CSE211Digital Design

Akdeniz University

Week04-05: Boolean Algebra and Logic Gates

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Course program

Week 01	2-Oct-23 Introduction
Week 02	9-Oct-23 Digital Systems and Binary Numbers I
Week 03	16-Oct-23 Digital Systems and Binary Numbers II
Week 04	23-Oct-23 Boolean Algebra and Logic Gates I
Week 05	30-Oct-23 Boolean Algebra and Logic Gates II
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Week 09	27-Nov-23 Karnaugh Maps
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Previous chapter terms to know

- DVD (Digital Versatile Disc)
- GUI (Graphical User Interface)
- BCD (Binary Coded Decimal)
- ASCII (American Standard Code for Information Interchange)
- STX (Start of Text)
- ETX (End of Text)
- CR (Carriage Return)
- HDL (Hardware Description Language)

Boolean Algebra

- Boolean algebra is a mathematical system for the manipulation of variables that can have one of two values.
 - In formal logic, these values are "true" and "false."
 - ■In digital systems, these values are "on" and "off," 1 and 0, or "high" and "low."
- Boolean expressions are created by performing operations on Boolean variables.
 - Common Boolean operators include AND, OR, and NOT.

Boolean Algebra – Truth Table

- A Boolean operator can be completely described using a truth table.
- The truth table for the Boolean operators AND and OR are shown at the right.
- The AND operator is also known as a Boolean product. The OR operator is the Boolean sum.

X AND Y					
	X	Y	XY		
	0	0	0		
	0	1	0		
	1	0	0		
	1	1	1		

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X	Y	X+Y
0	0	0
0	1	1
1	0	1
1	1	1

X OR Y

Boolean Algebra – Truth Table

- The truth table for the Boolean NOT operator is shown at the right.
- The NOT operation is most often designated by an overbar. It is sometimes indicated by a prime mark (') or an "elbow" (¬).

NO.	ТХ
X	\overline{x}
0	1
1	0

Boolean Algebra – Boolean Functions

- A Boolean function has:
 - At least one Boolean variable,
 - At least one Boolean operator, and
 - At least one input from the set {0,1}.
- It produces an output that is also a member of the set {0,1}.

Now you know why the binary numbering system is so handy in digital systems.

Boolean Algebra – Boolean Functions

The truth table for the Boolean function:

$$F(x,y,z) = x\overline{z}+y$$
 is shown at the right.

To make evaluation of the Boolean function easier, the truth table contains extra (shaded) columns to hold evaluations of subparts of the function.

$$F(x,y,z) = x\overline{z}+y$$

x	У	Z	z	χZ	xz+y
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	0	1

Boolean Algebra – Rules of precedence

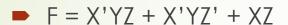
- As with common arithmetic, Boolean operations have rules of precedence
- The NOT operator has highest priority, followed by AND and then OR.
- This is how we chose the (shaded) function subparts in our table.

$$F(x,y,z) = x\overline{z} + y$$

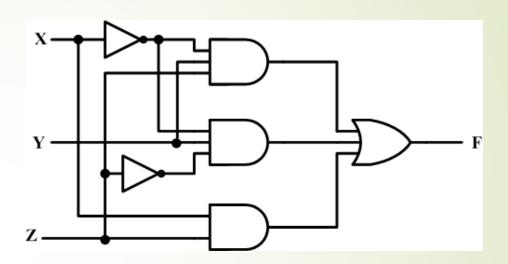
x	У	z	z	χZ	x z +y
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	0	1

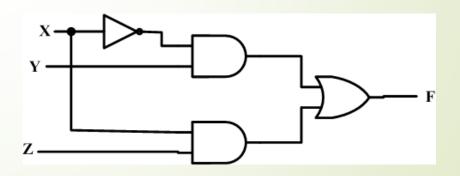
- Digital computers contain circuits that implement Boolean functions.
- The simpler that we can make a Boolean function, the smaller the circuit that will result.
 - Simpