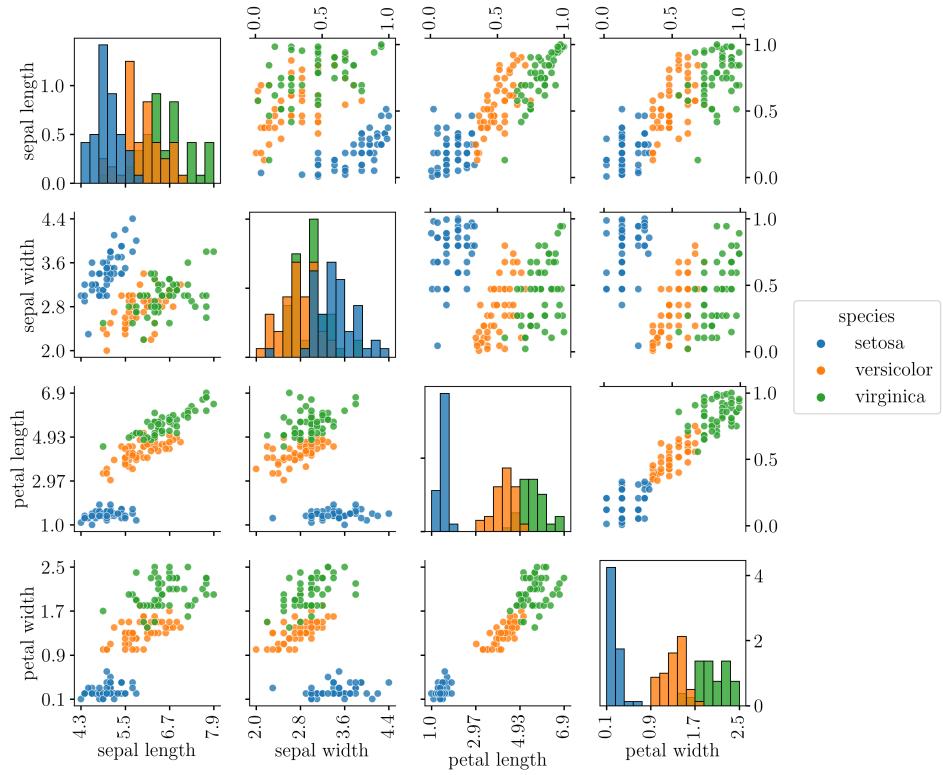


Figure 3.3 Copulogram of the iris flower dataset with colors assigned by the iris species.

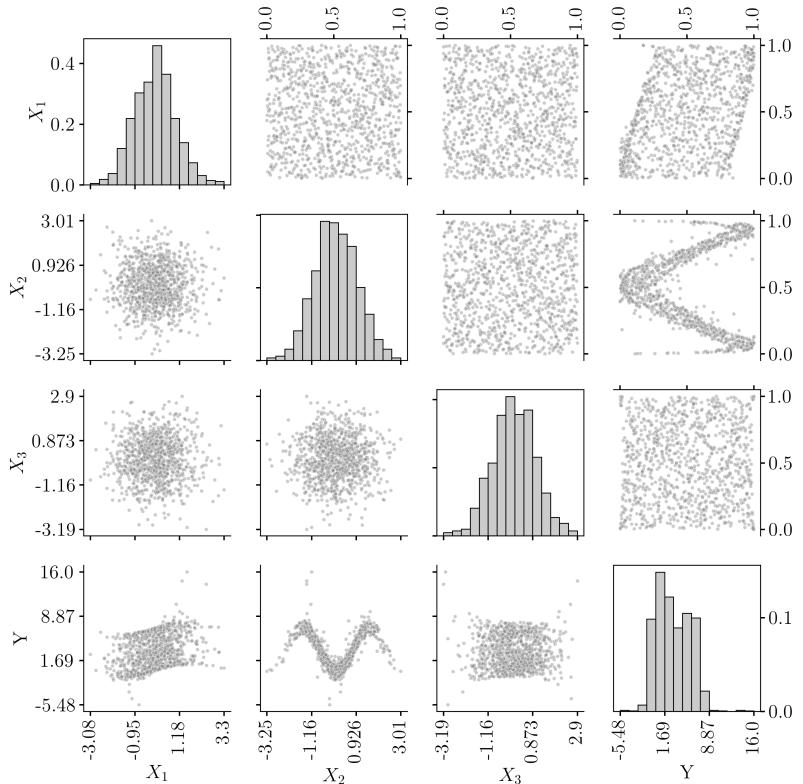


Figure 3.4 Copulogram of Monte Carlo sample (with size $n = 10^3$) of the inputs and outputs of the modified Ishigami problem.

3.4 Semiparametric inference of the South Brittany metocean conditions

Metocean conditions have been long studied in costal and offshore engineering. Inferring multivariate probabilistic models on metocean data became essential in wind energy.

Numerous approaches are proposed in the literature to fit a model on environmental data. Among them, let us mention the use of parametric methods as the conditional modeling (e.g., Bitner-Gregersen 2015; Vanem et al. 2023), or the construction of vine copulas (e.g., Vanem 2016; Montes-Iturriaga and Heredia-Zavoni 2016; Lin and Dong 2019). Nonparametric methods as the KDE were also applied in this context (e.g., Han et al. 2018). The nonparametric techniques generally struggle to model the distributions' tails, even if the tails are essential to qualify structures for ultimate events. However, they are highly flexible and often easier to implement than parametric methods.

In this section, a semiparametric inference strategy is presented, composing some well-known parametric models for the marginals (e.g., Weibull distribution for the wind speed), with a highly flexible dependence modeling by the EBC. A metocean dataset is used to showcase the empirical Bernstein copula and its representation by the copulogram. This dataset from the ANEMOC (Digital Atlas of Ocean and Coastal Sea States atlas, Raoult et al., 2018) gathers 32 years of preprocessed data (at an hourly resolution) from a location off the coast of South Brittany, France. A subset of 10^4 points is randomly selected among the ANEMOC data, which will be used to realize the semiparametric inference. The full code developed for this inference study is available on a GitHub repository³.

3.4.1 Inference of the marginals

The variables studied to describe the environmental conditions match the ones defined in Table 2.2. Unfortunately, the turbulence is provided by the ANEMOC database and is therefore not fitted. A straightforward inference is performed on the data, resulting in the models presented in Table 3.1. The wind and wave directions are fitted by KDE to catch their multimodal behavior while the other variables by MLE on various parametric models. Note that some variations of KDE with kernels specific to circular data could be interesting to ensure the continuity of the model at the bounds (Bai et al., 1989).

The results of the marginals' inferences, plotted in Fig. 3.5 against histograms, are visually satisfying. Statistical testing is not necessary in our case since the actual topic of discussion is related to the inference of the dependence. Considering these marginals, a study of the copula inference can be developed.

³GitHub repository: https://github.com/efekhari27/thesis/blob/main/numerical_experiments/chapter3/south_brittany_inference.ipynb

Name	Notation	Fitted model	KS p-value ($\alpha = 5\%$)
Wind speed	U	Weibull ($\beta = 11.4, \alpha = 2.2, \gamma = 0$)	0.238
Wind direction	θ_{wind}	KDE	–
Significant wave height	H_s	Inverse Normal ($\mu = 2.3, \lambda = 6.8$)	0.533
Wave period	T_p	Weibull ($\beta = 9.3, \alpha = 3.3, \gamma = 2$)	0.00021
Wave direction	θ_{wave}	KDE	–

Table 3.1 Marginal inference results of the South Brittany metocean data.

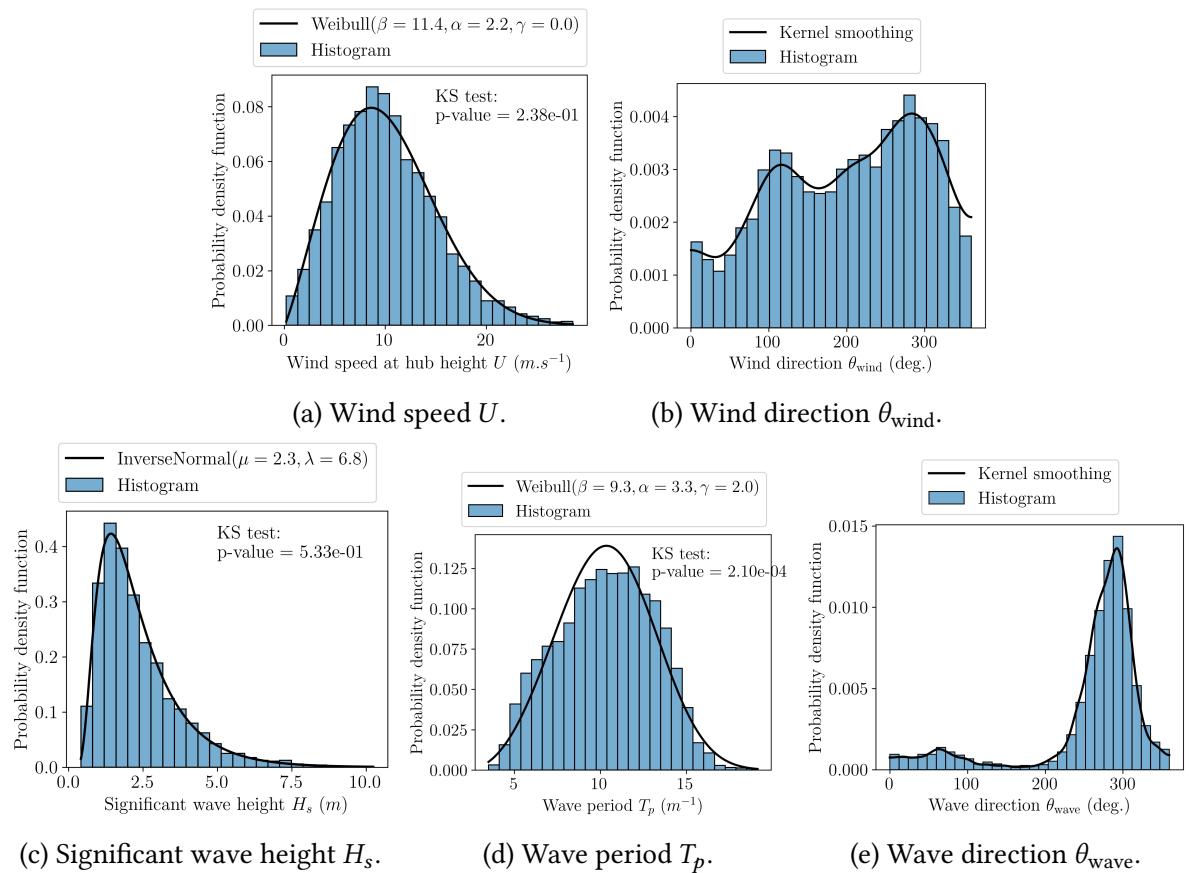


Figure 3.5 Marginal inference results of the South Brittany metocean data.

3.4.2 Nonparametric inference of the dependence

The aim of this section is to complete the set of marginals fitted previously by the inference of a copula. A nonparametric estimation using the empirical Bernstein copula is studied in this section. To validate the goodness-of-fit of the EBC, the ANEMOC dataset (with size $N = 10^4$) is randomly split into two: a learning set \mathbf{X}_n and a validation set \mathbf{X}'_n . A joint distribution $\widehat{F}_{\mathbf{X}}$ is then built using \mathbf{X}_n , by combining the marginals fitted earlier with an empirical copula $B_{\mathbf{m}}(C_n)$, such that:

$$\widehat{F}_{\mathbf{X}}(\mathbf{x}) = B_{\mathbf{m}}(C_n) \left(\widehat{F}_{X_1}(x_1), \dots, \widehat{F}_{X_d}(x_d) \right). \quad (3.20)$$

Where \widehat{F}_{X_j} stand for the model of the marginal j , just inferred in Section 3.4.1. While C_n is the empirical copula associated with the sample \mathbf{X}_n , and all the polynomial orders of the EBC are equal, $\mathbf{m} = \{m, \dots, m\}, m \in \mathbb{N}$.

Then, one could compare a sample $\widehat{\mathbf{X}}_n$, generated from the fitted joint distribution $\widehat{F}_{\mathbf{X}}$, with the learning set \mathbf{X}_n . However, to prevent an overfit from the semiparametric model, the comparison is rather done between \mathbf{X}_n and the independent validation set \mathbf{X}'_n . The statistic used is the maximum mean discrepancy, $\text{MMD}(\widehat{\mathbf{X}}_n, \mathbf{X}'_n)$, initially introduced for multivariate two-sample testing (a specific presentation of the MMD and its estimation is developed in Appendix B). For a given fitted joint distribution, this procedure is repeated 100 times to take into account the sampling variability.

In Fig. 3.6, MMD distributions are represented for different values of the EBC polynomial order, with $m \in \{5, 10, 20, 50, 100, 1000\}$. The smallest the values of this dissimilarity measure, the closer the samples should be. Even if further developments could be implemented to improve the MMD estimation, these results are sufficient to set the EBC tuning at $m = 100$. Considering this setup, Fig. 3.7 represents the copulogram of a sample $\widehat{\mathbf{X}}_n$ (in red), side by side with the copulogram of the learning set \mathbf{X}_n (in blue). This semiparametric approach offers a lot of flexibility, which is essential when inferring such complete dependence structures. As any nonparametric method, it should be used with caution when inferring distributions' tails.

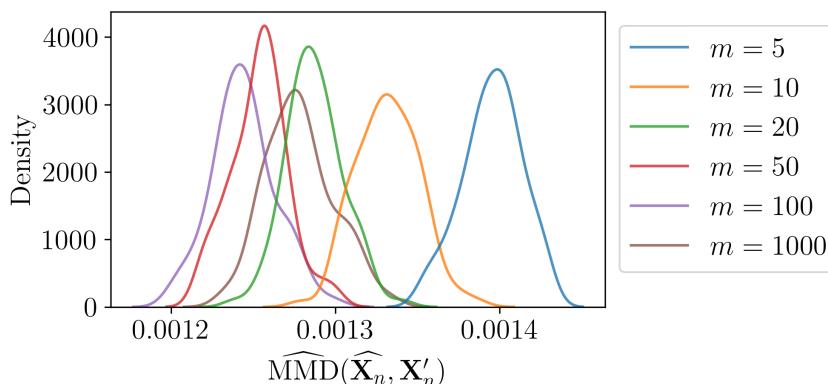
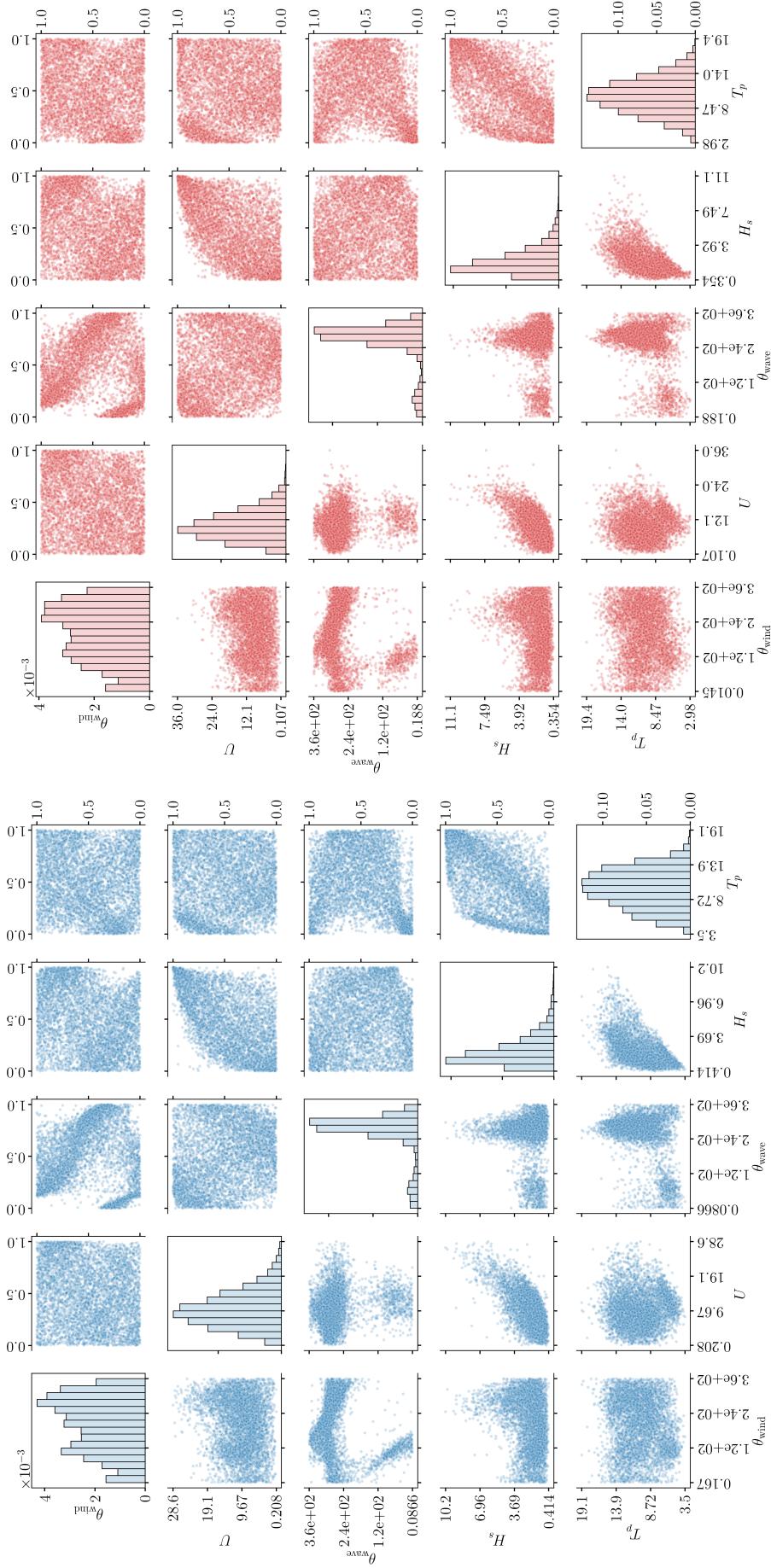


Figure 3.6 Empirical distributions of the maximum mean discrepancy between the validation sample \mathbf{X}' and the sample $\widehat{\mathbf{X}}_n \stackrel{\text{i.i.d.}}{\sim} \widehat{F}_{\mathbf{X}}$ (repeated for 100 samples $\widehat{\mathbf{X}}_n$).



(b) Monte Carlo sample (size $n = 5000$), generated from the semiparametric model fitted on the ANEMOC data (EBC $B_{\text{m}}(C_n)$ with $\{m_j = 100\}_{j=1}^d$).

(a) South Brittany ANEMOC data (size $n = 5000$).

Figure 3.7 Copulogram of the South Brittany metocean data.

3.4.3 Summary and discussion

In this section, a semiparametric inference strategy was illustrated on a metocean dataset from a site off the coast of South Brittany, France. This approach is pragmatic and offers a lot of flexibility. In our case, the unimodal marginals are fitted by MLE while the multimodal ones are fitted by KDE. Considering the complexity of the dependence, the EBC showed interesting results for large size samples. Its capacity to extrapolate in the tails could be further studied ([Heredia-Zavoni and Montes-Iturriaga, 2022](#)) but this tool is an appropriate solution for general inference (for example needed for fatigue assessment).

In the monograph of [Joe and Kurowicka \(2011\)](#), the use of nonparametric methods is briefly discussed p.250. The author recommends using nonparametric copulas when the marginals are well-behaved, but the dependence structure is nonlinear.

3.5 Quantifying and clustering the wake-induced perturbations within a wind farm

After defining a probabilistic model based on ambient metocean data, the present section studies the impact of the wake on the wind conditions within a farm. The wake arises from extracting kinetic energy from the wind, leading to a decrease in wind speed and an increase in turbulence downstream of the turbines. In a wind farm, the wake mostly depends on the turbines' layout, the ambient wind speed, and the ambient turbulence intensity. The resulting heterogeneous wind field in a wind farm can be simulated by numerical models with different fidelities (as discussed in Section [2.2.2](#)).

In our case, simplified wake models (sometimes called “dynamic wake meandering”, or “engineering” models, see e.g., [Doubrawa et al., 2020](#)) are used to simulate the wind speed deficit and the added turbulence at each turbine. To recover a wake-perturbed wind distribution at each turbine, the ambient wind distribution is propagated through a wake model. Having different wind distributions naturally impacts the loading and should be considered during fatigue assessment at a farm scale. Such heterogeneity is mostly considered by international standards via empirical coefficients (also called “effective turbulence”). [Doubrawa et al. \(2023\)](#) compares the effective turbulence approach with dynamic wake meandering models and studies their impact on loading.

In practice, fatigue assessment on a turbine represents a computational effort. As a consequence of wake modeling, each turbine presents a different wind distribution, which implies repeating a fatigue assessment for every turbine. To make this computation tractable at a farm scale, the present section aims at building clusters of turbines similarly affected by the wake. Then, fatigue assessment can be computed on a few turbines, each representing a cluster of turbines facing similar wake-modified wind loading. The maximum mean discrepancy (MMD) is used as a statistical dissimilarity measure to compare the perturbed distributions induced by the wake.

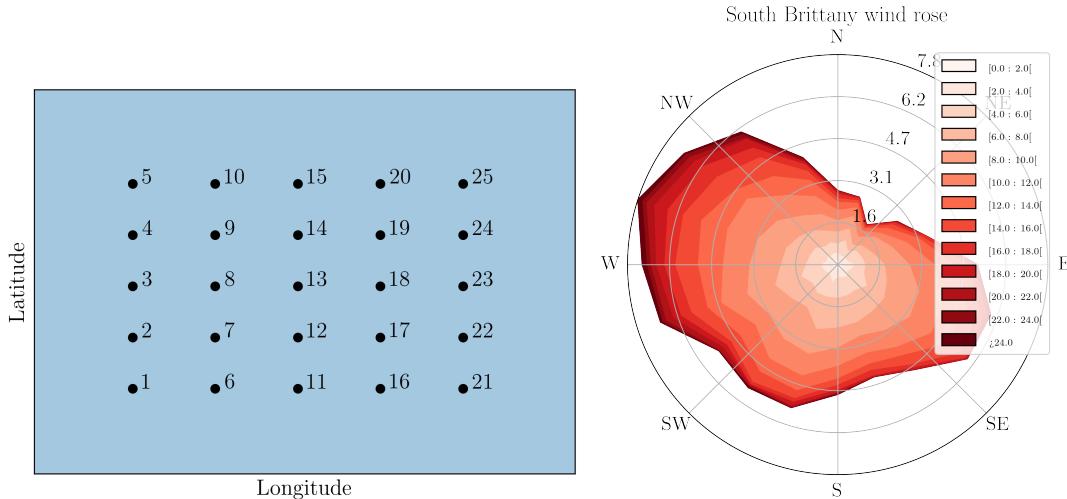


Figure 3.8 South Brittany wind farm layout, the vertical direction does not represent the exact north (left). South Brittany wind rose from the ANEMOC data (right).

This new approach is applied on a theoretical wind farm representing the recent call for tenders off the coast of South Brittany, France. The turbines considered are a modified version of the floating offshore wind turbine (FOWT) IEA-15MW (described in [Kim et al., 2022](#)). Figure 3.8 illustrates the layout of the 25 FOWTs modeled in the following, the coordinates are normalized by the rotor's diameter D . The spacing between turbines in this regular layout is equivalent to seven rotor diameters in the dominant wind direction and five rotor diameters in the orthogonal direction (i.e., crosswind).

To develop this approach on the South Brittany farm, the present section is structured as follows: first the wake model and its corresponding uncertainty propagation are presented, then MMDs are estimated between the resulting empirical joint wind distributions, and finally a simple clustering gathers turbines perceiving similar wake and set their representative turbines. In the end, the 25 FOWT are split into four groups, whose representant can be used for fatigue assessment of the hole group.

3.5.1 Uncertainty propagation on a wake model

Wake model definition When simulating the wake effect of floating wind turbines, different studies (either using dynamic wake meandering [Wise and Bachynski \(2020\)](#) or LES [Johlas et al. \(2020\)](#)), showed the importance of modeling the floaters' position (translation and rotation). Therefore, an engineering wake model based on the Farmshadow™ software (developed by IFPEN) was coupled with a hydro-static calculation to predict the floater's position, as well as the wind speed and turbulence intensity. In this model, the floaters are considered to be rigid, and all degrees of freedom are considered (surge, sway and heave for the three translations and roll, pitch and yaw for the three rotations).

FarmShadow™ uses engineering wake models to simulate the wind field throughout the whole farm, starting from the most upstream WT and working downwards. More precisely, the

model used includes the “super-gaussian” approach for speed deficit ([Blondel and Cathelain, 2020](#)), a waked-induced turbulence according to [Ishihara and Qian \(2018\)](#), and superimposition following the linear sum approach defined by [Zong and Porté-Agel \(2020\)](#). Further assumptions regarding the hydro-statics loading, the effect of the mooring lines, and the wake model are defined in [Lovera et al. \(2023\)](#).

Monte Carlo uncertainty propagation The wake model described earlier takes as input a set of variables describing the ambient wind conditions $\mathbf{x} \in \mathbb{R}^3$ and computes the perturbed wind conditions at each WT represented by the vector $\mathbf{x}'_l, l \in (1, \dots, n_{WT})$, where $n_{WT} \in \mathbb{N}$ is the total number of turbines in the farm:

$$g : \mathbb{R}^3 \rightarrow \mathbb{R}^{3n_{WT}} \quad (3.21)$$

$$\mathbf{x} \longmapsto g(\mathbf{x}) = (\mathbf{x}'_1, \dots, \mathbf{x}'_{n_{WT}}) \quad (3.22)$$

The uncertainties associated with the ambient wind conditions are represented by a random vector \mathbf{X} following the distribution f_0 . A parametric model has been fitted in [Vanem et al. \(2023\)](#) using conditional probability density functions to capture the dependence structure, but the semi-parametric inference proposed in Section 3.4 could have been an alternative. Note that the missing turbulence intensity in the ANEMOC data from South Brittany were assumed to follow a lognormal distribution. The random vector \mathbf{X} gathers the following random variables:

- Mean wind speed (U) is the 10-min average horizontal wind speed at hub height.
- Wind turbulence intensity (TI) is the 10-min wind speed turbulence intensity at hub height.
- Wind direction (θ_{wind}) is the 10-min average wind direction.

In the following, the wind orientation θ_{wind} is supposed to be unaffected by the wake. The uncertainty propagation through the wake models provides a set of perturbed environmental distributions $f'_l, l \in (1, \dots, n_{WT})$. In practice, a Monte Carlo sample $\mathbf{X}_n = \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)} \sim \mathbf{X}$ (with size $n=6000$) is generated and evaluated by the wake model. Since the model has a low computational cost, Monte Carlo sampling was affordable while giving strong convergence guarantees.

Fig. 3.9 illustrates the perturbation of the wind distributions for three WT differently affected by the wake depending on their position in the farm (see Figure 3.8). One can notice that the distribution of WT 25 (in orange) is very close to the ambient distribution (in black), as expected since this WT being located on the edge of the farm and facing the dominant wind direction. Meanwhile, the distribution of WT 13 (in red) seems more affected by the wake, by getting a higher wind turbulence with lower wind speed. This analysis can be completed with the two marginals in Fig. 3.10a and Fig. 3.10b, both describing the ambient marginal distributions (in black) and wake-disturbed distributions. In general, a small wind speed deficit is indicated by the small shifts of the probability density functions to the left on Fig. 3.10a. Also, a small added

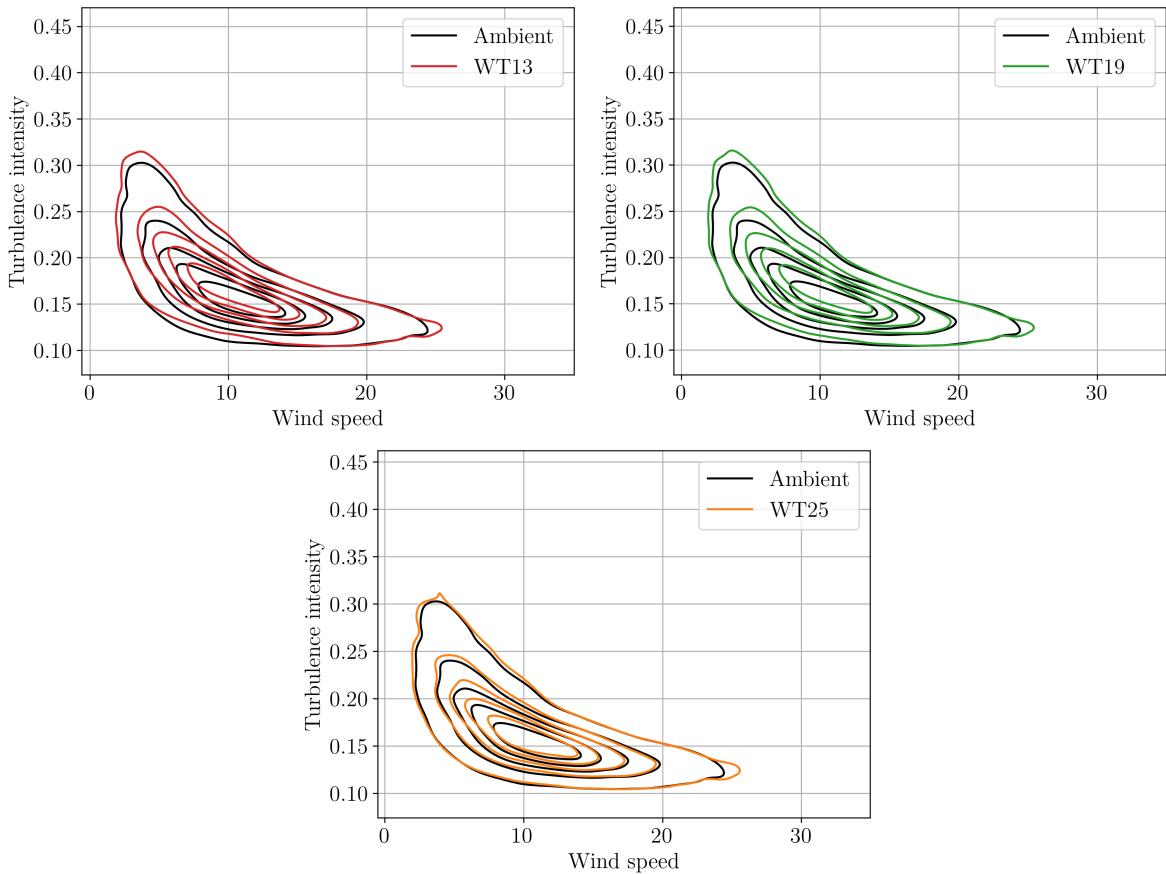


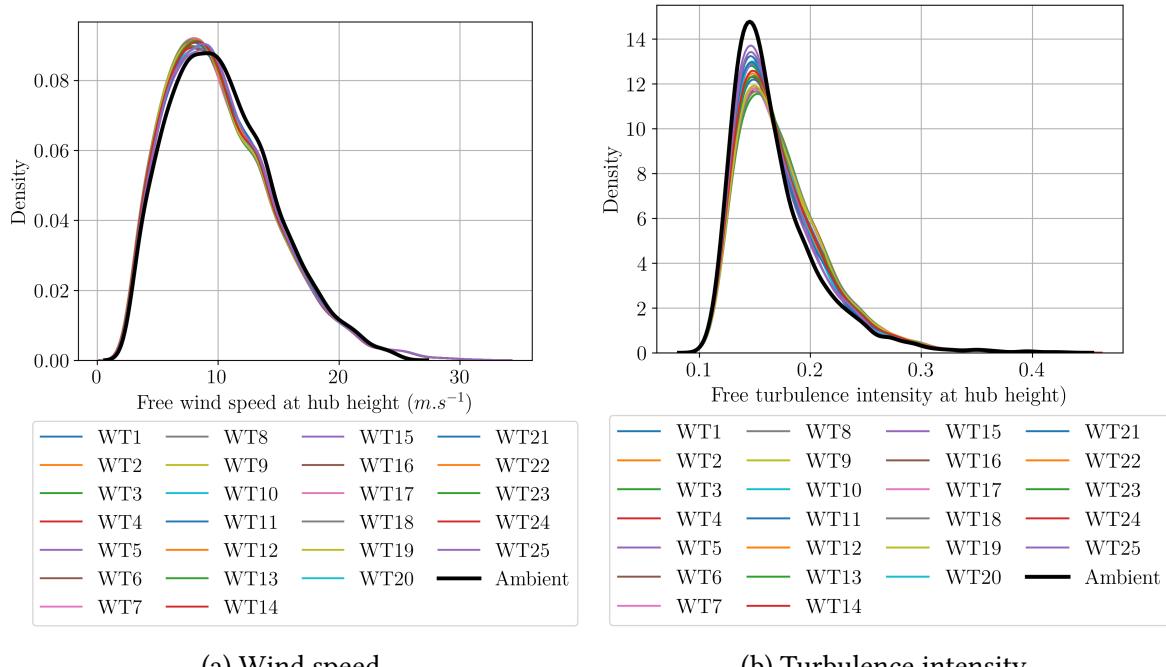
Figure 3.9 Joint distributions of the wake-perturbed wind conditions at WT 13, 19, and 25 (in color) compared with the ambient wind conditions (in black).

turbulence is noticeable with shifts of the PDFs to the right on Fig. 3.10b. A tool is needed to quantify the wind perturbations induced by the wake.

3.5.2 Statistical metric of wake-induced perturbations

The maximum mean discrepancy was introduced by [Gretton et al. \(2006\)](#) as a statistic for two-sample testing. After further work on this tool, authors such as [Sriperumbudur et al. \(2010\)](#) showed that the MMD is a distance between two distributions embedded in a specific function space. The MMD becomes a metric for specific kernels called “characteristic kernels”, which offer the following property: $\text{MMD}(\pi, \zeta) = 0 \iff \pi = \zeta$. The squared MMD has been used for multiple purposes and is further presented in Appendix B. In the following, the idea is to compare the ambient wind distribution f_0 to the wake-perturbed wind conditions f'_l at the WT l using the squared MMD.

Application to the South Brittany wind farm project Once the joint perturbed distributions of each WT evaluated on a large Monte Carlo sample, their squared MMD with the ambient wind conditions can be computed. Fig. 3.11 represents the squared MMD for each WT to quantify the wake-induced perturbation. The values of squared MMD presented in this



(a) Wind speed.

(b) Turbulence intensity.

Figure 3.10 Ambient (in black) and wake-perturbed (in color) distributions of wind distributions.

figure are estimated between two Monte Carlo samples with size $n = 6000$. A verification of their convergence is realized in terms of coefficient of variation.

3.5.3 Clustering the wake-induced perturbations

The aim of this section is to use the MMD as a metric to define clusters of turbines getting similar wind conditions. Instead of comparing the wake-perturbed distributions with the ambient one, let us compare all the pairs of wake-perturbed distributions. Considering all the WT $\{1, \dots, n_{WT}\}$ in the farm, and their respective wake-perturbed distributions $\{f_l\}_{l=1}^{n_{WT}}$, let us

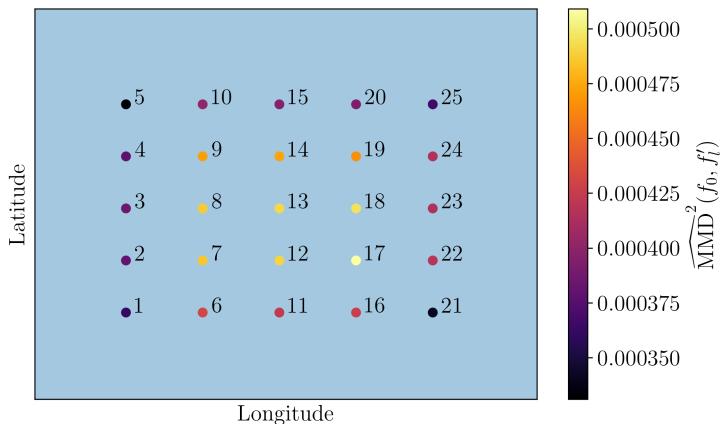


Figure 3.11 South Brittany layout and wake effects measured by the squared MMD on wind conditions. Note that the vertical direction does not represent the north direction.

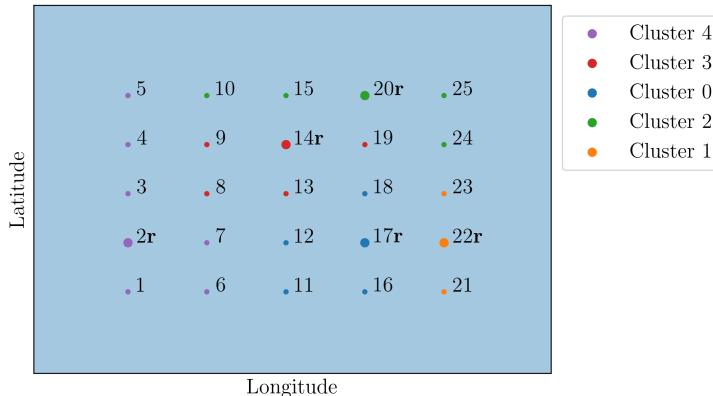


Figure 3.12 K-medoids clustering solution for five clusters. The representative elements of the clusters are tagged with the mention “r”.

define the symmetric matrix D of MMD between every pair of perturbed distributions such that $D_{i,j} = \widehat{\text{MMD}}^2(f'_i, f'_j)$.

Different unsupervised clustering techniques, such as hierarchical or centroid based methods were compared in Lovera et al. (2023). The “k-medoids” (Park and Jun, 2009) are a variation of the well known k-means that selects actual data points as centers (i.e., medoids). In our case, this method not only gathers turbines with the same wind conditions, but it also determines a representative turbine for each cluster.

Assuming a final number of clusters equal to five, Fig. 3.12 represents the clusters defined by the k-medoid method applied to the matrix D . Let us notice that the results are rather coherent with the main wind orientation illustrated in the wind rose (see Fig. 3.8). Interestingly, the conclusion that emerged from comparing the wake-perturbed distributions to the ambient one (see Fig. 3.11) are different from the ones obtained by comparing pairs of wake-perturbed distributions. Finally, the representative turbines are tagged with the mention “r” and could be used to perform a fatigue analysis.

3.5.4 Summary and discussion

This section studied the impact of the wake on the wind conditions in a farm. The ambient wind conditions were propagated on a low-fidelity wake model of a floating wind farm. Even if higher-fidelity models could simulate this phenomenon more accurately, their huge computational cost would not allow uncertainty propagation (e.g., several days for LES certain solver, with intensive parallelization). The present model has a reasonable error and has a short execution (i.e., about a few minutes on a regular computer). The resulting wake-perturbed distributions at each turbine were compared to each other using a kernel-based dissimilarity measure for distributions. Using this scalar metric of perturbations created by the wake, clusters were built. In a context of fatigue assessment at the farm scale, the computed clusters may reduce the computational effort by assessing one turbine per cluster.

3.6 Conclusion

In this chapter, different aspects of uncertainty quantification were applied to define the metoceanean conditions in a wind farm. First, to infer a joint environmental distribution, a semiparametric approach was applied to a large dataset from a site in South Brittany, France. This semiparametric approach combines parametric models of the marginals with nonparametric models of the copula such as the highly flexible empirical Bernstein copula. Second, to take into account the heterogeneous wind conditions inside a farm, a dynamic wake meandering model was developed for the South Brittany farm. Then, the ambient joint environmental distribution was propagated through this model to recover a wake-perturbed environmental distribution per turbine.

The copulogram is a new data visualization tool based on the empirical copula, aiming at improving the description of nonlinear dependencies in datasets.

The maximum mean discrepancy is a kernel-based dissimilarity measure between multivariate distributions which was used throughout this chapter. Either for testing the goodness-of-fit of the semiparametric model fitted using the EBC, or as a measure of the perturbation occasioned by the wake on the wind conditions. In this context, the MMD allowed building clusters of turbines witnessing the same wind conditions. As perspectives, the robustness of this metric to the choice of kernel could be further investigated as well as the goodness-of-fit of the EBC over distributions' tails. After defining the probabilistic model of the environmental conditions, this uncertainty is ready to be propagated in a multiphysics wind turbine model as DIEGO.

Chapter **4**

Kernel-based central tendency estimation

4.1	Introduction	114
4.2	Treatment of uncertainties on the Teesside wind farm	116
4.2.1	Numerical simulation model	117
4.2.2	Measured environmental data	117
4.2.3	Non parametric fit with empirical Bernstein copula	121
4.2.4	Fatigue assessment	121
4.3	Numerical integration procedures for mean damage estimation	123
4.3.1	Quadrature rules and quasi-Monte Carlo methods	124
4.3.2	Kernel herding sampling	125
4.3.3	Bayesian quadrature	128
4.4	Numerical experiments	132
4.4.1	Illustration on analytical toy-cases	132
4.4.2	Application to the Teesside wind turbine fatigue estimation	133
4.5	Conclusion	137

This chapter is adapted from the following reference:

E. Fekhari, V. Chabridon, J. Muré and B. Iooss (2023). “Given-data probabilistic fatigue assessment for offshore wind turbines using Bayesian quadrature”. In: *Data-Centric Engineering*, In press.

4.1 Introduction

As a sustainable and renewable energy source, offshore wind turbines (OWT) are likely to take a growing share of the global electric mix. However, to be more cost-effective, wind farm projects tend to move further from the coast, exploiting stronger and steadier wind resources. Going further offshore, wind turbines are subject to more severe and uncertain environmental conditions (i.e., wind and waves). In such conditions, their structural integrity should be certified. To do so, numerical simulation and probabilistic tools have to be used. In fact, according to [Graf et al. \(2016\)](#), for new environmental conditions or new turbine models, international standards such as [IEC-61400-1 \(2019\)](#) from the International Electrotechnical Commission and [DNV-ST-0437 \(2016\)](#) from Det Norske Veritas recommend performing over 2×10^5 simulations distributed over a grid. However, numerical simulations are computed by a costly hydro-servo-aero-elastic wind turbine model, making the design process time-consuming. In the following, the simulated output cyclic loads studied are aggregated over the simulation period to assess the mechanical fatigue damage at hot spots of the structure. To compute the risks associated with wind turbines throughout their lifespan, one can follow the steps of the universal framework for the treatment of uncertainties presented in the introduction of this manuscript Fig. 1. After specifying the problem (Step A), one can quantify the uncertainties related to site-specific environmental conditions represented by the random vector $\mathbf{X} \in \mathcal{D}_X \subset \mathbb{R}^d, d \in \mathbb{N}^*$ (Step B). Then, one can propagate them through the OWT simulation model (Step C) denoted by $g : \mathcal{D}_X \rightarrow \mathbb{R}, \mathbf{X} \mapsto Y = g(\mathbf{X})$, and estimate a relevant quantity of interest $\psi(Y) = \psi(g(\mathbf{X}))$ (e.g., a mean, a quantile, a failure probability). An accurate estimation of the quantity of interest $\psi(Y)$ relies on both a relevant quantification of the input uncertainty and an efficient sampling method.

Regarding Step B, when dealing with uncertain environmental conditions, a specific difficulty often arises from the complex dependence structure such variables may exhibit. Here, two cases may occur: either measured data are directly available (i.e., the “given-data” context) or a theoretical parametric form for the joint input probability distribution can be postulated. Such existing parametric joint distributions often rely on prior data fitting combined with expert knowledge. For example, several parametric approaches have been proposed in the literature to derive such formulations, ranging from fitting conditional distributions [Vanem et al., 2023](#)) to using vine copulas ([Li and Zhang, 2020](#)). When a considerable amount of environmental data is available, nonparametric approaches as the empirical Bernstein copula were studied in the Chapter 3 to capture complex dependence structures. Alternatively, an idea is to directly use

the data as an empirical representation of input uncertainties in order to avoid an additional inference error.

Step C usually focuses on propagating the input uncertainties in order to estimate the quantity of interest. Depending on the nature of $\psi(Y)$, one often distinguishes between two types of uncertainty propagation: a central tendency estimation (e.g., focusing on the output mean value or the variance) and a tail estimation (e.g., focusing on a high-order quantile or a failure probability). When uncertainty propagation aims at central tendency estimation, the usual methods can be split into two groups. First, those relying on sampling, i.e., mainly Monte Carlo sampling (Graf et al., 2016), quasi-Monte Carlo sampling (Müller and Cheng, 2018), geometrical subsampling (Kanner et al., 2018), or deterministic quadrature rules (Van den Bos, 2020). All these methods estimate the quantity directly on the numerical simulator's outputs. Second, those that rely on the use of surrogate models (or metamodels, see Fig. 1) to emulate the costly numerical model by a statistical model. Among a large panel of surrogates, one can mention, regarding wind energy applications, the use of polynomial chaos expansions (Dimitrov et al., 2018; Murcia et al., 2018), Gaussian process regression (Huchet, 2019; Teixeira et al., 2019a; Slot et al., 2020; Wilkie and Galasso, 2021), or artificial neural networks (Bai et al., 2023). When uncertainty propagation aims at studying the tail of the output distribution such as in risk or reliability assessment, one usually desires to estimate a quantile or a failure probability. In the wind energy literature, failure probability estimation has been largely studied, e.g., in time-independent reliability assessment (Zwick and Muskulus, 2015; Slot et al., 2020; Wilkie and Galasso, 2021) or regarding time-dependent problems (Abdallah et al., 2019; Lataniotis, 2019).

During the overall process described in Fig. 1, modelers and analysts often need to determine whether inputs are influential or not in order to prioritize their effort (in terms of experimental data collecting, simulation budget, or expert elicitation). Sometimes, they want to get a better understanding of the OWT numerical models' behavior or to enhance the input uncertainty modeling. All these questions are intimately related to the topic of sensitivity analysis (Saltelli et al., 2008; Da Veiga et al., 2021) and can be seen as an “inverse analysis” denoted by Step C' in Fig. 1. In the wind energy literature, one can mention, among others, some works related to Spearman's rank correlation analysis and the use of the Morris method in Velarde et al. (2019); Petrovska (2022). Going to variance-based analysis, the direct calculation of Sobol' indices after fitting a polynomial chaos surrogate model has been proposed in many works (e.g., in Murcia et al., 2018) while the use of distributional indices (e.g., based on the Kullback–Leibler divergence) has been investigated by Teixeira et al. (2019b).

The present chapter focuses on the problem of uncertainty propagation, and more specifically, on the mean fatigue damage estimation (i.e., $\psi(Y) = \mathbb{E}[g(\mathbf{X})]$). Such a problem is usually encountered, by engineers, during the design phase. Most of the time, current standards as well as common engineering practices make them use regular grids (Huchet, 2019). Altogether, one can describe three alternative strategies: (i) direct sampling on the numerical model (e.g., using Monte Carlo), (ii) sampling on a static surrogate model (e.g., using Gaussian process regression), or (iii) using an “active learning” strategy (i.e., progressively adding evaluations of the numerical

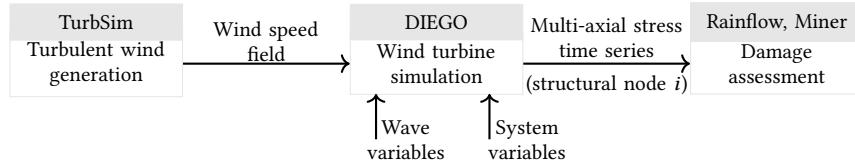


Figure 4.1 Diagram of the chained OWT simulation model.

model to enhance the surrogate model fitting process). In practice, fitting a surrogate model in the context of OWT fatigue damage can be challenging due to the nonlinearity of the code. Moreover, the surrogate model validation procedure complexifies the process. Finally, active learning strategies restrict the potential number of parallel simulations, which limits the use of HPC facilities. Thus, the main contribution of this chapter is to explore different ways to propagate uncertainties by directly evaluating the numerical model (i.e., without any surrogate model) with a relevant tradeoff between computational cost and accuracy. In the specific context of wind turbine fatigue damage, this work shows how to propagate uncertainties arising from a complex input distribution through a costly wind turbine simulator. The proposed work consists of evaluating the advantages and limits of kernel herding as a tool for given-data, fast, and fully-distributable uncertainty propagation in OWT simulators. Additionally, this sampling method is highly flexible, allowing one to complete an existing design of experiments. Such a property can be crucial in practice when the analyst is asked to include some specific points to the design (e.g., characteristic points describing the system's behavior required by experts or by standards, see [Huchet, 2019](#)).

The present chapter is organized as follows. Section 4.2 will present the industrial use case related to a wind farm operating in Teesside, UK. Then, Section 4.3 will introduce various kernel-based methods for central tendency estimation. Section 4.4 will analyze the results of numerical experiments obtained on both analytical and industrial cases. Finally, the last section will present some discussions and draw some conclusions.

4.2 Treatment of uncertainties on the Teesside wind farm

An OWT is a complex system interacting with its environment. To simulate the response of this system against a set of environmental solicitations, multi-physics numerical models are developed. In the present chapter, the considered use case consists of a chain of three numerical codes executed sequentially. As illustrated in Fig. 4.1, a simulation over a time period is the sequence of, first, a turbulent wind speed field generation, then a wind turbine simulation (computing various outputs including mechanical stress), and finally, a post-processing phase to assess the fatigue damage of the structure.

4.2.1 Numerical simulation model

This subsection generally describes the modeling hypotheses considered in the industrial use case, further details regarding wind turbines modeling a provided in Chapter 2 of this manuscript. The first block of the chain consists of a turbulent wind field simulator called “TurbSim” (developed by [Jonkman, 2009](#) from the National Renewable Energy Laboratory, USA) that uses, as a turbulence model, a Kaimal spectrum ([Kaimal et al., 1972](#)) (as recommended by the [IEC-61400-1, 2019](#)). Moreover, to extrapolate the wind speed vertically, the shear is modeled by a power law. Since the wind field generation shows inherent stochasticity, each 10-minute long simulation is repeated with different pseudo-random seeds and one averages the estimated damage over these repetitions. This question has been widely studied by some authors, (e.g., [Slot et al., 2020](#)), who concluded that the six repetitions recommended by the [IEC-61400-1 \(2019\)](#) may be insufficient to properly average this stochasticity. Thus, in the following, the simulations are repeated eleven times (picking an odd number also directly provides the median value over the repetitions). This number of repetitions was chosen to suit the maximum number of simulations and the storage capacity of the generated simulations.

As a second block, one finds the “DIEGO” software (for “Dynamique Intégrée des Éoliennes et Génératrices Offshore”¹) which is developed by EDF R&D ([Kim et al., 2022](#)) to simulate the aero-hydro-servo-elastic behavior of OWTs. It takes the turbulent wind speed field generated by TurbSim as input and computes the dynamical behavior of the system (including the multiaxial mechanical stress at different nodes of the structure). For the application of interest here, the control system is modeled by the open-source DTU controller ([Hansen and Henriksen, 2013](#)), and no misalignment between the wind and the OWT is assumed. As for the waves, they are modeled in DIEGO using a JONSWAP spectrum (named after the 1975 Joint North Sea Wave Project). The considered use case here consists of a DIEGO model of a Siemens SWT 2.3MW bottom-fixed turbine on a monopile foundation (see the datasheet in Table 4.1), currently operating in Teesside, UK (see the wind farm layout and wind turbine diagram in Fig. 4.2). Although wind farms are subject to the wake effect, affecting the behavior and performance of some turbines in the farm, this phenomenon is not considered in this chapter. To avoid numerical perturbations and reach the stability of the dynamical system, our simulation period is extended to 1000 seconds and the first 400 seconds are cropped in the post-processing step. This chained OWT numerical simulation model has been deployed on an EDF R&D HPC facility to benefit from parallel computing speed up (a single simulation on one CPU takes around 20 minutes).

4.2.2 Measured environmental data

During the lifespan of a wind farm project, environmental data is collected at different phases. In order to decide on the construction of a wind farm, meteorological masts, and wave buoys are usually installed on a potential site for a few years. After its construction, each wind turbine

¹In English, “Integrated Dynamics of Wind Turbines and Offshore Generators”.

Table 4.1 Teesside Offshore Wind turbine datasheet

Siemens SWT-2.3-93	
Rated power	2.3 MW
Rotor diameter	93 m
Hub height	83 m
Cut-in, cut-out wind speed	4 m/s, 25 m/s

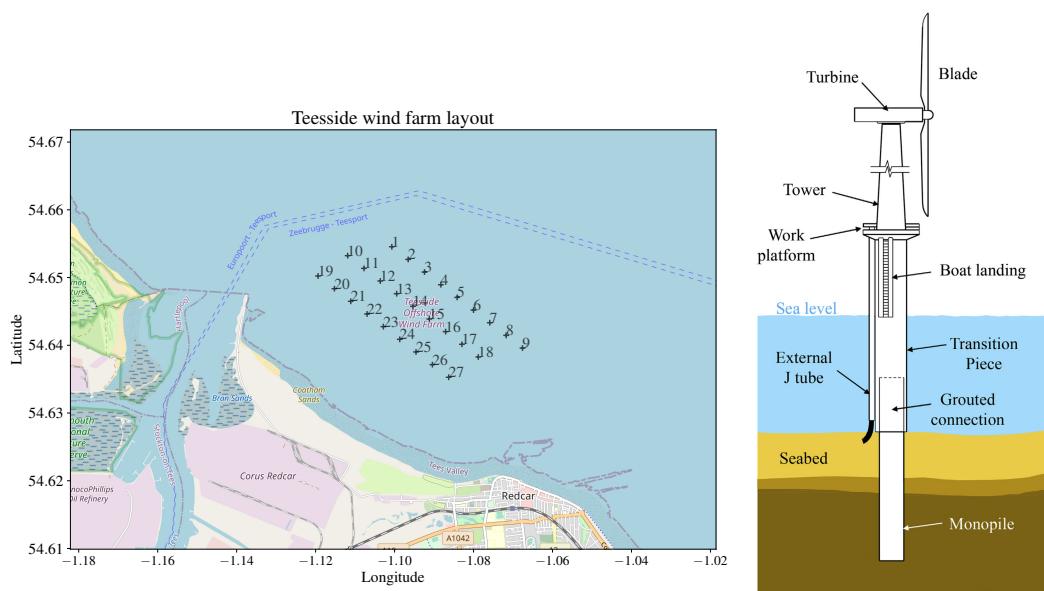


Figure 4.2 Teesside wind farm layout (left). Monopile OWT diagram (Chen et al., 2018) (right)

Variable	Notation	Unit	Description
Mean wind speed	U	m.s^{-1}	10-min. average horizontal wind speed
Wind turbulence	σ_U	m.s^{-1}	10-min. wind speed standard deviation
Wind direction ²	θ_{wind}	deg.	10-min. average wind direction
Significant wave height	H_s	m	Significant wave height
Peak wave period	T_p	s	Peak 1-hour spectral wave period
Wave direction	θ_{wave}	deg.	10-min. average wave direction

Table 4.2 Description of the environmental data.

is equipped with monitoring instruments (e.g., cup anemometers). In total, five years of wind data have been collected on the turbines which are not affected by the wake on this site. Their acquisition system (usually called “SCADA” for “Supervisory Control And Data Acquisition”) has a sampling period of ten minutes. The wave data arise from a buoy placed in the middle of the farm. These data describe the physical features listed in Table 4.2. A limitation of the present study is that it controller-induced uncertainty (like wind misalignment) is not considered.

The Teesside farm is located close to the coast, making the environmental conditions very different depending on the direction (see the wind farm layout in Fig. 4.2). Since measures are also subject to uncertainties, a few checks were made to ensure that the data were physically consistent. Truncation bounds were applied since this study is focused on central tendency estimation (i.e., mean behavior) rather than extreme values. In practice, this truncation only removes extreme data points (associated with storm events). In addition, a simple trigonometric transform is applied to each directional feature to take into account their cyclic structure. Finally, the remaining features are rescaled (i.e., using a min-max normalization).

Teesside’s environmental data is illustrated by its copulogram in Fig. 4.3, a graphical tool presented in Section 3.3 to visualize multivariate data. The copulogram exhibits the marginals with univariate kernel density estimation plots (in the diagonal), and the dependence structure with scatter plots in the normalized rank space (in the upper triangle). Looking at data in the rank space instead of the initial space allows one to observe the ordinal associations between variables. The scatter plots of normalized ranks are actually a representation of the empirical copula density. Two independent variables will present a uniformly distributed scatter plot in the rank space. In the lower triangular matrix, the scatter plots are set in the physical space, merging the effects of the marginals and the dependencies (as in the usual visualization offered by the matrix plot). Since the dependence structure is theoretically modeled by an underlying copula, this plot is called *copulogram*, generalizing the well-known “correlogram” to nonlinear dependencies. It gives a synthetic and empirical decomposition of the dataset that was implemented in a new open-source Python package named `copulogram`³.

On Fig. 4.3, a large sample $\mathcal{S} \subset \mathcal{D}_X$ (with size $N = 10^4$) is randomly drawn from the entire Teesside data (with size $N_{\text{Teesside}} = 2 \times 10^5$), and plotted in grey. In the same figure, the orange

²Note that the two directional variables could be replaced by a wind-wave misalignment variable for a bottom-fixed technology, however, our framework can be directly transposed to floating models.

³GitHub repository: <https://github.com/efekhari27/copulogram>

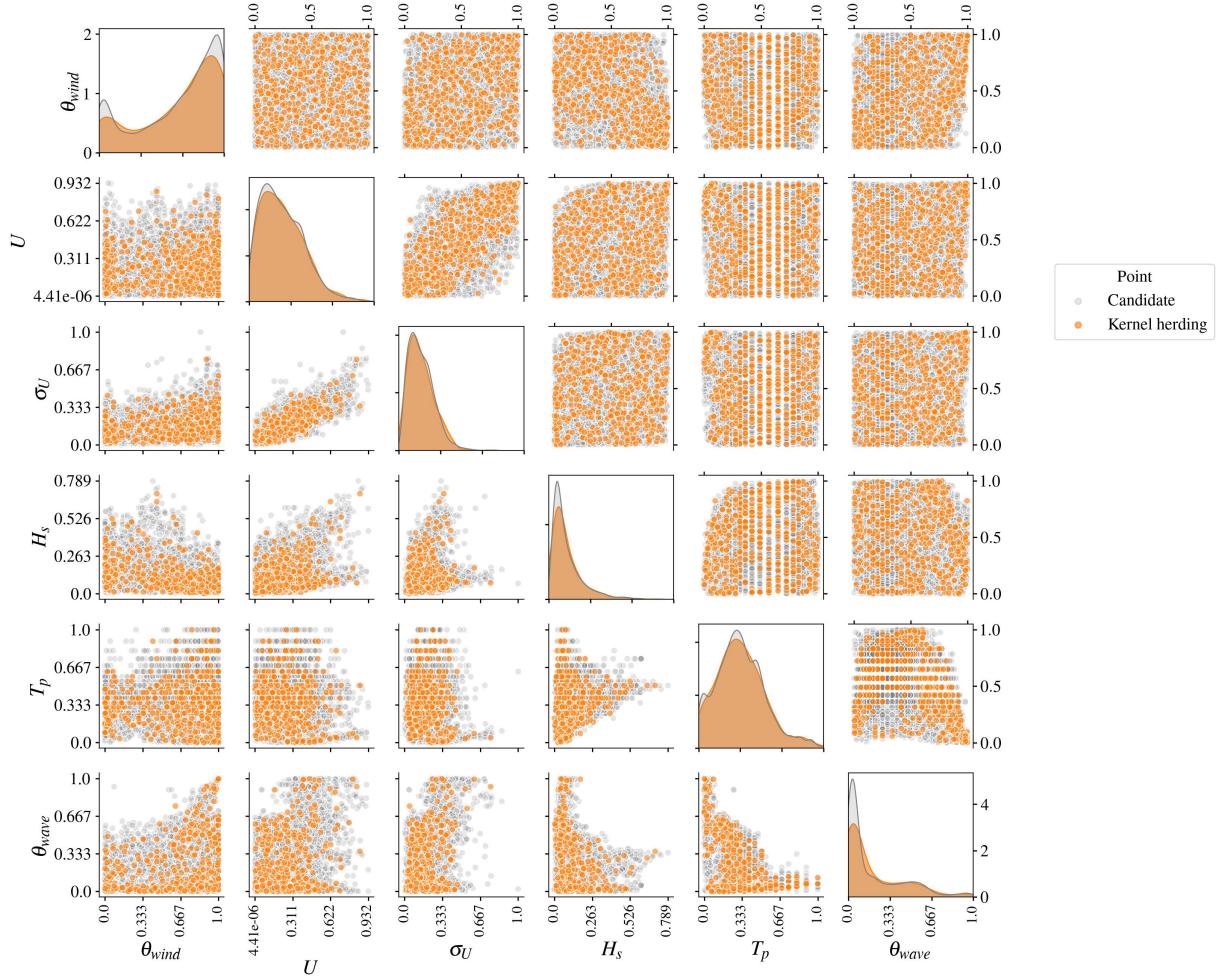


Figure 4.3 Copulogram of the Teesside measured data ($N = 10^4$ in grey), kernel herding subsample ($n = 500$ in orange). Marginals are represented by univariate kernel density estimation plots (diagonal), the dependence structure with scatter plots in the rank space (upper triangle). Scatter plots on the bottom triangle are set in the physical space.

matrix plot is a subsample of the sample \mathcal{S} , selected by kernel herding, a method minimizing some discrepancy measure with the sample \mathcal{S} that will be presented in Section 4.3). For this example, generating the kernel herding subsample takes under one minute, which is negligible compared with the simulation time of OWT models. Visually, this orange subsample seems to be representative of the original sample both in terms of marginal distributions and dependence structure. In the following study, the large samples \mathcal{S} will be considered as an empirical representation of the distribution of the random vector $\mathbf{X} \in \mathcal{D}_{\mathbf{X}}$, with probability density function $f_{\mathbf{X}}$, and called *candidate set*. Kernel herding allows direct subsampling from this large and representative dataset, instead of fitting a joint distribution and generating samples from it. Indeed, fitting a joint distribution would introduce an additional source of error in the uncertainty propagation process. Note that a proper parametric model fit would be challenging for complex dependence structures such as the one plotted on Fig. 4.3. As examples of works

that followed this path, one can mention the work of [Li and Zhang \(2020\)](#) who built a parametric model of a similar multivariate distribution using vine copulas.

For a similar purpose and to avoid some limits imposed by the parametric framework, a nonparametric approach coupling empirical Bernstein copula fitting with kernel density estimation of the marginals is proposed in subSection [4.2.3](#).

4.2.3 Non parametric fit with empirical Bernstein copula

Instead of directly subsampling from a dataset such as the one from Fig. [4.3](#), one could first infer a multivariate distribution and generate a sample from it. However, accurately fitting such complex multivariate distributions is challenging. The amount of available data is large enough to make nonparametric inference a viable option.

The Sklar theorem ([Durante and Sempi, 2015](#)) states that the multivariate distribution of any random vector $\mathbf{X} \in \mathbb{R}^d, d \in \mathbb{N}^*$ can be broken down into two objects:

1. A set of univariate marginal distributions to describe the behavior of the individual variables;
2. A function describing the dependence structure between all variables, called a *copula*.

This theorem states that considering a random vector $\mathbf{X} \in \mathbb{R}^d$, with its cumulative distribution function F and its marginals $\{F_i\}_{i=1}^d$, there exists a copula $C : [0, 1]^d \rightarrow [0, 1]$, such that:

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)). \quad (4.1)$$

It allows us to divide the problem of fitting a joint distribution into two independent problems: fitting the marginals and fitting the copula. The empirical Bernstein copula is a nonparametric copula approximation method introduced and applied on similar data in Chapter [3](#). Provided a large enough learning set \mathbf{X}_n (over five years in the present case), the EBC combined with kernel density estimation for the marginals enable to fit well the environmental joint distribution related to the dataset in Fig. [4.3](#). Moreover, the densities of the EBC are available in an explicit form, making Monte Carlo or quasi-Monte Carlo generation easy. As discussed in Chapter [3](#), this method is sensitive to the chosen polynomial orders $\{m_j\}_{j=1}^d$ and the learning set size.

4.2.4 Fatigue assessment

As described in Fig. [4.1](#), a typical DIEGO simulation returns a 10-minute multiaxial stress time series at each node $i \in \mathbb{N}$ of the 1D meshed structure. Since classical fatigue laws are established for uniaxial stresses, the first step is to compute one equivalent Von Mises stress time series at each structural node. The present section recalls the main concepts but fatigue assessment is further discussed in Section [2.3.5](#).

The foundation and the tower of an OWT are a succession of tubes with various sections connected by bolted or welded joints. Our work focuses on the welded joints at the mudline

level, identified as a critical area for the structure. This hypothesis is confirmed in the literature by different contributions, see e.g., the results of Müller and Cheng (2018) in Figure 12, or Katsikogiannis et al. (2021). At the top of the turbine, the fatigue is commonly studied at the blade root, which was not studied here since the blades in composite material have different properties (see e.g., Dimitrov (2013)). Note that the OWT simulations provide outputs allowing us to similarly study any node along the structure (without any additional computational effort).

To compute fatigue in a welded joint, the external circle of the welding ring is discretized for a few azimuth angles $\theta \in \mathbb{R}_+$ (see the red points in the monopile cross-section on the right in Fig. 4.4). The equivalent Von Mises stress time series is then reported on the external welding ring for an azimuth θ . According to Li and Zhang (2020) and our own experience, the most critical azimuth angles are roughly aligned with the main wind and wave directions (whose distributions are illustrated in Fig. 4.4). Looking at these illustrations, the wind and wave conditions have a very dominant orientation, which is explained by the closeness of the wind farm to the shore. Then, it is assumed that azimuth angles in these directions will be more solicited, leading to higher fatigue damage. To assess fatigue damage, rainflow counting (Dowling, 1972) first identifies the stress cycles and their respective amplitudes (using the implementation of the ASTM E1049-85 rainflow cycle counting algorithm from the Python package `rainflow`⁴). For each identified stress cycle of amplitude $s \in \mathbb{R}_+$, the so-called “Stress vs. Number of cycles” curve (also called the “SN curve” or “Wöhler curve”) allows one to estimate the number N_c of similar stress cycles necessary to reach fatigue ruin. The SN curve, denoted by $W(\cdot)$ is an affine function in the log-log scale with slope $-m$ and y-intercept a :

$$N_c(s) = as^{-m}, a \in \mathbb{R}_+, m \in \mathbb{R}_+. \quad (4.2)$$

Finally, a usual approach to compute the damage is to aggregate the fatigue contribution of each stress cycle identified using Miner’s rule. Damage occurring during a 10-minute operating time is simulated and then scaled up to the OWT lifetime. More details regarding damage assessment and the Wöhler curve used are available in Section 2.4.6 from (DNV-RP-C203, 2016). For a realization $\mathbf{x} \in \mathcal{D}_{\mathbf{X}}$ of environmental conditions, at a structural node i , an azimuth angle θ ; $k \in \mathbb{N}$ stress cycles of respective amplitude $\{s_{i,\theta}^{(j)}(\mathbf{x})\}_{j=1}^k$ are identified. Then, Miner’s rule (Fatemi and Yang, 1998) defines the damage function $g_{i,\theta}(\mathbf{x}) : \mathcal{D}_{\mathbf{X}} \rightarrow \mathbb{R}_+$ by:

$$g_{i,\theta}(\mathbf{x}) = \sum_{j=1}^k \frac{1}{N_c(s_{i,\theta}^{(j)}(\mathbf{x}))}. \quad (4.3)$$

As defined by the DNV standards for OWT fatigue design (DNV-RP-C203, 2016), the quantity of interest in the present chapter is the “mean damage” $d_c^{i,\theta}$ (also called “cumulative damage”),

⁴<https://github.com/iamlikeme/rainflow>

computed at a node i , for an azimuth angle θ :

$$d_c^{i,\theta} = \mathbb{E}[g_{i,\theta}(\mathbf{X})] = \int_{\mathcal{D}_{\mathbf{X}}} g_{i,\theta}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}. \quad (4.4)$$

To get a preview of the distribution of this output random variable $g_{i,\theta}(\mathbf{X})$, a histogram of a large Monte Carlo simulation ($N_{\text{ref}} = 2000$) is represented in Fig. 4.5 (with a log scale). In this case, the log-damage histogram presents a little asymmetry but it is frequently modeled by a normal distribution (see e.g., Teixeira et al. (2019b)).

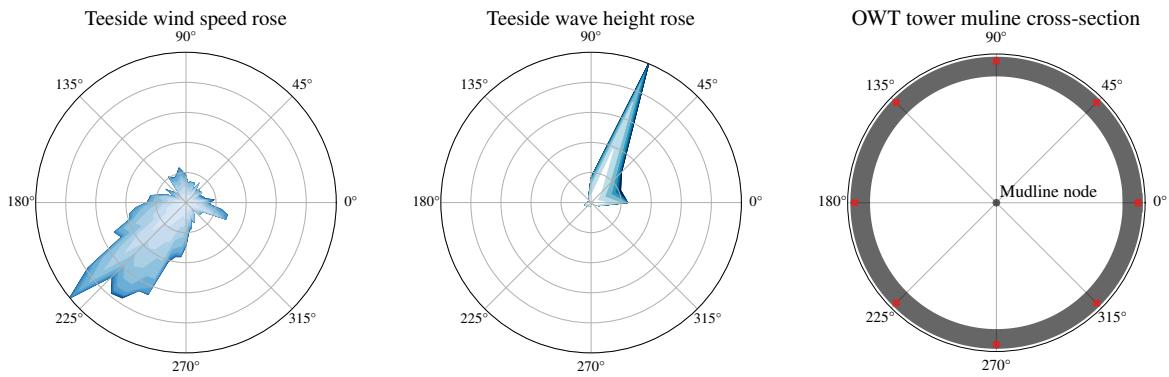


Figure 4.4 Angular distribution of the wind and waves with a horizontal cross-section of the OWT structure and the mudline. Red crosses represent the discretized azimuths for which the fatigue is computed

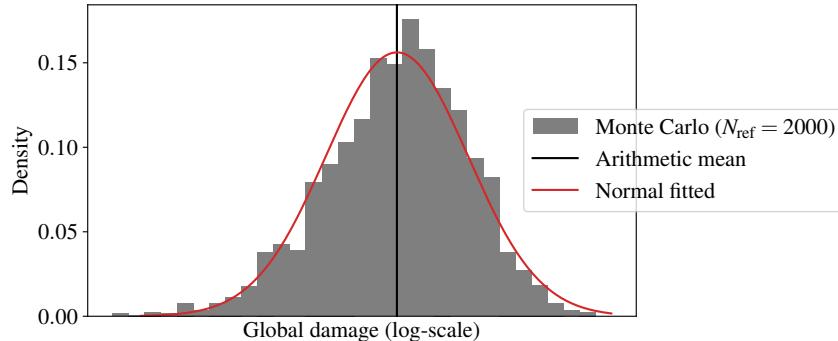


Figure 4.5 Histogram of the log-damage, at mudline, azimuth 45 deg. (Monte Carlo reference sample)

4.3 Numerical integration procedures for mean damage estimation

The present section explores different methods aiming at approximating the integral of a function against a probability measure. In the case of OWT mean damage estimation, these

methods can be used for defining efficient design load cases. This problem is equivalent to the central tendency estimation of $\mathbf{Y} = g(\mathbf{X})$, the image of the environmental random variable \mathbf{X} by the damage function $g(\cdot) : \mathcal{D}_X \rightarrow \mathbb{R}$ (see e.g., Eq. (4.4)). Considering a measurable space $\mathcal{D}_X \subset \mathbb{R}^d, d \in \mathbb{N}^*$, associated with a known probability measure π , this section studies the approximation of integrals of the form $\int_{\mathcal{D}_X} g(\mathbf{x})d\pi(\mathbf{x})$.

4.3.1 Quadrature rules and quasi-Monte Carlo methods

Numerical integration authors may call this generic problem *probabilistic integration* (Briol et al., 2019). In practice, this quantity of interest is estimated on an n -sized set of damage realizations $\mathbf{y}_n = \{g(\mathbf{x}^{(1)}), \dots, g(\mathbf{x}^{(n)})\}$ of an input sample $\mathbf{X}_n = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\}$. A weighted arithmetic mean of the realizations $\{g(\mathbf{x}^{(1)}), \dots, g(\mathbf{x}^{(n)})\}$ is called a *quadrature rule* with a set of unconstrained weights $\mathbf{w}_n = \{w_1, \dots, w_n\} \in \mathbb{R}^n$:

$$I_\pi(g) := \int_{\mathcal{D}_X} g(\mathbf{x})d\pi(\mathbf{x}) \approx \sum_{i=1}^n w_i g(\mathbf{x}^{(i)}). \quad (4.5)$$

Our numerical experiment framework often implies that the function g is costly to evaluate, making the realization number limited. For a given sample size n , our goal is to find a set of tuples $\{\mathbf{x}^{(i)}, w_i\}_{i=1}^n$ (i.e., quadrature rule), giving the best approximation of our quantity. In the literature, a large panel of numerical integration methods has been proposed to tackle this problem. For example, Van den Bos (2020) recently developed a quadrature rule based on arbitrary sample sets which was applied to a similar industrial OWT use case.

Alternatively, sampling methods rely on generating a set of points \mathbf{X}_n drawn from the input distribution to compute the arithmetic mean of their realizations (i.e., uniform weights $\{w_i = \frac{1}{n}\}_{i=1}^n$). Among them, low-discrepancy sequences, also called “quasi-Monte Carlo” sampling (e.g., Sobol’, Halton, Faure sequences) are known to improve the standard Monte Carlo convergence rate and will be used as a deterministic reference method in the following numerical experiments (Leobacher and Pillichshammer, 2014).

Quantization of probability measures and quadrature When dealing with probabilistic integration such as Eq. (4.5), a quadrature rule is a finite representation of a continuous measure π by a discrete measure $\zeta_n = \sum_{i=1}^n w_i \delta(\mathbf{x}^{(i)})$ (weighted sum of Dirac distributions at the design points \mathbf{X}_n). In the literature, this procedure is also called *quantization* of a continuous measure π . Overall, numerical integration is a particular case of probabilistic integration against a uniform input measure. For uniform measures, the Koksma-Hlawka inequality (Morokoff and Caflisch, 1995) provides a useful upper bound on the absolute error of a quadrature rule:

$$\left| \int_{[0,1]^d} g(\mathbf{x})d\mathbf{x} - \frac{1}{n} \sum_{i=1}^n g(\mathbf{x}^{(i)}) \right| \leq V(g) D_n^*(\mathbf{X}_n). \quad (4.6)$$

As presented in Oates (2021), $V(g) = \sum_{\mathbf{u} \subseteq \{1, \dots, p\}} \int_{[0,1]^{\mathbf{u}}} \left| \frac{\partial^{\mathbf{u}} g}{\partial \mathbf{x}_{\mathbf{u}}}(\mathbf{x}_{\mathbf{u}}, 1) \right| d\mathbf{x}$, quantifies the complexity of the integrand, while $D_n^*(\mathbf{X}_n)$ evaluates the discrepancy to uniformity of the design \mathbf{X}_n . Therefore, the Koksma-Hlawka inequality shows that the quadrature rule's accuracy relies on the good quantization of π by \mathbf{X}_n . For a uniform measure π , the star discrepancy $D_n^*(\mathbf{X}_n)$ is a metric assessing how far from uniformity a sample \mathbf{X}_n is. When generalizing to a non-uniform measure, a good quantization of π should also lead to a good approximation of the quantity.

4.3.2 Kernel herding sampling

Quasi-Monte Carlo sampling methods widely rely on a metric of uniformity, called *discrepancy*. To go beyond uniform measures, Appendix B introduces a kernel-based discrepancy, generalizing the discrepancy concept to non-uniform measures. This tool, named the maximum mean discrepancy (MMD) allows comparing multivariate distributions by embedding them in a specific function space. In this manuscript, the MMD was employed as a tool for statistical testing and quantifying the perturbations of distributions in Chapter 3, and for sensitivity analysis in Section 1.6.

Herin, the MMD is used to build a quadrature rule by sampling from a known measure. In other words, to quantize a known target measure π by a design sample \mathbf{X}_n . For practical reasons, the design construction is done sequentially. Sequential strategies have also been used to learn and validate regression models for statistical learning (see Fekhari et al. (2023)). Moreover, since each realization is supposed to be obtained at the same unitary cost, the quadrature weights are first fixed as uniform during the construction of the design \mathbf{X}_n .

Kernel herding (KH), proposed by Chen et al. (2010), is a sampling method that offers a quantization of the measure π by minimizing a squared MMD when adding points iteratively. With a current design \mathbf{X}_n and its corresponding discrete distribution with uniform weights $\zeta_n = \frac{1}{n} \sum_{i=1}^n \delta(\mathbf{x}^{(i)})$, a KH iteration is as an optimization of the following criterion, selecting the point $\mathbf{x}^{(n+1)} \in \mathcal{D}_X$:

$$\mathbf{x}^{(n+1)} \in \arg \min_{\mathbf{x} \in \mathcal{D}_X} \left\{ \text{MMD} \left(\pi, \frac{1}{n+1} \left(\delta(\mathbf{x}) + \sum_{i=1}^n \delta(\mathbf{x}^{(i)}) \right) \right)^2 \right\}. \quad (4.7)$$

In the literature, two formulations of this optimization problem can be found. The first one uses the Frank-Wolfe algorithm (or “conditional gradient algorithm”) to compute a linearization of the problem under the convexity hypothesis (see Lacoste-Julien et al. (2015) and Briol et al. (2015) for more details). The second one is a straightforward greedy optimization. Due to the combinatorial complexity, the greedy formulation is tractable for sequential construction only.

Let us develop the MMD criterion from Eq. (B.7):

$$\text{MMD} \left(\pi, \frac{1}{n+1} \left(\delta(\mathbf{x}) + \sum_{i=1}^n \delta(\mathbf{x}^{(i)}) \right) \right)^2 = \varepsilon_\pi - \frac{2}{n+1} \sum_{i=1}^{n+1} P_\pi \left(\mathbf{x}^{(i)} \right) + \frac{1}{(n+1)^2} \sum_{i,j=1}^{n+1} k \left(\mathbf{x}^{(i)}, \mathbf{x}^{(j)} \right) \quad (4.8a)$$

$$= \varepsilon_\pi - \frac{2}{n+1} \left(P_\pi(\mathbf{x}) + \sum_{i=1}^n P_\pi \left(\mathbf{x}^{(i)} \right) \right) \quad (4.8b)$$

$$+ \frac{1}{(n+1)^2} \left(\sum_{i,j=1}^n k \left(\mathbf{x}^{(i)}, \mathbf{x}^{(j)} \right) + 2 \sum_{i=1}^n k \left(\mathbf{x}^{(i)}, \mathbf{x} \right) - k(\mathbf{x}, \mathbf{x}) \right). \quad (4.8c)$$

In the previously developed expression, only a few terms actually depend on the next optimal point $\mathbf{x}^{(n+1)}$ since the target energy, denoted by ε_π , and $k(\mathbf{x}, \mathbf{x}) = \sigma^2$ are constant (by taking a stationary kernel). Therefore, the greedy minimization of the MMD can be equivalently written as:

$$\mathbf{x}^{(n+1)} \in \arg \min_{\mathbf{x} \in \mathcal{D}_X} \left\{ \frac{1}{n+1} \sum_{i=1}^n k \left(\mathbf{x}^{(i)}, \mathbf{x} \right) - P_\pi(\mathbf{x}) \right\} = \arg \min_{\mathbf{x} \in \mathcal{D}_X} \left\{ \frac{n}{n+1} P_{\zeta_n}(\mathbf{x}) - P_\pi(\mathbf{x}) \right\}. \quad (4.9)$$

Remark 3. For the sequential and uniformly weighted case, the formulation in Eq. (4.9) is almost similar to the Frank-Wolfe formulation. Our numerical experiments showed that these two versions generate very close designs, especially as n becomes large. [Pronzato and Rendas \(2021a\)](#) express the Frank-Wolfe formulation in the sequential and uniformly weighted case as follows:

$$\mathbf{x}^{(n+1)} \in \arg \min_{\mathbf{x} \in \mathcal{D}_X} \left\{ P_{\zeta_n}(\mathbf{x}) - P_\pi(\mathbf{x}) \right\}. \quad (4.10)$$

Remark 4. In practice, the optimization problem is solved by a brute-force approach on a fairly dense finite subset $\mathcal{S} \subseteq \mathcal{D}_X$ of candidate points with size $N \gg n$ that emulates the target distribution, also called the “candidate set”. This sample is also used to estimate the target potential $P_\pi(\mathbf{x}) \approx \frac{1}{N} \sum_{i=1}^N k \left(\mathbf{x}^{(i)}, \mathbf{x} \right)$.

The diagram illustrated in Fig. 4.6 summarizes the main steps of a kernel herding sampling algorithm. One can notice that the initialization can either be done using a median point (maximizing the target potential) or from any existing design of experiments. This second configuration showed to be practical when the analyst must include some characteristic points in the design (e.g., points with a physical interpretation).

As explained previously, choosing the kernel defines the function space on which the worst-case function is found (see Eq. (B.5)). Therefore, this sampling method is sensitive to the kernel’s choice. A kernel is defined, both by the choice of its parametric family (e.g., Matérn, squared exponential) and the choice of its tuning. The so-called “support points” method developed by [Mak and Joseph \(2018\)](#) is a special case of kernel herding that uses the

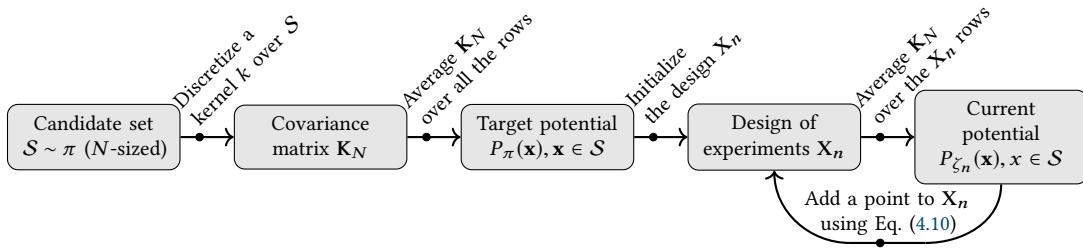


Figure 4.6 Greedy kernel herding algorithm

Energy-distance	$k_E(\mathbf{x}, \mathbf{x}') = \frac{1}{2} (\ \mathbf{x}\ + \ \mathbf{x}'\ - \ \mathbf{x} - \mathbf{x}'\)$
Squared exponential	$k_G(\mathbf{x}, \mathbf{x}') = \prod_{i=1}^p k_{\theta_i}(x_i - x'_i)$
Matérn ($\nu = 5/2$)	$k_M(\mathbf{x}, \mathbf{x}') = \prod_{i=1}^p k_{5/2, \theta_i}(x_i - x'_i)$
	$k_{5/2, \theta}(x - x') = \left(1 + \frac{\sqrt{5}}{\theta} x - x' + \frac{5}{3\theta^2} (x - x')^2\right) \exp\left(-\frac{\sqrt{5}}{\theta} x - x' \right)$

Table 4.3 Kernels considered in the following numerical experiments.

characteristic and parameter-free “energy-distance” kernel (introduced by Székely and Rizzo (2013)). In the following numerical experiments, the energy-distance kernel will be compared with an isotropic tensor product of a Matérn kernel (with regularity parameter $\nu = 5/2$ and correlation lengths θ_i), or a squared exponential kernel (with correlation lengths θ_i) defined in Table 4.3. Since the Matérn and squared exponential kernels are widely used for Gaussian process regression (Rasmussen and Williams, 2006), they were naturally picked to challenge the energy-distance kernel. The correlation lengths for the squared exponential and Matérn kernels are set using the heuristic given in Fekhari et al. (2023), $\theta_i = n^{-1/d}, i \in \{1, \dots, d\}$.

Fig. 4.7 represents the covariance structure of the three kernels. One can notice that the squared exponential and Matérn $\nu = 5/2$ kernels are closer to one another than they are to the energy-distance. In fact, as ν tends to infinity, the Matérn kernel tends toward the squared exponential kernel (which has infinitely differentiable sample paths, see Rasmussen and Williams (2006)). For these two stationary kernels, the correlation length controls how fast the correlation between two points decreases as their distance from one another increases.

Meanwhile, the energy distance is not stationary (but still positive and semi-definite). Therefore, its value does not only depend on the distance between two points but also on the norm of each of the points. Interestingly, the energy-distance kernel is almost similar to the kernel used by Hickernell (1998) to define a widely-used space-filling metric called the centered L^2 -discrepancy. A presentation of these kernel-based discrepancies from the design of experiment point of view is also provided in Chapter Two from Fang et al. (2018).

To illustrate the kernel herding sampling of a complex distribution, Fig. 4.8 shows three nested samples (orange crosses for different sizes $n \in \{10, 20, 40\}$) of a mixture of Gaussian distributions with complex nonlinear dependencies (with density represented by the black isoprobability contours). In this example, the method seems to build a parsimonious design

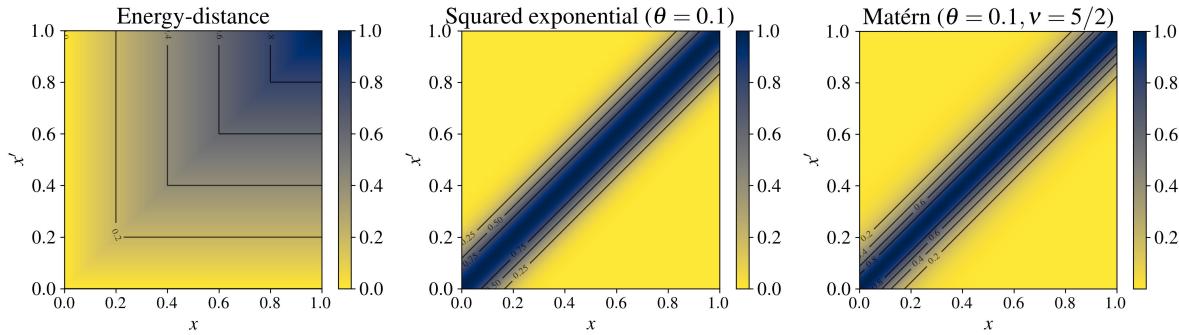


Figure 4.7 Kernel illustrations (left to right: energy-distance, squared exponential, and Matérn 5/2)

between each mode of the distribution (by subsampling directly without any transformation). The candidate set (in light grey) was generated by a large quasi-Monte sample of the underlying Gaussian mixture. In this two-dimensional case, this candidate set is sufficient to estimate the target potential P_π . However, the main bottleneck of kernel herding is the estimation of the potentials, which becomes costly in high dimensions.

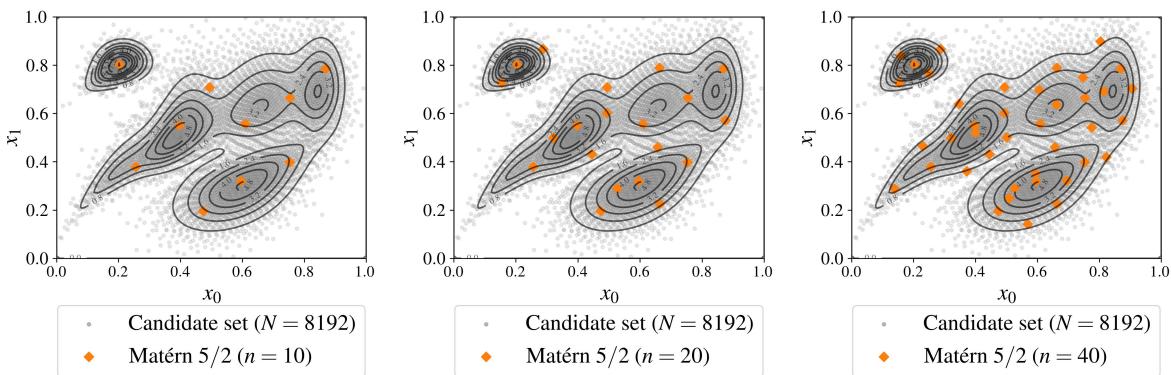


Figure 4.8 Sequential kernel herding for increasing design sizes ($n \in \{10, 20, 40\}$) built on a candidate set of $N = 8196$ points drawn from a complex Gaussian mixture π

Other approaches take advantage of the progressive knowledge acquired sequentially from the outputs to select the following points in the design. These methods are sometimes called “active learning” or “adaptive strategies” (Fuhg. et al., 2021). Many of them rely on a sequentially updated Gaussian process (or Kriging) metamodel. To solve a probabilistic integration problem, the concept of Bayesian quadrature is introduced in the following.

4.3.3 Bayesian quadrature

Gaussian processes for Bayesian quadrature Kernel methods and Gaussian processes present a lot of connections and equivalences, thoroughly reviewed by Kanagawa et al. (2018). In numerical integration, Gaussian processes have been used to build quadrature rules in the seminal paper of O’Hagan (1991), introducing the concept of *Bayesian quadrature* (BQ). Let us

recall the probabilistic integration problem $I_\pi(g) = \int_{\mathcal{D}_X} g(\mathbf{x}) d\pi(\mathbf{x})$ (stated in Eq. (4.5)). From a general point of view, this quantity could be generalized by composing g with another function ψ (e.g., other moments, quantiles, exceedance probabilities). The quantity of interest then becomes, $I_\pi(\psi(g))$, for example when ψ is a monomial, it gives a moment of the output distribution.

Let us assume, adopting a Bayesian point of view, that G is a stochastic process describing the uncertainty affecting the knowledge about the true function g . Let G be a Gaussian process (GP) prior with a zero trend (denoted by $\mathbf{0}$) to ease the calculation, and a stationary covariance kernel (denoted by $k(\cdot, \cdot)$). The conditional posterior $G_n := (G|\mathbf{y}_n) \sim \text{GP}(\eta_n, s_n^2)$ has been conditioned on the function observations $\mathbf{y}_n = [g(\mathbf{x}^{(1)}), \dots, g(\mathbf{x}^{(n)})]^\top$ computed from the input design \mathbf{X}_n and is fully defined by the well-known “Kriging equations” (see e.g., [Rasmussen and Williams \(2006\)](#)):

$$\begin{cases} \eta_n(\mathbf{x}) &:= \mathbf{k}_n^\top(\mathbf{x}) \mathbf{K}_n^{-1} \mathbf{y}_n \\ s_n^2(\mathbf{x}) &:= k_n(\mathbf{x}, \mathbf{x}) - \mathbf{k}_n^\top(\mathbf{x}) \mathbf{K}_n^{-1} \mathbf{k}_n(\mathbf{x}) \end{cases} \quad (4.11)$$

where $\mathbf{k}_n(\mathbf{x})$ is the column vector of the covariance kernel evaluations $[k_n(\mathbf{x}, \mathbf{x}^{(1)}), \dots, k_n(\mathbf{x}, \mathbf{x}^{(n)})]$ and \mathbf{K}_n is the $(n \times n)$ variance-covariance matrix such that the (i, j) -element is $\{\mathbf{K}_n\}_{i,j} = k_n(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$.

In BQ, the main object is the random variable $I_\pi(G_n)$. According to [Briol et al. \(2019\)](#), its distribution on \mathbb{R} is the pushforward of G_n through the integration operator $I_\pi(\cdot)$, sometimes called *posterior distribution*:

$$I_\pi(G_n) = \int_{\mathcal{D}_X} (G(\mathbf{x}) | \mathbf{y}_n) d\pi(\mathbf{x}) = \int_{\mathcal{D}_X} G_n(\mathbf{x}) d\pi(\mathbf{x}). \quad (4.12)$$

[Fig. 4.9](#) provides a one-dimensional illustration of the Bayesian quadrature of an unknown function (dashed black curve) against a given input measure π (with corresponding grey distribution at the bottom). For an arbitrary design, one can fit a Gaussian process model, interpolating the function observations (black crosses). Then, multiple trajectories of this conditioned Gaussian process G_n are drawn (orange curves) whilst its mean function, also called “predictor”, is represented by the red curve. Therefore, the input measure π is propagated through the conditioned Gaussian process to obtain the random variable $I_\pi(G_n)$, with distribution represented on the right plot (brown curve). Again on the right plot, remark how the mean of this posterior distribution (brown line) is closer to the reference output expected value (dashed black line) than the arithmetic mean of the observations (black line). This plot was inspired by the paper of [Huszár and Duvenaud \(2012\)](#).

Optimal weights computed by Bayesian quadrature Taking the random process G_n as Gaussian conveniently implies that its posterior distribution $a_\pi(G_n)$ is also Gaussian. This comes from the linearity of the infinite sum of realizations of a Gaussian process. The posterior distribution is described in a closed form through its mean and variance by applying Fubini’s theorem (see the supplementary materials from [Briol et al. \(2019\)](#) for the proof regarding the

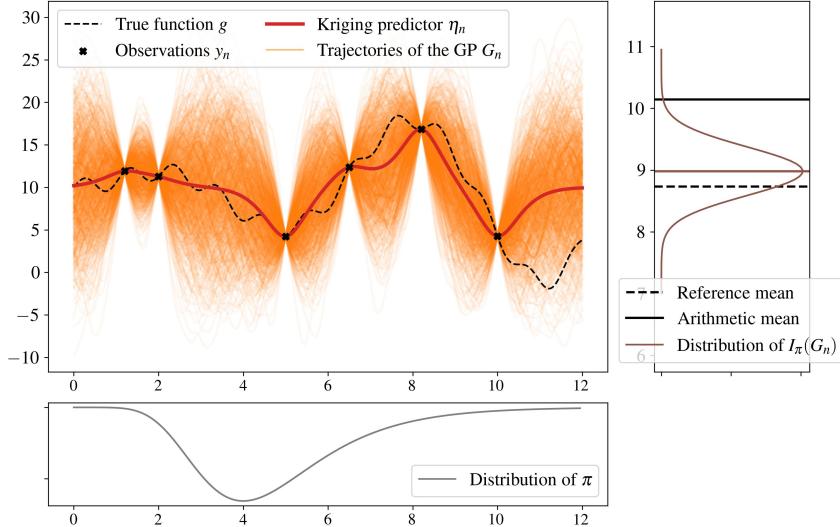


Figure 4.9 Bayesian quadrature on a one-dimensional case

variance):

$$\bar{y}_n^{\text{BQ}} := \mathbb{E}[I_\pi(G_n)|\mathbf{y}_n] = \int_{\mathcal{D}_X} \eta_n(\mathbf{x}) d\pi(\mathbf{x}) = \left[\int_{\mathcal{D}_X} \mathbf{k}_n^\top(\mathbf{x}) d\pi(\mathbf{x}) \right] \mathbf{K}_n^{-1} \mathbf{y}_n = P_\pi(\mathbf{X}_n) \mathbf{K}_n^{-1} \mathbf{y}_n, \quad (4.13)$$

$$\left(\sigma_n^{\text{BQ}} \right)^2 := \text{Var}(I_\pi(G_n)) = \iint_{\mathcal{D}_{X^2}} k_n(\mathbf{x}, \mathbf{x}') d\pi(\mathbf{x}) d\pi(\mathbf{x}') = \varepsilon_\pi - P_\pi(\mathbf{X}_n) \mathbf{K}_n^{-1} P_\pi(\mathbf{X}_n)^\top. \quad (4.14)$$

Where $P_\pi(\mathbf{X}_n)$ is the row vector of potentials $\left[\int k_n(\mathbf{x}, \mathbf{x}^{(1)}) d\pi(\mathbf{x}), \dots, \int k_n(\mathbf{x}, \mathbf{x}^{(n)}) d\pi(\mathbf{x}) \right]$, and ε_π is given in Eq. (1). As in the one-dimensional example presented in Fig. 4.9, the expected value of $I_\pi(G_n)$ expressed in Eq. (4.13) is a direct estimator of the quantity of interest Eq. (4.5). The so-called “Bayesian quadrature estimator” appears to be a simple linear combination of the observations by taking the row vector of “optimal weights” as:

$$\mathbf{w}_{\text{BQ}} := P_\pi(\mathbf{X}_n) \mathbf{K}_n^{-1} \quad (4.15)$$

For any given sample, an optimal set of weights can be computed, leading to the mean of the posterior distribution. Remark here that this enhancement depends on the evaluation of the inverse variance-covariance matrix \mathbf{K}_n^{-1} , which can present numerical difficulties, either when design points are too close, making the conditioning bad. Moreover, a prediction interval on the BQ estimator can be computed since the posterior distribution is Gaussian, with a variance expressed in closed-form in Eq. (4.14). The expressions in Eq. (4.13) and Eq. (4.14) were extended to Gaussian processes in the case of constant and linear trends in Pronzato and Zhigljavsky (2020). In the following numerical experiments, the expression with a hypothesis of constant

trend β_n is used, which leads to:

$$\mathbb{E}[I_\pi(G_n)] = \beta_n + P_\pi(\mathbf{X}_n)\mathbf{K}_n^{-1}(\mathbf{y}_n - \beta_n \mathbf{1}_n). \quad (4.16)$$

Then, an a posteriori 95% prediction interval around the mean Bayesian estimator is directly given by:

$$\bar{y}_n^{\text{BQ}} \in \left[\bar{y}_n^{\text{BQ}} - 2\sigma_n^{\text{BQ}}, \bar{y}_n^{\text{BQ}} + 2\sigma_n^{\text{BQ}} \right]. \quad (4.17)$$

Variance-based Bayesian quadrature rule The link between the posterior variance and the squared MMD has been first made by [Huszár and Duvenaud \(2012\)](#) in their Proposition 1: the expected variance in the Bayesian quadrature $\text{Var}(I_\pi(G_n))$ is the MMD between the target distribution π and $\zeta_n = \sum_{i=1}^n \mathbf{w}_{\text{BQ}}^{(i)} \delta(\mathbf{x}^{(i)})$. The proof is reproduced below (as well as in Proposition 6.1 from [Kanagawa et al. \(2018\)](#)):

$$\text{Var}(I_\pi(G_n)) = \mathbb{E} \left[(I_\pi(G_n) - I_{\zeta_n}(G_n))^2 \right] \quad (4.18a)$$

$$= \mathbb{E} \left[\left(\langle G_n, P_\pi \rangle_{\mathcal{H}(k)} - \langle G_n, P_{\zeta_n} \rangle_{\mathcal{H}(k)} \right)^2 \right] \quad (4.18b)$$

$$= \mathbb{E} \left[\langle G_n, P_\pi - P_{\zeta_n} \rangle_{\mathcal{H}(k)}^2 \right] \quad (4.18c)$$

$$= \|P_\pi - P_{\zeta_n}\|_{\mathcal{H}(k)}^2 \quad (4.18d)$$

$$= \text{MMD}(\pi, \zeta_n)^2. \quad (4.18e)$$

Note that the transition from equation (27c) to (27d) relies on the property stating that if G is a standard Gaussian process then $\forall g \in \mathcal{H}(k) : \langle G, g \rangle_{\mathcal{H}(k)} \sim \mathcal{N}(0, \|g\|_{\mathcal{H}(k)}^2)$. The method that sequentially builds a quadrature rule by minimizing this variance is called by the authors “Sequential Bayesian Quadrature” (SBQ). According to the previous proof, this criterion can be seen as an optimally-weighted version of the kernel herding criterion, as stated in the title of the paper from [Huszár and Duvenaud \(2012\)](#). Later, [Briol et al. \(2015\)](#) proved the weak convergence of $I_\pi(G_n)$ towards the target integral. Closer to wind turbines applications, [Huchet \(2019\)](#) and [Huchet et al. \(2019\)](#) introduced the “Adaptive Kriging Damage Assessment” method: a Kriging-based method for mean damage estimation that is very close to SBQ. However, This type of method inherits the limits from both KH and BQ since it searches for optimal design points among a candidate set and computes an inverse variance-covariance matrix. These numerical operations both scale hardly in high dimension.

Remark 5. Every quadrature method introduced in this section has been built without any observation of the possibly costly function g . Therefore, they cannot be categorized as active learning approaches. Contrarily, [Kanagawa and Hennig \(2019\)](#) presents a set of methods for BQ with transformations (i.e., adding a positivity constraint on the function g), which are truly active learning methods.

4.4 Numerical experiments

This section presents numerical results computed on two different analytical toy cases, respectively in dimension 2 (toy case #1) and dimension 10 (toy case #2), with easy-to-evaluate functions $g(\cdot)$ and associated input distributions π . Therefore, reference values can easily be computed with great precision. For each toy case, a large reference Monte Carlo sample ($N_{\text{ref}} = 10^8$) is taken. This first benchmark compares the mean estimation of toy cases given by a quasi-Monte Carlo technique (abbreviated by QMC in the next figures) which consists herein using a Sobol' sequence, and kernel herding with the three kernels defined in Table 4.3. Notice that the quasi-Monte Carlo designs are first generated on the unit hypercube and then, transformed using the generalized Nataf transformation to follow the target distribution (Lebrun and Dutfoy, 2009). Additionally, the performances of kernel herding for both uniform and optimally-weighted Eq. (4.16) estimators are compared.

All the following results and methods (i.e., the kernel-based sampling and BQ methods) have been implemented in a new open-source Python package named `otkerneldesign`⁵. This development mostly relies on the open source software OpenTURNS⁶ (“Open source initiative for the Treatment of Uncertainties, Risks’N Statistics”) devoted to uncertainty quantification and statistical learning (Baudin et al., 2017). Finally, note that the numerical experiments for the toy cases are available in the Git repository named `ctbenchmark`⁷.

4.4.1 Illustration on analytical toy-cases

The toy cases were chosen to cover a large panel of complex probabilistic integration problems, completing the ones from Fekhari et al. (2022). To assess the complexity of numerical integration problems, Owen (2003) introduced the concept of the “effective dimension” of an integrand function (number of the variables that actually impact the integral). The author showed that functions built on sums yield a low effective dimension (unlike functions built on products). In the same vein, Kucherenko et al. (2011) build three classes of integrand sorted by difficulty depending on their effective dimension:

- *class A*: problem with a few dominant variables.
- *class B*: problem without unimportant variables, and important low-order interaction terms.
- *class C*: problems without unimportant variables, and important high-order interaction terms.

The 10-dimensional “GSobol function” (toy case #2) with a set of coefficient $\{a_i = 2\}_{i=1}^{10}$ has an effective dimension equal to 10 and belongs to the hardest class C from Kucherenko et al. (2011).

⁵<https://efekhari27.github.io/otkerneldesign/master/index.html>

⁶<https://openturns.github.io/www/>

⁷<https://github.com/efekhari27/ctbenchmark>

Table 4.4 Analytical toy-cases

Toy-case #1	$dim = 2$	$g_1(\mathbf{x}) = x_1 + x_2$	Gaussian mixture from Fig. 4.8
Toy-case #2	$dim = 10$	$g_2(\mathbf{x}) = \prod_{i=1}^{10} \frac{ 4x_i - 2 + a_i}{1 + a_i}, \{a_i = 2\}_{i=1}^{10}$	Gaussian $\mathcal{N}(\mathbf{0.5}, \mathbf{I}_{10})$

In the case of the two-dimensional Gaussian mixture problem, the complexity is carried by the mixture of Gaussian distributions with highly nonlinear dependencies. Probabilistic integration results are presented in Fig. 4.10 (toy case #1) and Fig. 4.11 (toy case #2). Kernel herding samples using the energy-distance kernel are in red, while quasi-Monte Carlo samples built from Sobol' sequences are in grey. Convergences of the arithmetic means are plotted on the left and MMDs on the right. The respective BQ estimators of the means are plotted in dashed lines.

Remark 6. Different kernels are used in these numerical experiments. First, the generation kernel, used by the kernel herding algorithm to generate designs (with the heuristic tuning defined in Section 4.3.2). Second, the BQ kernel allows computation of the optimal weights (arbitrarily set up as a Matérn 5/2 with the heuristic tuning). Third, the evaluation kernel, which must be common to allow a fair comparison of the computed MMD results (same as the BQ kernel).

Results analysis for toy case #1. Convergence plots are provided in Fig. 4.10. KH consistently converges faster than quasi-Monte Carlo in this case, especially for small sizes in terms of MMD. BQ weights tend to reduce the fluctuations in the mean convergence, which ensures better performance for any size. Overall, applying the weights enhances the convergence rate.

Results analysis for toy case #2. Convergence plots are provided in Fig. 4.11. Although quasi-Monte Carlo is known to suffer the “curse of dimensionality”, KH does not outperform it drastically in this example. In fact, KH with uniform weights performs worse than quasi-Monte Carlo while optimally-weighted KH does slightly better. Moreover, the results confirm that $MMD_{BQ} < MMD_{unif}$ for all our experiments. The application of optimal weights to the quasi-Monte Carlo sample slightly improves the estimation in this case. Note that the prediction interval around the BQ estimator is not plotted for the sake of readability.

In these two toy cases, the MMD is shown to quantify numerical integration convergence well, which illustrates the validity of the inequality given in Eq. (B.4c), similar to the Koksma-Hlawka inequality (as recalled in Eq. (4.6)).

4.4.2 Application to the Teesside wind turbine fatigue estimation

Let us summarize the mean damage estimation strategies studied in this chapter. The diagram represented in Fig. 4.12 describes the different workflows computed. The simplest workflow is represented by the grey horizontal sequence. It directly subsamples a design of experiments from a large and representative dataset (previously referred to as candidate set). This workflow simply estimates the mean damage by computing an arithmetic average of the outputs.

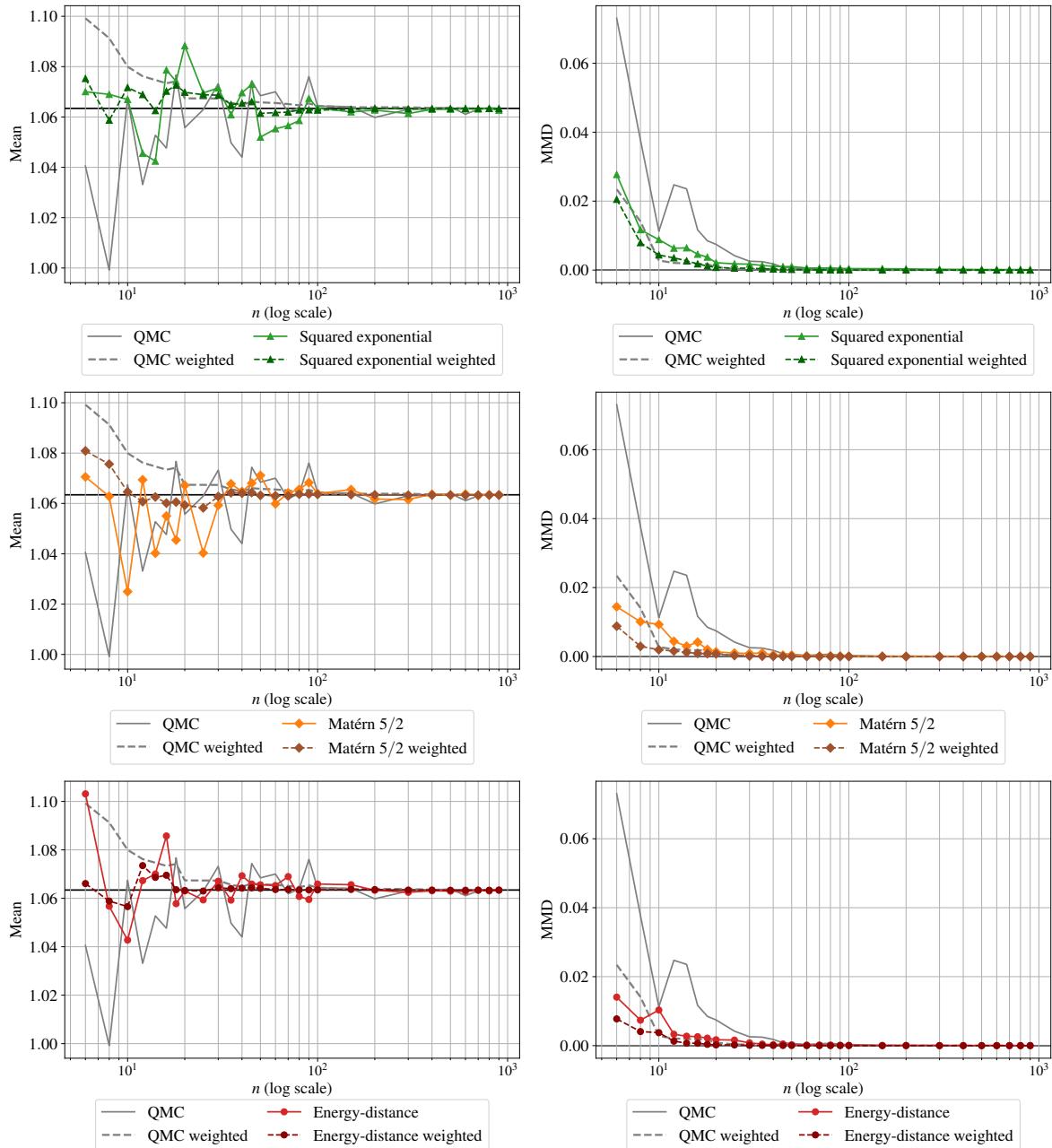


Figure 4.10 Analytical benchmark results on the toy-case #1

Alternatively, one can respectively fit a joint distribution and sample from it. In our case, this distribution is only known empirically via the candidate set. Since its dependence structure is complex (see Fig. 4.3), a parametric method might fit the distribution poorly (and therefore lead to a poor estimation of the quantity). Then, a nonparametric fit using the empirical Bernstein copula (introduced in Section 4.2.3) coupled with a kernel density estimation on each marginal is applied to the candidate set (with the EBC parameter $m = 100 > m_{\text{MISE}}$ to avoid bias, see Lasserre (2022) p.117). The sampling on this hybrid joint distribution is realized with a quasi-Monte Carlo method. A Sobol' low-discrepancy sequence generates a uniform sample in the unit hypercube, which can then be transformed according to a target distribution. Remember

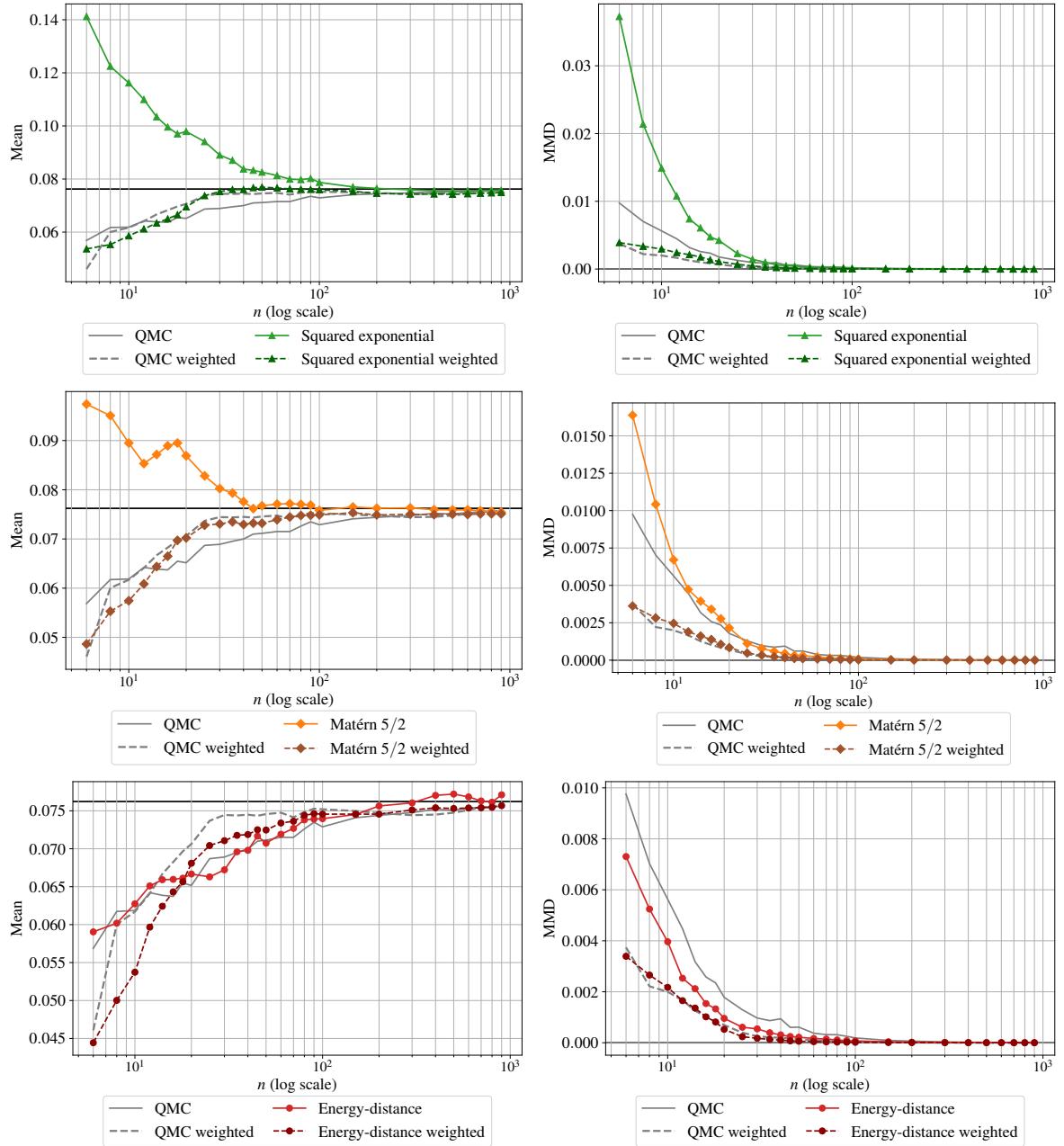


Figure 4.11 Analytical benchmark results on the toy-case #2

that quasi-Monte Carlo sampling is also sensitive to the choice of a low-discrepancy sequence, each presenting different properties (e.g., Sobol', Halton, Faure, etc.). Finally, the estimation by an arithmetic mean can be replaced by an optimally weighted mean. To do so, optimal weight must be computed, using the formulas introduced in Eq. (4.15).

The copulogram in Fig. 4.13 illustrates the intensity of the computed damages, proportionally to the color scale. Note that the numerical values of the damage scale are kept confidential since it models the state of an operating asset. Before analyzing the performance of the KH on this industrial application, let us notice that the copulogram Fig. 4.13 seems to be in line with the global sensitivity analysis presented in Murcia et al. (2018) and Li and Zhang (2020).

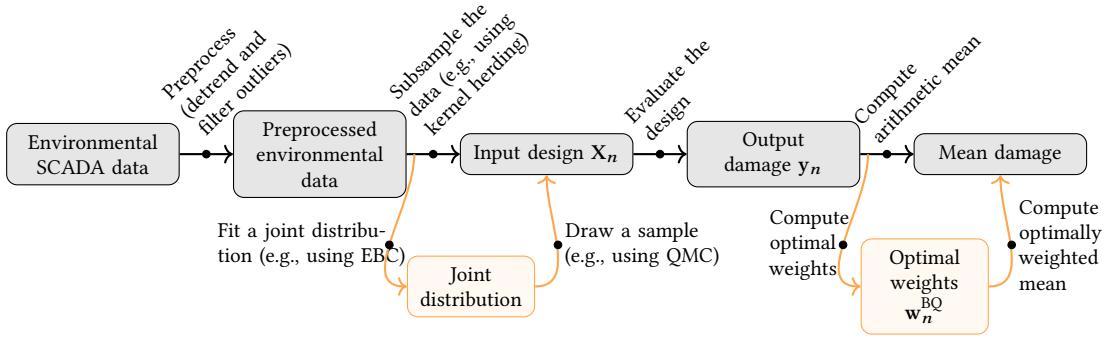


Figure 4.12 Mean damage estimation workflows for the industrial use case. The orange parts represent optional alterations to the workflow: the first one is an alternative to input data subsampling where the underlying distribution is sampled from, the second one improves mean damage calculation by using optimal weights over the output data

In particular, the fact that the scatter plot of mean wind speed vs. turbulence wind speed is the main factor explaining the variance of the output $Y = g(\mathbf{X})$. Judging from these references, the numerical model does not seem to have a highly effective dimension, however, the input dependence structure is challenging and the damage assessment induces strong nonlinearities (see Eq. (4.2)).

The results presented are compared in the following to a large reference Monte Carlo sample (size 2000) with a confidence interval computed by bootstrap (see Fig. 4.14). This reference is represented by a horizontal line intersecting with the most converged Monte Carlo estimation. Once again, the mean damage scale is hidden for confidentiality reasons, but all the plots are represented for the same vertical scale. The performance of the KH is good: it quickly converges towards the confidence interval of the Monte Carlo obtained with the reference sample. In addition, the Bayesian quadrature estimator also offers a posteriori prediction interval, which can reassure the user. The BQ prediction intervals are smaller than the ones obtained by bootstrap on the reference Monte Carlo sample.

To provide more representative results, note that a set of scale parameters is computed with a kriging procedure to define the kernel used to compute BQ intervals. Since other methods do not generate independent samples, bootstrapping them is not legitimate. Contrarily to the other kernels, we observe that the energy-distance kernel presents a small bias with the MC reference for most of the azimuth angles computed in this experiment. Meanwhile, combining nonparametric fitting with quasi-Monte Carlo sampling also delivers good results as long as the fitting step does not introduce a bias. In our case, any potential bias due to poor fitting would be the result of a poorly tuned empirical Bernstein copula. Fortunately, Nagler et al. (2017) formulated recommendations regarding how to tune empirical Bernstein copulas. We follow these recommendations in the present work.

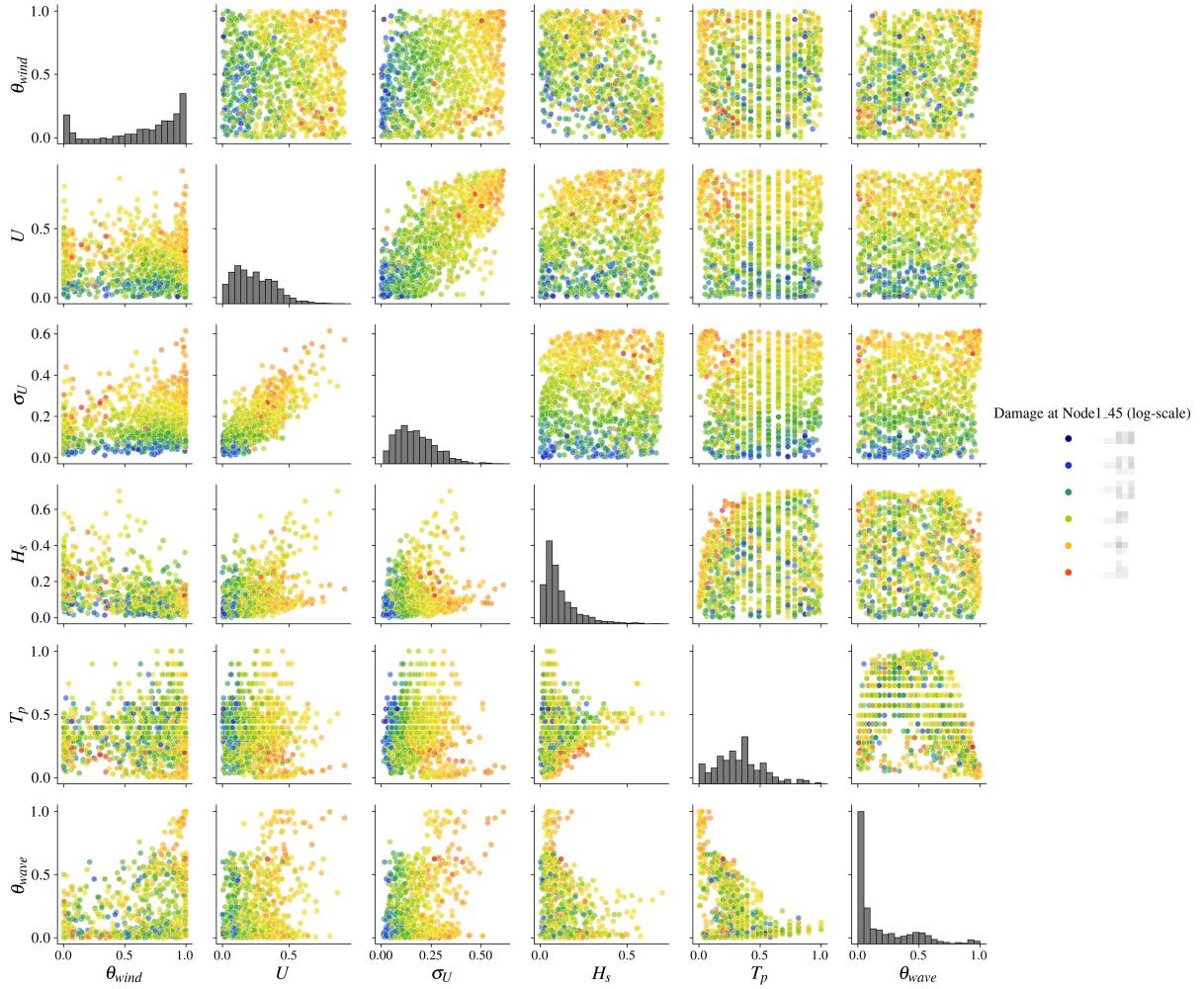


Figure 4.13 Copulogram of the kernel herding design of experiments with corresponding outputs in color (log-scale) on the Teesside case ($n = 10^3$). The color scale ranges from blue for the lowest values to red for the largest. Marginals are represented by histograms (diagonal), the dependence structure with scatter plots in the ranked space (upper triangle). Scatter plots on the bottom triangle are set in the physical space.

4.5 Conclusion

Wind energy assets are subject to highly uncertain environmental conditions. Uncertainty propagation through numerical models is performed to ensure their structural integrity (and energy production). For this case, the method recommended by the standards (regular grid sampling) is intractable for even moderate-fidelity simulators. In practice, such an approach can lead to poor uncertainty propagation, especially when facing simulation budget constraints.

In the present chapter, a real industrial wind turbine fatigue estimation use case is investigated, considering site-specific data. As a perspective, other sites with different environmental conditions could be studied. This use case induces two practical constraints: first, usual active learning methods are hard to set up on such a model (mainly due to the nonlinearity of the variable of interest), and they restrict the use of high-performance computing facilities; second,

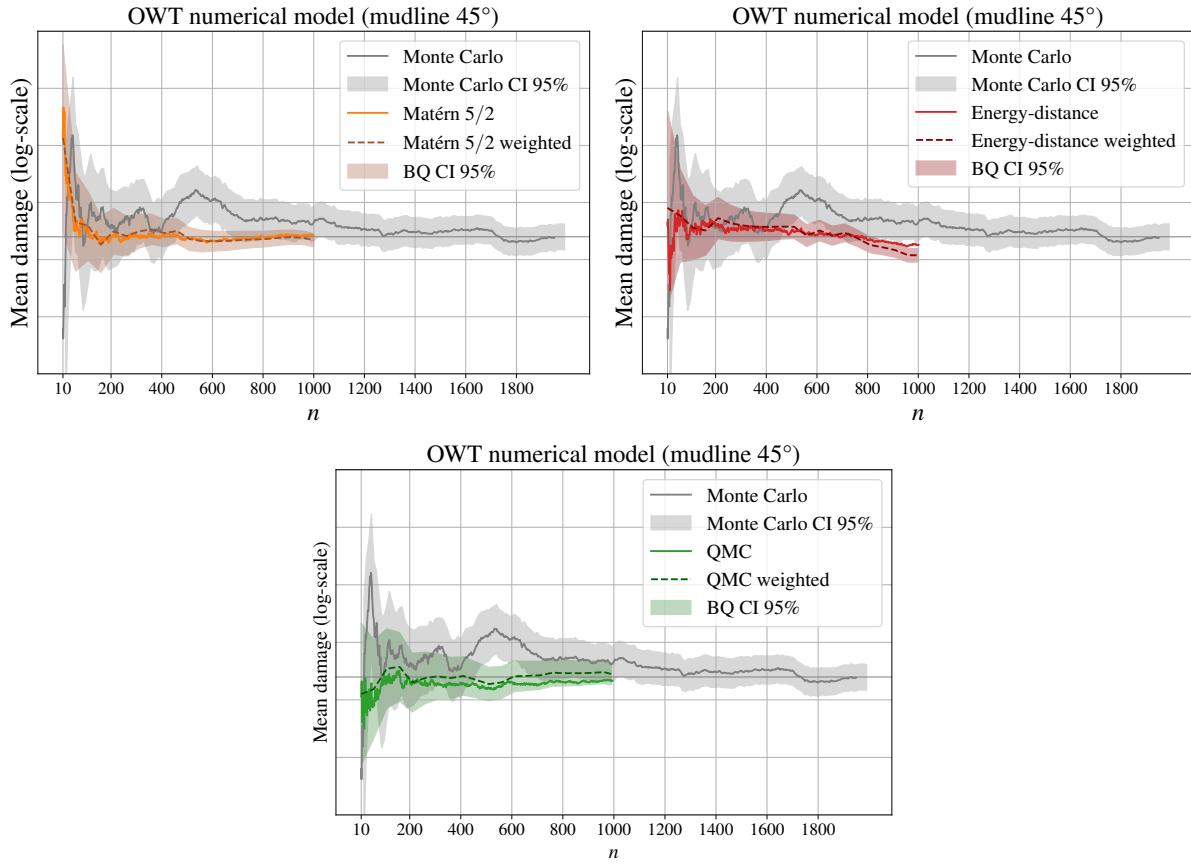


Figure 4.14 Mean estimation convergence (at the mudline, azimuth $\theta = 45$ deg.) on the Teesside case. Monte Carlo confidence intervals are all computed by bootstrap

the input distribution of the environmental conditions presents a complex dependence structure which is hard to infer with common parametric approaches.

In this work, the association of kernel herding sampling with Bayesian quadrature for central tendency estimation is explored theoretically and numerically. This method fits with the practical constraints induced by the industrial use case. To be more specific, the kernel herding method easily subsamples the relevant points directly from a given dataset (here, from the measured environmental data). Moreover, the method is fully compatible with intensive high-performance computer use. Moreover, the present work outlined an upper bound based on the maximum mean discrepancy (MMD) on numerical integration absolute error. Kernel herding and Bayesian quadrature both aim at finding the quadrature rule minimizing the MMD, and therefore the absolute integration error. The numerical experiments confirm that the MMD is an appropriate criterion since it leads to results being better or equivalent to quasi-Monte Carlo sampling. Finally, the proposed numerical benchmark relies on a Python package, called `otkerneldesign`, which implements the methods and allows anyone to reproduce the results.

The limits of the proposed method are reached when the input dimension of the problem increases, requiring a larger candidate set and therefore a larger covariance matrix. Moreover,

the numerical experiments show that the method can be sensitive to the choice of the kernel and its tuning (although good practices can be derived). From a methodological viewpoint, further interpretation of the impact of the different kernels could be explored. Besides, extensions of kernel herding sampling for quantile estimation could be investigated, in a similar fashion as the work on randomized quasi-Monte Carlo for quantiles proposed by [Kaplan et al. \(2019\)](#). Kernel herding could also be used to quantize conditional distributions, using the so-called “conditional kernel mean embedding” concept reviewed by [Klebanov et al. \(2020\)](#). Finally, regarding the industrial use case, the next step should be to perform a reliability analysis by considering another group of random variables (related to the wind turbine) or to explore the possibilities offered by reliability-oriented sensitivity analysis in the context of kernel-based indices, as studied in [Marrel and Chabridon \(2021\)](#).

Chapter **5**

Kernel-based surrogate models validation

5.1	Introduction	142
5.2	Predictivity assessment criteria for an ML model	143
5.2.1	The predictivity coefficient	144
5.2.2	Weighting the test sample	145
5.3	Test-set construction	147
5.3.1	Fully-Sequential Space-Filling design	147
5.3.2	Support points	148
5.3.3	Kernel herding	150
5.3.4	Numerical illustration	151
5.4	Numerical results I: construction of a training set and a test set	151
5.4.1	Test-cases	154
5.4.2	Results and analysis	155
5.5	Numerical results II: splitting a dataset into a training set and a test set	161
5.5.1	Industrial test-case CATHARE	161
5.5.2	Benchmark results and analysis	162
5.6	Conclusion	165

This chapter is adapted from the following reference:

E. Fekhari, B. Iooss, J. Muré, L. Pronzato and M.J. Rendas (2023). “Model predictivity assessment: incremental test-set selection and accuracy evaluation”. In: *Studies in Theoretical and Applied Statistics*, pages 315–347. Springer.

5.1 Introduction

The development of methods to validate and certify the predictivity of supervised learning models is essential to the industry. Estimating the predictivity of these models can either be done by cross-validation or using a suitably selected test sample (as introduced in Section 1.7). Both in a given-data context (i.e., machine learning) or a simulated data context (i.e., computer experiment), guarantees on the validation procedure are increasingly asked. Certain risk-averse industries (e.g., nuclear) impose to establish these guarantees from independent test-sets, i.e., data set that has not been used either to train or to select the learning model (Borovicka et al., 2012; Xu and Goodacre, 2018; Iooss, 2021). Using the prediction residuals on this test set, an independent evaluation of the proposed leaning model can be done, enabling the estimation of relevant performance metrics, such as the mean-squared error for regression problems, or the misclassification rate for classification problems.

The present chapter introduces methods to choose a “good” test set, either within a given dataset or within the input space of the model, as recently motivated in Iooss (2021); Joseph and Vakayil (2022). The construction of test sets is studied as an uncertainty propagation of the learning model’s error, on which an average error may be estimated using the Bayesian quadrature methods introduced in Chapter 4 for mean estimation.

A first choice concerns the size of the test set. No optimal choice exists, and, when only a finite dataset is available, classical machine learning (ML) handbooks (Hastie et al., 2009; Goodfellow et al., 2016) provide different heuristics on how to split it, e.g., 80%/20% between the training and test samples, or 50%/25%/25% between the training, validation (used for model selection) and test samples. This point is not formally addressed in the following (see Xu and Goodacre, 2018 for a numerical study of this issue). A second issue concerns how the test sample is picked within the input space. The simplest, and most common way to build a test sample is to extract an independent Monte Carlo sample (Hastie et al., 2009). For small test sets, these randomly chosen points may fall too close to the training points or leave large areas of the input space unsampled, and a more constructive method to select points inside the input domain is therefore preferable. Similar concerns motivate the use of space-filling designs when choosing a small set of runs for computationally expensive computer experiments on which a model will be identified (Fang et al., 2006; Pronzato and Müller, 2012).

When the test set must be a subset of an initial dataset, the problem amounts to selecting a certain number of points within a finite collection of points. A review of classical methods for solving this issue is given in Borovicka et al. (2012). For example, the CADEX and DUPLEX

algorithms (Kennard and Stone, 1969; Snee, 1977) can sequentially extract points from a database to include them in a test sample, using an inter-point distance criterion.

Several algorithms have also been proposed for the case where points need to be added to an already existing training sample. When the goal is to assess the quality of a model learned using a known training set, one may be tempted to locate the test points the furthest away from the training samples, such that, in some sense, the union of the training and test sets is space-filling. As this chapter shows, test sets built in this manner do enable a good assessment of the quality of models learned with the training set if the observed residuals are appropriately weighted. Moreover, the incremental augmentation of a design can be useful when the assessed model turns out to be of poor quality, or when an additional computational budget is available after a first study (Sheikholeslami and Razavi, 2017; Shang and Apley, 2020). Different empirical strategies have been proposed for incremental space-filling design (Iooss et al., 2010; Crombecq et al., 2011; Li et al., 2017), which basically entail the addition of new points in the zones poorly covered by the current design. Shang and Apley (2020) have recently proposed an improvement of the CADEX algorithm, called the “fully-sequential space-filling” (FSSF) design. Nogales Gómez et al. (2021) also proposed an improved version of such design enforcing boundary avoidance. Although they are developed for different purposes, nested space filling designs (Qian et al., 2009) and sliced space filling designs (Qian and Wu, 2009) can also be used to build sequential designs.

This work provides insights into these subjects in two main directions: (*i*) definition of a new predictivity criteria through an optimal weighting of the test points residuals, and (*ii*) use of test sets built by incremental space-filling algorithms, namely FSSF, support points Mak and Joseph (2018) and kernel herding Chen et al. (2010), the latter two algorithms being typically used to provide a representative sample of a desired theoretical or empirical distribution. Besides, this chapter presents a numerical benchmark analysis comparing the behavior of the three algorithms on a selected set of test cases and an industrial case.

This chapter is organized as follows. Section 5.2 defines the predictivity criterion considered and proposes different methods for its estimation. Section 5.3 presents the algorithms used for test-point selection. Our numerical results are presented in Sections 5.4 and 5.5: in Section 5.4 a test set is freely chosen within the entire input space, while in Section 5.5 an existing data set can be split into a training sample and a test set. Finally, Section 5.6 concludes and outlines some perspectives.

5.2 Predictivity assessment criteria for an ML model

In this section, a new criterion to assess the predictive performance of a model is proposed, enhancing a standard model quality metric by suitably weighting the errors observed on the test set. Let us denote by $\mathcal{X} \subset \mathbb{R}^d$ the space of the input variables $\mathbf{x} = (x_1, \dots, x_d)$ of the model. Then let $y(\mathbf{x}) \in \mathbb{R}$ (resp. $y(\mathbf{x}') \in \mathbb{R}$) be the observed output at point $\mathbf{x} \in \mathcal{X}$ (resp. $\mathbf{x}' \in \mathcal{X}$). Considering the training sample denoted by $(\mathbf{X}_m, \mathbf{y}_m)$, with $\mathbf{y}_m = [y(\mathbf{x}^{(1)}), \dots, y(\mathbf{x}^{(m)})]^\top$. The test sample is

denoted by $(\mathbf{X}_n, \mathbf{y}_n) = (\mathbf{x}^{(i)}, y(\mathbf{x}^{(i)}))_{1 \leq i \leq n}$. Remember that the intersection between these two samples is empty, $\mathbf{X}_m \cap \mathbf{X}_n = \emptyset$.

5.2.1 The predictivity coefficient

Let us denote by $\eta_m(\mathbf{x})$ the prediction at point \mathbf{x} of a model learned using $(\mathbf{X}_m, \mathbf{y}_m)$ (Hastie et al., 2009; Rasmussen and Williams, 2006). A classical measure for assessing the predictive ability of η_m , in order to evaluate its validity, is the *predictivity coefficient*. Considering the probability measure μ that weights how comparatively important it is to accurately predict y over the different regions of \mathcal{X} . For example the input could be a random vector with known distribution: in that case, this distribution would be a reasonable choice for μ . The true (i.e., ideal) value of the predictivity is defined as the following normalization of the Integrated Square Error (ISE):

$$Q_\mu^2(\eta_m) = 1 - \frac{\text{ISE}_\mu(\eta_m)}{\text{Var}_\mu(\eta_m)}, \quad (5.1)$$

where

$$\begin{aligned} \text{ISE}_\mu(\eta_m) &= \int_{\mathcal{X}} [y(\mathbf{x}) - \eta_m(\mathbf{x})]^2 d\mu(\mathbf{x}), \\ \text{Var}_\mu(\eta_m) &= \int_{\mathcal{X}} \left[y(\mathbf{x}) - \int_{\mathcal{X}} y(\mathbf{x}') d\mu(\mathbf{x}') \right]^2 d\mu(\mathbf{x}). \end{aligned}$$

The ideal predictivity $Q_{\text{ideal}}^2(\mu)$ is usually estimated by its empirical version calculated over the test sample $(\mathbf{X}_n, \mathbf{y}_n)$, see (Da Veiga et al., 2021, p. 32):

$$\widehat{Q}_n^2 = 1 - \frac{\sum_{i=1}^n [y(\mathbf{x}^{(i)}) - \eta_m(\mathbf{x}^{(i)})]^2}{\sum_{i=1}^n [y(\mathbf{x}^{(i)}) - \bar{y}_n]^2}, \quad (5.2)$$

where $\bar{y}_n = (1/n) \sum_{i=1}^n y(\mathbf{x}^{(i)})$ denotes the empirical mean of the observations in the test sample. Note that the calculation of \widehat{Q}_n^2 only requires access to the predictor $\eta_m(\cdot)$. To compute \widehat{Q}_n^2 , one does not need to know the training set which was used to build $\eta_m(\cdot)$. This estimator \widehat{Q}_n^2 is the *coefficient of determination* (also called “Nash-Sutcliffe criterion” Nash and Sutcliffe, 1970), which is a standard notion in regression studies (Kleijnen and Sargent, 2000; Iooss et al., 2010). It compares the prediction errors obtained with the model η_m with those obtained when prediction equals the empirical mean of the observations. Thus, the closer \widehat{Q}_n^2 is to one, the more accurate the surrogate model is (for the test set considered). On the contrary, \widehat{Q}_n^2 close to zero (negative values are possible too) indicates poor predictions abilities, as there is little improvement compared to prediction by the simple empirical mean of the observations. The next section shows how a suitable weighting of the residual on the training sample may be key to improving the estimation of \widehat{Q}_n^2 .

5.2.2 Weighting the test sample

The simplest way to estimate the $\text{ISE}_\mu(\mathbf{X}_m, \mathbf{y}_m)$ (present on the numerator of the predictivity coefficient) is by computing the arithmetic mean of the squared residuals evaluated on the test set \mathbf{X}_n . Writing the equivalent discrete measure $\xi_n = \frac{1}{n} \sum_{i=1}^n \delta_{\mathbf{x}^{(i)}}$, with $\delta_{\mathbf{x}}$ the Dirac measure at \mathbf{x} , this estimate can be expressed as:

$$\text{ISE}_{\xi_n}(\eta_m) = \frac{1}{n} \sum_{i=1}^n \left[y(\mathbf{x}^{(i)}) - \eta_m(\mathbf{x}^{(i)}) \right]^2.$$

When the points $\mathbf{x}^{(i)}$ of the test set \mathbf{X}_n are distant from the points of the training set \mathbf{X}_m , the squared prediction errors $|y(\mathbf{x}^{(i)}) - \eta_m(\mathbf{x}^{(i)})|^2$ tend to represent the worst possible error situations, and $\text{ISE}_{\xi_n}(\eta_m)$ tends to overestimate $\text{ISE}_\mu(\eta_m)$. In this section, we postulate a statistical model for the prediction errors in order to be able to quantify this potential bias when sampling the residual process, enabling its subsequent correction.

Let us assume that the prediction error $\delta_m(\mathbf{x}) = y(\mathbf{x}) - \eta_m(\mathbf{x})$ is a realization of a Gaussian Process (GP) with mean $\widehat{\delta}_m(\mathbf{x})$ and covariance kernel $\sigma^2 K_{|m}$, $\delta_m(\mathbf{x}) \sim \text{GP}(\widehat{\delta}_m(\mathbf{x}), \sigma^2 K_{|m})$

$$\begin{cases} \widehat{\delta}_m(\mathbf{x}) = \mathbf{k}_m^\top(\mathbf{x}) \mathbf{K}_m^{-1} (\mathbf{y}_m - \boldsymbol{\eta}_m), \\ \sigma^2 K_{|m}(\mathbf{x}, \mathbf{x}') = \mathbb{E}[\delta_m(\mathbf{x})\delta_m(\mathbf{x}')] = \sigma^2 [K(\mathbf{x}, \mathbf{x}') - \mathbf{k}_m^\top(\mathbf{x}) \mathbf{K}_m^{-1} \mathbf{k}_m(\mathbf{x}')] . \end{cases} \quad (5.3)$$

Where $\boldsymbol{\eta}_m = [\eta_m(\mathbf{x}^{(1)}), \dots, \eta_m(\mathbf{x}^{(m)})]^\top$, $\mathbf{k}_m(\mathbf{x})$ is the column vector $[K(\mathbf{x}, \mathbf{x}^{(1)}), \dots, K(\mathbf{x}, \mathbf{x}^{(m)})]^\top$, and \mathbf{K}_m is the $m \times m$ covariance matrix whose element (i, j) is given by $\{\mathbf{K}_m\}_{i,j} = K(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$, with K a positive definite kernel. Note that in the case of a learning model η_m which interpolates the observations \mathbf{y}_m , the errors observed at the learning points \mathbf{X}_m equal zero, leading finally to the posterior $\text{GP}(0, \sigma^2 K_{|m})$ for $\delta_m(\mathbf{x})$.

The prediction model error above allows us to study how well $\text{ISE}_\mu(\eta_m)$ is estimated using a test set \mathbf{X}_n . The expected squared error when estimating $\text{ISE}_\mu(\eta_m)$ by $\text{ISE}_{\xi_n}(\eta_m)$, is defined as $\overline{\Delta}^2(\xi_n, \mu; \eta_m)$:

$$\begin{aligned} \overline{\Delta}^2(\xi_n, \mu; \eta_m) &= \mathbb{E} \left[(\text{ISE}_{\xi_n}(\eta_m) - \text{ISE}_\mu(\eta_m))^2 \right] \\ &= \mathbb{E} \left[\left(\int_{\mathcal{X}} \delta_m^2(\mathbf{x}) d(\xi_n - \mu)(\mathbf{x}) \right)^2 \right] \\ &= \mathbb{E} \left[\int_{\mathcal{X}^2} \delta_m^2(\mathbf{x}) \delta_m^2(\mathbf{x}') d(\xi_n - \mu)(\mathbf{x}) d(\xi_n - \mu)(\mathbf{x}') \right]. \end{aligned}$$

Tonelli's theorem gives

$$\overline{\Delta}^2(\xi_n, \mu; \eta_m) = \int_{\mathcal{X}^2} \mathbb{E}[\delta_m^2(\mathbf{x}) \delta_m^2(\mathbf{x}')] d(\xi_n - \mu)(\mathbf{x}) d(\xi_n - \mu)(\mathbf{x}').$$

Since $\mathbb{E}[U^2V^2] = 2(\mathbb{E}[UV])^2 + \mathbb{E}[U^2]\mathbb{E}[V^2]$ for any one-dimensional normal centered random variables U and V . The expression can then be written as:

$$\bar{\Delta}^2(\xi_n, \mu; \eta_m) = \int_{\mathcal{X}^2} 2\mathbb{E}[\delta_m(\mathbf{x})\delta_m(\mathbf{x}')]^2 + \mathbb{E}[\delta_m^2(\mathbf{x})]\mathbb{E}[\delta_m^2(\mathbf{x}')] d(\xi_n - \mu)(\mathbf{x})d(\xi_n - \mu)(\mathbf{x}') \quad (5.4a)$$

$$\bar{\Delta}^2(\xi_n, \mu; \eta_m) = \int_{\mathcal{X}^2} \bar{K}_{|m}(\mathbf{x}, \mathbf{x}') d(\xi_n - \mu)(\mathbf{x})d(\xi_n - \mu)(\mathbf{x}') \quad (5.4b)$$

$$\bar{\Delta}^2(\xi_n, \mu; \eta_m) = \sigma^2 \text{MMD}_{\bar{K}_{|m}}^2(\xi_n, \mu). \quad (5.4c)$$

Interestingly, the last expression is equivalent to the maximum mean discrepancy (previously introduced in this manuscript and further defined in Appendix B) between μ and ξ_n for a kernel $\bar{K}_{|m}(\cdot, \cdot)$. Note that σ^2 only appears as a multiplying factor in Eq. (5.4b), with the consequence that σ^2 does not impact the choice of a suitable ξ_n . The resulting kernel $\bar{K}_{|m}(\cdot, \cdot)$ is differently defined whether (i) the learning model $\eta_m(\mathbf{x})$ interpolates \mathbf{y}_m or not (ii):

$$\begin{cases} (i) \Rightarrow \bar{K}_{|m}(\mathbf{x}, \mathbf{x}') := 2\bar{K}_{|m}^2(\mathbf{x}, \mathbf{x}') + K_{|m}(\mathbf{x}, \mathbf{x})K_{|m}(\mathbf{x}', \mathbf{x}') \\ (ii) \Rightarrow \bar{K}_{|m}(\mathbf{x}, \mathbf{x}') := 2 \left[K_{|m}(\mathbf{x}, \mathbf{x}') + 2\hat{\delta}_m(\mathbf{x})\hat{\delta}_m(\mathbf{x}') \right] K_{|m}(\mathbf{x}, \mathbf{x}') \\ \quad + \left[\hat{\delta}_m^2(\mathbf{x}) + K_{|m}(\mathbf{x}, \mathbf{x}) \right] \left[\hat{\delta}_m^2(\mathbf{x}') + K_{|m}(\mathbf{x}', \mathbf{x}') \right]. \end{cases} \quad (5.5)$$

The main idea is to replace ξ_n , uniform on \mathbf{X}_n , by a nonuniform measure ζ_n supported on \mathbf{X}_n , $\zeta_n = \sum_{i=1}^n w_i \delta_{\mathbf{x}^{(i)}}$ with weights $\mathbf{w}_n = (w_1, \dots, w_n)^\top$ chosen such that the estimation error $\bar{\Delta}^2(\zeta_n, \mu; \eta_m)$, and thus $d_{\bar{K}_{|m}}^2(\zeta_n, \mu)$, is minimized. The squared MMD for the kernel $\bar{K}_{|m}$ between μ and the weighted measure ζ_n can be expressed as:

$$\text{MMD}_{\bar{K}_{|m}}^2(\zeta_n, \mu) = \varepsilon_{\bar{K}_{|m}, \mu} - 2\mathbf{w}_n^\top P_{\bar{K}_{|m}, \mu}(\mathbf{X}_n) + \mathbf{w}_n^\top \bar{\mathbf{K}}_{|m}(\mathbf{X}_n) \mathbf{w}_n, \quad (5.6)$$

where $P_{\bar{K}_{|m}, \mu}(\mathbf{X}_n)$ is the vector of potentials $\left[\int_{\mathcal{X}} \bar{K}_{|m}(\mathbf{x}, \mathbf{x}^{(1)}) d\pi(\mathbf{x}), \dots, \int_{\mathcal{X}} \bar{K}_{|m}(\mathbf{x}, \mathbf{x}^{(n)}) d\pi(\mathbf{x}) \right]^\top$, and $\bar{\mathbf{K}}_{|m}(\mathbf{X}_n)$ is the $n \times n$ covariance matrix such that $\{\bar{\mathbf{K}}_{|m}(\mathbf{X}_n)\}_{i,j} = \bar{K}_{|m}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$, $\forall i, j = 1, \dots, n$, and $\varepsilon_{\bar{K}_{|m}, \mu} = \int_{\mathcal{X}^2} \bar{K}_{|m}(\mathbf{x}, \mathbf{x}') d\mu(\mathbf{x})d\mu(\mathbf{x}')$. The squared MMD defined in Eq. (5.6) is minimized for the following optimal weights \mathbf{w}_n^* :

$$\mathbf{w}_n^* = \bar{\mathbf{K}}_{|m}^{-1}(\mathbf{X}_n) \mathbf{p}_{\bar{K}_{|m}, \mu}(\mathbf{X}_n). \quad (5.7)$$

Therefore, an optimally weighted estimator of the predictivity coefficient supported on the test set \mathbf{X}_n is defined as:

$$\widehat{Q}_{n*}^2 = 1 - \frac{\sum_{i=1}^n w_i^* [y(\mathbf{x}^{(i)}) - \eta_m(\mathbf{x}^{(i)})]^2}{\frac{1}{n} \sum_{i=1}^n [y(\mathbf{x}^{(i)}) - \bar{y}_n]^2}, \quad (5.8)$$

with $\bar{y}_n = (1/n) \sum_{i=1}^n y(\mathbf{x}^{(m+i)})$. Notice that the weights w_i^* do not depend on the variance parameter σ^2 of the GP model. Moreover, this approach does no constraint the weights, which

is works best in our experience than the different constrained versions (e.g., non-negativity, summing to one) studied in [Pronzato and Rendas \(2021b\)](#).

Remark 7. The optimal estimator \widehat{Q}_{n*}^2 focused on weighting numerator of the coefficient of determination defined in [\[ccc\]](#). However, the variance's estimator on the denominator can also be optimally weighted. Let us write an alternative version of \widehat{Q}_{n*}^2

$$\widehat{Q}'_n^2 = 1 - \frac{\sum_{i=1}^n [y(\mathbf{x}^{(i)}) - \eta_m(\mathbf{x}^{(i)})]^2}{\sum_{i=1}^n [y(\mathbf{x}^{(i)}) - \bar{y}_m]^2}, \quad (5.9)$$

which compares the performance on the test set of η_m and $\bar{y}_m = (1/m) \sum_{i=1}^m y(\mathbf{x}^i)$. Using similar developments as previously, is possible to also apply a weighting procedure to the denominator of \widehat{Q}'_n^2 , in order to make it resemble its integral version $V'_\mu(y_m) = \int_{\mathcal{X}} [y(\mathbf{x}) - \bar{y}_m]^2 d\mu(\mathbf{x})$ (see [Fekhari et al., 2023](#)).

5.3 Test-set construction

In the previous section the test set was assumed as given, and a method was proposed to estimate the $\text{ISE}_\mu(\eta_m)$ (with the learning model η_m , built on \mathbf{X}_m) by an optimally weighted sum of the residuals. The objective in this section is to construct a test set of size n , denoted by $\mathbf{X}_n = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\} \subset \mathcal{X}$. To evaluate the predictivity of a model learned on a training set \mathbf{X}_m , a strategy is to place the test points the furthest away from the training set, to obtain a space-filling design when gathering leaning and test set. The sampling methods used for building the test set should then be space-filling and incremental. The advantage of using an iterative construction is that it can be stopped as soon as the predictivity estimation is considered sufficiently accurate. In case the conclusion is that model predictions are not reliable enough, the full design $\mathbf{X}_{m+n} = \mathbf{X}_m \cup \mathbf{X}_n$ and the associated observations \mathbf{y}_{m+n} can be used to update the model. This updated model can then be tested at additional design points, elements of a new test set to be constructed. This section introduces different space-filling methods, later compared for test set construction.

5.3.1 Fully-Sequential Space-Filling design

The Fully-Sequential Space-Filling forward-reflected (FSSF-fr) algorithm [Shang and Apley \(2020\)](#) relies on the CADEX algorithm [Kennard and Stone \(1969\)](#) (also called the “coffee-house” method [Müller \(2007\)](#)). It constructs a sequence of nested designs in a bounded set \mathcal{X} by sequentially selecting a new point \mathbf{x} as far away as possible from the $\mathbf{x}^{(i)}$ previously selected. New inserted points are selected within a set of candidates \mathcal{S} which may coincide with \mathcal{X} or be a finite subset of \mathcal{X} (which simplifies the implementation, only this case is considered here). The improvement of FSSF-fr when compared to CADEX is that new points are selected *at the same time* far from the previous design points as well as far from the boundary of \mathcal{X} .

The algorithm is as follows:

1. Choose \mathcal{S} , a finite set of candidate points in \mathcal{X} , with size $N \gg n$ in order to allow a fairly dense covering of \mathcal{X} . When $\mathcal{X} = [0, 1]^d$, [Shang and Apley \(2020\)](#) recommends taking \mathcal{S} equal to the first $N = 1000d + 2n$ points of a Sobol sequence in \mathcal{X} .
2. Choose the first point $\mathbf{x}^{(1)}$ randomly in \mathcal{S} and define $\mathbf{X}_1 = \{\mathbf{x}^{(1)}\}$.
3. At iteration i , with $\mathbf{X}_i = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(i)}\}$, select
$$\mathbf{x}^{(i+1)} \in \arg \max_{\mathbf{x} \in \mathcal{S} \setminus \mathbf{X}_i} \left[\min \left(\min_{j \in \{1, \dots, i\}} \|\mathbf{x} - \mathbf{x}^{(j)}\|, \sqrt{2}d \text{dist}(\mathbf{x}, R(\mathbf{x})) \right) \right], \quad (5.10)$$
where $R(\mathbf{x})$ is the symmetric of \mathbf{x} with respect to its nearest boundary of \mathcal{X} , and set $\mathbf{X}_{i+1} = \mathbf{X}_i \cup \mathbf{x}^{(i+1)}$.
4. Stop the algorithm when \mathbf{X}_n has the required size.

The role of the reflected point $R(\mathbf{x})$ is to avoid selecting $\mathbf{x}^{(i+1)}$ too close to the boundary of \mathcal{X} , which is a major problem with standard coffee-house, especially when $\mathcal{X} = [0, 1]^d$ with d large. While the standard coffee-house (greedy packing) algorithm simply uses $\mathbf{x}^{(i+1)} \in \arg \max_{\mathbf{x} \in \mathcal{S} \setminus \mathbf{X}_i} \min_{j \in \{1, \dots, i\}} \|\mathbf{x} - \mathbf{x}^{(j)}\|$. The factor $\sqrt{2}d$ in Eq. (5.10) proposed in [Shang and Apley \(2020\)](#) sets a balance between distance to the design \mathbf{X}_i and distance to the boundary of \mathcal{X} . Another scaling factor, depending on the target design size n is proposed in [Nogales Gómez et al. \(2021\)](#).

FSSF-fr is entirely based on geometric considerations and implicitly assumes that the selected set of points should cover \mathcal{X} evenly.

However, in the context of uncertainty quantification the distribution μ of the model inputs is frequently not uniform. It is then desirable to select a test set representative of μ . Which can be achieved through the inverse transform of the CDF: FSSF-fr constructs \mathbf{X}_n in the unit hypercube $[0, 1]^d$, and an “isoprobabilistic” transform $T : [0, 1]^d \rightarrow \mathcal{X}$ is then applied to the points in \mathbf{X}_i , T being such that, if U is a random variable uniform on $[0, 1]^d$, then $T(U)$ follows the target distribution μ . The transformation can be applied to each input separately when μ is the product of its marginals, but is more complicated in other cases, see ([Lemaire et al., 2009](#), Chap. 4). Note that FSSF-fr operates in the bounded set $[0, 1]^d$ even if the support of μ is unbounded. The other two algorithms presented in this section are able to directly choose points representative of a given distribution μ and do not need to resort to such a transformation.

5.3.2 Support points

Support points [Mak and Joseph \(2018\)](#) are such that their associated empirical distribution ξ_n has minimum Maximum-Mean-Discrepancy (MMD) with respect to μ for the energy-distance kernel of Székely and Rizzo [Székely and Rizzo \(2013\)](#),

$$K_E(\mathbf{x}, \mathbf{x}') = \frac{1}{2} (\|\mathbf{x}\| + \|\mathbf{x}'\| - \|\mathbf{x} - \mathbf{x}'\|). \quad (5.11)$$

The squared MMD between ξ_n and μ for the distance kernel equals

$$\text{MMD}_{K_E}^2(\xi_n, \mu) = \frac{2}{n} \sum_{i=1}^n \mathbb{E}\|\mathbf{x}^{(i)} - \zeta\| - \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\| - \mathbb{E}\|\zeta - \zeta'\|, \quad (5.12)$$

where ζ and ζ' are independently distributed with μ ; see [Sejdinovic et al. \(2013\)](#). A key property of the energy-distance kernel is that it is characteristic [Sriperumbudur et al. \(2010\)](#): for any two probability distributions μ and ξ on \mathcal{X} , $d_{K_E}^2(\mu, \xi)$ equals zero if and only if $\mu = \xi$, and so it defines a norm in the space of probability distributions. Compared to more heuristic methods for solving quantization problems, support points benefit from the theoretical guarantees of MMD minimization in terms of convergence of ξ_n to μ as $n \rightarrow \infty$.

As $\mathbb{E}\|\mathbf{x}^{(i)} - \zeta\|$ is not known explicitly, in practice μ is replaced by its empirical version μ_N for a given large-size sample $(\mathbf{x}'^{(k)})_{k=1 \dots N}$. The support points \mathbf{X}_n^s are then given by

$$\mathbf{X}_n^s \in \arg \min_{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}} \left(\frac{2}{nN} \sum_{i=1}^n \sum_{k=1}^N \|\mathbf{x}^{(i)} - \mathbf{x}'^{(k)}\| - \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\| \right). \quad (5.13)$$

The function to be minimized can be written as a difference of functions convex in \mathbf{X}_n , which yields a difference-of-convex program. In [Mak and Joseph \(2018\)](#), a majorization-minimization procedure, efficiently combined with resampling, is applied to the construction of large designs (up to $n = 10^4$) in high dimensional spaces (up to $d = 500$). The examples treated clearly show that support points are distributed in a way that matches μ more closely than Monte-Carlo and quasi-Monte Carlo samples.

The method was used to split a dataset into a training set and a test set in [Joseph and Vakayil \(2022\)](#), where the N points \mathbf{X}_N in Eq. (5.13) are those from the dataset. Then \mathbf{X}_n^s gives the test set and the other $N - n$ points are used for training. There is a serious additional difficulty though, as choosing \mathbf{X}_n^s among the dataset corresponds to a difficult combinatorial optimization problem. A possible solution is to perform the optimization in a continuous domain \mathcal{X} and then choose \mathbf{X}_n^s that corresponds to the closest points in \mathbf{X}_N (for the Euclidean distance) to the continuous solution obtained [Joseph and Vakayil \(2022\)](#).

The direct determination of support points through Eq. (5.13) does not allow the construction of a nested sequence of test sets. One possibility would be to solve Eq. (5.13) sequentially, one point at a time, in a continuous domain, and then select the closest point within \mathbf{X}_N as the one to be included in the test set. We shall use a different approach here, based on the greedy minimization of the MMD Eq. (5.12) for the candidate set $\mathcal{S} = \mathbf{X}_N$: at iteration i , the algorithm chooses

$$\mathbf{x}_{i+1}^s \in \arg \min_{\mathbf{x} \in \mathcal{S}} \left(\frac{1}{N} \sum_{k=1}^N \|\mathbf{x} - \mathbf{x}'^{(k)}\| - \frac{1}{i+1} \sum_{j=1}^i \|\mathbf{x} - \mathbf{x}^{(j)}\| \right). \quad (5.14)$$

The method requires the computation of the $N(N-1)/2$ distances $\|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\|$, $i, j = 1, \dots, N$, $i \neq j$, which hinders its applicability to large-scale problems (a test-case with $N = 1000$ is presented

in Section 5.5). Note that [Joseph and Vakayil \(2022\)](#) only studies the split of a given data set into a learning and test set while this chapter builds support points on the input space \mathcal{X} .

Greedy MMD minimization can be applied to other kernels than the distance kernel Eq. (5.11), see [Teymur et al. \(2021\)](#); [Pronzato \(2022\)](#). In the next section the closely related method of kernel herding is recalled (KH) ([Chen et al., 2010](#)), after its presentation in Chapter 4 of the present manuscript.

5.3.3 Kernel herding

As introduced in Section 5.3.3, [Lacoste-Julien et al. \(2015\)](#) proposed a linearization of the MMD minimization using the Frank-Wolfe algorithm. Let us define a positive definite kernel K on $\mathcal{X} \times \mathcal{X}$, and consider $\xi_i = (1/i) \sum_{j=1}^i \delta_{\mathbf{x}^{(j)}}$ as the empirical measure for \mathbf{X}_i . In the sequential and uniformly weighted case, this iteration i of kernel herding is expressed as a difference of potentials:

$$\mathbf{x}_{i+1} \in \arg \min_{\mathbf{x} \in \mathcal{S}} [P_{\xi_i}(\mathbf{x}) - P_{\mu}(\mathbf{x})], \quad (5.15)$$

with $\mathcal{S} \subseteq \mathcal{X}$ a given candidate set and $P_{\xi_i}(\mathbf{x}) = (1/i) \sum_{j=1}^i K(\mathbf{x}, \mathbf{x}^{(j)})$.

Once the targeted measure substituted by an empirical measure μ_N based on a sample $(\mathbf{x}'^{(k)})_{k=1 \dots N}$ then complete estimation becomes: $P_{\mu_N}(\mathbf{x}) = (1/N) \sum_{k=1}^N K(\mathbf{x}, \mathbf{x}'^{(k)})$, which gives

$$\mathbf{x}_{i+1} \in \arg \min_{\mathbf{x} \in \mathcal{S}} \left[\frac{1}{i} \sum_{j=1}^i K(\mathbf{x}, \mathbf{x}^{(j)}) - \frac{1}{N} \sum_{k=1}^N K(\mathbf{x}, \mathbf{x}'^{(k)}) \right].$$

When K is the energy-distance kernel Eq. (5.11) the greedy support points from Eq. (5.14) are recovered with a factor $1/i$ instead of $1/(i+1)$ in the second sum.

The candidate set \mathcal{S} in Eq. (5.15) is arbitrary and can be chosen as in Section 5.3.1. A neat advantage of kernel herding over support points is that the potential $P_{\mu}(\mathbf{x})$ is sometimes explicitly available. When $\mathcal{S} = \mathbf{X}_N$, this avoids the need to calculate the $N(N-1)/2$ distances $\|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\|$ and thus allows application to very large sample sizes. This is the case in particular when \mathcal{X} is the cross product of one-dimensional sets $\mathcal{X}_{[i]}$, $\mathcal{X} = \mathcal{X}_{[1]} \times \dots \times \mathcal{X}_{[d]}$, μ is the product of its marginals $\mu_{[i]}$ on the $\mathcal{X}_{[i]}$, K is the product of one-dimensional kernels $K_{[i]}$, and the one-dimensional integral in $P_{\mu_{[i]}}(\mathbf{x})$ is known explicitly for each $i \in \{1, \dots, d\}$. Indeed, for $\mathbf{x} = (x_1, \dots, x_d) \in \mathcal{X}$, $P_{\mu}(\mathbf{x}) = \prod_{i=1}^d P_{\mu_{[i]}}(x_i)$ (see [Pronzato and Zhigljavsky, 2020](#)). When K is the product of Matérn kernels with regularity parameter $5/2$ and correlation lengths θ_i , $K(\mathbf{x}, \mathbf{x}') = \prod_{i=1}^d K_{5/2, \theta_i}(x_i - x'_i)$, with

$$K_{5/2, \theta}(x - x') = \left(1 + \frac{\sqrt{5}}{\theta} |x - x'| + \frac{5}{3\theta^2} (x - x')^2 \right) \exp \left(-\frac{\sqrt{5}}{\theta} |x - x'| \right), \quad (5.16)$$

the one-dimensional potentials are given in Appendix B for $\mu_{[i]}$ uniform on $[0, 1]$ or $\mu_{[i]}$ the standard normal $\mathcal{N}(0, 1)$. When no observation is available, which is the common situation at the design stage, the correlation lengths have to be set to heuristic values. The values of the

correlation lengths empirically show a significant influence over the design. A reasonable choice for $\mathcal{X} = [0, 1]^d$ is $\theta_i = n^{-1/d}$ for all i , with n the target number of design points (see Pronzato and Zhigljavsky, 2020).

5.3.4 Numerical illustration

As first numerical illustration, the FSSF-fr (denoted FSSF in the following), support points and kernel herding algorithms were applied in the situation where a given initial design of size m has to be completed by a series of additional points $\mathbf{x}^{(m+1)}, \dots, \mathbf{x}^{(m+n)}$. The objective is to obtain a full design \mathbf{X}_{m+n} that is a good quantization of a given distribution μ .

Figures 5.1 and 5.2 correspond to μ uniform on $[0, 1]^2$ and μ the standard normal distribution $\mathcal{N}(\mathbf{0}, \mathbf{I}_2)$, with \mathbf{I}_2 the 2-dimensional identity matrix, respectively. All methods are applied to the same candidate set \mathcal{S} .

The initial designs \mathbf{X}_m are chosen in the class of space-filling designs, well suited to initialize sequential learning strategies (Santner et al., 2003). When μ is uniform, the initial design is a maximin Latin hypercube design (introduced in Section 1.4.2) with $m = 10$ and the candidate set is given by the $N = 2^{12}$ first points \mathbf{S}_N of a Sobol sequence in $[0, 1]$. When μ is normal, the inverse probability transform method is first applied to \mathbf{S}_N and \mathbf{X}_m (this does not raise any difficulty here as μ is the product of its marginals). The candidate points \mathcal{S} are marked in gray on Fig. 5.1 and Fig. 5.2 and the initial design is indicated by the red crosses. The index i of each added test point $\mathbf{x}^{(m+i)}$ is indicated (the font size decreases with i). In such a small dimension ($d = 2$), a visual appreciation gives the impression that the three methods have comparable performance. However, FSSF tends to choose points closer to the boundary of \mathcal{S} than the other two, and that support points seem to sample more freely the holes of \mathbf{X}_m than kernel herding, which seems to be closer to a space-filling continuation of the training set. In the next section, these designs are used for estimating the quality of the predictivity metric.

5.4 Numerical results I: construction of a training set and a test set

This section presents numerical results obtained on three different test-cases, in dimension 2 (test-cases 1 and 2) and 8 (test-case 3), for which $y(\mathbf{x}) = f(\mathbf{x})$ with $f(\mathbf{x})$ has an easy to evaluate analytical expression, see Section 5.4.1. This allows a good estimation of Q_μ^2 (see Eq. (5.1)) by a large Monte Carlo sample (with size $M = 10^6$), which will serve as reference when assessing the performance of each of the other estimators.

The validation designs are built by FSSF, support points and kernel herding, presented in Sections 5.3.1, 5.3.2, and 5.3.3, and the performances obtained are compared for each one, considering the uniform and the weighted estimator of Section 5.2.2.

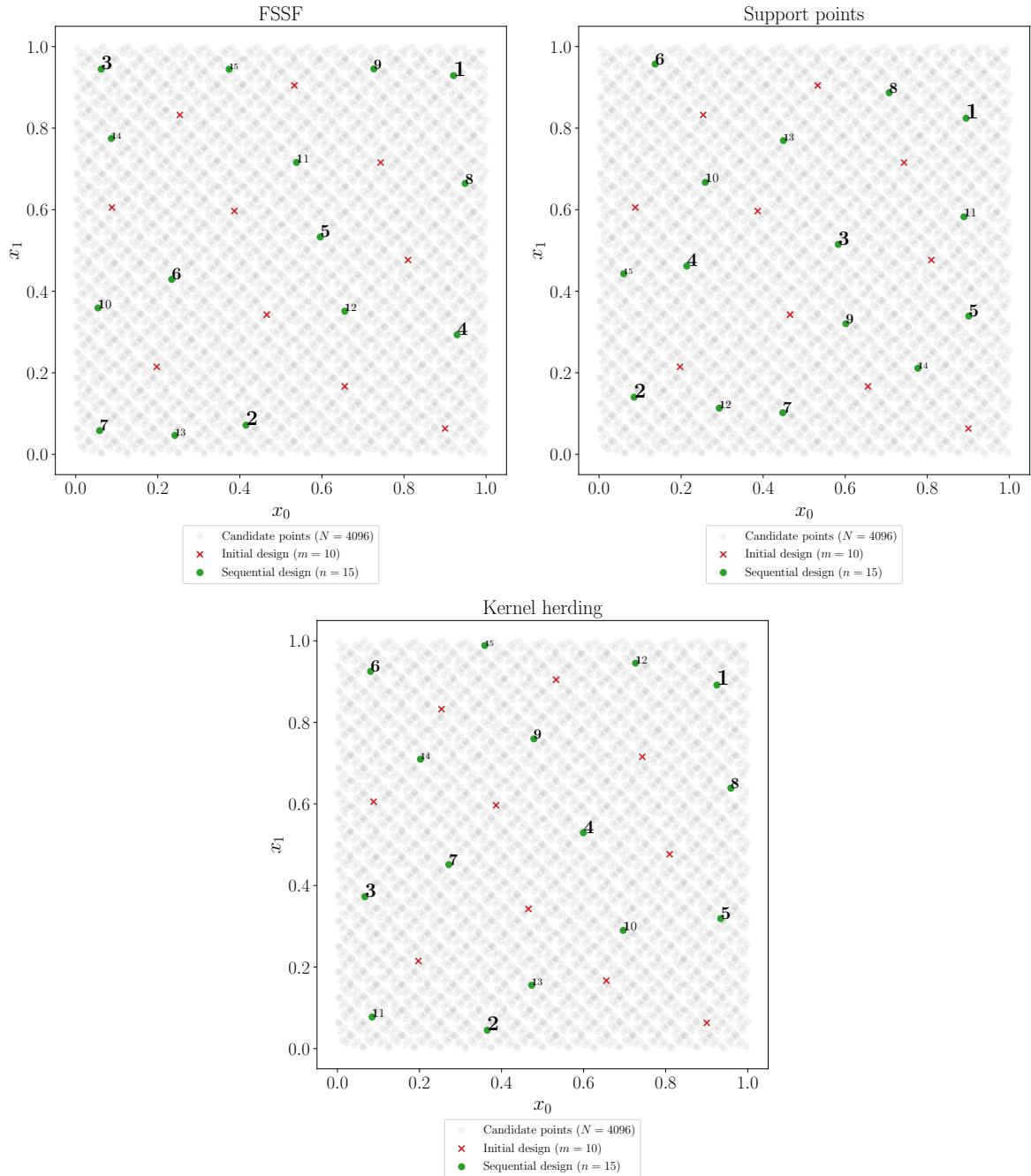


Figure 5.1 Additional points (ordered, green) complementing an initial design (red crosses), μ is uniform on $[0, 1]$, the candidate points are in gray.

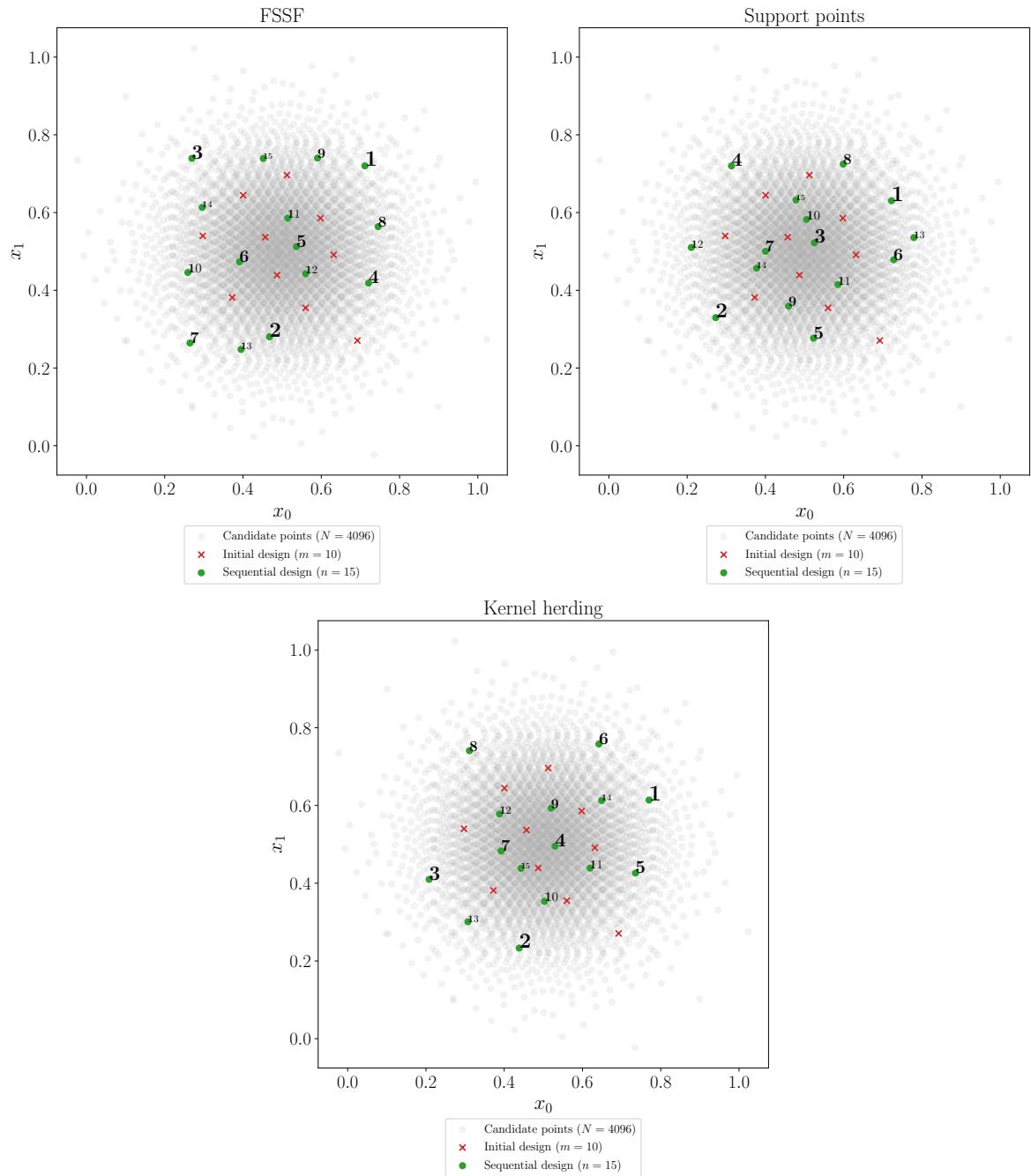


Figure 5.2 Additional points (ordered, green) complementing an initial design (red crosses), μ normal, the candidate points are in gray.

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5.4.1 Test-cases

The training design \mathbf{X}_m and the set \mathcal{S} of potential test set points are as in Section 5.3.4. For test-cases 1 and 3, μ is the uniform measure on $\mathcal{X} = [0, 1]^d$, with $d = 2$ and $d = 8$, respectively; \mathbf{X}_m is a maximin Latin hypercube design in \mathcal{X} , and \mathcal{S} corresponds to the first N points \mathbf{S}_N of Sobol' sequence in \mathcal{X} , complemented by the 2^d vertices. In the second test-case, $d = 2$, μ is the normal distribution $\mathcal{N}(\mathbf{0}, \mathbf{I}_2)$, and the sets \mathbf{X}_m and \mathbf{S}_N must be transformed as explained in section 5.3.1. There are $N = 2^{14}$ candidate points for test-cases 1 and 2 and $N = 2^{15}$ for test-case 3 (this value is rather moderate for a problem in dimension 8, but using a larger N yields numerical difficulties for support points; see Section 5.3.2).

For each test-case, a GP regression model is fitted to the m observations using ordinary Kriging [Rasmussen and Williams \(2006\)](#) (a GP model with constant mean), with an anisotropic Matérn kernel with regularity parameter $5/2$, and the correlation lengths θ_i are estimated by maximum likelihood via a truncated Newton algorithm. All calculations were done using the Python package OpenTURNS for uncertainty quantification [Baudin et al. \(2017\)](#). The kernel used for kernel herding is different and corresponds to the tensor product of one-dimensional Matérn kernels Eq. (5.16), so that the potentials $P_\mu(\cdot)$ are known explicitly (see Appendix B); the correlations lengths are set to $\theta = 0.2$ in test-cases 1 and 3 ($d = 2$) and to $\theta = 0.7$ in test-case 3 ($d = 8$).

Assuming that a model is classified, in terms of the estimated value of its predictivity index Q^2 as “poor fitting” if $Q^2 \in [0.6, 0.8]$, “reasonably good fitting”, when $Q^2 \in (0.8, 0.9]$, and “very good fitting” if $Q^2 > 0.9$, for each test-case, three different sizes m of the training set are selected such that the corresponding models cover all three possible situations. For all test-cases, the impact of the size n of the test set is studied in the range $n \in \{4, \dots, 50\}$.

Test-case 1. This test function is $f_1(\mathbf{x}) = h(2x_1 - 1, 2x_2 - 1)$, $(x_1, x_2) \in \mathcal{X} = [0, 1]^2$, with

$$h(u_1, u_2) = \frac{\exp(u_1)}{5} - \frac{u_2}{5} + \frac{u_2^6}{3} + 4u_2^4 - 4u_2^2 + \frac{7u_1^2}{10} + u_1^4 + \frac{3}{4u_1^2 + 4u_2^2 + 1}.$$

Color coded 3d and contour plots of f_1 for $\mathbf{X} \in \mathcal{X}$ are shown on the left panel of Figure 5.3, showing that the function is rather smooth, even if its behavior along the boundaries of \mathcal{X} , in particular close to the vertices, may present difficulties for some regression methods. The size of the training set for this function are: $m \in \{5, 15, 30\}$.

Test-case 2. The second test function, plotted in the right panel of Figure 5.3 for $\mathbf{x} \in [0, 1]^2$, is

$$f_2(\mathbf{x}) = \cos\left(5 + \frac{3}{2}x_1\right) + \sin\left(5 + \frac{3}{2}x_1\right) + \frac{1}{100}\left(5 + \frac{3}{2}x_1\right)\left(5 + \frac{3}{2}x_2\right).$$

Training set sizes for this test-case are $m \in \{8, 15, 30\}$.

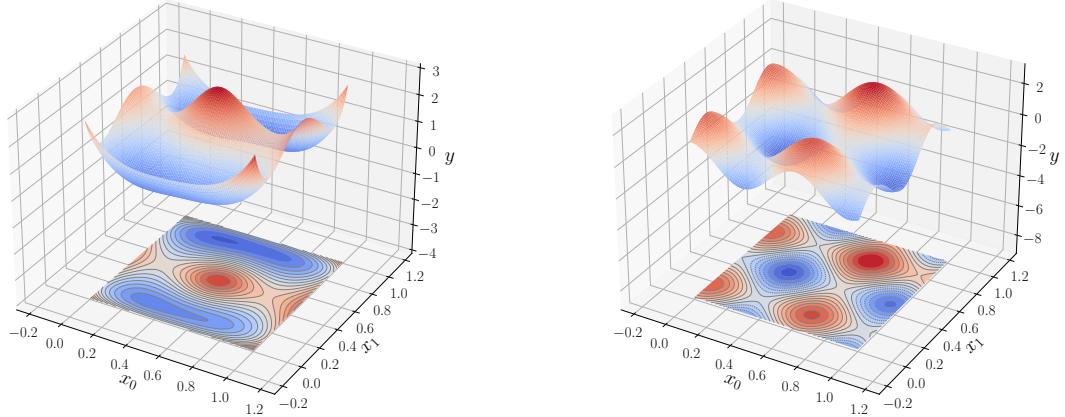


Figure 5.3 Left: $f_1(\mathbf{x})$ (test-case 1); right: $f_2(\mathbf{x})$ (test-case 2); $\mathbf{x} \in \mathcal{X} = [0, 1]^2$.

Test-case 3. The third function is the so-called “gSobol” function, defined over $\mathcal{X} = [0, 1]^8$ by

$$f_3(\mathbf{x}) = \prod_{i=1}^8 \frac{|4x_i - 2| + a_i}{1 + a_i}, \quad a_i = i^2.$$

This parametric function is very versatile as both the dimension of its input space and the coefficients a_i can be freely chosen. The sensitivity to input variables is determined by the a_i : the larger a_i is, the less f is sensitive to x_i . Larger training sets are considered for this test-case: $m \in \{15, 30, 100\}$.

5.4.2 Results and analysis

The numerical results obtained in this section are presented in Figures 5.4, 5.5, and 5.6. Each figure corresponds to one of the test-cases and gathers three sub-figures, corresponding to test sets with sizes m yielding poor (left), reasonably good (centre) or very good (right) fittings.

The baseline value of Q_{MC}^2 , calculated with 10^6 Monte-Carlo points, is indicated by the black diamonds (the black horizontal lines). Assuming that the error of Q_{MC}^2 is much smaller than the errors of all other estimators, and compare the distinct methods through their ability to approximate Q_{MC}^2 . For each sequence of nested test-sets ($n \in \{4, \dots, 50\}$), the observed values of \widehat{Q}_n^2 (equation Eq. (5.2)) and Q_{n*}^2 (equation Eq. (5.8)), are plotted as the solid and dashed lines, respectively.

The figures also show the value Q_{LOO}^2 obtained by Leave-One-Out (LOO) cross validation, which is indicated at the left of each figure by a red diamond (values smaller than 0.25 are not shown). Note that, contrarily to the other methods considered, for LOO the test set is not disjoint from the training set, and thus the method does not satisfy the conditions set in the Introduction. As the complete model-fitting procedure is repeated for each training sample

of size $m - 1$, including the maximum-likelihood estimation of the correlation lengths of the Matérn kernel, the closed-form expressions of Dubrule (1983) cannot be used, making the computations rather intensive. The three figures show, and as expected, that the Q_{LOO}^2 tends to under-estimate Q_{ideal}^2 : by construction of the training set, LOO cross validation relies on model predictions at points $\mathbf{x}^{(i)}$ far from the other $m - 1$ design points used to build the model, and thus tends to systematically overestimate the prediction error at $\mathbf{x}^{(i)}$. The underestimation of Q_{ideal}^2 can be particularly severe when m is small, the training set being then necessarily sparse; see Figure 5.4 where $Q_{LOO}^2 < 0.3$ for $m = 5$ and 15.

Let us first concentrate on the non-weighted estimators (solid curves). The two MMD-based constructions, support points (in orange) and kernel herding (in blue), generally produce better validation designs than FSSF (green curves), leading to values of \widehat{Q}_n^2 that approach Q_{ideal}^2 quicker as n increases. This is particularly noticeable for “good” and “very good” models (central and rightmost panels of all three figures). This supports the idea that test sets should complement the training set \mathbf{X}_m by populating the holes it leaves in \mathcal{X} while at the same time be able to mimic the target distribution μ , this second objective being more difficult to achieve for FSSF than for the MMD-based constructions.

Comparison of the two MMD based estimators reveals that support points tend to under-estimate ISE, leading to an over-confident assessment of the model predictivity, while kernel herding displays the expected behavior, with a negative bias that decreases with n . The reason for the positive bias of estimates based on support points designs is not fully understood, but may be linked to the fact that support points tend to place validation points at “mid-range” from the designs (and not at the furthest points like FSSF or kernel herding), see central and rightmost panels in Figure 5.1, and thus residuals at these points are themselves already better representatives of the local average errors.

Let us consider now the impact of the GP-based weighting of the residuals when estimating Q^2 (by Q_{n*}^2), which is related to the relative training-set/validation-set geometry (the manner in which the two designs are entangled in ambient space). The improvement resulting of applying residual weighting is apparent on all panels of the three figures, the dashed curves lying closer to Q_{ideal}^2 than their solid counterparts; see in particular kernel herding (blue curve) in Figure 5.4 and FSSF (green curve) in Figure 5.5. Unexpectedly, the estimators based on support points seem to be rather insensitive to residual weighting, the dashed and solid orange curves being most of the time close to each other (and in any case, much closer than the green and blue ones). While the reason for this behavior deserves a deeper study, the fact that the support point designs – see Figure 5.1 – sample in a better manner the range of possible training-to-validation distances, being in some sense less space-filling than both FSSF and kernel herding, is again a plausible explanation for this weaker sensitivity to residual weighting.

Consider now comparison of the behavior across test-cases. Setting aside the strikingly singular situation of test-case 2, for which kernel herding displays a pathological (bad) behavior for the “very good” model, and all methods present an overall good behavior, the details of the

tested function do not seem to play an important role concerning the relative merits of the estimators and validation designs.

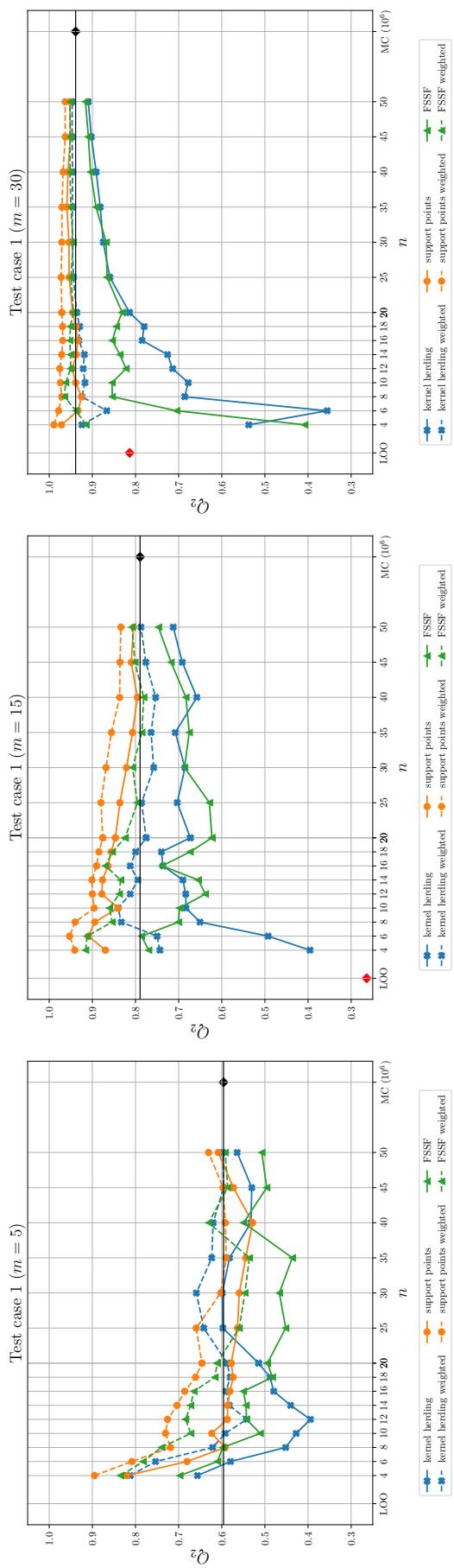


Figure 5.4 Test-case 1: predictivity assessment of a poor (left), good (center) and very good (right) model with kernel herding, support points and FSSF test sets.

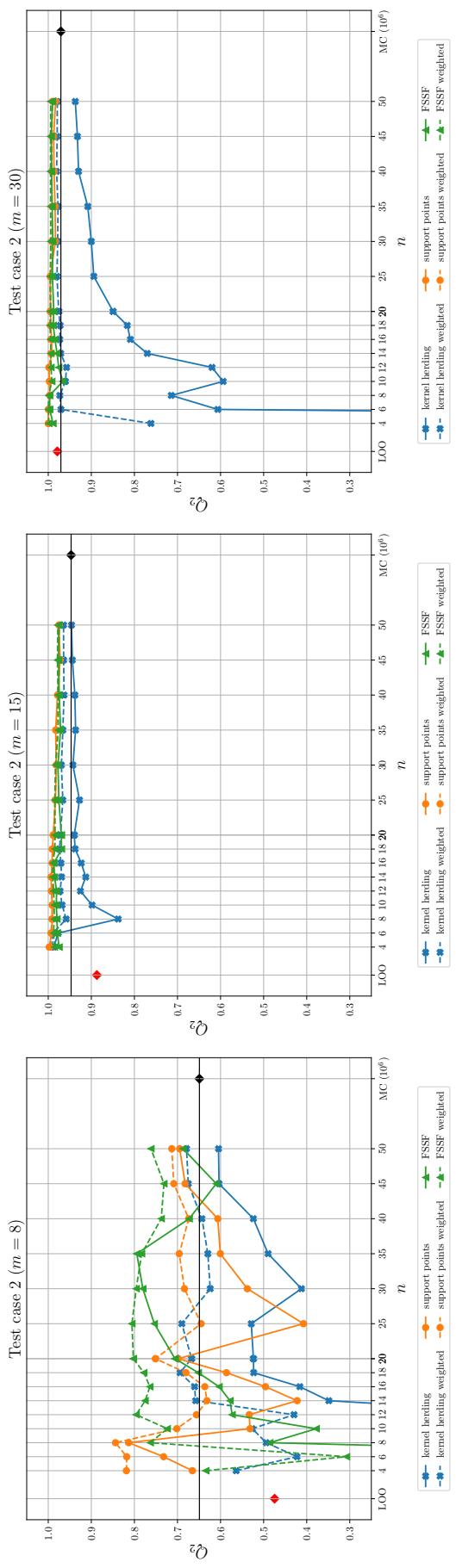
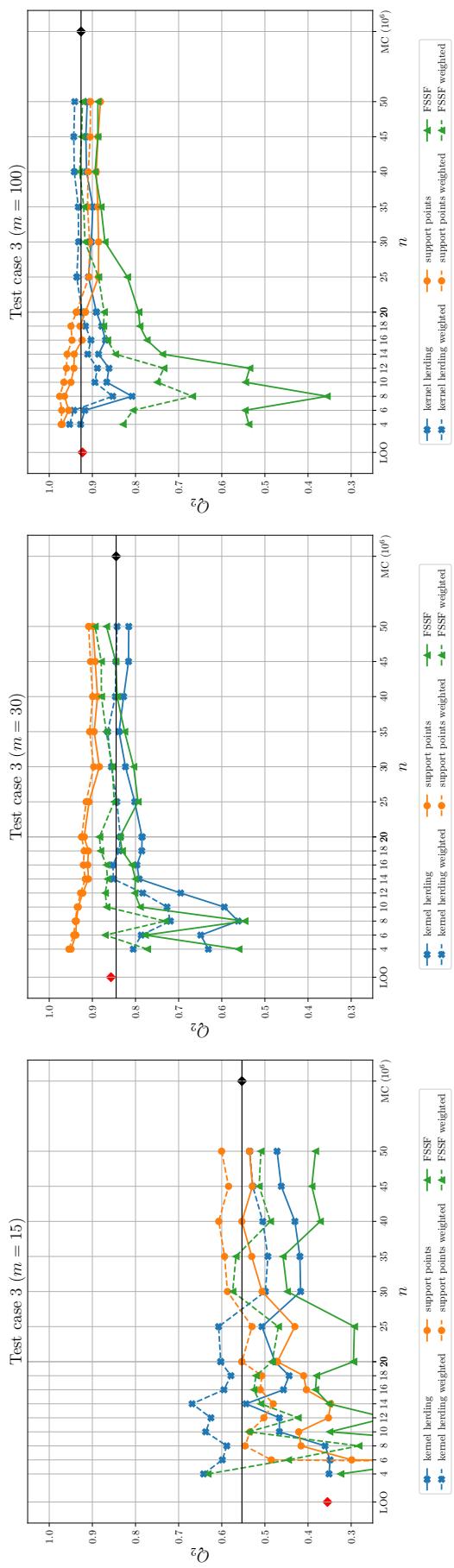


Figure 5.5 Test-case 2: predictivity assessment of a poor (left), good (center) and very good (right) model with kernel herding, support points and FSSF test sets.



Let us finally observe how the methods behave for models of distinct quality (m leading to poor, good or very good models), comparing the three panels in each figure. On the left panels, m is too small for the model η_m to be accurate, and all methods and test-set sizes are able to detect this. For models of practical interest (good and very good), the test sets generated with support points and kernel herding allow a reasonably accurate estimation of Q^2 with a few points. Note, incidentally, that except for test-case 2 (where the interplay with a non-uniform measure μ complicates the analysis), it is in general easier to estimate the quality of the very good model (right-most panel) than that of the good model (central panel), indicating that the expected complexity (the entropy) of the residual process should be a key factor determining how large the validation set must be. In particular, it may be that larger values of m allow for smaller values of n .

5.5 Numerical results II: splitting a dataset into a training set and a test set

This section illustrates the performance of the different designs and estimators considered in this chapter when applied in the context of an industrial application, to split a given dataset of size N into training and test sets, with m and n points respectively, $m + n = N$. In contrast with [Joseph and Vakayil \(2022\)](#), the observations $y(\mathbf{x}^{(i)})$, $i = 1, \dots, N$, are not used in the splitting mechanism, meaning that it can be performed before the observations are collected and that there cannot be any selection bias related to observations (indeed, the use of observation values in a MMD-based splitting criterion may favor the allocation of the most different observations to different sets, training versus validation).

A ML model is fitted to the training data, and the data collected on the test-set are used to assess the predictivity of the model. The influence of the ratio $r_n = n/N = 1 - m/N$ on the quality assessment is investigated. Random Cross-Validation (RCV) is also considered, where n points are chosen at random among the N points of the dataset: for each n , there are $\binom{N}{n}$ possible choices, and $R = 1000$ designs were randomly selected among them. A model is fitted on the m complementary points ($m = N - n$), which yields an empirical distribution of Q^2 values for each ratio n/N considered.

5.5.1 Industrial test-case CATHARE

The test-case corresponds to the computer code CATHARE2 (for “Code Avancé de ThermoHydraulique pour les Accidents de Réacteurs à Eau”), which models the thermal-hydraulic behavior inside nuclear pressurized water reactors [Geffraye et al. \(2011\)](#). The studied scenario simulates a hypothetical large-break loss of primary coolant accident for which the output of interest is the peak cladding temperature [de Crécy et al. \(2008\)](#); [Iooss et al. \(2010\)](#). The complexity of this application lies in the large run-time of the computer model (of the order of twenty minutes) and in the high dimension of the input space: the model involves 53 input parameters z_i , cor-

responding mostly to constants of physical laws, but also coding initial conditions, material properties and geometrical modeling. The z_i were independently sampled according to normal or log-normal distributions (see axes histograms in Fig. 5.7 corresponding to 10 inputs). These characteristics make this test-case challenging in terms of construction of a surrogate model and validation of its predictivity.

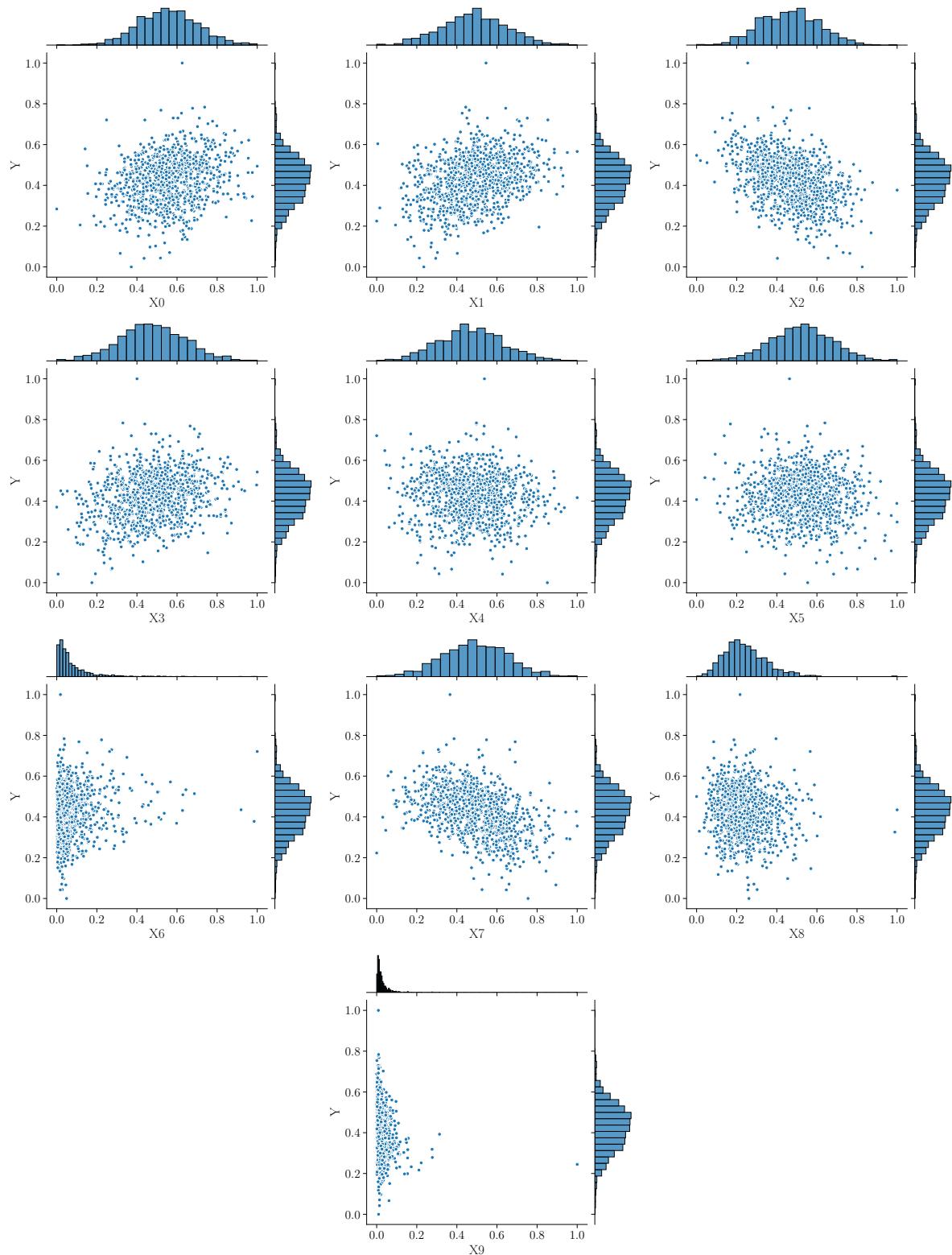
We have access to an existing Monte Carlo sample Z_N of $N = 1\,000$ points in \mathbb{R}^{53} , that corresponds to 53 independent random input configurations; see Iooss et al. (2010) for details. The output of the CATHARE2 code at these N points is also available. To reduce the dimensionality of this dataset, we first performed a sensitivity analysis Da Veiga et al. (2021) to eliminate inputs that do not impact the output significantly. This dimension-reduction step relies on the Hilbert-Schmidt Independence Criterion (HSIC), which is known as a powerful tool to perform input screening from a single sample of inputs and output values without reference to any specific ML regression model ???. HSIC-based statistical tests and their associated p -values are used to identify (with a 5%-threshold) inputs on which the output is significantly dependent (and therefore, also those of little influence). They were successfully applied to similar datasets from thermal-hydraulic applications in ???. The screened dataset only includes 10 influential inputs, over which the candidate set X_N used for the construction of the test-set X_n (and therefore of the complementary training set X_{N-n}) is defined. An input-output scatter plot is presented in Figure 5.7, showing that indeed the retained factors are correlated with the code output. The marginal distributions are shown as histograms along to the axes of the plots.

To include RCV in the methods to be compared, we need to be able to construct many (here, $R = 1\,000$) different models η_m for each considered design size m . Since Gaussian Process regression proved to be too expensive for this purpose, we settled for the comparatively cheaper Partial Least Squares (PLS) method ?, which retains acceptable accuracy. For each given training set, the model obtained is a sum of monomials in the 10 input variables. Note that models constructed with different training sets may involve different monomials and have different numbers of monomial terms.

5.5.2 Benchmark results and analysis

Figure 5.8 compares various ways of extracting an n -point test set from an N -point dataset to estimate model predictivity, for different splitting ratios $n/N \in \{0.1, 0.15, 0.2, \dots, 0.9\}$.

Consider RCV first. For each value of $r_n = n/N$, the empirical distribution of Q_{RCV}^2 obtained from $R = 10^3$ random splittings of X_N into $X_m \cup X_n$ is summarized by a boxplot. Depending on r_n , we can roughly distinguish three behaviors. For $0.1 \leq r_n \lesssim 0.3$ the distribution is bi-modal, with the lower mode corresponding to unlucky test-set selections leading to poor performance evaluations. When $0.3 \lesssim n/N \lesssim 0.7$, the distribution looks uni-modal, revealing a more stable performance evaluation. Note that this is (partly) in line with the recommendations discussed in section ???. For $r_n \gtrsim 0.7$, the variance of the distribution increases with r_n : many unlucky training sets lead to poor models. Note that the median of the empirical distribution slowly

Figure 5.7 Test-case CATHARE: inputs output scatter plots ($N = 10^3$)

decreases as r_n increases, which is consistent with the intuition that the model predictivity should decrease when the size of the training set decreases.

For completeness, we also show by a red diamond on the left of Figure 5.8 the value of Q_{LOO}^2 computed by LOO cross-validation. In principle, being computed using the entire dataset, this value should establish an upper bound on the quality of models computed with smaller training sets. This is indeed the case for small training sets (rightmost values in the figure), for which the predictivity estimated by LOO is above the majority of the predictivity indexes calculated. But at the same time, we know that LOO cross-validation tends to overestimate the errors, which explains the higher predictivity estimated by some other methods when $m = N - n$ is large enough.

Compare now the behavior of the two MMD-based algorithms of Section ??, \widehat{Q}_n^2 (unweighted) and Q_{n*}^2 (weighted) are plotted using solid and dashed lines, respectively, for both kernel herding (in blue) and support points (in orange). FSSF test-sets are not considered, as the application of an iso-probabilistic transformation imposes knowledge of the input distribution, which is not known for this example. Compare first the unweighted versions of the two MMD-based estimators. For small values of the ratio r_n , $0.1 \lesssim r_n \lesssim 0.45$, the relative behavior of support points and kernel herding coincides with what we observed in the previous section, support points (solid orange line) estimating a better performance than kernel herding (solid blue line), which, moreover, is close to the median of the empirical distribution of Q_{RCV}^2 . However, for $r_n \geq 0.5$, the dominance is reversed, support points estimating a worse performance than kernel herding.

As r_n increases up to $r_n \lesssim 0.7$ the solid orange and blue curves crossover, and it is now \widehat{Q}_n^2 for kernel herding that approximates the RCV empirical median, while the value obtained with support points underestimates the predictivity index. Also, note that for (irrealistic) very large values of r_n both support points and kernel herding estimate lower Q^2 values, which are smaller than the median of the RCV estimates.

Let us now focus on the effect of residual weighting, i.e., in estimators Q_{n*}^2 which use the weights computed by the method of Section 5.2.2, shown in dashed lines in Figure 5.8. First, note that while for kernel herding weighting leads, as in the previous section, to higher estimates of the predictivity (compare solid and dashed blue lines), this is not the case for support points (solid and dashed orange curves), which, for small split ratios, produces smaller estimates when weighting is introduced. In the large r_n region, the behavior is consistent with what we saw previously, weighting inducing an increase of the estimated predictivity. It is remarkable – and rather surprising – that Q_{n*}^2 for support points (the dashed orange line) does not present the discontinuity of the uncorrected curve.

The sum $\sum_{i=1}^n w_i^*$ of the optimal weights of support points and kernel herding Eq. (5.7) is shown in Figure 5.9 (orange and blue curves, respectively). The slow increase with n/N of the sum of kernel-herding weights (blue line) is consistent with the increase of the volume of the input region around each validation point when the size of the training set decreases. The behavior of the sum of weights is more difficult to interpret for support points (orange line)

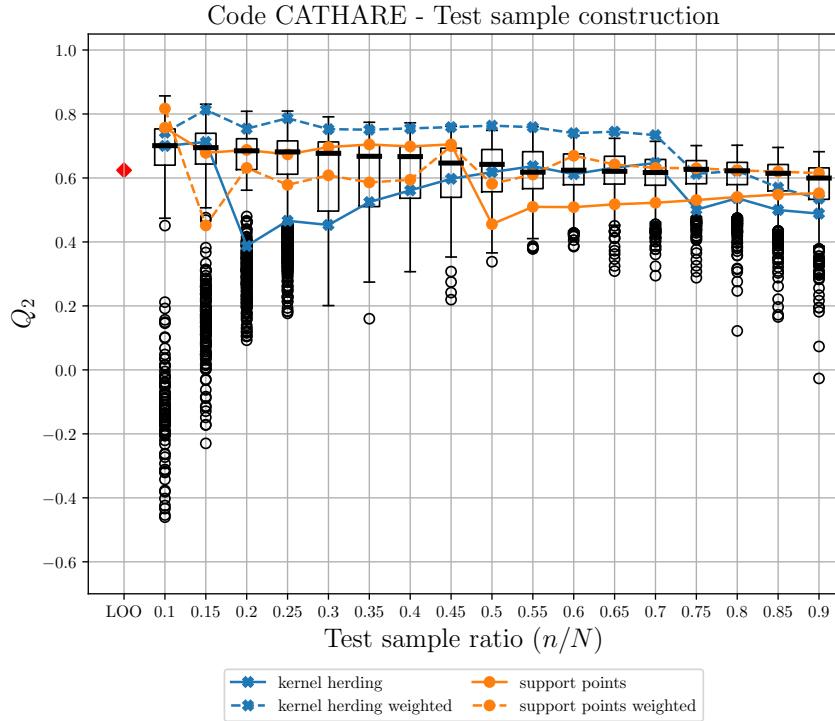


Figure 5.8 Test-case CATHARE: estimated Q^2 . The box-plots are for random cross-validation, the red diamond (left) is for Q^2_{LOO} .

but is consistent with the behavior of Q^2_{n*} on Figure 5.8. Note that the energy-distance kernel Eq. (5.11) used for support points cannot be used for the weighting method of Section 5.2.2 as K_E is not positive definite but only conditionally positive definite. A full understanding of the observed curves would require a deeper analysis of the geometric characteristics of the designs generated by the two MMD methods, in particular of their interleaving with the training designs, which is not compatible with the space constraints of this manuscript.

While a number of unanswered points remain, in particular how deeply the behaviors observed may be affected by the poor predictivity resulting from the chosen PLS modeling methodology, the example presented in this section shows that the construction of test sets via MMD minimization and estimation of the predictivity index using the weighted estimator Q^2_{n*} is promising as an efficient alternative to RCV: at a much lower computational cost, it builds performance estimates based on independent data the model developers may not have access to. Moreover, kernel herding proved, in the examples studied in this manuscript, to be a more reliable option for designing the test set, exhibiting a behavior that is consistent with what is expected, and very good estimation quality when the residuals over the design points are appropriately weighted.

5.6 Conclusion

Our study shows that ideas and tools from the design of experiments framework can be transposed to the problem of test-set selection. This chapter explored approaches based on support

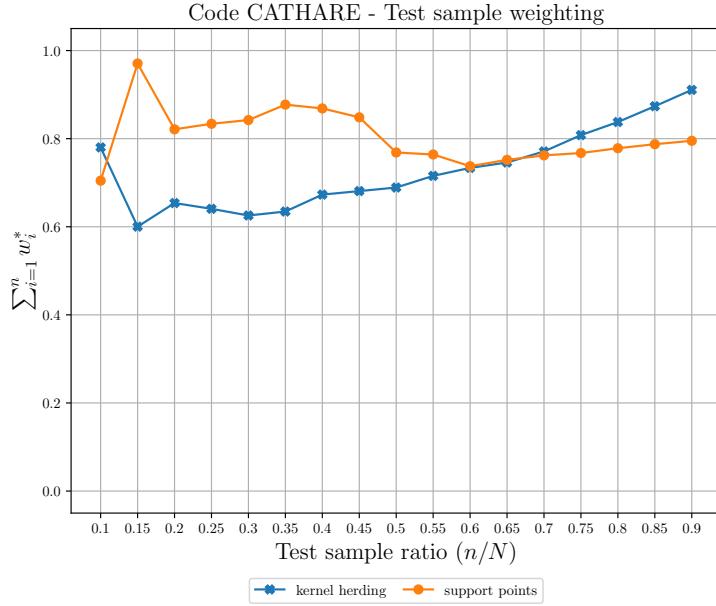


Figure 5.9 Test-case CATHARE: sum of the weights Eq. (5.7).

points, kernel herding and FSSF, considering the incremental construction of a test set (*i*) either as a particular space-filling design problem, where design points should populate the holes left in the design space by the training set, or (*i*) from the point of view of partitioning a given dataset into a training set and a test set.

A numerical benchmark has been performed for a panel of test-cases of different dimensions and complexity. Additionally to the usual predictivity coefficient, a new weighted metric (see Pronzato and Rendas (2021b)) has been proposed and shown to improve assessment of the predictivity of a given model for a given test set.

This weighting procedure appears very efficient for interpolators, like Gaussian process regression models, as it corrects the bias when the points in the test set used to predict the errors are far from the training points. For the first three test-cases (Section ??), pairing one iterative design method with the weight-corrected estimator of the predictivity coefficient Q^2 shows promising results as the estimated Q^2 characteristic is close to the true one even for test-sets of moderate size.

Weighting can also be applied to models that do not interpolate the training data. For the industrial test-case of Section ??, the true Q^2 value is unknown, but the weight-corrected estimation Q_{n*}^2 of Q^2 is close to the value estimated by Leave-One-Out cross validation and to the median of the empirical distribution of Q^2 values obtained by random k -fold cross-validation. At the same time, estimation by Q_{n*}^2 involves a much smaller computational cost than cross-validation methods, and uses a dataset fully independent of the one used to construct the model.

To each of the design methods considered to select a test set a downside can be attached. FSSF requires knowledge of the input distribution to be able to apply an iso-probabilistic transformation if necessary; it tends to select many points along the boundary of the candidate

set considered. Support points require the computation of the $N(N - 1)/2$ distances between all pairs of candidate points, which implies important memory requirements for large N ; the energy-distance kernel on which the method relies cannot be used for the weighting procedure. Finally, the efficient implementation of kernel herding relies on analytical expressions for the potentials P_μ , see Appendices A and B, which are available for particular distributions (like the uniform and the normal) and kernels (like Matérn) only. The great freedom in the choice of the kernel K gives a lot of flexibility, but at the same time implies that some non-trivial decisions have to be made; also, the internal parameters of K , such as its correlation lengths, must to be specified. Future work should go beyond empirical rules of thumb and study the influence of these choices.

We have only computed numerical tests with independent inputs. Kernel herding and support points are both well suited for probability measures not being equal to the product of their marginals, which is a frequent case in real datasets. We have also only considered incremental constructions, as they allow to stop the validation procedure as soon as the estimation of the model predictivity is deemed sufficiently accurate, but it is also possible to select several points at once, using support points [Mak and Joseph \(2018\)](#), or MMD minimization in general ?.

Further developments around this work could be as follows. Firstly, the incremental construction of a test set could be coupled with the definition of an appropriate stopping rule, in order to decide when it is necessary to continue improving the model (possibly by supplementing the initial design with the test set, which seems well suited to this). The MMD $d_{\bar{K}|_m}(\zeta_n^*, \mu)$ of Section 5.2.2 could play an important role in the derivation of such a rule. Secondly, the approach presented gives equal importance to all the d inputs. However, it seems that inputs with a negligible influence on the output should receive less attention when selecting a test set. A preliminary screening step that identifies the important inputs would allow the test-set selection algorithm to be applied on these variables only. For example, when a $\mathbf{X}_N \subset \mathbb{R}^d$ dataset is to be partitioned into $\mathbf{X}_m \cup \mathbf{X}_n$, one could use only $d' < d$ components to define the partition, but still use all d components to build the model and estimate its (weighted) Q^2 . Note, however, that this would imply a slight violation of the conditions mentioned in introduction, as it renders the test set dependent on the function observations.

Finally, in some cases the probability measure μ is known up to a normalizing constant. The use of a Stein kernel then makes the potential $P_{K,\mu}$ identically zero ??, which would facilitate the application of kernel herding. Also, more complex problems involve functional inputs, like temporal signals or images, or categorical variables; the application of the methods presented to kernels specifically designed for such situations raises challenging issues.

PART III:

CONTRIBUTIONS TO RARE EVENT ESTIMATION

La résignation est un suicide quotidien.

H. BALZAC

Nonparametric rare event estimation

6.1	Introduction	172
6.1.1	Background	173
6.2	Bernstein adaptive nonparametric conditional sampling	174
6.2.1	Empirical Bernstein copula	174
6.2.2	Bernstein adaptive nonparametric conditional sampling algorithm . .	176
6.3	Numerical experiments	177
6.3.1	Results analysis	179
6.4	Application to wind turbine fatigue reliability	180
6.5	Conclusion	180

6.1 Introduction

Reliability analysis of a system is often associated with rare event probability estimation. Considering that the system's performance is modeled by a deterministic scalar function $g : \mathcal{D}_x \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$, called *limit-state function* and a critical threshold on the system's output $y_{\text{th}} \in \mathbb{R}$, one can define the *failure domain* as $\mathcal{F}_x := \{\mathbf{x} \in \mathcal{D}_x | g(\mathbf{x}) \leq y_{\text{th}}\}$. Uncertain inputs are represented by a continuous random vector $\mathbf{X} \in \mathcal{D}_x$ assumed to be distributed according to its joint probability density function (PDF) f_X . In this context, uncertainty propagation consists in composing the random vector \mathbf{X} by the function g to get an output variable of interest $Y = g(\mathbf{X}) \in \mathbb{R}$. A usual risk measure in reliability analysis is the *failure probability*, denoted by p_f , and defined as the probability that the system exceeds the threshold y_{th} : Rare event problems are usually solved in the so-called *standard normal space* after applying an “iso-probabilistic transformation” which can be either the Rosenblatt or the generalized Nataf one (Lebrun, 2013). Additionally, the limit-state function g can be viewed as an input-output “black-box” model which can be costly to evaluate (e.g., a complex numerical model), making the failure probability estimation nontrivial. When the limit-state function is a costly computer model, one can build a surrogate model and use specific active learning methods (see, e.g., Moustapha et al. (2022)). However, using surrogate models is not always possible for practical engineering applications as they might introduce another level of approximation, which can be prohibitive from safety auditing. Moreover, their validation as well as their behavior with respect to large input dimension case make also their use quite complex (see, e.g., (Marrel et al., 2022)).

Going back to the rare event estimation literature, one can consider two major types of techniques for failure probability calculation (Morio and Balesdent, 2015): (i) Geometric approaches, such as the *first-/second-order reliability method* (FORM/SORM) whose aim is to approximate the limit-state function by a first-/second-order Taylor expansion at the most probable failure point; (ii) Simulation-based techniques such as the *crude Monte Carlo* method. Unfortunately, FORM/SORM methods do not provide a lot of statistical information as they are purely geometric approaches. Meanwhile, estimating a rare event probability by crude Monte Carlo becomes rapidly intractable. To overcome this limit, advanced simulation techniques have been developed: among others, one can mention several “variance reduction methods” such as the non-adaptive and adaptive versions of the *Importance Sampling* (Rubinstein and Kroese, 2008) (either parametric, using the Cross-Entropy method ?, or nonparametric Morio (2011)) and splitting techniques (Cérou et al., 2012) such as the *Subset Simulation* (SS) Au and Beck (2001). In these techniques, the idea is to write the rare event p_f as a product of larger conditional probabilities, each one of them being easier to estimate. To generate intermediary conditional samples, this method uses Markov chain Monte Carlo (MCMC) sampling, which presents numerous versions (Papaioannou et al., 2015). However, MCMC algorithms are known to be highly tunable algorithms which produce non-i.i.d. samples, which consequently, cannot be used for direct statistical estimation (e.g., failure probability or sensitivity indices (Da Veiga et al., 2021)).

The present work proposes a new rare event estimation method, adopting the same sequential structure as SS while using a strictly different sampling mechanism to generate conditional samples. This method intends to fit the intermediary conditional distributions with a nonparametric tool called the *Empirical Bernstein Copula*. Contrarily to SS, the proposed method named “Bernstein adaptive nonparametric conditional sampling” (BANCS), generates i.i.d. samples of the intermediary conditional distributions. For instance, a practical use of such i.i.d. samples can be to estimate dedicated reliability-oriented sensitivity indices (see, e.g., Chabridon et al. (2021); Marrel and Chabridon (2021)).

In this paper, Section 2 will recall the methodology of subset sampling and probabilistic modeling. Then, Section 3 will introduce the BANCS method for rare event estimation. Section 4 will apply this method to three toy-cases and analyze the results with respect to SS performances. Then, the last section present some conclusions and research perspectives.

6.1.1 Background

Subset sampling

Subset sampling splits the failure event \mathcal{F}_x into an intersection of $k_\#$ intermediary events $\mathcal{F}_x = \cap_{k=1}^{k_\#} \mathcal{F}_{[k]}$. Each are nested such that $\mathcal{F}_{[1]} \supset \dots \supset \mathcal{F}_{[k_\#]} = \mathcal{F}_x$. The failure probability is then expressed as a product of conditional probabilities:

$$p_f = \mathbb{P}(\mathcal{F}_x) = \mathbb{P}(\cap_{k=1}^{k_\#} \mathcal{F}_{[k]}) = \prod_{k=1}^{k_\#} \mathbb{P}(\mathcal{F}_{[k]} | \mathcal{F}_{[k-1]}). \quad (6.1)$$

From a practical point of view, the analyst tunes the algorithm by setting the intermediary probabilities $\mathbb{P}(\mathcal{F}_{[k]} | \mathcal{F}_{[k-1]}) = p_0, \forall k \in \{1, \dots, k_\#\}$. Then, the corresponding quantiles $q_{[1]}^{p_0} > \dots > q_{[k_\#]}^{p_0}$ are estimated for each conditional subset samples $X_{[k],N}$ of size N . Note that the initial quantile is estimated by crude Monte Carlo sampling on the input PDF f_x . Following conditional subset samples are generated by MCMC sampling of $f_x(x | \mathcal{F}_{[k-1]})$, using as seeds initialisation points the $n = Np_0$ samples given by $A_{[k],n} = \{X_{[k-1]}^{(j)} \subset X_{[k-1],N} | g(X_{[k-1]}^{(j)}) > \hat{q}_{[k-1]}^{\alpha}\}_{j=1}^n$. This process is repeated until an intermediary quantile exceeds the threshold: $\hat{q}_{[k_\#]}^{p_0} < y_{\text{th}}$. Finally, the failure probability is estimated by:

$$p_f \approx \hat{p}_f^{\text{SS}} = p_0^{k_\# - 1} \frac{1}{N} \sum_{j=1}^N \mathbb{1}_{\{g(x) \leq y_{\text{th}}\}}(X_{[k_\#],N}^{(j)}). \quad (6.2)$$

In practice, the subset sample size should be large enough to properly estimate intermediary quantiles, which leads Au and Beck (2001) to recommend setting $p_0 = 0.1$. SS efficiency depends on the proper choice and tuning of the MCMC algorithm (Papaioannou et al., 2015). Our work uses the SS implementation from OpenTURNS¹ (Baudin et al., 2017) which integrates a

¹<https://openturns.github.io/www/index.html>

component-wise Metropolis-Hastings algorithm. As an alternative to generating samples on a conditional distribution by MCMC, one could try to fit this conditional distribution.

Multivariate modeling using copulas

The Sklar theorem ([Joe, 1997](#)) affirms that the multivariate distribution of any random vector $\mathbf{X} \in \mathbb{R}^d$ can be broken down into two objects:

1. A set of univariate marginal distributions to describe the behavior of the individual variables;
2. A function describing the dependence structure between all variables, called a copula.

This theorem states that considering a random vector $\mathbf{X} \in \mathbb{R}^d$, with its distribution F and its marginals $\{F_i\}_{i=1}^d$, there exists a copula $C : [0, 1]^d \rightarrow [0, 1]$, such that:

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_p(x_d)). \quad (6.3)$$

It allows us to divide the problem of fitting a joint distribution into two independent problems: fitting the marginals and fitting the copula. Note that when the joint distribution is continuous, this copula is unique. Provided a dataset, this framework allows to combine a parametric (or nonparametric) fit of marginals with a parametric (or nonparametric) fit of the copula. When the distribution's dimension is higher than two, one can perform a parametric fit using vine copulas ([Joe and Kurowicka, 2011](#)), implying the choice of multiple types of parametric copulas. Otherwise, nonparametric fit by multivariate kernel density estimation (KDE) presents a computational burden as soon as the dimension increases ([Chabridon et al., 2021](#)). Since univariate marginals are usually well-fitted with nonparametric tools (e.g., KDE), let us introduce an effective nonparametric method for copula fitting.

6.2 Bernstein adaptive nonparametric conditional sampling

6.2.1 Empirical Bernstein copula

Copulas are continuous and bounded functions defined on a compact set (the unit hypercube). Bernstein polynomials allow to uniformly approximate as closely as desired any continuous and real-valued function defined on a compact set (Weierstrass approximation theorem). Therefore, they are good candidates to approximate unknown copulas. This concept was introduced as *empirical Bernstein copula* (EBC) by [Sancetta and Satchell \(2004\)](#) for applications in economics and risk management. Later on, [Segers et al. \(2017\)](#) offered further asymptotic studies. Formally, the multivariate Bernstein polynomial for a function $C : [0, 1]^d \rightarrow \mathbb{R}$ on a grid over the unit

hypercube $G := \left\{ \frac{0}{m_1}, \dots, \frac{m_1}{m_1} \right\} \times \dots \times \left\{ \frac{0}{m_d}, \dots, \frac{m_d}{m_d} \right\}$, $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$, writes:

$$B_{\mathbf{m}}(C)(\mathbf{u}) := \sum_{t_1=0}^{m_1} \dots \sum_{t_d=0}^{m_d} C\left(\frac{t_1}{m_1}, \dots, \frac{t_d}{m_d}\right) \prod_{j=1}^d P_{m_j, t_j}(u_j), \quad (6.4)$$

with $\mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d$, and the Bernstein polynomial $P_{m,t}(u) := \frac{t!}{m!(t-m)!} u^m (1-u)^{t-m}$. Notice how the grid definition implies the polynomial's order. When C is a copula, then $B_{\mathbf{m}}(C)$ is called “Bernstein copula”. Therefore, the empirical Bernstein copula is an application of the Bernstein polynomial in Eq. (6.4) to the so-called “empirical copula”.

In practice, considering a sample $\mathbf{X}_n = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\} \in \mathbb{R}^{np}$ and the associated ranked sample $\mathbf{R}_n = \{\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(n)}\}$, the corresponding empirical copula writes:

$$C_n(\mathbf{u}) := \frac{1}{n} \sum_{i=0}^n \prod_{j=1}^p \mathbb{1} \left\{ \frac{r_j^{(i)}}{n} \leq u_j \right\}, \quad (6.5)$$

with $\mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d$. In the following, the polynomial order is set as equal in each dimension: $\{m_i = m\}_{j=1}^d$. Theoretically, the tuning parameter can be optimized to minimize an “Mean Integrated Squared Error” (MISE), leading to a bias-variance tradeoff. Formally, the MISE of the empirical Bernstein copula $B_{\mathbf{m}}(C_n)$ is defined as follows:

$$\mathbb{E}[\|B_{\mathbf{m}}(C_n) - C\|_2^2] = \mathbb{E} \left[\int_{\mathbb{R}^d} (B_{\mathbf{m}}(C_n)(\mathbf{u}) - C(\mathbf{u}))^2 \right]. \quad (6.6)$$

Then, [Sancetta and Satchell \(2004\)](#) prove in their Theorem 3 that:

- $B_{\mathbf{m}}(C_n)(\mathbf{u}) \rightarrow C(\mathbf{u})$ for any $u_j \in]0, 1[$ if $\frac{m^{d/2}}{n} \rightarrow 0$, when $m, n \rightarrow \infty$.
- The optimal order of the polynomial in terms of MISE is: $m \lesssim m_{\text{IMSE}} = n^{2/(d+4)}$, $\forall u_j \in]0, 1[$. The sign \lesssim means “less than or approximately”.

Let us remark that in the special case $m = n$, also called the “Beta copula” in [Segers et al. \(2017\)](#), the bias is very small while the variance gets large. To illustrate the previous theorem, [Lasserre \(2022\)](#) represents the evolution of the m_{IMSE} for different dimensions and sample sizes (see Fig. 6.1). In high dimension, the values of m_{IMSE} tend towards one, which is equivalent to the independent copula. Therefore, high-dimensional problems should be divided into a product of smaller problems on which the EBC is tractable. Provided a large enough learning set \mathbf{X}_n , KDE fitting of marginals combined with EBC fitting of the copula delivers good results even on complex dependence structures. Moreover, EBC provides an explicit expression, making a Monte Carlo generation of i.i.d. samples simple. In the following, this nonparametric tool is used to fit the intermediary conditional distributions present in subset sampling.

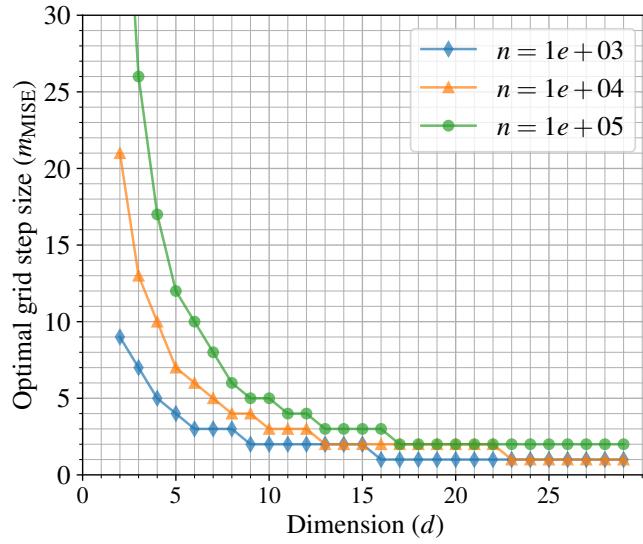


Figure 6.1 Evolution of m_{IMSE} for different dimensions and sample sizes.

6.2.2 Bernstein adaptive nonparametric conditional sampling algorithm

This new method reuses the main idea from SS while employing a different approach to generate conditional samples. Instead of using MCMC sampling, the conditional distribution is firstly fitted by a nonparametric procedure, before sampling on this nonparametric model. As described in Algorithm 1, conditional sampling is done on a distribution composed by merging marginals $\{\widehat{F}_i\}_{i=1}^d$ fitted by KDE, with a copula $B_m(C_n)$ fitted by EBC. Fig. 6.2 illustrate the nonparametric fit and conditional sampling in BANCS method on a two-dimensional reliability problem (later introduced as “toy-case #1”). At iteration k , after estimating the intermediary quantile $\widehat{q}_{[k]}^{p_0}$, a nonparametric model is fitted on $A_{[k+1],n}$ and used to generate the next N -sized subset sample $X_{[k+1],N}$. Note that the BANCS method does not require iso-probabilistic transform.

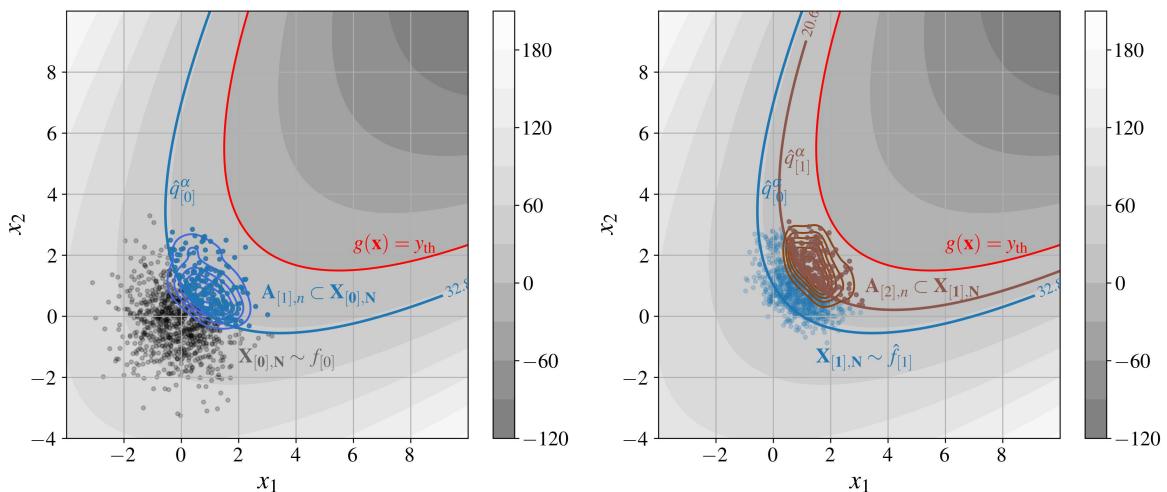


Figure 6.2 BANCS on toy-case #1: illustration of conditional sampling and nonparametric fit at the first and second iterations.

Algorithm 1 Bernstein adaptive nonparametric conditional sampling (BANCS).

» Inputs:

f_X , joint PDF of the inputs
 $g(\cdot)$, limit-state function
 $y_{\text{th}} \in \mathbb{R}$, threshold defining the failure event
 N , number of samples per iteration
 $m \in \mathbb{N}$, parameter of the EBC fitting
 $p_0 \in]0, 1[$, empirical quantile order (rarity parameter)

» Algorithm:

```

Set  $k = 0$  and  $f_{[0]} = f_X$ 
Sample  $\mathbf{X}_{[0], N} = \{\mathbf{X}_{[0]}^{(j)}\}_{j=1}^N \stackrel{\text{i.i.d.}}{\sim} f_{[0]}$ 
Evaluate  $G_{[0], N} = \{g(\mathbf{X}_{[0]}^{(j)})\}_{j=1}^N$ 
Estimate the empirical  $p_0$ -quantile  $\hat{q}_{[0]}^{p_0}$  of the set  $G_{[0], N}$ 
while  $\hat{q}_{[k]}^{p_0} > y_{\text{th}}$  do
    Subsample  $\mathbf{A}_{[k+1], n} = \{\mathbf{X}_{[k]}^{(j)} \subset \mathbf{X}_{[k], N} | g(\mathbf{X}_{[k]}^{(j)}) > \hat{q}_{[k]}^{p_0}\}_{j=1}^n$ 
    Fit marginals of the subset  $\mathbf{A}_{[k+1], n}$  by KDE  $\{\hat{F}_i\}_{i=1}^d$ 
    Fit the copula of the subset  $\mathbf{A}_{[k+1], n}$  by EBC  $B_m(C_n)$ 
    Build a CDF  $\hat{F}_{[k+1]}(\mathbf{x}) = B_m(C_n)(\hat{F}_1(x_1), \dots, \hat{F}_d(x_d))$ 
    Sample  $\mathbf{X}_{[k+1], N} = \{\mathbf{X}_{[k+1]}^{(j)}\}_{j=1}^N \stackrel{\text{i.i.d.}}{\sim} \hat{f}_{[k+1]}$ 
    Evaluate  $G_{[k+1], N} = \{g(\mathbf{X}_{[k+1]}^{(j)})\}_{j=1}^N$ 
    Estimate the empirical  $p_0$ -quantile  $\hat{q}_{[k+1]}^{p_0}$  of  $G_{[k+1], N}$ 
    Set  $k = k + 1$ 
end while
Set total iteration number  $k_{\#} = k - 1$ 
Estimate  $\hat{p}_f = (1 - p_0)^{k_{\#}} \cdot \frac{1}{N} \sum_{j=1}^N \mathbb{1}_{\{g(\mathbf{X}_{[k_{\#}]}^{(j)}) \geq y_{\text{th}}\}} (\mathbf{X}_{[k_{\#}]}^{(j)})$ 
 $\hat{p}_f$ , estimate of  $p_f$ 
```

As discussed in the previous section, EBC fitting is tuned by the Bernstein polynomial of order m , implying a bias-variance trade off. In Fig. 6.2, conditional distributions fitted by EBC (blue and brown isolines) seem to present a slight bias since they overlay the quantiles. However, reducing this bias implies decreasing the tuning parameter m , until $m = 1$, which is equivalent to an independent copula. Tools to control the goodness of fit of nonparametric conditional distributions are also available. As an example, let us consider the fitted conditional distribution at the first iteration (visible in Fig. 6.2). Its quantile-quantile plot in Fig. 6.3 shows a good fit of the two marginals by KDE. Then, the goodness of fit of copulas can be evaluated by Kendall's plot, represented in Fig. 6.4. This fit is also good, even if a slight bias is again visible.

6.3 Numerical experiments

In the following analytical numerical experiments, the intermediary probabilities were set to $p_0 = 0.1$, allowing a fair comparison with subset sampling. Then, the subset sample size is set to $N = 10^4$, in order to get a reasonable sample size $n = Np_0 = 10^3$ to perform the nonparametric fitting. EBC tuning is setup to minimize the MISE in Eq. (6.6): $m = 1 + n^{\frac{2}{d+4}}$. In order to take into account the variability of the method's results, each experiment is repeated 100 times, allowing

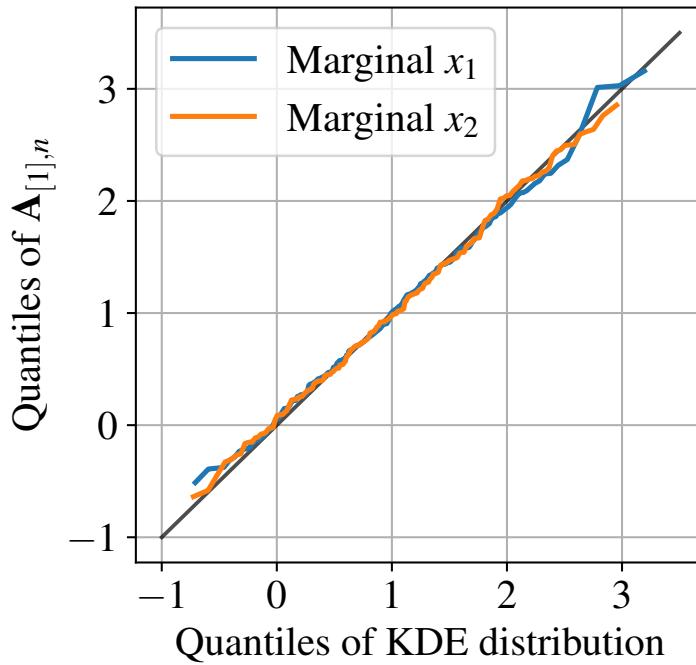


Figure 6.3 QQ-plot for KDE of marginals of the conditional distribution from Fig. 6.2.

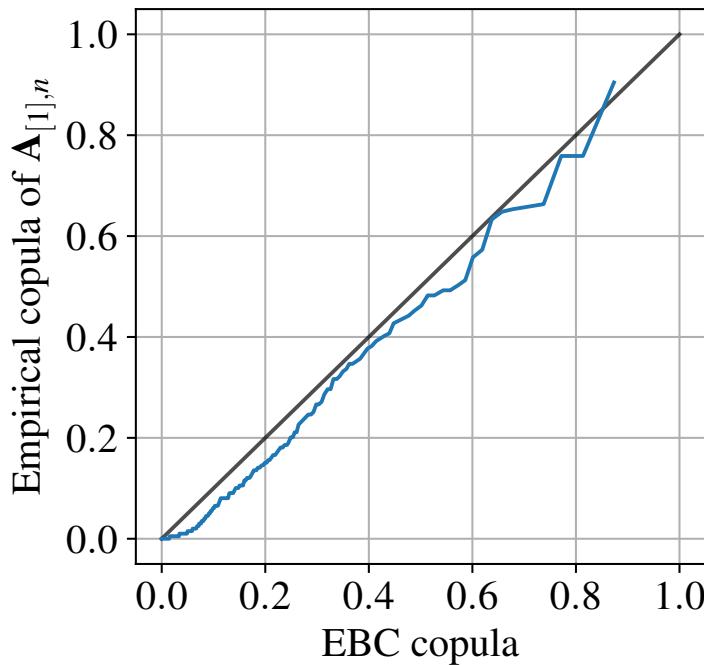


Figure 6.4 Kendall plot for EBC on the copula of a conditional distribution from Fig. 6.2.

the computation of a coefficient of variation $\widehat{\delta} = \frac{\sigma_{\widehat{P}_f}}{\mu_{\widehat{P}_f}}$. Note that an implementation of the BANCS method and the following numerical experiments are available in a Git repository².

²<https://github.com/efekhari27/icasp14>

Toy-case #1: Parabolic reliability problem Let us define the parabolic reliability problem, considering the function $g_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$:

$$g_1(\mathbf{x}) = (x_1 - x_2)^2 - 8(x_1 + x_2 - 5), \quad (6.7)$$

with the input random vector $\mathbf{X} = (X_1, X_2)$ following a standard 2-dimensional normal distribution. The reliability problem consists in evaluating: $p_{f,1} = \mathbb{P}(g_1(\mathbf{X}) \leq 0) = 1.31 \times 10^{-4}$.

Toy-case #2: Four-branch reliability problem Let us define the four-branch reliability problem (originally proposed by [Waarts \(2000\)](#)), considering the following function $g_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$:

$$g_2(\mathbf{x}) = \min \left(\begin{array}{l} 5 + 0.1(x_1 - x_2)^2 - \frac{(x_1 + x_2)}{\sqrt{2}} \\ 5 + 0.1(x_1 - x_2)^2 + \frac{(x_1 + x_2)}{\sqrt{2}} \\ (x_1 - x_2) + \frac{9}{\sqrt{2}} \\ (x_2 - x_1) + \frac{9}{\sqrt{2}} \end{array} \right), \quad (6.8)$$

with the input random vector $\mathbf{X} = (X_1, X_2)$ following a standard 2-dimensional normal distribution. The reliability problem consists in evaluating: $p_{f,2} = \mathbb{P}(g_2(\mathbf{X}) \leq 0) = 2.21 \times 10^{-4}$.

Toy-case #3: high-dimensional reliability problem Let us define the higher-dimensional reliability problem (proposed by [Yun et al. \(2018\)](#)), considering the following function $g_3 : \mathbb{R}^7 \rightarrow \mathbb{R}$:

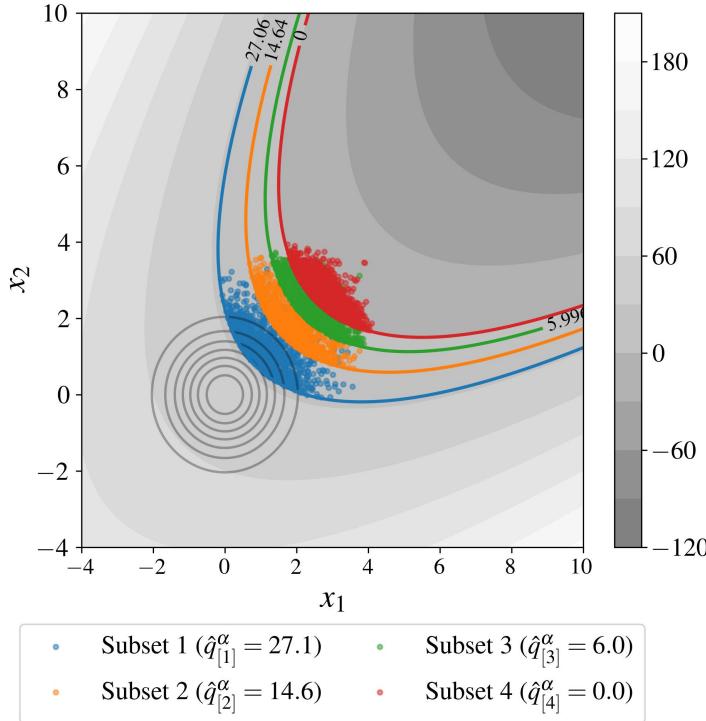
$$g_3(\mathbf{x}) = 15.59 \times 10^4 - \frac{x_1 x_3^2}{2x_3^2} \frac{x_2^4 - 4x_5 x_6 x_7^2 + x_4(x_6 + 4x_5 + 2x_6 x_7)}{x_4 x_5(x_4 + x_6 + 2x_6 x_7)}, \quad (6.9)$$

with the input random vector $\mathbf{X} = (X_1, \dots, X_7)$, following a product of normal distributions defined in [Yun et al. \(2018\)](#). The reliability problem consists in evaluating: $p_{f,3} = \mathbb{P}(g_3(\mathbf{X}) \leq 0) = 8.10 \times 10^{-3}$.

6.3.1 Results analysis

Results of our numerical experiments are presented graphically (for 2-dimensional problems) in Figures 6.5 and 6.6, and numerically in Table 6.1. In the same fashion as the previous illustrations, the figures represent the intermediary quantiles $\widehat{q}_{[k]}^{p_0}$ estimated over conditional samples of size $N = 10^4$. Moreover, samples $\mathbf{A}_{[k+1],n}$ exceeding these quantiles are also represented in the same color. Notice how the last estimated quantile is set to the problem threshold $y_{th} = 0$. To capture the dispersion of BANCS estimation, 100 repetitions were realized. Let us notice that for each toy-case, BANCS well estimates the failure probabilities' orders of magnitude. Yet the numerical values in Table 6.1 consistently present a positive bias, leading to an overestimated failure probability. This bias is partially explained by the EBC tuning choice and could be reduced at the expense of a slightly higher variance.

The variance obtained with the repetitions is quite large. Although, part of it is due to the fact that the algorithm might compute a different total number of subsets (e.g., toy-case #1 is



either solved in four or five subsets). Overall, considering the EBC tuning from Eq. (6.6), BANCS performs worst than SS on toy-cases #1 and #2 but performs as well as SS on the toy-case #3. This might be due to the fact that toy-case #3 has a higher input dimension. However, one can note that SS coefficient of variation is computed by an approximation, tending to underestimate the true coefficient of variation (see e.g., Papaioannou et al. (2015)).

Table 6.1 Results of the numerical experiments (subset sample size $N = 10^4$, $p_0 = 0.1$).

	d	p_f^{ref}	\hat{p}_f^{BANCS}	$\hat{\delta}^{\text{BANCS}}$	\hat{p}_f^{SS}	$\hat{\delta}^{\text{SS}}$
Toy-case #1	2	1.31×10^{-4}	2.67×10^{-4}	24%	1.30×10^{-4}	9%
Toy-case #2	2	2.21×10^{-4}	4.23×10^{-4}	7%	2.24×10^{-4}	6%
Toy-case #3	7	8.10×10^{-3}	9.32×10^{-3}	15%	8.92×10^{-3}	6%

6.4 Application to wind turbine fatigue reliability

6.5 Conclusion

Subset Simulation uses MCMC sampling to generate its intermediary conditional samples. However, MCMC algorithms tends to be complex to tune and does not generate i.i.d. conditional samples. In this work, a new method is proposed, replacing MCMC sampling with a simpler procedure. An intermediary conditional distribution is first fitted by a nonparametric approach, mixing kernel density estimation for fitting the marginals and Empirical Bernstein Copula (EBC)

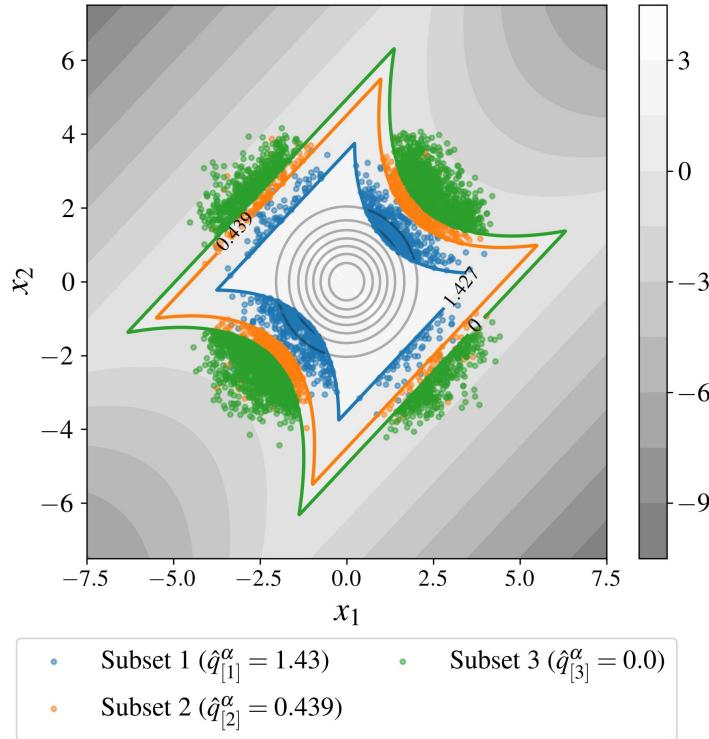


Figure 6.6 BANCS sampling steps on toy-case #2.

for fitting the copula. Then, the resulting allows to perform direct Monte Carlo sampling. This method is named “Bernstein adaptive nonparametric conditional sampling” (BANCS) and is applied to three toy-cases (two 2-dimensional and one 7-dimensional) and compared with SS.

The method shows promising results, even though a small positive bias consistently appears. This issue results from EBC tuning, creating a bias-variance tradeoff in the copula fit. Theoretical works offer optimal tuning, allowing us to find the optimal compromise. In our numerical experiments, an empirical estimation of BANCS variance is computed over a set of repetitions. BANCS estimated coefficient of variation is higher than SS approximated coefficient of variation. This work can be further explored by building an approximation of BANCS variance and confidence interval. One major advantage remains that the samples generated at each iteration are i.i.d. leading to a possible use of these samples to perform global reliability-oriented sensitivity analysis (Marrel and Chabridon, 2021) in order to detect and analyze the most influential input variables leading to failure.

Chapter **7**

Sequential reliability oriented sensitivity analysis

7.1	Introduction	184
7.2	PLI + application on the case	184
7.3	HSIC for GSA	184
7.4	HSIC for TSA & CSA	184
7.5	Sequential ROSA	184
7.6	Application to wind turbine fatigue reliability	184
7.7	Conclusion	184

On risk levels: <https://onlinelibrary.wiley.com/doi/epdf/10.1002/we.2610> [Partie plus prospective]

7.1 Introduction

7.2 PLI + application on the case

7.3 HSIC for GSA

7.4 HSIC for TSA & CSA

7.5 Sequential ROSA

7.6 Application to wind turbine fatigue reliability

7.7 Conclusion

Conclusion and perspectives

Bibliography

- Abdallah, I., Lataniotis, C., and Sudret, B. (2019). Parametric hierarchical kriging for multi-fidelity aero-servo-elastic simulators – Application to extreme loads on wind turbines. *Probabilistic Engineering Mechanics*, 55:67 – 77.
- Abdo, T. and Rackwitz, R. (1991). A new beta-point algorithm for large time-invariant and time-variant reliability problems. In *Reliability and Optimization of Structural Systems' 90: Proceedings of the 3rd IFIP WG 7.5 Conference*, pages 1–12.
- Abdullah, M., Yatim, A., Tan, C., and Saidur, R. (2012). A review of maximum power point tracking algorithms for wind energy systems. *Renewable and sustainable energy reviews*, 16(5):3220–3227.
- Abtini, M. (2018). *Plans prédictifs à taille fixe et séquentiels pour le krigeage*. PhD thesis, Ecole Centrale Lyon.
- Ahmed, S., Hawlader, B., and Roy, K. (2015). Finite element modeling of large diameter monopiles in dense sand for offshore wind turbine foundations. In *International Conference on Offshore Mechanics and Arctic Engineering*, volume 56475, page V001T10A009. American Society of Mechanical Engineers.
- Ajenjo, A. (2023). *Info-gap robustness assessment of reliability evaluations for the safety of critical industrial systems*. PhD thesis, Université Bourgogne Franche-Comté.
- Ajenjo, A., Ardillon, E., Chabridon, V., Iooss, B., Cogan, S., and Sadoulet-Reboul, E. (2022). An info-gap framework for robustness assessment of epistemic uncertainty models in hybrid structural reliability analysis. *Structural Safety*, 96:102196.
- Al-Solihat, M. and Nahon, M. (2018). Flexible multibody dynamic modeling of a floating wind turbine. *International Journal of Mechanical Sciences*, 142:518–529.
- Ameryoun, H., Schoefs, F., Barillé, L., and Thomas, Y. (2019). Stochastic modeling of forces on jacket-type offshore structures colonized by marine growth. *Journal of Marine Science and Engineering*, 7(5):158.
- Ang, G., Ang, A. H.-S., and Tang, W. (1992). Optimal importance-sampling density estimator. *Journal of engineering mechanics*, 118(6):1146–1163.
- Ardillon, E., Paskyabi, M., Cousin, A., Dimitrov, N., Dupoirion, M., Eldevik, S., Fekhari, E., Ferreira, C., Guiton, M., Jezequel, B., et al. (2023). Turbine loading and wake model uncertainty. Technical report, European Union.
- Au, S.-K. and Beck, J. L. (2001). Estimation of small failure probabilities in high dimensions by subset simulation. *Probabilistic Engineering Mechanics*, 16(4):263–277.

- Auder, B., De Crecy, A., Iooss, B., and Marques, M. (2012). Screening and metamodeling of computer experiments with functional outputs. Application to thermal–hydraulic computations. *Reliability Engineering & System Safety*, 107:122–131.
- Bachoc, F. (2013). Cross validation and maximum likelihood estimations of hyper-parameters of Gaussian processes with model misspecification. *Computational Statistics & Data Analysis*, 66:55–69.
- Bai, H., Shi, L., Aoues, Y., Huang, C., and Lemosse, D. (2023). Estimation of probability distribution of long-term fatigue damage on wind turbine tower using residual neural network. *Mechanical Systems and Signal Processing*, 190:110101.
- Bai, Z., Radhakrishna-Rao, C., and Zhao, L. (1989). Kernel estimators of density function of directional data. In *Multivariate statistics and probability*, pages 24–39.
- Baker, E., Barbillon, P., Fadikar, A., Gramacy, R., Herbei, R., Higdon, D., Huang, J., Johnson, L., Mondal, A., Pires, B., et al. (2020). Stochastic simulators: An overview with opportunities. arXiv preprint arXiv:2002.01321.
- Baudin, M., Dutfoy, A., Iooss, B., and Popelin, A. (2017). Open TURNS: An industrial software for uncertainty quantification in simulation. In Ghanem, R., Higdon, D., and Owhadi, H., editors, *Springer Handbook on Uncertainty Quantification*, pages 2001–2038. Springer.
- Beauregard, E., Bérille, E., Berrabah, N., Berthelot, M., Burrows, J., Capaldo, M., Cornet, S., Costan, V., Duchet, M., Dufossé, E., Dupont, E., Franchet, M., Gouze, E., Grau, A., Joly, A., Kell, N., de Laleu, V., Latraube, F., Lovera, A., de Bazelaire, A., Monnot, E., Nogaro, G., Pagot, J., Pérony, R., Peyrard, C., Piguet, C., Régnier, A., Santibanez, E., Senn, C., Smith, C., Soriano, F., Stephan, P., Terte, N., Veyan, P., Vizireanu, D., and Yeow, L. (2022). *L'éolien en mer : un défi pour la transition énergétique*. Lavoisier.
- Bect, J., Bachoc, F., and Ginsbourger, D. (2019). A supermartingale approach to Gaussian process based sequential design of experiments. *Bernoulli*, 25(4A):2883 – 2919.
- Bect, J., Ginsbourger, D., Li, L., Picheny, V., and Vazquez, E. (2012). Sequential design of computer experiments for the estimation of a probability of failure. *Statistics and Computing*, 22:773–793.
- Beer, M., Ferson, S., and Kreinovich, V. (2013). Imprecise probabilities in engineering analyses. *Mechanical systems and signal processing*, 37(1-2):4–29.
- Beirlant, J., Goegebeur, Y., Segers, J., and Teugels, J. (2006). *Statistics of extremes: theory and applications*. John Wiley & Sons.
- Bénard, C., Biau, G., Da Veiga, S., and Scornet, E. (2022). SHAFF: Fast and consistent SHAPley eFficit estimates via random Forests. In *Proceedings of The 25th International Conference on Artificial Intelligence and Statistics*, volume 151, pages 5563–5582.
- Benoumechiara, N. and Elie-Dit-Cosaque, K. (2019). Shapley effects for sensitivity analysis with dependent inputs: bootstrap and kriging-based algorithms. *ESAIM: Proceedings and Surveys*, 65:266–293.
- Binois, M., Huang, J., Gramacy, R., and Ludkovski, M. (2019). Replication or exploration? Sequential design for stochastic simulation experiments. *Technometrics*, 61(1):7–23.
- Bitner-Gregersen, E. (2015). Joint met-ocean description for design and operations of marine structures. *Applied Ocean Research*, 51:279–292.
- Bjerager, P. (1988). Probability integration by directional simulation. *Journal of Engineering Mechanics*, 114(8):1285–1302.

- Blatman, G. and Sudret, B. (2011). Adaptive sparse polynomial chaos expansion based on least angle regression. *Journal of computational Physics*, 230(6):2345–2367.
- Blondel, F. and Cathelain, M. (2020). An alternative form of the super-Gaussian wind turbine wake model. *Wind Energy Science*, 5(3):1225–1236.
- Borovicka, T., Jirina, M. J., Kordik, P., and Jirina, M. (2012). Selecting representative data sets. In Karahoca, A., editor, *Advances in data mining, knowledge discovery and applications*, pages 43–70. INTECH.
- Bossanyi, E. (2003). Individual blade pitch control for load reduction. *Wind Energy: An International Journal for Progress and Applications in Wind Power Conversion Technology*, 6(2):119–128.
- Bouezmarni, T., Ghouch, E., and Taamouti, A. (2013). Bernstein estimator for unbounded copula densities. *Statistics & Risk Modeling*, 30(4):343–360.
- Bourinet, J.-M. (2018). *Reliability analysis and optimal design under uncertainty-Focus on adaptive surrogate-based approaches*. PhD thesis, Université Clermont Auvergne.
- Branlard, E. (2017). *Wind turbine aerodynamics and vorticity-based methods: Fundamentals and recent applications*. Springer.
- Breiman, L. (1996). Bagging predictors. *Machine learning*, 24:123–140.
- Briol, F., Oates, C., Girolami, M., Osborne, M., and Sejdinovic, D. (2019). Probabilistic Integration: A Role in Statistical Computation? *Statistical Science*, 34:1 – 22.
- Briol, F.-X. (2019). *Statistical computation with kernels*. PhD thesis, University of Warwick.
- Briol, F.-X., Oates, C., Girolami, M., and Osborne, M. (2015). Frank-Wolfe Bayesian Quadrature: Probabilistic Integration with Theoretical Guarantees. In *Advances in Neural Information Processing Systems*.
- Bucher, C. (1988). Adaptive sampling—an iterative fast Monte Carlo procedure. *Structural safety*, 5(2):119–126.
- Bugallo, M., Elvira, V., Martino, L., Luengo, D., Miguez, J., and Djuric, P. (2017). Adaptive importance sampling: The past, the present, and the future. *IEEE Signal Processing Magazine*, 34(4):60–79.
- Bui, H. and Bakhoday-Paskyabi, M. (2022). Mesoscale Simulation of Open Cellular Convection: Roles of Model Resolutions and Physics Parameterizations. In *Journal of Physics: Conference Series*, volume 2362, page 012006.
- Burton, T., Jenkins, N., Bossanyi, E., Sharpe, D., and Graham, M. (2021). *Wind energy handbook*. John Wiley & Sons.
- Caillé, F., Bozonnet, P., Perdrizet, T., Poirette, Y., and Melis, C. (2017). Model test and simulation comparison for an inclined-leg TLP dedicated to floating wind. In *International Conference on Offshore Mechanics and Arctic Engineering*, volume 57786, page V010T09A070. American Society of Mechanical Engineers.
- Cermelli, C., Leroux, C., Díaz-Domínguez, S., and Peiffer, A. (2018). Experimental measurements of WindFloat 1 prototype responses and comparison with numerical model. In *International Conference on Offshore Mechanics and Arctic Engineering*, volume 51319, page V010T09A050.

- Cérou, F., Del Moral, P., Furon, T., and Guyader, A. (2012). Sequential monte carlo for rare event estimation. *Statistics and computing*, 22:795–808.
- Cérou, F., Guyader, A., and Rousset, M. (2019). Adaptive multilevel splitting: Historical perspective and recent results. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 29(4):043108.
- Chabridon, V. (2018). *Reliability-oriented sensitivity analysis under probabilistic model uncertainty—Application to aerospace systems*. PhD thesis, Université Clermont Auvergne.
- Chabridon, V., Balesdent, M., Perrin, G., Morio, J., Bourinet, J.-M., and Gayton, N. (2021). Global reliability-oriented sensitivity analysis under distribution parameter uncertainty. *Mechanical Engineering under Uncertainties: From Classical Approaches to Some Recent Developments*, pages 237–277.
- Chen, T., Wang, X., Yuan, G., and Liu, J. (2018). Fatigue bending test on grouted connections for monopile offshore wind turbines. *Marine Structures*, 60:52–71.
- Chen, Y., Welling, M., and Smola, A. (2010). Super-samples from kernel herding. In *Proceedings of the Twenty-Sixth Conference on Uncertainty in Artificial Intelligence*, pages 109 – 116. AUAI Press.
- Cordle, A. and Jonkman, J. (2011). State of the art in floating wind turbine design tools. In *ISOPE International Ocean and Polar Engineering Conference*.
- Cortes, C. and Vapnik, V. (1995). Support-vector networks. *Machine learning*, 20:273–297.
- Cottin, C. and Pfeifer, D. (2014). From Bernstein polynomials to Bernstein copulas. *J. Appl. Funct. Anal.*, 9(3-4):277–288.
- Cousin, A. (2021). *Optimisation sous contraintes probabilistes d'un système complexe : Application au dimensionnement d'une éolienne offshore flottante*. PhD thesis, Institut Polytechnique de Paris.
- Crombecq, K., Laermans, E., and Dhaene, T. (2011). Efficient space-filling and non-collapsing sequential design strategies for simulation-based modelling. *European Journal of Operational Research*, 214:683–696.
- Da Veiga, S. (2015). Global sensitivity analysis with dependence measures. *Journal of Statistical Computation and Simulation*, 85:1283 – 1305.
- da Veiga, S. (2021). Kernel-based anova decomposition and shapley effects – application to global sensitivity analysis.
- Da Veiga, S., Gamboa, F., Iooss, B., and Prieur, C. (2021). *Basics and Trends in Sensitivity Analysis: Theory and Practice in R*. Society for Industrial and Applied Mathematics.
- Dai, J., Hu, Y., Liu, D., and Long, X. (2011). Aerodynamic loads calculation and analysis for large scale wind turbine based on combining BEM modified theory with dynamic stall model. *Renewable Energy*, 36(3):1095–1104.
- Damblin, G., Couplet, M., and Iooss, B. (2013). Numerical studies of space-filling designs: Optimization of Latin Hypercube Samples and subprojection properties. *Journal of Simulation*, 7.
- de Crécy, A., Bazin, P., Glaeser, H., Skorek, T., Joufcla, J., Probst, P., Fujioka, K., Chung, B., Oh, D., Kyncl, M., Pernica, R., Macek, J., Meca, R., Macian, R., D'Auria, F., Petrucci, A., Batet, L., Perez, M., and Reventos, F. (2008). Uncertainty and sensitivity analysis of the LOFT L2-5 test: Results of the BEMUSE programme. *Nuclear Engineering and Design*, 12:3561–3578.

- De Lozzo, M. and Marrel, A. (2016). New improvements in the use of dependence measures for sensitivity analysis and screening. *Journal of Statistical Computation and Simulation*, 86(15):3038–3058.
- de Rocquigny, E., Devictor, N., and Tarantola, S. (2008). *Uncertainty in industrial practice: a guide to quantitative uncertainty management*. John Wiley & Sons.
- Deheuvels, P. (1979). La fonction de dépendance empirique et ses propriétés. Un test non paramétrique d'indépendance. *Bulletins de l'Académie Royale de Belgique*, 65(1):274–292.
- Demange-Chrst, J., Bachoc, F., and Morio, J. (2023). Shapley effect estimation in reliability-oriented sensitivity analysis with correlated inputs by importance sampling. *International Journal for Uncertainty Quantification*, 13(3).
- Der Kiureghian, A. and Dakessian, T. (1998). Multiple design points in first and second-order reliability. *Structural Safety*, 20:37–49.
- Der Kiureghian, A. and Ditlevsen, O. (2009). Aleatory or epistemic? Does it matter? *Structural Safety*, 31(2):105–112.
- Dimitrov, N. (2013). *Structural reliability of wind turbine blades: Design methods and evaluation*. PhD thesis, Technical University of Denmark.
- Dimitrov, N., Kelly, M., Vignaroli, A., and Berg, J. (2018). From wind to loads: wind turbine site-specific load estimation with surrogate models trained on high-fidelity load databases. *Wind Energy Science*, 3:767 – 790.
- Dimitrov, N. and Natarajan, A. (2017). Application of simulated lidar scanning patterns to constrained Gaussian turbulence fields for load validation. *Wind Energy*, 20(1):79–95.
- Dimitrov, N., Natarajan, A., and Mann, J. (2017). Effects of normal and extreme turbulence spectral parameters on wind turbine loads. *Renewable Energy*, 101:1180–1193.
- Dirlik, T. and Benasciutti, D. (2021). Dirlik and tovo-benasciutti spectral methods in vibration fatigue: a review with a historical perspective. *Metals*, 11(9):1333.
- DNV-OS-J103 (2013). DNV-OS-J103: Design of floating wind turbine structures. Technical report, Det Norske Veritas.
- DNV-RP-C203 (2016). DNV-RP-C203: Fatigue design of offshore steel structures. Technical report, Det Norske Veritas.
- DNV-ST-0126 (2018). Dnv-st-0126: Support structures for wind turbines. *Copenhagen, Denmark: DNV*.
- DNV-ST-0437 (2016). DNV-ST-0437: Loads and site conditions for wind turbines. Technical report, Det Norske Veritas.
- Doubrawa, P., Churchfield, M., Godvik, M., and Sirnivas, S. (2019). Load response of a floating wind turbine to turbulent atmospheric flow. *Applied Energy*, 242:1588–1599.
- Doubrawa, P., Quon, E., Martinez-Tossas, L., Shaler, K., Debnath, M., Hamilton, N., Herges, T., Maniaci, D., Kelley, C., Hsieh, A., et al. (2020). Multimodel validation of single wakes in neutral and stratified atmospheric conditions. *Wind Energy*, 23(11):2027–2055.
- Doubrawa, P., Shaler, K., and Jonkman, J. (2023). Difference in load predictions obtained with effective turbulence vs. a dynamic wake meandering modeling approach. *Wind Energy Science Discussions*, 2023:1–28.

- Dowling, N. E. (1972). Fatigue Failure Predictions for Complicated Stress-Strain Histories. *Journal of Materials, JMLSA*, 7:71 – 87.
- Drexler, S. and Muskulus, M. (2021). Reliability of an offshore wind turbine with an uncertain S-N curve. *Journal of Physics: Conference Series*, 2018:012014.
- Driscoll, F., Jonkman, J., Robertson, A., Sirnivas, S., Skaare, B., and Nielsen, F. (2016). Validation of a FAST model of the statoil-hywind demo floating wind turbine. *Energy Procedia*, 94:3–19. 13th Deep Sea Offshore Wind R&D Conference, EERA DeepWind’2016.
- Dubrule, O. (1983). Cross validation of kriging in a unique neighborhood. *Journal of the International Association for Mathematical Geology*, 15(6):687–699.
- Durante, F. and Sempi, C. (2015). *Principles of copula theory*. CRC press.
- Echard, B., Gayton, N., and Lemaire, M. (2011). AK-MCS: An active learning reliability method combining Kriging and Monte Carlo Simulation. *Structural Safety*, 33:145–154.
- Ehre, M., Papaioannou, I., and Straub, D. (2020). A framework for global reliability sensitivity analysis in the presence of multi-uncertainty. *Reliability Engineering & System Safety*, 195:106726.
- Ewans, K. C., Bitner-Gregersen, E. M., and Guedes-Soares, C. (2004). Estimation of Wind-Sea and Swell Components in a Bimodal Sea State. *Journal of Offshore Mechanics and Arctic Engineering*, 128(4):265–270.
- Fan, J. and Lv, J. (2010). A selective overview of variable selection in high dimensional feature space. *Statistica Sinica*, 20(1):101.
- Fang, K., Liu, M.-Q., Qin, H., and Zhou, Y.-D. (2018). *Theory and application of uniform experimental designs*, volume 221. Springer.
- Fang, K.-T., Li, R., and Sudjianto, A. (2006). *Design and Modeling for Computer Experiments*. Chapman & Hall/CRC.
- Fatemi, A. and Yang, L. (1998). Cumulative fatigue damage and life prediction theories: a survey of the state of the art for homogeneous materials. *International Journal of Fatigue*, 20(1):9–34.
- Fazeres-Ferradosa, T., Chambel, J., Taveira-Pinto, F., Rosa-Santos, P., Taveira-Pinto, F., Giannini, G., and Haerens, P. (2021). Scour protections for offshore foundations of marine energy harvesting technologies: A review. *Journal of Marine Science and Engineering*, 9(3):297.
- Fekhari, E., Baudin, M., Chabridon, V., and Jebroun, Y. (2021). otbenchmark: an open source Python package for benchmarking and validating uncertainty quantification algorithms. In *Proceedings of the 4th International Conference on Uncertainty Quantification in Computational Sciences and Engineering*, pages 218 – 231.
- Fekhari, E., Iooss, B., Chabridon, V., and Muré, J. (2022). Efficient techniques for fast uncertainty propagation in an offshore wind turbine multi-physics simulation tool. In *Proceedings of the 5th International Conference on Renewable Energies Offshore*, pages 837–846.
- Fekhari, E., Iooss, B., Muré, J., Pronzato, L., and Rendas, J. (2023). Model predictivity assessment: incremental test-set selection and accuracy evaluation. In *Studies in Theoretical and Applied Statistics*, pages 315–347. Springer.
- Fernández-Godino, M., Park, C., Kim, N.-H., and Haftka, R. (2016). Review of multi-fidelity models. arXiv preprint arXiv:1609.07196.

- Fleming, P., King, J., Bay, C., Simley, E., Mudafort, R., Hamilton, N., Farrell, A., and Martinez-Tossas, L. (2020). Overview of FLORIS updates. In *Journal of Physics: Conference Series*, volume 1618, page 022028.
- Forrester, A., Sobester, A., and Keane, A. (2008). *Engineering design via surrogate modelling: a practical guide*. John Wiley & Sons.
- Freyssinet, C., Rey, V., Schoefs, F., and Moro, T. (2023). Bayesian calibration of a non linear damage model of steel structures with random material property: Sensitivity analysis and reliability assessment. *Engineering Structures*, 295:116853.
- Fuhg, J., Fau, A., and Nackenhorst, U. (2021). State-of-the-Art and Comparative Review of Adaptive Sampling Methods for Kriging. *Archives of Computational Methods in Engineering*, 28:2689–2747.
- Galparsoro, I., Menchaca, I., Garmendia, J., Borja, A., Maldonado, A., Iglesias, G., and Bald, J. (2022). Reviewing the ecological impacts of offshore wind farms. *npj Ocean Sustainability*, 1(1):1.
- Geffraye, G., Antoni, O., Farvacque, M., Kadri, D., Lavialle, G., Rameau, B., and Ruby, A. (2011). CATHARE2 V2.5_2: A single version for various applications. *Nuclear Engineering and Design*, 241:4456–4463.
- Genest, C., Nešlehová, J. G., and Rémillard, B. (2017). Asymptotic behavior of the empirical multilinear copula process under broad conditions. *Journal of Multivariate Analysis*, 159:82–110.
- Geoga, C., Anitescu, M., and Stein, M. (2020). Scalable Gaussian process computations using hierarchical matrices. *Journal of Computational and Graphical Statistics*, 29(2):227–237.
- Geyer, S., Papaioannou, I., and Straub, D. (2019). Cross entropy-based importance sampling using Gaussian densities revisited. *Structural Safety*, 76:15–27.
- Giles, M. (2008). Multilevel Monte Carlo Path Simulation. *Operations Research*, 56:607–617.
- Gionfra, N. (2018). *Stratégies de commande distribuée pour l'optimisation de la production des fermes éoliennes*. PhD thesis, Université Paris-Saclay; La Sapienza, Università di Roma.
- Glad, I., Hjort, N., and Ushakov, N. (2007). Mean-squared error of kernel estimators for finite values of the sample size. *Journal of Mathematical Sciences*, 146(4):5977–5983.
- Gobet, E., Lerasle, M., and Métivier, D. (2022). Mean estimation for randomized quasi Monte Carlo method. hal-03631879v3.
- González-Barrios, J. and Hoyos-Argüelles, R. (2021). Estimating checkerboard approximations with sample d-copulas. *Communications in Statistics-Simulation and Computation*, 50(12):3992–4027.
- Goodfellow, I., Bengio, Y., and Courville, A. (2016). *Deep learning*. The MIT Press.
- Gottschall, J., Gribben, B., Stein, D., and Würth, I. (2017). Floating lidar as an advanced offshore wind speed measurement technique: current technology status and gap analysis in regard to full maturity. *Wiley Interdisciplinary Reviews: Energy and Environment*, 6(5):e250.
- Graf, P., Stewart, G., Lackner, M., Dykes, K., and Veers, P. (2016). High-throughput computation and the applicability of Monte Carlo integration in fatigue load estimation of floating offshore wind turbines. *Wind Energy*, 19(5):861–872.

- Gramacy, R. (2020). *Surrogates: Gaussian process modeling, design, and optimization for the applied sciences*. CRC press.
- Grasedyck, L., Kressner, D., and Tobler, C. (2013). A literature survey of low-rank tensor approximation techniques. *GAMM-Mitteilungen*, 36(1):53–78.
- Gretton, A., Borgwardt, K. M., Rasch, M., Schölkopf, B., and Smola, A. (2006). A Kernel Method for the Two-Sample-Problem. In *Proceedings of the 19th International Conference on Neural Information Processing Systems*, pages 513–520.
- Guédé, Z., Sudret, B., and Lemaire, M. (2007). Life-time reliability based assessment of structures submitted to thermal fatigue. *International Journal of Fatigue*, 29(7):1359–1373.
- Guignier, L., Courbois, A., Mariani, R., and Choisnet, T. (2016). Multibody Modelling of Floating Offshore Wind Turbine Foundation for Global Loads Analysis. In *International Ocean and Polar Engineering Conference*, pages ISOPE-I-16–392.
- Han, Q., Hao, Z., Hu, T., and Chu, F. (2018). Non-parametric models for joint probabilistic distributions of wind speed and direction data. *Renewable Energy*, 126:1032–1042.
- Hansen, M. and Henriksen, L. (2013). *Basic DTU Wind Energy controller*. Number 0028 in DTU Wind Energy E. DTU Wind Energy.
- Hansen, N., Auger, A., Ros, R., Mersmann, O., Tušar, T., and Brockhoff, D. (2021). COCO: A platform for comparing continuous optimizers in a black-box setting. *Optimization Methods and Software*, 36(1):114–144.
- Hartigan, J. (1975). Printer graphics for clustering. *Journal of Statistical Computation and Simulation*, 4(3):187–213.
- Hasselmann, K., Barnett, T., Bouws, E., Carlson, H., Cartwright, D., Enke, K., Ewing, J., Gienapp, A., Hasselmann, D., Kruseman, P., et al. (1973). Measurements of wind-wave growth and swell decay during the Joint North Sea Wave Project (JONSWAP). *Ergaenzungsheft zur Deutschen Hydrographischen Zeitschrift, Reihe A*.
- Hastie, T., Tibshirani, R., and Friedman, J. (2009). *The elements of statistical learning: data mining, inference, and prediction*, volume 2. Springer.
- Hawchar, L., El Soueid, C.-P., and Schoefs, F. (2017). Principal component analysis and polynomial chaos expansion for time-variant reliability problems. *Reliability Engineering & System Safety*, 167:406–416.
- Hegseth, J. and Bachynski, E. (2019). A semi-analytical frequency domain model for efficient design evaluation of spar floating wind turbines. *Marine Structures*, 64:186–210.
- Heier, S. (2014). *Grid integration of wind energy: onshore and offshore conversion systems*. John Wiley & Sons.
- Heinrich, J. and Weiskopf, D. (2013). State of the art of parallel coordinates. *Eurographics (state of the art reports)*, pages 95–116.
- Heredia-Zavoni, E. and Montes-Iturrizaga, R. (2022). Environmental contours using nonparametric copulas. *Ocean Engineering*, 266:112971.
- Hickernell, F. (1998). A generalized discrepancy and quadrature error bound. *Mathematics of computation*, 67(221):299–322.

- Hirvoas, A. (2021). *Development of a data assimilation method for the calibration and continuous update of wind turbines digital twins*. PhD thesis, Université Grenoble Alpes.
- Hoeffding, W. (1948). A class of statistics with asymptotically normal distributions. *Annals of Mathematical Statistics*, 19(3):293–325.
- Holm-Jørgensen, K. (2009). *Nonlinear multibody dynamics of wind turbines*. PhD thesis, Aalborg University.
- Huchet, Q. (2019). *Kriging based methods for the structural damage assessment of offshore wind turbines*. PhD thesis, Université Blaise Pascal.
- Huchet, Q., Matstrand, C., Beaurepaire, P., Relun, N., and Gayton, N. (2019). AK-DA: An efficient method for the fatigue assessment of wind turbine structures. *Wind Energy*, 22(5):638–652.
- Huszár, F. and Duvenaud, D. (2012). Optimally-Weighted Herding is Bayesian Quadrature. In *Proceedings of the Twenty-Eighth Conference on Uncertainty in Artificial Intelligence*, pages 377 – 386.
- IEC-61400-1 (2019). IEC 61400-1: Wind energy generation systems - Part 1: Design requirements. Technical report, International Electrotechnical Commission (IEC).
- Iooss, B. (2021). Sample selection from a given dataset to validate machine learning models. In *Proceedings of 50th Meeting of the Italian Statistical Society (SIS2021)*, pages 88–93, Pisa, Italy.
- Iooss, B., Boussouf, L., Feuillard, V., and Marrel, A. (2010). Numerical studies of the meta-model fitting and validation processes. *International Journal of Advances in Systems and Measurements*, 3:11–21.
- Ishihara, T. and Qian, G. W. (2018). A new Gaussian-based analytical wake model for wind turbines considering ambient turbulence intensities and thrust coefficient effects. *Journal of Wind Engineering and Industrial Aerodynamics*, 177:275–292.
- Janssen, P., Swanepoel, J., and Veraverbeke, N. (2012). Large sample behavior of the Bernstein copula estimator. *Journal of Statistical Planning and Inference*, 142(5):1189 – 1197.
- Jensen, J. and Skelton, K. (2018). Wind turbine blade recycling: Experiences, challenges and possibilities in a circular economy. *Renewable and Sustainable Energy Reviews*, 97:165–176.
- Jensen, N. (1983). A note on wind generator interaction. Technical report, DTU.
- Jiang, Z. (2021). Installation of offshore wind turbines: A technical review. *Renewable and Sustainable Energy Reviews*, 139:110576.
- Joe, H. (1997). *Multivariate Models and Multivariate Dependence Concepts*. Chapman and Hall.
- Joe, H. (2014). *Dependence modeling with copulas*. CRC press.
- Joe, H. and Kurowicka, D. (2011). *Dependence modeling: vine copula handbook*. World Scientific.
- Johlas, H., Martínez-Tossas, L., Lackner, M., Schmidt, D., and Churchfield, M. (2020). Large eddy simulations of offshore wind turbine wakes for two floating platform types. In *Journal of Physics: Conference Series*, volume 1452, page 012034.
- Jones, D., Schonlau, M., and Welch, W. (1998). Efficient global optimization of expensive black-box functions. *Journal of Global optimization*, 13:455–492.
- Jonkman, B. (2009). Turbsim User's Guide: Version 1.50. Technical report, NREL.

- Joseph, V., Gul, E., and Ba, S. (2015). Maximum projection designs for computer experiments. *Biometrika*, 102.
- Joseph, V. R. and Vakayil, A. (2022). SPLIT: An optimal method for data splitting. *Technometrics*, 64(2):166–176.
- Kaimal, J., Wyngaard, J., Izumi, Y., and Coté, O. (1972). Spectral characteristics of surface-layer turbulence. *Quarterly Journal of the Royal Meteorological Society*, 98(417):563–589.
- Kallehave, D., Byrne, B. W., LeBlanc T., C., and M., K. K. (2015). Optimization of monopiles for offshore wind turbines. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 373(2035):20140100.
- Kanagawa, M. and Hennig, P. (2019). Convergence Guarantees for Adaptive Bayesian Quadrature Methods. In *Advances in Neural Information Processing Systems*, volume 32.
- Kanagawa, M., Hennig, P., Sejdinovic, D., and Sriperumbudur, B. (2018). Gaussian Processes and Kernel Methods: A Review on Connections and Equivalences. arXiv:1807.02582.
- Kanner, S., Aubault, A., Peiffer, A., and Yu, B. (2018). Maximum dissimilarity-based algorithm for discretization of metocean data into clusters of arbitrary size and dimension. In *International Conference on Offshore Mechanics and Arctic Engineering*, volume 51319.
- Kaplan, Z., Li, Y., Nakayama, M., and Tuffin, B. (2019). Randomized quasi-Monte Carlo for quantile estimation. In *2019 Winter Simulation Conference (WSC)*, pages 428–439.
- Katsikogiannis, G., Sørum, S., Bachynski, E., and Amdahl, J. (2021). Environmental lumping for efficient fatigue assessment of large-diameter monopile wind turbines. *Marine Structures*, 77:102939.
- Kennard, R. and Stone, L. (1969). Computer aided design of experiments. *Technometrics*, 11:137–148.
- Kim, T., Natarajan, A., Lovera, A., Julian, E., Peyrard, E., Capaldo, M., Huwart, G., Bozonnet, P., and Guiton, M. (2022). A comprehensive code-to-code comparison study with the modified IEA15MW-UMaine Floating Wind Turbine for H2020 HIPERWIND project. *Journal of Physics: Conference Series*, 2265(4):042006.
- Kiriliouk, A., Segers, J., and Tsukahara, H. (2021). *Resampling procedures with empirical beta copulas*, pages 27–53. Springer.
- Klebanov, I., Schuster, I., and Sullivan, T. (2020). A rigorous theory of conditional mean embeddings. *SIAM Journal on Mathematics of Data Science*, 2(3):583–606.
- Kleijnen, J. and Sargent, R. (2000). A methodology for fitting and validating metamodels in simulation. *European Journal of Operational Research*, 120:14–29.
- Koehler, J. and Owen, A. (1996). 9 Computer experiments. In *Design and Analysis of Experiments*, volume 13 of *Handbook of Statistics*, pages 261–308. Elsevier.
- Koutsourelakis, P. (2004). Reliability of structures in high dimensions. Part II. Theoretical validation. *Probabilistic engineering mechanics*, 19(4):419–423.
- Kucherenko, S., Feil, B., Shah, N., and Mauntz, W. (2011). The identification of model effective dimensions using global sensitivity analysis. *Reliability Engineering & System Safety*, 96:440–449.

- Kucherenko, S. and Iooss, B. (2017). *Derivative-Based Global Sensitivity Measures*, pages 1241–1263. Springer International Publishing, Cham.
- Kucherenko, S., Rodriguez-Fernandez, M., Pantelides, C., and Shah, N. (2009). Monte Carlo evaluation of derivative-based global sensitivity measures. *Reliability Engineering & System Safety*, 94:1135–1148. Special Issue on Sensitivity Analysis.
- Kurtz, N. and Song, J. (2013). Cross-entropy-based adaptive importance sampling using Gaussian mixture. *Structural Safety*, 42:35–44.
- Lacoste-Julien, S., Lindsten, F., and Bach, F. (2015). Sequential Kernel Herding: Frank-Wolfe Optimization for Particle Filtering. In *Proceedings of the Eighteenth International Conference on Artificial Intelligence and Statistics*, volume 38, pages 544–552.
- Lamboni, M., Iooss, B., Popelin, A.-L., and Gamboa, F. (2013). Derivative-based global sensitivity measures: General links with Sobol' indices and numerical tests. *Mathematics and Computers in Simulation*, 87:45–54.
- Larsen, G. C., Madsen, H., Thomsen, K., and Larsen, T. (2008). Wake meandering: a pragmatic approach. *Wind Energy: An International Journal for Progress and Applications in Wind Power Conversion Technology*, 11(4):377–395.
- Lasserre, J.-B. (2023). Chebyshev and equilibrium measure vs bernstein and lebesgue measure.
- Lasserre, M. (2022). *Apprentissages dans les réseaux bayésiens à base de copules non-paramétriques*. PhD thesis, Sorbonne Université.
- Lataniotis, C. (2019). *Data-driven uncertainty quantification for high-dimensional engineering problems*. PhD thesis, ETH Zürich.
- Laurie, D. (1997). Calculation of Gauss-Kronrod quadrature rules. *Mathematics of Computation*, 66(219):1133–1145.
- Le, T., Eiksund, G., Strøm, P., and Saue, M. (2014). Geological and geotechnical characterisation for offshore wind turbine foundations: A case study of the Sheringham Shoal wind farm. *Engineering Geology*, 177:40–53.
- Le Maître, O. and Knio, O. (2010). *Spectral methods for uncertainty quantification: with applications to computational fluid dynamics*. Springer Science & Business Media.
- Le Riche, R. and Picheny, V. (2021). Revisiting Bayesian optimization in the light of the COCO benchmark. *Structural and Multidisciplinary Optimization*, 64(5):3063–3087.
- Lebrun, R. (2013). *Contributions à la modélisation de la dépendance stochastique*. PhD thesis, Université Paris-Diderot – Paris VII. (in English).
- Lebrun, R. and Dutfoy, A. (2009). A generalization of the Nataf transformation to distributions with elliptical copula. *Probabilistic Engineering Mechanics*, 24(2):172–178.
- L'Ecuyer, P. (2018). Randomized Quasi-Monte Carlo: An Introduction for Practitioners. In *Monte Carlo and Quasi-Monte Carlo Methods*, pages 29–52, Cham. Springer International Publishing.
- Lemaire, M., Chateauneuf, A., and Mitteau, J.-C. (2009). *Structural reliability*. John Wiley & Sons.
- Lemaître, P., Sergienko, E., Arnaud, A., Bousquet, N., Gamboa, F., and Iooss, B. (2015). Density modification-based reliability sensitivity analysis. *Journal of Statistical Computation and Simulation*, 85(6):1200–1223.

- Leobacher, G. and Pillichshammer, F. (2014). *Introduction to quasi-Monte Carlo integration and applications*. Springer.
- Li, W., Lu, L., Xie, X., and Yang, M. (2017). A novel extension algorithm for optimized Latin hypercube sampling. *Journal of Statistical Computation and Simulation*, 87:2549–2559.
- Li, X., Mikusiński, P., and Taylor, M. D. (1998). Strong approximation of copulas. *Journal of Mathematical Analysis and Applications*, 225(2):608–623.
- Li, X. and Zhang, W. (2020). Long-term fatigue damage assessment for a floating offshore wind turbine under realistic environmental conditions. *Renewable Energy*, 159:570–584.
- Li, Y., Kang, L., and Hickernell, F. (2020). Is a transformed low discrepancy design also low discrepancy? *Contemporary Experimental Design, Multivariate Analysis and Data Mining: Festschrift in Honour of Professor Kai-Tai Fang*, pages 69–92.
- Lieudarde, H.-P. (1982). *La Pratique des essais de fatigue – méthodes expérimentales et analyse des résultats*. Pyc édition.
- Lin, Y. and Dong, S. (2019). Wave energy assessment based on trivariate distribution of significant wave height, mean period and direction. *Applied Ocean Research*, 87:47–63.
- Lovera, A. (2019). *Cyclic lateral design for offshore monopiles in weak rocks*. PhD thesis, Université Paris Est.
- Lovera, A., Fekhari, E., Jézéquel, B., Dupoirion, M., Guiton, M., and Ardillon, E. (2023). Quantifying and clustering the wake-induced perturbations within a wind farm for load analysis. In *Journal of Physics: Conference Series*, volume 2505, page 012011. IOP Publishing.
- MacKinnon, D., Afewerki, S., and Karlsen, A. (2022). Technology legitimation and strategic coupling: A cross-national study of floating wind power in Norway and Scotland. *Geoforum*, 135:1–11.
- Mak, S. and Joseph, V. (2018). Support points. *The Annals of Statistics*, 46:2562 – 2592.
- Mann, J. (1998). Wind field simulation. *Probabilistic engineering mechanics*, 13(4):269–282.
- Mara, T. A. and Tarantola, S. (2012). Variance-based sensitivity indices for models with dependent inputs. *Reliability Engineering & System Safety*, 107:115–121.
- Marelli, S. and Sudret, B. (2014). UQLab: A framework for uncertainty quantification in Matlab. In *Vulnerability, uncertainty, and risk: quantification, mitigation, and management*, pages 2554–2563.
- Marrel, A. and Chabridon, V. (2021). Statistical developments for target and conditional sensitivity analysis: Application on safety studies for nuclear reactor. *Reliability Engineering & System Safety*, 214:107711.
- Marrel, A., Iooss, B., and Chabridon, V. (2022). The ICSCREAM methodology: Identification of penalizing configurations in computer experiments using screening and metamodel – Applications in thermal-hydraulics. *Nuclear Science and Engineering*, 196:301–321.
- Marrel, A., Iooss, B., Laurent, B., and Roustant, O. (2009). Calculations of Sobol indices for the Gaussian process metamodel. *Reliability Engineering & System Safety*, 94(3):742–751.
- Marty, A., Berhault, C., Damblans, G., Facq, J.-V., Gaurier, B., Germain, G., Soulard, T., and Schoefs, F. (2021). Experimental study of hard marine growth effect on the hydrodynamical behaviour of a submarine cable. *Applied Ocean Research*, 114:102810.

- Matha, D., Schlipf, M., Pereira, R., and Jonkman, J. (2011). Challenges in simulation of aerodynamics, hydrodynamics, and mooring-line dynamics of floating offshore wind turbines. In *ISOPE International Ocean and Polar Engineering Conference*.
- Matsumoto, M. and Nishimura, T. (1998). Mersenne twister: a 623-dimensionally equidistributed uniform pseudo-random number generator. *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, 8(1):3–30.
- Mckay, M., Beckman, R., and Conover, W. (1979). A Comparison of Three Methods for Selecting Vales of Input Variables in the Analysis of Output From a Computer Code. *Technometrics*, 21:239 – 245.
- Mei, X. and Xiong, M. (2021). Effects of second-order hydrodynamics on the dynamic responses and fatigue damage of a 15 mw floating offshore wind turbine. *Journal of Marine Science and Engineering*, 9(11):1232.
- Meyers, J., Bottasso, C., Dykes, K., Fleming, P., Gebraad, P., Giebel, G., Göçmen, T., and Van Wingerden, J. (2022). Wind farm flow control: prospects and challenges. *Wind Energy Science Discussions*, 2022:1–56.
- Milano, D. (2021). *Numerical prototype for floating offshore wind turbines*. PhD thesis, The University of Edinburgh.
- Montes-Iturrizaga, R. and Heredia-Zavoni, E. (2016). 'multivariate environmental contours using c-vine copulas'. *Ocean engineering*, 118:68–82.
- Morio, J. (2011). Non-parametric adaptive importance sampling for the probability estimation of a launcher impact position. *Reliability Engineering and System Safety*, 96(1):178–183.
- Morio, J. and Balesdent, M. (2015). *Estimation of Rare Event Probabilities in Complex Aerospace and Other Systems: A Practical Approach*. Woodhead Publishing, Elsevier.
- Morokoff, W. J. and Caflisch, R. E. (1995). Quasi-Monte Carlo Integration. *Journal of Computational Physics*, 122(2):218–230.
- Morris, M. (1991). Factorial sampling plans for preliminary computational experiments. *Technometrics*, 33:161–174.
- Moustapha, M., Marelli, S., and Sudret, B. (2022). Active learning for structural reliability: Survey, general framework and benchmark. *Structural Safety*, 96:102174.
- Müller, W. G. (2007). *Collecting Spatial Data*. Springer, 3rd edition.
- Murcia, J., Réthoré, P., Dimitrov, N., Natarajan, A., Sørensen, J., Graf, P., and Kim, T. (2018). Uncertainty propagation through an aeroelastic wind turbine model using polynomial surrogates. *Renewable Energy*, 119:910–922.
- Müller, K. and Cheng, P. (2018). Application of a Monte Carlo procedure for probabilistic fatigue design of floating offshore wind turbines. *Wind Energy Science*, 3:149 – 162.
- Nagababu, G., Srinivas, B., Kachhwaha, S., Puppala, H., and Kumar, S. (2023). Can offshore wind energy help to attain carbon neutrality amid climate change? A GIS-MCDM based analysis to unravel the facts using CORDEX-SA. *Renewable Energy*, page 119400.
- Nagler, T., Schellhase, C., and Czado, C. (2017). Nonparametric estimation of simplified vine copula models: comparison of methods. *Dependence Modeling*, 5:99–120.

- Nash, J. and Sutcliffe, J. (1970). River flow forecasting through conceptual models part I-A discussion of principles. *Journal of Hydrology*, 10(3):282–290.
- Nelsen, R. (2006). *An introduction to copulas*. Springer.
- Nogales Gómez, A., Pronzato, L., and Rendas, M.-J. (2021). Incremental space-filling design based on coverings and spacings: improving upon low discrepancy sequences. *Journal of Statistical Theory and Practice*, 15(4):77.
- Oates, C. and Sullivan, T. (2019). A modern retrospective on probabilistic numerics. *Statistics and computing*, 29(6):1335–1351.
- Oates, C. J. (2021). Minimum Discrepancy Methods in Uncertainty Quantification. Lecture Notes at École Thématique sur les Incertitudes en Calcul Scientifique (ETICS21), <https://www.gdr-mascotnum.fr/etics.html>.
- Oberkampf, W. and Roy, C. (2010). *Verification and validation in scientific computing*. Cambridge university press.
- O'Hagan, A. (1991). Bayes-Hermite quadrature. *Journal of Statistical Planning and Inference*, 29:245–260.
- Otter, A., Murphy, J., Pakrashi, V., Robertson, A., and Desmond, C. (2022). A review of modelling techniques for floating offshore wind turbines. *Wind Energy*, 25(5):831–857.
- Owen, A. (2003). The dimension distribution and quadrature test functions. *Statistica Sinica*, 13:1–17.
- Owen, A. (2013). Monte Carlo theory, methods and examples.
- Owen, A. (2014). Sobol'indices and Shapley value. *SIAM/ASA Journal on Uncertainty Quantification*, 2(1):245–251.
- Owen, A. and Zhou, Y. (2000). Safe and effective importance sampling. *Journal of the American Statistical Association*, 95(449):135–143.
- Papaioannou, I., Betz, W., Zwirglmaier, K., and Straub, D. (2015). MCMC algorithms for Subset Simulation. *Probabilistic Engineering Mechanics*, 41:89–103.
- Papaioannou, I., Geyer, S., and Straub, D. (2019). Improved cross entropy-based importance sampling with a flexible mixture model. *Reliability Engineering & System Safety*, 191:106564.
- Papaioannou, I. and Straub, D. (2021). Variance-based reliability sensitivity analysis and the FORM α -factors. *Reliability Engineering & System Safety*, 210:107496.
- Papakonstantinou, K., N., H., and Eshra, E. (2023). Hamiltonian MCMC methods for estimating rare events probabilities in high-dimensional problems. *Probabilistic Engineering Mechanics*, 74:103485.
- Park, H. and Jun, C. (2009). A simple and fast algorithm for k-medoids clustering. *Expert systems with applications*, 36:3336–3341.
- Petit, S. (2022). *Improved Gaussian process modeling : Application to Bayesian optimization*. PhD thesis, Université Paris-Saclay.
- Petrovska, E. (2022). *Fatigue life reassessment of monopile-supported offshore wind turbine structures*. PhD thesis, University of Edinburgh.

- Plischke, E. and Borgonovo, E. (2019). Copula theory and probabilistic sensitivity analysis: Is there a connection? *European Journal of Operational Research*, 277(3):1046–1059.
- Póczos, B., Ghahramani, Z., and Schneider, J. (2012). Copula-Based Kernel Dependency Measures. In *Proceedings of the 29th International Conference on International Conference on Machine Learning*, ICML’12, pages 1635–1642.
- Popko, W., Robertson, A., Jonkman, J., Wendt, F., Thomas, P., Müller, K., Kretschmer, M., Hagen, T., Galinos, C., Le Dreff, J.-B., et al. (2021). Validation of numerical models of the offshore wind turbine from the alpha ventus wind farm against full-scale measurements within OC5 Phase III. *Journal of Offshore Mechanics and Arctic Engineering*, 143(1):012002.
- Powell, M. (1994). *A direct search optimization method that models the objective and constraint functions by linear interpolation*. Springer.
- Pronzato, L. (2022). Performance analysis of greedy algorithms for minimising a Maximum Mean Discrepancy. preprint, <https://hal.inria.fr/hal-03114891/>.
- Pronzato, L. and Müller, W. (2012). Design of computer experiments: space filling and beyond. *Statistics and Computing*, 22:681–701.
- Pronzato, L. and Rendas, M. (2021a). Validation design I: construction of validation designs via kernel herding. preprint, <https://arxiv.org/abs/2112.05583>.
- Pronzato, L. and Rendas, M.-J. (2021b). Validation design I: construction of validation designs via kernel herding. Preprint.
- Pronzato, L. and Zhigljavsky, A. (2020). Bayesian quadrature and energy minimization for space-filling design. *SIAM/ASA Journal on Uncertainty Quantification*, 8:959 – 1011.
- Pulselli, R., Maccanti, M., Bruno, M., Sabbetta, A., Neri, E., Patrizi, N., and Bastianoni, S. (2022). Benchmarking marine energy technologies through LCA: Offshore floating wind farms in the mediterranean. *Frontiers in Energy Research*, 10:902021.
- Qian, P., Ai, M., and Wu, C. (2009). Construction of Nested Space-Filling Designs. *Annals of Statistics*, 37:3616–3643.
- Qian, P. and Wu, C. (2009). Sliced space filling designs. *Biometrika*, 96:945–956.
- Qiu, J., Wu, Q., Ding, G., Xu, Y., and Feng, S. (2016). A survey of machine learning for big data processing. *EURASIP Journal on Advances in Signal Processing*, 2016:1–16.
- Ragan, P. and Manuel, L. (2007). Comparing estimates of wind turbine fatigue loads using time-domain and spectral methods. *Wind engineering*, 31(2):83–99.
- Rahman, S. (2016). The f-sensitivity index. *SIAM/ASA journal on uncertainty quantification*, 4(1):130–162.
- Raoult, C., Joly, A., Andreevsky, M., and Joly-Lauzel, A. (2018). Anemoc-3: amélioration de la base de données d'état de mer anemoc-2 par prise en compte des effets de la marée anemoc-3: improving the anemoc-2 sea state database by adding tide effects. In *16e journées de l'hydrodynamique*.
- Rasmussen, C. and Williams, C. (2006). *Gaussian processes for machine learning*, volume 1. Springer.
- Reale, C., Tott-Buswell, J., and Prendergast, L. (2021). Impact of geotechnical uncertainty on the preliminary design of monopiles supporting offshore wind turbines. *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part B: Mechanical Engineering*, 7(4):040903.

- Ren, Z., Verma, A., Li, Y., Teuwen, J., and Jiang, Z. (2021). Offshore wind turbine operations and maintenance: A state-of-the-art review. *Renewable and Sustainable Energy Reviews*, 144:110886.
- Robertson, A., Gueydon, S., Bachynski, E., Wang, L., Jonkman, J., Alarcon, D., Amet, E., Beardsell, A., Bonnet, P., Boudet, B., et al. (2020). OC6 Phase I: Investigating the underprediction of low-frequency hydrodynamic loads and responses of a floating wind turbine. In *Journal of Physics: Conference Series*, volume 1618, page 032033.
- Rocher, B., Schoefs, S., François, M., Salou, A., and Caouissin, A.-L. (2020). A two-scale probabilistic time-dependent fatigue model for offshore steel wind turbines. *International Journal of Fatigue*, 136:105620.
- Rockafellar, R. and Royset, J. (2015). Engineering Decisions under Risk Averseness. *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering*, 1.
- Rollón de Pinedo, A., Couplet, M., Iooss, B., Marie, N., Marrel, A., Merle, E., and Sueur, R. (2021). Functional outlier detection by means of h-mode depth and dynamic time warping. *Applied Sciences*, 11(23):11475.
- Rongé, E., Peyrard, C., Venugopal, V., Xiao, Q., Johanning, L., and Benoit, M. (2023). Evaluation of second and third-order numerical wave-loading models for floating offshore wind TLPs. *Ocean Engineering*, 288:116064.
- Rose, D. (2015). *Modeling and estimating multivariate dependence structures with the Bernstein copula*. PhD thesis, Ludwig Maximilian University of Munich.
- Rott, A., Doekemeijer, B., Seifert, J., van Wingerden, J., and Kühn, M. (2018). Robust active wake control in consideration of wind direction variability and uncertainty. *Wind energy science*, 3(2):869–882.
- Rozsas, A. and Slobbe, A. (2019). Repository and black-box reliability challenge 2019.
- Rubinstein, R. Y. and Kroese, D. P. (2004). *The cross-entropy method: a unified approach to combinatorial optimization, Monte-Carlo simulation, and machine learning*, volume 133. Springer.
- Rubinstein, R. Y. and Kroese, D. P. (2008). *Simulation and the Monte Carlo Method*. Wiley, Second ed. edition.
- Saltelli, A., Ratto, M., Andres, T., Campolongo, F., Cariboni, J., Gatelli, D., Saisana, M., and Tarantola, S. (2008). *Global sensitivity analysis: the primer*. John Wiley & Sons.
- Sancetta, A. and Satchell, S. (2004). The Bernstein copula and its applications to modeling and approximations of multivariate distributions. *Econometric Theory*, 20(3):535–562.
- Santner, T., Williams, B., and Notz, W. (2003). *The Design and Analysis of Computer Experiments*. Springer.
- Saporta, G. (2006). *Probabilités, analyse des données et statistique*. Editions technip.
- Saranyasoontorn, K., Manuel, L., and Veers, P. (2004). A Comparison of Standard Coherence Models for Inflow Turbulence With Estimates from Field Measurements. *Journal of Solar Energy Engineering*, 126(4):1069–1082.
- Schöbi, R. and Sudret, B. (2017). Structural reliability analysis for p-boxes using multi-level meta-models. *Probabilistic Engineering Mechanics*, 48:27–38.

- Schobi, R., Sudret, B., and Wiart, J. (2015). Polynomial-chaos-based Kriging. *International Journal for Uncertainty Quantification*, 5(2).
- Schoefs, F. and Tran, T. (2022). Reliability updating of offshore structures subjected to marine growth. *Energies*, 15(2):414.
- Schütz, W. (1996). A history of fatigue. *Engineering Fracture Mechanics*, 54(2):263–300.
- Segers, J., Sibuya, M., and Tsukahara, H. (2017). The empirical beta copula. *Journal of Multivariate Analysis*, 155:35–51.
- Sejdinovic, D., Sriperumbudur, B., Gretton, A., and Fukumizu, K. (2013). Equivalence of distance-based and RKHS-based statistics in hypothesis testing. *The Annals of Statistics*, 41:2263–2291.
- Sempreviva, A., Barthelmie, R., and Pryor, S. (2008). Review of methodologies for offshore wind resource assessment in European seas. *Surveys in Geophysics*, 29:471–497.
- Shahriari, B., Swersky, K., Wang, Z., Adams, R., and De Freitas, N. (2015). Taking the human out of the loop: A review of Bayesian optimization. *Proceedings of the IEEE*, 104(1):148–175.
- Shang, B. and Apley, D. (2020). Fully-sequential space-filling design algorithms for computer experiments. *Journal of Quality Technology*, 53:1 – 24.
- Shapley, L. S. et al. (1953). A value for n-person games.
- Sheikholeslami, R. and Razavi, S. (2017). Progressive Latin hypercube sampling: An efficient approach for robust sampling-based analysis of environmental models. *Environmental Modelling & Software*, 93:109–126.
- Shields, M., Beiter, P., Nunemaker, J., Cooperman, A., and Duffy, P. (2021). Impacts of turbine and plant upsizing on the leveled cost of energy for offshore wind. *Applied Energy*, 298:117189.
- Simley, E., Fleming, P., and King, J. (2020). Design and analysis of a wake steering controller with wind direction variability. *Wind Energy Science*, 5(2):451–468.
- Slot, R. M., Sørensen, J. D., Sudret, B., Svenningsen, L., and Thøgersen, M. L. (2020). Surrogate model uncertainty in wind turbine reliability assessment. *Renewable Energy*, 151:1150 – 1162.
- Snee, R. (1977). Validation of regression models: Methods and examples. *Technometrics*, 19:415–428.
- Sobolí, I. (1993). Sensitivity estimates for nonlinear mathematical models. *Mathematical Modelling and Computational Experiments*, 1:407.
- Sobol, I. and Gresham, A. (1995). On an alternative global sensitivity estimators. *Proceedings of SAMO*, pages 40–42.
- Soize, C. and Ghanem, R. (2004). Physical systems with random uncertainties: chaos representations with arbitrary probability measure. *SIAM Journal on Scientific Computing*, 26:395–410.
- Song, E., Nelson, B. L., and Staum, J. (2016). Shapley effects for global sensitivity analysis: Theory and computation. *SIAM/ASA Journal on Uncertainty Quantification*, 4(1):1060–1083.
- Sriperumbudur, B., Gretton, A., Fukumizu, K., Schölkopf, B., and Lanckriet, G. (2010). Hilbert Space Embeddings and Metrics on Probability Measures. *Journal of Machine Learning Research*, 11:1517–1561.
- Straub, D. (2014). *Engineering risk assessment*, pages 333–362. Springer.

- Sudret, B. (2008). Global sensitivity analysis using polynomial chaos expansions. *Reliability engineering & system safety*, 93(7):964–979.
- Sudret, B. (2013). *Probabilistic Design of Structures Submitted to Fatigue*, chapter 5, pages 223–263. John Wiley & Sons, Ltd.
- Sullivan, T. (2015). *Introduction to uncertainty quantification*, volume 63. Springer.
- Sun, F., Wang, Y., and Xu, H. (2019). Uniform projection designs. *The Annals of Statistics*, 47:641 – 661.
- Suresh, S. (1998). *Fatigue of materials*. Cambridge university press.
- Sutherland, H. (2000). A summary of the fatigue properties of wind turbine materials. *Wind Energy: An International Journal for Progress and Applications in Wind Power Conversion Technology*, 3(1):1–34.
- Székely, G. J. and Rizzo, M. L. (2013). Energy statistics: A class of statistics based on distances. *Journal of Statistical Planning and Inference*, 143:1249 – 1272.
- Tabandeh, A., Jia, G., and Gardoni, P. (2022). A review and assessment of importance sampling methods for reliability analysis. *Structural Safety*, 97:102216.
- Teixeira, R., Nogal, M., and O'Connor, A. (2021). Adaptive approaches in metamodel-based reliability analysis: A review. *Structural Safety*, 89:102019.
- Teixeira, R., Nogal, M., O'Connor, A., Nichols, J., and Dumas, A. (2019a). Stress-cycle fatigue design with Kriging applied to offshore wind turbines. *International Journal of Fatigue*, 125:454–467.
- Teixeira, R., O'Connor, N., and Nogal, M. (2019b). Probabilistic sensitivity analysis of offshore wind turbines using a transformed Kullback-Leibler divergence. *Structural Safety*, 81:101860.
- Teymur, O., Gorham, J., Riabiz, M., and Oates, C. (2021). Optimal Quantisation of Probability Measures Using Maximum Mean Discrepancy. In *Proceedings of The 24th International Conference on Artificial Intelligence and Statistics*, volume 130, pages 1027 – 1035.
- Torre, E., Marelli, S., Embrechts, P., and Sudret, B. (2019). A general framework for data-driven uncertainty quantification under complex input dependencies using vine copulas. *Probabilistic Engineering Mechanics*, 55:1–16.
- Trefethen, L. (2008). Is Gauss quadrature better than Clenshaw–Curtis? *SIAM review*, 50(1):67–87.
- Van den Bos, L. (2020). *Quadrature Methods for Wind Turbine Load Calculations*. PhD thesis, Delft University of Technology.
- Van der Hoven, I. (1957). Power spectrum of horizontal wind speed in the frequency range from 0.0007 to 900 cycles per hour. *Journal of Atmospheric Sciences*, 14(2):160–164.
- Van Kuik, G., Peinke, J., Nijssen, R., Lekou, D., Mann, J., Sørensen, J., Ferreira, C., van Wingerden, J., Schlipf, D., Gebraad, P., et al. (2016). Long-term research challenges in wind energy—a research agenda by the European Academy of Wind Energy. *Wind energy science*, 1(1):1–39.
- Vanem, E. (2016). Joint statistical models for significant wave height and wave period in a changing climate. *Marine Structures*, 49:180–205.

- Vanem, E., Fekhari, E., Dimitrov, N., Kelly, M., Cousin, A., and Guiton, M. (2023). A joint probability distribution model for multivariate wind and wave conditions. In *International Conference on Offshore Mechanics and Arctic Engineering*, volume 86847, page V002T02A013. American Society of Mechanical Engineers.
- Veers, P. (1988). Three-dimensional wind simulation. Technical report, Sandia National Labs., Albuquerque, NM (USA).
- Veers, P., Dykes, K., Lantz, E., Barth, S., Bottasso, C., Carlson, O., Clifton, A., Green, J., Green, P., Holttinen, H., et al. (2019). Grand challenges in the science of wind energy. *Science*, 366(6464):eaau2027.
- Velarde, J., Kramhøft, C., and Sørensen, J. (2019). Global sensitivity analysis of offshore wind turbine foundation fatigue loads. *Renewable Energy*, 140:177 – 189.
- Waarts, P. (2000). *Structural reliability using finite element methods: an appraisal of directional adaptive response surface sampling (DARS)*. PhD thesis, Technical University of Delft, The Netherlands.
- Walter, C. (2015). Moving particles: A parallel optimal multilevel splitting method with application in quantiles estimation and meta-model based algorithms. *Structural Safety*, 55:10–25.
- Wand, M. and Jones, M. (1994). *Kernel smoothing*. CRC press.
- Wang, Z. and Song, J. (2016). Cross-entropy-based adaptive importance sampling using von Mises-Fisher mixture for high dimensional reliability analysis. *Structural Safety*, 59:42–52.
- Warnock, T. (1972). Computational investigations of low-discrepancy point sets. In *Applications of number theory to numerical analysis*, pages 319–343. Elsevier.
- Waskom, M. (2021). Seaborn: statistical data visualization. *Journal of Open Source Software*, 6(60):3021.
- Wilkie, D. and Galasso, C. (2021). Gaussian process regression for fatigue reliability analysis of offshore wind turbines. *Structural Safety*, 88:102020.
- Wise, A. and Bachynski, E. (2020). Wake meandering effects on floating wind turbines. *Wind Energy*, 23(5):1266–1285.
- X., Z. (2022). *Surrogate Modeling for Stochastic Simulators Using Statistical Approaches*. PhD thesis, Chair of Risk, Safety and Uncertainty Quantification, ETH Zurich.
- Xu, Y. and Goodacre, R. (2018). On splitting training and validation set: A comparative study of cross-validation, bootstrap and systematic sampling for estimating the generalization performance of supervised learning. *Journal of Analysis and Testing*, 2:249–262.
- Yun, W., Lu, Z., Zhang, Y., and Jiang, X. (2018). An efficient global reliability sensitivity analysis algorithm based on classification of model output and subset simulation. *Structural Safety*, 74:49–57.
- Zhan, L., Letizia, S., and Iungo, G. (2020). Optimal tuning of engineering wake models through lidar measurements. *Wind Energy Science*, 5(4):1601–1622.
- Zhang, P. (1996). Nonparametric importance sampling. *Journal of the American Statistical Association*, 91(435):1245–1253.
- Zong, H. and Porté-Agel, F. (2020). A momentum-conserving wake superposition method for wind farm power prediction. *Journal of Fluid Mechanics*, 889:A8.
- Zwick, D. and Muskulus, M. (2015). The simulation error caused by input loading variability in offshore wind turbine structural analysis. *Wind Energy*, 18:1421 – 1432.

Appendix **A**

Univariate distribution fitting

This appendix recalls the main methods to infer a univariate distribution considering a n -sized i.i.d sample $X_n = \{x^{(1)}, \dots, x^{(n)}\} \in \mathbb{R}^n$. The goal is to use this finite set of observations of the random variable X to approach its underlying distribution by an estimated distribution. The inference techniques are split into two main groups, the methods assuming that the underlying distribution belongs to a family of parametric distributions are called parametric. Otherwise, the fitting method falls into the nonparametric group. Nonparametric methods often require a larger amount of data but allow more flexibility. In fact, nontrivial distributions (e.g., multimodal) might be easier to model using nonparametric approaches. To assess the quality of this estimation regarding the sample, a panel of goodness-of-fit methods are proposed [add ref], this appendix recalls a few of them. Note that the following tools can be used to estimate the marginals of a multivariate distribution.

A.1 Main parametric methods

Moments method

The moments method aims at looking for a parametric distribution with density $f_X(\theta)$, whose first moments (e.g., $m(\theta)$ and $\sigma^2(\theta)$) match the empirical moments of the sample X_n (e.g., \widehat{m}_{X_n} and $\widehat{\sigma}^2$). After computing the empirical moments:

$$\widehat{m}_n = \frac{1}{n} \sum_{i=1}^n x^{(i)}, \quad \widehat{\sigma}_n^2 = \frac{1}{n-1} \sum_{i=1}^n (x^{(i)} - \widehat{m}_{X_n})^2, \quad (\text{A.1})$$

one can solve the system of equations ($m(\theta) = \widehat{m}_n$; $\sigma^2(\theta) = \widehat{\sigma}_n^2$) to determine the optimal set of parameters θ in this situation. Some families of distributions are more suited to this method (i.e., \mathcal{N}) because of the analytical expression of their moments. Moreover, this technique is sensitive to the possible biases in the estimation of the sample moments.

Maximum likelihood estimation

Maximum likelihood estimation (MLE) is a popular alternative to the moments method. Similarly, it aims at maximizing a given correspondence metric between the dataset X_n and a parametric distribution with density $f_X(\theta)$. This metric is the *likelihood* function, defined as:

$$\mathcal{L}(\theta|X_n) = \prod_{i=1}^n f_X(x^{(i)}; \theta), \quad (\text{A.2})$$

with the PDF taking the set of parameters θ written: $f_X(x^{(i)}; \theta)$. For numerical reasons, the optimization is often performed on the natural logarithm of the likelihood function, called *log-likelihood*. The goal is then finding the optimal vector $\hat{\theta}^*$ of parameters minimizing the following expression:

$$\hat{\theta}^* = \arg \min_{\theta \in \mathcal{D}_\theta} \left(- \sum_{i=1}^n \ln(f_X(x^{(i)}; \theta)) \right). \quad (\text{A.3})$$

Remark that the quick analytical results from the moment method can be used as a starting point of the MLE optimization. [Asymptotic behaviors of this method are described in: add ref] [This method can be applied to censored data in the field of survival analysis. Add ref]

Example 1. Considering a small set of observations $X_n = \{1, 2, 3, 4, 6\}$, the following figure xx represents

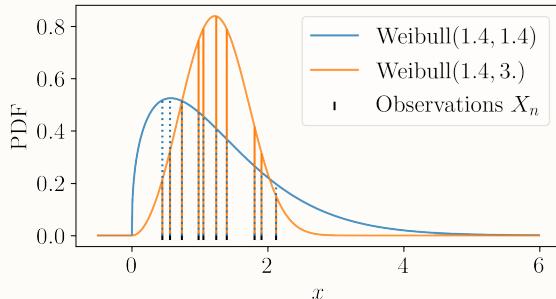


Figure A.1 Adequation of two different Weibull models using their likelihood with a sample of observations (black crosses).

A.2 Main nonparametric methods

Empirical CDF and histogram

The empirical CDF is a cumulative stair-shaped representation of the sorted sample X_n :

$$\widehat{F}_X(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{x^{(i)} \leq x\}}. \quad (\text{A.4})$$

A histogram consists in sorting and gathering the observations in a sample X_n into a finite number of categories. These categories are called bins and each regroups the same number of observations (identical binwidth). The number of bins is the only tuning parameter of this method. Its definition has a great impact on the visual consistency of the plot, therefore, many rules exist to define it. Note that the empirical CDF can be seen as a cumulative histogram with the number of bins equal to the number of observations.

Kernel density estimation

Kernel density estimation (KDE) is a nonparametric method, it estimates a PDF by weighing a sample of observations X_n with kernels. After setting a kernel $k : \mathbb{R} \rightarrow \mathbb{R}_+$ and a scaling parameter $h > 0$, also called bandwidth, the kernel density estimator is defined as:

$$\hat{f}_X(x) = \frac{1}{nh} \sum_{i=1}^n k\left(\frac{x - x^{(i)}}{h}\right) \quad (\text{A.5})$$

Different types of kernels are used for KDE, such as the uniform, triangular, squared exponential or Epachnikov. The choice of bandwidth results in a bias-variance trade-off, that has been extensively discussed in the literature (Wand and Jones, 1994).

Example 2. Considering a small set of observations $X_n = \{1, 2, 3, 4, 6\}$, the following figure xx represents three fits obtained by.

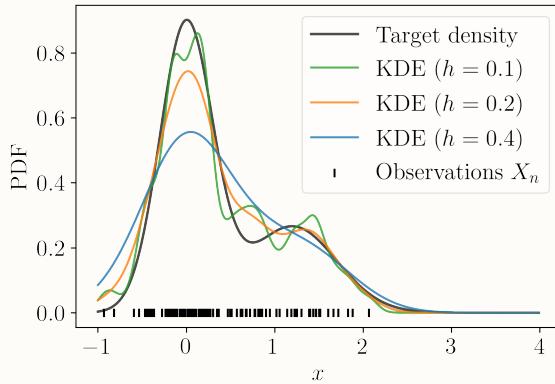


Figure A.2 Fit of a bimodal density by KDE using different tuning parameters.

Main goodness-of-fit methods

Penalized likelihood criteria

Two quantitative goodness-of-fit criteria are commonly used to assess parametric inference: the *Akaike information criterion* (AIC) and the *Bayesian information criterion* (BIC). The likelihood as a goodness-of-fit criterion should only be applied to the same family of distributions. Otherwise, the comparison would unfairly advantage distributions with a large number of degrees of

freedom. The two following criteria are metrics based on the likelihood with a correction related to the number of degrees of freedom of the distribution. Moreover, let us remind that more flexible models will require more data to provide a robust estimation.

The AIC and BIC are expressed as follows:

$$\text{AIC} = \frac{-2\ln(\mathcal{L}(\theta|X_n))}{n} + \frac{2q}{n}, \quad \text{BIC} = \frac{-2\ln(\mathcal{L}(\theta|X_n))}{n} + \frac{q\ln(n)}{n}, \quad (\text{A.6})$$

with the likelihood $\mathcal{L}(\theta|X_n)$ and the number of distribution's number degrees of freedom denoted q . The second term adds a penalty depending on the number of parameters. The best inference will be given by the model with the smallest AIC or BIC. Note that an additional correction can be applied in a small data context.

Kolmogorov-Smirnov adequacy test

Quantile-quantile plot

The quantile-quantile plot (also called QQ-plot) is a graphical tool providing a qualitative check of the goodness of fit. It compares the CDF of the fitted model with the empirical CDF of the sample X_n . To do so, it represents a scatterplot of the empirical quantiles (i.e., the ranked observations), against the quantiles of the fitted model at the levels $\{\alpha^{(i)}\}_{i=1}^n = \{\widehat{F}_X(x^{(i)})\}_{i=1}^n$. The following Fig. A.3 is a QQ-plot of the model fitted in [Example xx]. The closer the scatter plot gets to the first bisector line the best the fit is.

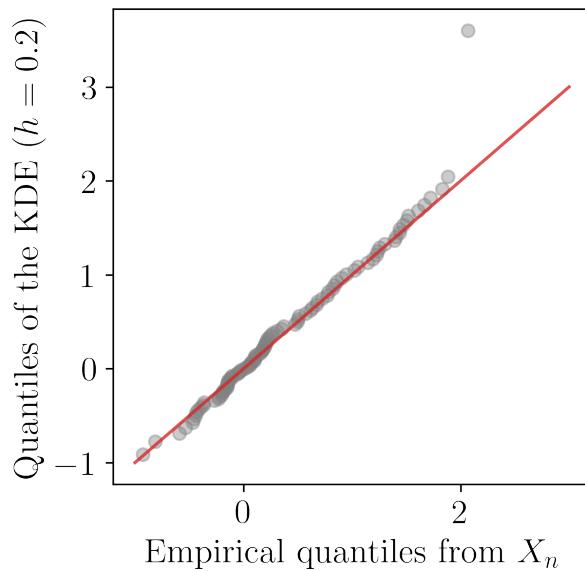


Figure A.3 QQ-plot between the data from Example 2 and a KDE model.

Appendix **B**

Dissimilarity measures between probability distributions

Beyond the discrepancy measure to the uniform distribution, this section introduces different dissimilarity measures between probability distributions.

B.1 Csizár f -divergences

[General definition]

[Numerous examples depending on the function chosen: see the book culte]

[Link between KL and mutual information] [Further inputs in the review from Rahman, maybe some in the PhD subject from A.Dufroy.] [Problems generated in the estimation]

B.2 Integral probability metrics

[general definition]

[Numerous examples see the book culte]

[No closed form expression unlike the f -divergence but the use of RKHS goes around this issue.]

B.2.1 Kernel discrepancy

Quasi-Monte Carlo sampling methods widely rely on a uniformity metric, called *discrepancy*. This section first presents the link between discrepancy and numerical integration. Then it introduces a kernel-based discrepancy, generalizing the concept to non-uniform measures. This tool is eventually used to build a sequential quadrature rule by subsampling from a finite dataset.

Reproducing kernel Hilbert space and kernel mean embedding To generalize the Koksma-Hlawka inequality to any probability measure, let us assume that the integrand g lives in a specific function space $\mathcal{H}(k)$. $\mathcal{H}(k)$ is a *reproducing kernel Hilbert space* (RKHS), which is an

inner product space of functions $g : \mathcal{D}_X \rightarrow \mathbb{R}$. Considering a symmetric and positive definite function $k : \mathcal{D}_X \times \mathcal{D}_X \rightarrow \mathbb{R}$, later called a “reproducing kernel” or simply a “kernel”, an RKHS verifies the following axioms:

- The “feature map” $\phi : \mathcal{D}_X \rightarrow \mathcal{H}(k); \phi(\mathbf{x}) = k(\cdot, \mathbf{x})$ belongs to the RKHS: $k(\cdot, \mathbf{x}) \in \mathcal{H}(k), \forall \mathbf{x} \in \mathcal{D}_X$;
- The “reproducing property”: $\langle g, k(\cdot, \mathbf{x}) \rangle_{\mathcal{H}(k)} = g(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{D}_X, \forall g \in \mathcal{H}(k)$.

Note that it can be shown that every positive semi-definite kernel defines a unique RKHS (and vice versa) with a feature map ϕ , such that $k(\mathbf{x}, \mathbf{x}') = \langle \phi(\mathbf{x}), \phi(\mathbf{x}') \rangle_{\mathcal{H}(k)}$. This framework allows us to embed a continuous or discrete probability measure in an RKHS, as illustrated in Fig. B.1. For any measure π , let us define its *kernel mean embedding* (Sejdinovic et al., 2013), also called “potential” $P_\pi(\mathbf{x})$ in Pronzato and Zhigljavsky (2020), associated with the kernel k as:

$$P_\pi(\mathbf{x}) := \int_{\mathcal{D}_X} k(\mathbf{x}, \mathbf{x}') d\pi(\mathbf{x}'). \quad (\text{B.1})$$

Respectively, the potential $P_{\zeta_n}(\mathbf{x})$ of a discrete distribution $\zeta_n = \sum_{i=1}^n w_i \delta(\mathbf{x}^{(i)})$, $w_i \in \mathbb{R}$ associated with the kernel k can be written as:

$$P_{\zeta_n}(\mathbf{x}) = \int_{\mathcal{D}_X} k(\mathbf{x}, \mathbf{x}') d\zeta_n(\mathbf{x}') = \sum_{i=1}^n w_i k(\mathbf{x}, \mathbf{x}^{(i)}). \quad (\text{B.2})$$

The potential $P_\pi(\mathbf{x})$ of the targeted measure π will be referred to as “target potential” and the potential $P_{\zeta_n}(\mathbf{x})$ associated with the discrete distribution ζ_n called “current potential” when its support is the current design X_n . When $P_{\zeta_n}(\mathbf{x})$ is close to $P_\pi(\mathbf{x})$, it can be interpreted as ζ_n being an adequate quantization or representation of π (which leads to a good estimation of a quantity such as $I_\pi(g)$ from Eq. (4.5)). Potentials can be computed in closed forms for specific pairs of distribution and associated kernel. Summary tables of some of these cases are detailed in Briol (2019) (section 3.4), Pronzato and Zhigljavsky (2020) (section 4), and extended in Fekhari et al. (2023). However, in most cases, the target potentials must be estimated on a large and representative sample, typically a large quasi-Monte Carlo sample of π .

Definition 1. The *energy* of a measure π is defined as the integral of the potential P_π against the measure, which leads to the following scalar quantity:

$$\varepsilon_\pi := \int_{\mathcal{D}_X} P_\pi(\mathbf{x}) d\pi(\mathbf{x}) = \iint_{\mathcal{D}_X^2} k(\mathbf{x}, \mathbf{x}') d\pi(\mathbf{x}) d\pi(\mathbf{x}'). \quad (\text{B.3})$$

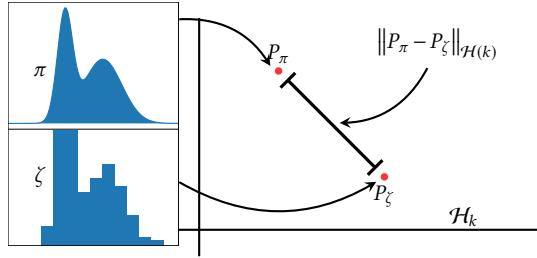


Figure B.1 Kernel mean embedding of a continuous and discrete probability distribution

Finally, using the reproducing property and writing the Cauchy-Schwarz inequality on the absolute quadrature error leads to the following inequality, similar to the Koksma-Hlawka inequality Eq. (4.6) (see Briol et al. (2019)):

$$\left| \sum_{i=1}^n w_i g(\mathbf{x}^{(i)}) - \int_{\mathcal{D}_X} g(\mathbf{x}) d\pi(\mathbf{x}) \right| = \left| \langle g, P_{\zeta_n}(\mathbf{x}) \rangle_{\mathcal{H}(k)} - \langle g, P_\pi(\mathbf{x}) \rangle_{\mathcal{H}(k)} \right| \quad (\text{B.4a})$$

$$= \left| \langle g, (P_{\zeta_n}(\mathbf{x}) - P_\pi(\mathbf{x})) \rangle_{\mathcal{H}(k)} \right| \quad (\text{B.4b})$$

$$\leq \|g\|_{\mathcal{H}(k)} \|P_\pi(\mathbf{x}) - P_{\zeta_n}(\mathbf{x})\|_{\mathcal{H}(k)}. \quad (\text{B.4c})$$

Maximum mean discrepancy A metric of discrepancy and quadrature error is offered by the *maximum mean discrepancy* (MMD). This distance between two probability distributions π and ζ is given by the worst-case error for any function within a unit ball of the Hilbert space $\mathcal{H}(k)$, associated with the kernel k :

$$\text{MMD}(\pi, \zeta) := \sup_{\|g\|_{\mathcal{H}(k)} \leq 1} \left| \int_{\mathcal{D}_X} g(\mathbf{x}) d\pi(\mathbf{x}) - \int_{\mathcal{D}_X} g(\mathbf{x}) d\zeta(\mathbf{x}) \right| \quad (\text{B.5})$$

According to the inequality in Eq. (B.4c), $\text{MMD}(\pi, \zeta) = \|P_\pi - P_\zeta\|_{\mathcal{H}(k)}$, meaning that the MMD fully relies on the difference of potentials. Moreover, Sriperumbudur et al. (2010) defines a kernel as “characteristic kernel” when the following equivalence is true: $\text{MMD}(\pi, \zeta) = 0 \Leftrightarrow \pi = \zeta$. This property makes the MMD a metric on \mathcal{D}_X . The squared MMD has been used for other purposes than numerical integration: e.g., statistical testing (Gretton et al., 2006), and global sensitivity analysis (Da Veiga, 2015). It can be written as follows:

$$\text{MMD}(\pi, \zeta)^2 = \|P_\pi(\mathbf{x}) - P_\zeta(\mathbf{x})\|_{\mathcal{H}(k)}^2 \quad (\text{B.6a})$$

$$= \langle (P_\pi(\mathbf{x}) - P_\zeta(\mathbf{x})), (P_\pi(\mathbf{x}) - P_\zeta(\mathbf{x})) \rangle_{\mathcal{H}(k)} \quad (\text{B.6b})$$

$$= \langle P_\pi(\mathbf{x}), P_\pi(\mathbf{x}) \rangle_{\mathcal{H}(k)} - 2 \langle P_\pi(\mathbf{x}), P_\zeta(\mathbf{x}) \rangle_{\mathcal{H}(k)} + \langle P_\zeta(\mathbf{x}), P_\zeta(\mathbf{x}) \rangle_{\mathcal{H}(k)} \quad (\text{B.6c})$$

$$= \iint_{\mathcal{D}_X^2} k(\mathbf{x}, \mathbf{x}') d\pi(\mathbf{x}) d\pi(\mathbf{x}') - 2 \iint_{\mathcal{D}_X^2} k(\mathbf{x}, \mathbf{x}') d\pi(\mathbf{x}) d\zeta(\mathbf{x}') + \iint_{\mathcal{D}_X^2} k(\mathbf{x}, \mathbf{x}') d\zeta(\mathbf{x}) d\zeta(\mathbf{x}'). \quad (\text{B.6d})$$

Taking a discrete distribution with uniform weights $\zeta_n = \frac{1}{n} \sum_{i=1}^n \delta(\mathbf{x}^{(i)})$, the squared MMD reduces to:

$$\text{MMD}(\pi, \zeta_n)^2 = \varepsilon_\pi - \frac{2}{n} \sum_{i=1}^n P_\pi(\mathbf{x}^{(i)}) + \frac{1}{n^2} \sum_{i,j=1}^n k(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}). \quad (\text{B.7})$$

Maximum discrepancy measure

A metric of discrepancy between distributions is introduced as the *maximum mean discrepancy* (MMD). This distance between two probability distributions μ and ζ is defined as the worst-case error for any function within a unit ball of a function space \mathcal{H} :

$$\text{MMD}(\mu, \zeta) := \sup_{\|g\|_{\mathcal{H}} \leq 1} \left| \int_{\mathcal{D}_X} g(\mathbf{x}) d\mu(\mathbf{x}) - \int_{\mathcal{D}_X} g(\mathbf{x}) d\zeta(\mathbf{x}) \right| = \|P_\mu(\mathbf{x}) - P_\zeta(\mathbf{x})\|_{\mathcal{H}}. \quad (\text{B.8})$$

To ease the calculation of the quantity, this metric was studied for a particular function space, offering specific properties. A *reproducing kernel Hilbert space* (RKHS), denoted $\mathcal{H}(k)$, is an inner product space $\mathcal{H}(k)$ of functions $g : \mathcal{D}_X \rightarrow \mathbb{R}$. It verifies the following axioms, considering a symmetric and positive definite function $k : \mathcal{D}_X \times \mathcal{D}_X \rightarrow \mathbb{R}$, later called a “reproducing kernel” or simply a “kernel”:

- The “feature map” $\phi : \mathcal{D}_X \rightarrow \mathcal{H}(k); \phi(\mathbf{x}) = k(\cdot, \mathbf{x}) \in \mathcal{H}(k), \forall \mathbf{x} \in \mathcal{D}_X$.
- The “reproducing property”: $\langle g, k(\cdot, \mathbf{x}) \rangle_{\mathcal{H}(k)} = g(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{D}_X, \forall g \in \mathcal{H}(k)$.

Every positive semi-definite kernel defines a unique RKHS (and vice versa) with a feature map ϕ , such that $k(\mathbf{x}, \mathbf{x}') = \langle \phi(\mathbf{x}), \phi(\mathbf{x}') \rangle_{\mathcal{H}(k)}$. Moreover, [Sriperumbudur et al. \(2010\)](#) defines a kernel as “characteristic kernel” when the following equivalence is true: $\text{MMD}_k(\mu, \zeta) = 0 \Leftrightarrow \mu = \zeta$. This property makes the MMD a metric on \mathcal{D}_X .

Then, a probability measure has a representation in the RKHS through its *kernel mean embedding* ([Sejdinovic et al., 2013](#)), also called “potential” $P_\mu(\mathbf{x})$ in [Pronzato and Zhigljavsky \(2020\)](#), defined as:

$$P_\mu(\mathbf{x}) := \int_{\mathcal{D}_X} k(\mathbf{x}, \mathbf{x}') d\mu(\mathbf{x}'). \quad (\text{B.9})$$

The reproducing property from the RKHS allows to express the squared MMD as expectations of kernels:

$$\text{MMD}_k(\mu, \zeta)^2 = \int_{\mathcal{D}_X} P_\mu(\mathbf{x}) d\mu(\mathbf{x}) - 2 \int_{\mathcal{D}_X} P_\mu(\mathbf{x}) d\zeta(\mathbf{x}) + \int_{\mathcal{D}_X} P_\zeta(\mathbf{x}) d\zeta(\mathbf{x}). \quad (\text{B.10})$$

[Add a sentence on estimation]

Analytical computation of potentials for Matérn kernels

As for tensor-product kernels, the potential is the product of the one-dimensional potentials, we only consider one-dimensional input spaces.

For μ the uniform distribution on $[0, 1]$ and K the Matérn kernel $K_{5/2,\theta}$ with smoothness $\nu = 5/2$ and correlation length θ , see (5.16), we get

$$P_{K_{5/2,\theta},\mu}(x) = \frac{16\theta}{3\sqrt{5}} - \frac{1}{15\theta}(S_\theta(x) + S_\theta(1-x)),$$

where

$$S_\theta(x) = \exp\left(-\frac{\sqrt{5}}{\theta}x\right)\left(5\sqrt{5}x^2 + 25\theta x + 8\sqrt{5}\theta^2\right).$$

The expressions $P_{K_{\nu,\theta},\mu}(x)$ for $\nu = 1/2$ and $\nu = 3/2$ can be found in ?.

When μ is the standard normal distribution $\mathcal{N}(0, 1)$, the potential $P_{K_{5/2,\theta},\mathcal{N}(0,1)}$ is $P_{K_{5/2,\theta},\mathcal{N}(0,1)}(x) = T_\theta(x) + T_\theta(-x)$, where

$$\begin{aligned} T_\theta(x) &= \frac{1}{6}\left(\frac{5}{\theta^2}x^2 + \left(3 - \frac{10}{\theta^2}\right)\frac{\sqrt{5}}{\theta}x + \frac{5}{\theta^2}\left(\frac{5}{\theta^2} - 2\right) + 3\right) \\ &\quad \times \operatorname{erfc}\left(\frac{\frac{\sqrt{5}}{\theta} - x}{\sqrt{2}}\right) \exp\left(\frac{5}{2\theta^2} - \frac{\sqrt{5}}{\theta}x\right) + \frac{1}{3\sqrt{2\pi}}\frac{\sqrt{5}}{\theta}\left(3 - \frac{5}{\theta^2}\right) \exp\left(-\frac{x^2}{2}\right). \end{aligned}$$

Advanced rare event estimation algorithms

C.1 Subset simulation (SS)

C.2 Nonparametric adaptive importance sampling (NAIS)

Appendix D

Uncertainty quantification practice with OpenTURNS

[Add short introduction to the motivation]

[Should I print the results?]

OpenTURNS 7 (Bivariate distribution). The following Python code proposes a minimalist OpenTURNS example of a probabilistic uncertainty modeling.

```
1 #!/usr/bin/python3
2 import openturns as ot
3 # Build multivariate distribution from marginals and copula
4 copula=ot.GumbelCopula(2.0)
5 marginals=[ot.Uniform(1.0, 2.0), ot.Normal(2.0, 3.0)]
6 distribution=ot.ComposedDistribution(marginals, copula)
7 # Compute first moments
8 mean_vector=distribution.getMean()
9 covariance_matrix=distribution.getCovariance()
10 # Compute CDF (respectively PDF)
11 x_cdf=distribution.computeCDF([1.5, 2.5]) # x=[1.5, 2.5]
12 a_quantile=distribution.computeQuantile([0.9]) # alpha=0.9
```

OpenTURNS 8 (Numerical integration). The following Python code presents a minimalistic OpenTURNS example to build multivariate quadrature rules.

```

1  #!/usr/bin/python3
2  import openturns as ot
3  marginals=[ot.Exponential(1.0), ot.Uniform(-1.0, 1.0)]
4  distribution=ot.ComposedDistribution(marginals)
5  # Build a 2D Gaussian quadrature
6  n_marginal=[4, 4] # Number of nodes per marginal
7  g_quad=ot.GaussProductExperiment(distribution, n_marginal)
8  g_nodes, weights=g_quad.generateWithWeights()
9  # Build a Monte Carlo design
10 n=16
11 mc_nodes=distribution.getSample(n)
12 # Build a quasi-Monte Carlo design
13 sequence=ot.HaltonSequence(2) # d=2
14 qmc_experiment=ot.LowDiscrepancyExperiment(sequence, distribution, n)
15 qmc_nodes=qmc_experiment.generate()
```

OpenTURNS 9 (Design of experiments). The following Python code is a minimalistic OpenTURNS example to build an LHS and an LHS optimized w.r.t. to a space-filling metric (here the L2-centered discrepancy) using the simulated annealing algorithm.

```

1  #!/usr/bin/python3
2  import openturns as ot
3  marginals=[ot.Uniform(0.0, 1.0), ot.Uniform(0.0, 1.0)]
4  distribution=ot.ComposedDistribution(marginals)
5  # Build a LHS
6  n=10
7  LHS_exp=ot.LHSExperiment(distribution, n)
8  LHS_design=LHS_exp.generate()
9  # Build an optimized LHS using L2-centered discrepancy
10 LHS_exp=ot.LHSExperiment(distribution, n)
11 SF_metric=ot.SpaceFillingC2()
12 SA_profile=ot.GeometricProfile(10., 0.95, 20000)
13 LHS_opt=ot.SimulatedAnnealingLHS(LHS_exp, SF_metric, SA_profile)
14 LHS_opt.generate()
15 LHS_design=LHS_opt.getResult().getOptimalDesign()
```

OpenTURNS 10 (Rare event estimation). The following Python code proposes a minimalist OpenTURNS implementation of rare event estimation algorithms.

```

1  #!/usr/bin/python3
2  import openturns as ot
3  marginals=[ot.Normal(0.0, 1.0), ot.Exponential(1.0)]
4  distribution=ot.ComposedDistribution(marginals)
5  # Build a limit-state function and failure event
6  g=ot.SymbolicFunction(["x1", "x2"], ["(x1 - x2) ^ 2"])
7  X=ot.RandomVector(distribution)
8  Y=ot.CompositeRandomVector(g, X)
9  th=0.0
10 failure_event=ot.ThresholdEvent(Y, ot.LessOrEqual(), th)
11 # Estimate pf using FORM
12 starting_p=distribution.getMean()
13 FORM_algo=ot.FORM(ot.Cobyla(), failure_event, starting_p)
14 FORM_algo.run()
15 FORM_results=FORM_algo.getResult()
16 design_point=FORM_results.getStandardSpaceDesignPoint()
17 FORM_pf=FORM_results.getEventProbability()
18 # Estimate pf using Monte Carlo
19 MC_exp=ot.MonteCarloExperiment()
20 MC algo=ot.ProbabilitySimulationAlgorithm(failure_event, MC_exp)
21 MC algo.run()
22 MC_results=MC algo.getResult()
23 MC_pf=MC_results.getProbabilityEstimate()
24 MC_pf_confidence=MC_results.getConfidenceLength(0.95)
25 # Estimate pf using importance sampling
26 aux_distribution=ot.Normal(design_point, [1.0, 1.0])
27 standard_event=ot.StandardEvent(failure_event)
28 IS_exp=ot.ImportanceSamplingExperiment(aux_distribution)
29 IS algo=ot.ProbabilitySimulationAlgorithm(standard_event, IS_exp)
30 IS algo.run()
31 IS_results=IS algo.getResult()
32 IS_pf=IS_results.getProbabilityEstimate()
33 IS_pf_confidence=IS_results.getConfidenceLength(0.95)
34 # Estimate pf using subset sampling
35 SS algo=ot.SubsetSampling(failure_event)
36 SS algo.run()
37 SS_results=SS algo.getResult()
38 SS_pf=SS_results.getProbabilityEstimate()
39 SS_pf_confidence=SS_results.getConfidenceLength(0.95)
```

OpenTURNS 11 (Sobol' indices). The following Python code gives a minimalistic OpenTURNS implementation of the Sobol' indices to assess global sensitivity analysis on the Ishigami analytical problem.

```

1  #!/usr/bin/python3
2  import openturns as ot
3  g=ot.SymbolicFunction(
4      ['x1', 'x2', 'x3'],
5      ['sin(x1) + 7.0 * sin(x2)^2 + 0.1 * x3^4 * sin(x1)']
6  )
7  X=ot.ComposedDistribution([ot.Uniform(-3.14, 3.14)] * 3)
8  size=1000
9  # Generate samples and evaluate their images
10 sie=ot.SobolIndicesExperiment(im.distributionX, size)
11 input_design=sie.generate()
12 output_design=im.model(input_design)
13 # Four estimators : Saltelli, Martinez, Jansen, and Mauntz-Kucherenko
14 SA=ot.JansenSensitivityAlgorithm(input_design, output_design, size)
15 sobol_first_order=SA.getFirstOrderIndices()
16 sobol_tolal=SA.getTotalOrderIndices()
```

OpenTURNS 12 (Gaussian process regression). The following Python code gives a minimalistic OpenTURNS implementation of an ordinary kriging model fitting.

```

1  #!/usr/bin/python3
2  import openturns as ot
3  g=ot.SymbolicFunction(['x'], ['x * sin(x) + sin(6 * x)'])
4  x_train=ot.Uniform(0., 12.).getSample(7) # n=7
5  y_train=g(x_train)
6  basis=ot.ConstantBasisFactory(1).build() # d=1
7  cov_model=ot.MaternModel([1.], 1.5)
8  algo=ot.KrigingAlgorithm(x_train, y_train, cov_model, basis)
9  algo.run()
10 kriging_results=algo.getResult()
11 kriging_predictor=kriging_results.getMetaModel()
```

Appendix E

Résumé étendu de la thèse

E.1 Introduction

Contexte industriel

L'enjeu actuel de la transition énergétique implique, entre autres, de réduire la part des énergies fossiles au sein du mix électrique mondial. Dans ce contexte, l'énergie éolienne en mer présente plusieurs avantages [Beauregard et al. \(2022\)](#). L'éolien en mer bénéficie notamment de vents plus constants que l'éolien terrestre, notamment dû à l'absence de relief, et offre la possibilité d'installer des éoliennes plus grandes donc plus puissantes. Depuis l'installation de la première ferme éolienne en mer à Vindeby, au Danemark, en 1991, l'industrie a connu une croissance rapide, avec une capacité totale de 56 GW exploitée dans le monde en 2021. Au fil du temps, la technologie éolienne en mer s'est améliorée, aboutissant à des succès importants tels que la signature de projets non subventionnés en Europe (en anglais *zero-subsidy bids*), pour lesquels l'électricité produite est directement vendue sur le marché de gros ([Beauregard et al., 2022](#)).

Cependant, malgré les progrès techniques indéniables, des limites industrielles émergent vis-à-vis de ces parcs éoliens en mer, posant ainsi de nombreux défis scientifiques. Pour atteindre les ambitieux objectifs de développement au niveau national et régional, la filière de l'éolien en mer fait face à plusieurs problèmes liés à l'augmentation de la taille des turbines. Ce changement d'échelle crée notamment des tensions liées à la logistique portuaire, aux besoins en ressources primaires et à la gestion durable du démantèlement futur. Ce secteur présente plusieurs défis techniques et scientifiques, qui requièrent l'utilisation conjointe de données mesurées et de simulations numériques d'éoliennes dans leur environnement. La recherche appliquée à l'éolien en mer fait intervenir plusieurs disciplines qui étudient notamment des sujets tels que la conception d'éoliennes flottantes, l'amélioration de l'estimation des ressources éoliennes, l'optimisation des opérations de maintenance et l'augmentation de la durée de vie utile des parcs. De manière générale, plusieurs décisions sont prises durant la vie d'une éolienne par son concepteur, installateur et exploitant, tout en ayant une connaissance partielle de certains phénomènes physiques. Par conséquent, modéliser et maîtriser les diverses sources

d'incertitudes associées à l'éolien en mer s'avère être un élément déterminant dans une industrie hautement concurrentielle.

Dans l'ensemble, l'industrie de l'éolien en mer a besoin de méthodes de traitement des incertitudes pour maîtriser les marges de sûreté et la gestion des actifs industriels (à la maille des composants, de l'éolienne et du parc dans son ensemble) (Van Kuik et al., 2016). Pour un développeur de projets éoliens, l'attention est d'abord portée sur l'amélioration du potentiel éolien des sites candidats en combinant différentes sources d'information et en modélisant la distribution multivariée des conditions environnementales au sein d'un parc éolien. Dans le cas de projets en éolien flottant, l'objectif est d'intégrer un aspect probabiliste dès la phase de conception (par exemple, du flotteur) afin de définir des solutions plus sûres, plus robustes et plus rentables. Pour un propriétaire d'un parc éolien, la gestion de la fin de vie est une autre problématique importante. Un propriétaire de parc éolien en fin de vie a le choix entre trois options : prolonger la durée de vie des actifs en exploitation, remplacer les éoliennes actuelles par des modèles plus récents, ou démanteler et vendre le parc éolien. Les deux premières solutions nécessitent d'évaluer la fiabilité de la structure et sa durée de vie résiduelle. Ces évaluations quantitatives sont examinées par des organismes de certification et des assureurs pour délivrer des permis d'exploitation. Pour fournir des évaluations rigoureuses des risques, la méthodologie générique de *traitement des incertitudes* est une démarche qui fait consensus dans les secteurs industriels confrontés à ce genre de problématique (de Rocquigny et al., 2008).

Méthodologie générique de traitement des incertitudes dans les outils de calcul scientifiques

La simulation numérique est une discipline qui a émergé avec l'avènement de l'informatique. Cette pratique produit des outils de calcul scientifique (OCS) qui permettent de simuler le comportement de système complexes compte tenu de conditions initiales définies par l'analyste. Les OCS sont vite devenus indispensables pour l'analyse, la conception, et la certification de systèmes complexes dans les cas où des expériences ou des mesures physiques sont coûteuses à obtenir, voire impossibles à réaliser. Cependant, ces modèles numériques s'intègrent dans une démarche déterministe : le résultat d'une simulation est associé à un vecteur de paramètres fixé en entrée. La question de la gestion des incertitudes associées aux entrées se pose rapidement lors de l'utilisation des OCS.

Le traitement des incertitudes vise à modéliser et à traiter les incertitudes autour d'un modèle numérique. Pour ce faire, une méthodologie générique a été proposée pour quantifier et analyser les incertitudes entre les variables d'entrée et de sortie d'un OCS (de Rocquigny et al., 2008). Une présentation des outils mathématiques utilisés dans ce domaine est proposée par Sullivan (2015). Cette approche apporte une meilleure compréhension d'un système, ce qui contribue à une prise de décision plus robuste.

La Figure E.1 illustre les étapes génériques de la méthodologie de quantification des incertitudes, qui sont brièvement décrites ci-après :

- **Étape A – Spécification du problème.** Cette étape consiste à déterminer le système étudié et construire un modèle numérique capable de simuler (précisément) son comportement. La spécification du problème implique également de définir l'ensemble des paramètres inhérents au modèle numérique. Ces paramètres comprennent aussi bien les variables d'entrée que les variables de sortie générées par la simulation. Dans ce document, le modèle numérique est considéré comme une boîte-noire, par opposition à des approches qui s'intègrent à l'intérieur des schémas de résolution numérique des équations de comportement du système (approches dites intrusives ([Le Maître and Knio, 2010](#))). En général, ces modèles numériques sont au préalable calibrés par rapport à des données mesurées et suivent un processus de validation et de vérification pour réduire les erreurs de modélisation ([Oberkampf and Roy, 2010](#)).
- **Étape B – Modélisation et quantification des incertitudes.** L'objectif de la deuxième étape est d'identifier et modéliser toutes les sources d'incertitude associées aux variables d'entrée. Dans la plupart des cas, cette modélisation est effectuée dans un cadre probabiliste.
- **Étape C – Propagation des incertitudes.** Lors de cette étape, les entrées incertaines sont propagées au travers du modèle de simulation numérique. Dès lors, la sortie du modèle numérique (habituellement de type scalaire) devient également incertaine. L'objectif est alors d'estimer une quantité d'intérêt, c'est-à-dire une statistique sur la variable aléatoire de sortie étudiée. La méthode de propagation de l'incertitude peut différer en fonction de la quantité d'intérêt visée (par exemple, la tendance centrale, un quantile, une probabilité d'événement rare, etc.).
- **Étape C' – Analyse de sensibilité.** En complément de la propagation d'incertitudes, une analyse de sensibilité peut être réalisée afin d'étudier le rôle attribué à chaque entrée incertaine dans la variabilité de la sortie d'intérêt.
- **Métamodélisation.** Compte tenu du coût de calcul élevé que représentent certaines simulations, des approches statistiques visent à émuler ces simulateurs coûteux partir d'un nombre limité de simulations. La quantification de l'incertitude peut alors être réalisée avec le modèle statique de substitution (ou métamodèle) pour un moindre coût de calcul. Cette étape optionnelle d'apprentissage statistique ne fait pas à proprement dit partie du traitement des incertitudes mais elle s'avère souvent essentielle pour permettre sa mise en œuvre pratique.

Verrous scientifiques et objectifs de la thèse

La maîtrise des risques et des incertitudes dans l'éolien est un enjeu majeur pour le groupe EDF en tant qu'exploitant. Cette thèse vise à adapter et appliquer, sur un cas d'usage issu de l'éolien en mer, une démarche globale de traitement des incertitudes. Ainsi, ce cas d'usage soulève des verrous scientifiques associés à ses particularités qui peuvent être décrites comme suit :

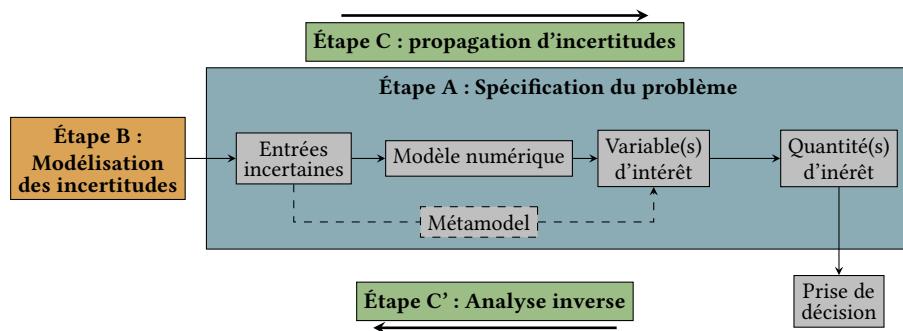


FIGURE E.1 Schéma générique de la quantification des incertitudes (de Rocquigny et al. (2008), adapté par Ajenjo (2023))

- Le code de simulation numérique autour duquel les travaux sont réalisés est constitué d'une chaîne de codes de calcul, exécutés en série. Cette chaîne s'articule en trois étapes : d'abord une génération temporelle et stochastique d'un champ de vitesse de vent et de houle, puis la simulation du comportement hydro-aéro-servo-élastique de l'éolienne et enfin une phase d'agrégation des résultats temporels pour obtenir des quantités d'intérêt scalaires ;
- La complexité de cet outil de calcul scientifique ainsi que le coût de calcul unitaire élevé (de l'ordre de 20 minutes par simulation) nécessite l'utilisation de méthodes d'échantillonnage performantes, ainsi que des systèmes de calcul haute performance. En plus de la complexité liée au modèle numérique, la modélisation des incertitudes en entrée présente, elle aussi, des difficultés. En effet, la loi conjointe des conditions environnementales liées à un site comporte une structure de dépendance complexe à capturer et à modéliser. L'étape d'inférence vis-à-vis des grandes quantités de données mesurées est d'autant plus importante que sa qualité impacte directement les conclusions de la propagation d'incertitudes.

Afin d'appliquer le schéma global de traitement des incertitudes au cas éolien, cette thèse vise à répondre aux problématiques suivantes :

- Q1.** *Comment précisément modéliser la structure de dépendance complexe associée aux lois conjointes de conditions environnementales ?* (⇒ Étape B)
- Q2.** *Comment réaliser une propagation d'incertitudes au travers d'une chaîne de simulation numérique coûteuse, uniquement basée sur une description empirique (données mesurées) des incertitudes en entrée ?* (⇒ Étape C)
- Q3.** *Comment estimer des probabilités d'événements rares associées à la ruine de structures éoliennes en mer ?* (⇒ Étape C)
- Q4.** *Comment évaluer et interpréter la sensibilité des entrées incertaines vis-à-vis des quantités d'intérêt liées à la fiabilité des structures (analyse de sensibilité fiabiliste) ?* (⇒ Étape C')

Les sections suivantes résument les travaux de thèse, tout en respectant la structure du manuscrit.

E.2 Résumés des chapitres relatifs à l'état de l'art des méthodes et outils mis en œuvre dans la thèse

Les deux premiers chapitres relateront l'état de l'art dans le domaine du traitement des incertitudes et de la modélisation numérique des systèmes éoliens.

Chapitre 1 – Traitement des incertitudes en simulation numérique

Ce chapitre vise à présenter un état de l'art concis des différentes thématiques en quantification des incertitudes (Sullivan, 2015). Après un rappel de quelques prérequis mathématiques, l'étape de spécification du modèle numérique (considéré comme étant une boîte-noire), ainsi que les variables d'entrée et de sortie est détaillée. Les différents types et sources d'incertitudes sont ensuite présentés, ainsi que leur modélisation dans un cadre probabiliste. La propagation des incertitudes dépend de la nature des quantités d'intérêt estimées, ainsi, une section aborde les méthodes de propagation pour l'étude en tendance centrale et une autre s'intéresse aux problèmes d'estimation de probabilités d'événements rares (statistiques liées aux queues de distributions). La section dédiée à la tendance centrale présente des méthodes d'intégration numérique, d'échantillonnage et de planification d'expériences (Fang et al., 2018). Celle consacrée aux probabilités d'événements rares présente des méthodes classiques issues du domaine de la fiabilité des structures (Lemaire et al., 2009; Morio and Balesdent, 2015).

Ce chapitre aborde également les principales méthodes d'analyse de sensibilité globale (Da Veiga et al., 2021). Ce domaine divise ses méthodes en deux grandes classes : les méthodes de criblage et les mesures d'importance. D'une part, les techniques de criblage, généralement mises en œuvre dans les problèmes de grande dimension, visent à identifier les variables n'ayant qu'un faible impact sur la variabilité de la sortie d'intérêt. D'autre part, les mesures d'importances visent, quant à elles, à attribuer de manière quantitative, pour chaque variable d'entrée, une part de variabilité de la sortie, permettant de proposer un classement des variables en fonction de leur influence.

Finalement, ce chapitre présente un panorama des familles de métamodèles communément utilisés en quantification des incertitudes (Forrester et al., 2008). Une attention particulière est apportée à la régression par processus gaussiens qui revient à conditionner un processus gaussien par un ensemble d'observations du code de simulation numérique. Une fois conditionné, le processus gaussien apporte une information plus riche que d'autres types de métamodèles. En effet, cette méthode propose conjointement un métamodèle (un prédicteur, ou moyenne du processus), et une fonction d'erreur (variance du processus). Certaines méthodes itératives (dites « actives ») exploitent cette information complémentaire pour enrichir progressivement le métamodèle et améliorer sa prédictivité. Ces techniques ont connu un franc succès dans les années 90 pour résoudre des problèmes d'optimisation de fonctions coûteuses (Jones et al., 1998). Depuis, leur utilisation s'est étendue à la résolution de problèmes de fiabilité des structures (Echard et al., 2011).

Chapitre 2 – Introduction à la modélisation et la conception de systèmes éoliens

La simulation d'une éolienne en mer implique la modélisation de plusieurs physiques en interaction avec des conditions environnementales de nature aléatoire. Ce chapitre introduit premièrement les méthodes spectrales utilisées pour générer des champs de vitesse de vent et de houle en appliquant des transformées de Fourier inverses (par exemple implémentées dans l'outil TurbSim ([Jonkman, 2009](#))). Ces champs de vitesses de vent simulés alimentent par la suite un outil de simulation multi-physique des éoliennes. Cette simulation intègre une modélisation simplifiée des interactions entre fluides et structures (méthode "BEMT" pour *blade element momentum theory*), une modélisation dynamique de la structure par des éléments finis de type poutre et une modélisation du contrôle-commande de l'éolienne [Milano \(2021\)](#). Ce code numérique produit en sortie des séries temporelles de plusieurs grandeurs physiques décrivant le comportement du système.

Cette thèse s'intéresse particulièrement à l'évaluation probabiliste du dommage en fatigue des structures éoliennes. Le dommage en fatigue est un phénomène qui détériore les propriétés mécaniques d'un matériau suite à sa sollicitation via un grand nombre de contraintes cycliques de faible amplitude. A l'heure actuelle, les standards [IEC-61400-1 \(2019\)](#); [DNV-ST-0437 \(2016\)](#) recommandent l'utilisation de coefficients de sécurité déterministes pour faire face à ce mode de défaillance. Une approche probabiliste permet d'enrichir l'analyse et parfois de mettre en évidence le conservatisme des marges de sûreté. Plusieurs travaux récents se sont intéressés à cette thématique en abordant des angles méthodologiques différents ([Huchet, 2019](#); [Lataniotis, 2019](#); [Cousin, 2021](#); [Hirvoas, 2021](#); [Petrovska, 2022](#)).

Dans ce contexte, ce chapitre liste les paramètres d'entrée de la chaîne de calcul considérés comme incertains par la suite. Ces variables aléatoires sont regroupées en deux groupes : le vecteur aléatoire lié à l'environnement (par exemple : la vitesse moyenne du vent, l'écart-type de la vitesse du vent, la direction du vent, la hauteur de houle, la période de houle, et la direction de houle), et le vecteur aléatoire lié au système (par exemple : l'erreur de d'alignement au vent du contrôleur, la rigidité du sol, les paramètres des courbes de calcul de fatigue).

E.3 Résumés des chapitres relatifs aux contributions méthodologiques et apports vis-à-vis des applications

Après avoir dressé l'état de l'art sur ce sujet, les prochains chapitres du manuscrit présentent les nouvelles contributions de la thèse. D'un point de vue méthodologique, un objet mathématique servira de fil conducteur au cours de ces travaux. La *maximum mean discrepancy* (MMD) [Oates \(2021\)](#) est une mesure de dissimilarité entre des lois de probabilité basée sur des noyaux qui est utilisée dans des contextes différents (tests statistiques [Gretton et al. \(2006\)](#), analyse de sensibilité [Da Veiga \(2015\)](#), échantillonnage [Pronzato and Zhigljavsky \(2020\)](#), etc.).

Chapitre 3 – Quantification des perturbations induites par les effets de sillage au sein d'un parc éolien

Ce chapitre étudie les perturbations sur les conditions environnementales à l'intérieur d'une ferme éolienne en mer induites par les effets de sillage (*wake effect* en anglais) [Larsen et al. \(2008\)](#). Un parc éolien en mer théorique au large de la côte sud de la Bretagne est considéré comme cas d'usage, et un modèle numérique simulant le sillage de ce parc est exploité. Ce modèle donne une prédition analytique du déficit en vitesse de vent et de la turbulence créés par le sillage, en tenant compte de l'influence de la position des flotteurs en raison des forces moyennes du vent. Une propagation de l'incertitude sur le modèle de sillage est réalisée, en considérant la loi conjointe des conditions environnementales ambiantes en entrée. Au final une distribution environnementale perturbée par le sillage est simulée pour chaque éolienne. Une mesure de dissimilarité (la MMD) est utilisée pour comparer les distributions perçues par chaque éolienne. Cette quantité permet de regrouper les éoliennes (phase de *clustering*) exposées à des conditions environnementales similaires, entraînant une réponse structurelle identiques. Compte tenu du coût de calcul élevé des simulations aéro-servo-hydro-élastiques des éoliennes en mer, cette étude préalable permet de réaliser une analyse de fiabilité à l'échelle d'une ferme éolienne sans répéter l'analyse pour chaque turbine. En fin de compte, seules quatre classes sont retenues pour représenter une ferme de 25 éoliennes. Ce travail a mené à la publication suivante :

✉ A. Lovera, [E. Fekhari](#), B. Jézéquel, M. Dupoiron, M. Guiton and E. Ardillon (2023). "Quantifying and clustering the wake-induced perturbations within a wind farm for load analysis". In : *Journal of Physics : Conference Series (WAKE 2023)*, Visby, Sweden.

Chapitre 4 – Méthodes à noyaux pour l'estimation de la tendance centrale

Ce chapitre présente une utilisation d'une mesure de dissimilarité basée sur des noyaux (la MMD) pour échantillonner suivant une loi de probabilité, méthode du "*kernel herding*" introduite par [Chen et al. \(2010\)](#). Cette technique de quadrature appartient à la famille dite des « quadratures Bayésiennes » ([Briol et al., 2019](#)) qui s'interprètent comme une généralisation des méthodes de quasi-Monte Carlo ([Li et al., 2020](#)). Le *kernel herding* est présenté en détails et plusieurs expériences numériques sur des fonctions analytiques illustrent son intérêt.

Les propriétés de cette méthode sont mises en valeur via une application industrielle dédiée à l'estimation de la moyenne du dommage en fatigue d'une structure éolienne. Cette quantité est déterminante dans le dimensionnement et la certification des éoliennes. Toutefois, son estimation par le biais de simulations numériques s'avère coûteuse. L'étude est réalisée sur un modèle d'une éolienne posée appartenant à une ferme installée en mer du Nord. Les incertitudes des conditions environnementales en entrée sont inférées sur des données mesurées in-situ.

Dans ce cadre, une comparaison numérique avec un échantillonnage Monte Carlo et quasi-Monte Carlo révèle la performance et les avantages pratiques du *kernel herding*. Cette méthode

permet notamment sous-échantillonner directement depuis une base de données environnementales importante, sans effectuer d'inférence (étape B). Ce travail a mené à la publication et au développement informatique suivant :

- ☞ E. Fekhari, V. Chabridon, J. Muré and B. Iooss (2023). “Given-data probabilistic fatigue assessment for offshore wind turbines using Bayesian quadrature”. In : *Data-Centric Engineering*, In press.
- ☞ Le module Python `ctbenchmark` standardise les expériences numériques liées à la quadrature Bayésienne et est disponible sur la plateforme GitHub.
- ☞ Le module Python `copulogram` propose une nouvelle représentation graphique de jeux de données multivariés et est disponible sur la plateforme de téléchargement Pypi.

Chapitre 5 – Méthodes à noyaux pour la validation de métamodèles

Ce chapitre propose une utilisation des méthodes d'échantillonage à base de noyaux dans le cadre de la validation de modèles d'apprentissage (ou métamodèles). L'estimation de la prédictivité des modèles d'apprentissage supervisé nécessite une évaluation de la fonction apprise sur un ensemble de points de test (non utilisés par lors de l'apprentissage). La qualité de l'évaluation dépend naturellement des propriétés de l'ensemble de test et de la statistique d'erreur utilisée pour estimer l'erreur de prédiction. Cette contribution propose d'une part d'utiliser des méthodes d'échantillonnage pour sélectionner de manière “optimale” un ensemble de test et d'autre part présente un nouveau critère de prédictivité qui pondère les erreurs observées pour obtenir une estimation globale de l'erreur. Une comparaison numérique entre plusieurs méthodes d'échantillonnage basées sur des approches géométriques ([Shang and Apley, 2020](#)) ou sur des méthodes à noyaux ([Chen et al., 2010](#); [Mak and Joseph, 2018](#)) est effectuée. Nos résultats montrent que les versions pondérées des méthodes à noyau offrent des performances supérieures. Une application aux efforts mécaniques simulées par un modèle éolien en mer est également présentée. Cette expérience illustre la pertinence pratique de cette technique comme alternative efficace aux techniques coûteuses de validation croisée. Ce travail a mené à la publication et au développement informatique suivant :

- ☞ E. Fekhari, B. Iooss, J. Muré, L. Pronzato and M.J. Rendas (2023). “Model predictivity assessment : incremental test-set selection and accuracy evaluation”. In : *Studies in Theoretical and Applied Statistics*, pages 315–347. Springer.
- ☞ Le module Python `otkerneldesign` est développé en collaboration avec J.Muré. Ce module dédié à la quadrature Bayésienne est documenté et disponible sur la plateforme de téléchargement Pypi.

Chapitre 6 – Estimation non-paramétrique de probabilités d'événements rares

L'estimation de probabilités d'événements rares est un problème courant dans la gestion des risques industriels, notamment dans le domaine de la fiabilité des structures Chabridon (2018). Pour ce faire, plusieurs techniques ont été proposées pour surmonter les limites connues de la méthode de Monte Carlo. Parmi elles, la méthode de “*subset sampling*” Au and Beck (2001) est une technique qui repose sur la décomposition de la probabilité de l'événement rare en un produit de probabilités conditionnelles moins rares (donc plus simples à estimer) associées à des événements de défaillance imbriqués. Cependant, cette technique repose sur la simulation conditionnelle à base de méthodes de Monte Carlo par chaînes de Markov (MCMC). Ces algorithmes permettent, à la convergence, de simuler selon la densité cible. Cependant, en pratique, ils produisent souvent des échantillons non indépendants et identiquement distribués (i.i.d.) en raison de la corrélation entre les chaînes de Markov. Ce chapitre propose une autre méthode pour échantillonner conditionnellement aux événements de défaillance imbriqués afin d'obtenir des échantillons dont la propriété d'être i.i.d. est préservée. La propriété d'indépendance des échantillons est particulièrement pertinente pour exploiter ces mêmes échantillons pour une analyse de sensibilité fiabiliste. L'algorithme proposé repose sur l'inférence non-paramétrique de la distribution conjointe conditionnelle en utilisant une estimation par noyau des marginales combinée à une inférence de la dépendance à l'aide de la copule empirique de Bernstein Sancetta and Satchell (2004). L'algorithme appelé “*Bernstein adaptive nonparametric conditional sampling*” (BANCS) est comparée à la méthode du *subset sampling* pour plusieurs problèmes de fiabilité des structures. Les premiers résultats sont encourageants, mais le contrôle du biais de l'estimateur doit être plus amplement investigué. Ce travail a mené à la publication et au développement informatique suivant :

☞ E. Fekhari, V. Chabridon, J. Muré and B. Iooss (2023). “Bernstein adaptive nonparametric conditional sampling : a new method for rare event probability estimation”. In : *Proceedings of the 13th International Conference on Applications of Statistics and Probability in Civil Engineering (ICASP 14)*, Dublin, Ireland.

☞ Le module Python `bancs` propose une implémentation de la méthode BANCS et est disponible sur la plateforme GitHub.

Chapitre 7 – Analyse de sensibilité fiabiliste adaptative

Ce chapitre traite d'analyse de sensibilité pour des mesures de risque (par exemple, une probabilité d'événement rare). L'analyse de sensibilité globale Da Veiga et al. (2021) attribue à chaque variable (ou groupe de variable) une part de variabilité globale de la sortie (le plus souvent à l'aide d'une décomposition fonctionnelle de la variance de la sortie). Cependant, les variables ayant un impact sur des quantités liées à une queue de distribution peuvent être très

différentes que celles ayant un impact sur la variabilité globale (pondérée par le poids associé au centre de la distribution). L’analyse de sensibilité fiabiliste (en anglais “*reliability-oriented sensitivity analysis*”, Chabridon (2018)) permet d’expliquer le rôle des entrées vis-à-vis de probabilités d’événements rares. L’idée de ce chapitre est d’étudier l’évolution de la sensibilité au fur et à mesure que l’échantillonnage se rapproche de l’événement rare. Cette analyse permet ainsi d’exploiter les paquets successifs d’échantillons conditionnels générés par l’algorithme BANCS (présenté dans le Chapitre 6). En post-traitement de l’estimation de la probabilité d’un événement rare, cette approche utilise une mesure d’importance à base de noyaux, nommée *Hilbert-Schmidt Independence Criterion*, pour évaluer la dynamique de la sensibilité fiabiliste Marrel and Chabridon (2021).

E.4 Conclusion

En résumé, cette thèse aborde plusieurs aspects du traitement des incertitudes à l’aide d’outils mathématiques à base de noyaux et présente un débouché industriel lié à l’enjeu de la maîtrise des risques des actifs éoliens en mer. Les contributions de cette thèse ont été principalement réalisées dans le cadre du projet européen HIPERWIND (*Highly advanced Probabilistic design and Enhanced Reliability methods for high-value, cost-efficient offshore wind.*), et de l’ANR INDEX (INcremental Design of EXperiments). Le sous-sections ci-après résument les communications, les publications dans revue à comité de lecture et les développements informatiques.

E.4.1 Communications et publications dans revues à comité de lecture

Book Chap.	<u>E. Fekhari</u> , B. Iooss, J. Muré, L. Pronzato and M.J. Rendas (2023). “Model predictivity assessment : incremental test-set selection and accuracy evaluation”. In : <i>Studies in Theoretical and Applied Statistics</i> , pages 315–347. Springer.
Jour. Pap.	<u>E. Fekhari</u> , V. Chabridon, J. Muré and B. Iooss (2023). “Given-data probabilistic fatigue assessment for offshore wind turbines using Bayesian quadrature”. In : <i>Data-Centric Engineering</i> , In press.
Int. Conf	<u>E. Fekhari</u> , B. Iooss, V. Chabridon, J. Muré (2022). “Numerical Studies of Bayesian Quadrature Applied to Offshore Wind Turbine Load Estimation”. In : <i>SIAM Conference on Uncertainty Quantification (SIAM UQ22)</i> , Atlanta, USA. (Talk)
	<u>E. Fekhari</u> , B. Iooss, V. Chabridon, J. Muré (2022). “Model predictivity assessment : incremental test-set selection and accuracy evaluation”. In : <i>22nd Annual Conference of the European Network for Business and Industrial Statistics (ENBIS 2022)</i> , Trondheim, Norway. (Talk)
	<u>E. Fekhari</u> , B. Iooss, V. Chabridon, J. Muré (2022). “Efficient techniques for fast uncertainty propagation in an offshore wind turbine multi-physics simulation tool”. In : <i>Proceedings of the 5th International Conference on Renewable Energies Offshore (RENEW 2022)</i> , Lisbon, Portugal. (Paper & Talk)
	<u>E. Fekhari</u> , V. Chabridon, J. Muré and B. Iooss (2023). “Bernstein adaptive nonparametric conditional sampling : a new method for rare event probability estimation” ¹ . In : <i>Proceedings of the 13th International Conference on Applications of Statistics and Probability in Civil Engineering (ICASP 14)</i> , Dublin, Ireland. (Paper & Talk)
	E. Vanem, <u>E. Fekhari</u> , N. Dimitrov, M. Kelly, A. Cousin and M. Guiton (2023). “A joint probability distribution model for multivariate wind and wave conditions”. In : <i>Proceedings of the ASME 2023 42th International Conference on Ocean, Offshore and Arctic Engineering (OMAE 2023)</i> , Melbourne, Australia. (Paper)
	A. Lovera, <u>E. Fekhari</u> , B. Jézéquel, M. Dupoirion, M. Guiton and E. Ardillon (2023). “Quantifying and clustering the wake-induced perturbations within a wind farm for load analysis”. In : <i>Journal of Physics : Conference Series (WAKE 2023)</i> , Visby, Sweden (Paper)
Nat. Conf.	<u>E. Fekhari</u> , B. Iooss, V. Chabridon, J. Muré (2022). “Kernel-based quadrature applied to offshore wind turbine damage estimation”. In : <i>Proceedings of the Mascot-Num 2022 Annual Conference (MASCOT NUM 2022)</i> , Clermont-Ferrand, France (Poster)
	<u>E. Fekhari</u> , B. Iooss, V. Chabridon, J. Muré (2023). “Rare event estimation using nonparametric Bernstein adaptive sampling”. In : <i>Proceedings of the Mascot-Num 2023 Annual Conference (MASCOT-NUM 2023)</i> , Le Croisic, France (Talk)
Invited Lec.	Le Printemps de la Recherche 2022, Nantes, France. “Traitement des incertitudes pour la gestion d’actifs éoliens”. (Talk)
	Journées Scientifiques de l’Eolien 2024, Saint-Malo, France. “Evaluation probabiliste de la fiabilité en fatigue des structures éoliennes en mer”. (Talk)

¹Cette contribution a été récompensée par le “CERRA Student Recognition Award”

E.4.2 Développements informatiques open source

`otkerneldesign`²

- Ce module Python génère des échantillons (aussi appelés plans d’expérience) en utilisant des méthodes à base de noyaux comme le *kernel herding* et les *support points*. Une implementation tensorisée qui améliore grandement les performances est également proposée. En complément, une méthode de pondération “optimale” à l’aide de quadrature Bayésienne est proposée.
- Ce module est développé en collaboration avec J. Muré, est documenté et disponible sur la plateforme de téléchargement Pypi.

`bancs`³

- Ce module Python offre une implémentation de la méthode “*Bernstein Adaptive Nonparametric Conditional Sampling*” mentionnée en Section E.3.
- Ce module est disponible sur la plateforme de GitHub et son utilisation est illustrée par des exemples analytiques.

`ctbenchmark`⁴

- Ce module Python standardise les comparaisons numériques réalisés pour étudier les méthodes de quadrature Bayésiennes.
- Le module et les expériences numériques sont disponibles sur un dépôt GitHub.

`copulogram`⁵

- Ce module Python propose une nouvelle représentation graphique de jeux de données multivariés appelée *copulogram*.
- Ce module, développé en collaboration avec V. Chabridon, est disponible sur la plateforme de téléchargement Pypi.

²Documentation :<https://efekhari27.github.io/otkerneldesign/master/>

³Dépôt: <https://github.com/efekhari27/bancs>

⁴Repository: <https://github.com/efekhari27/ctbenchmark>

⁵Repository: <https://github.com/efekhari27/copulogram>

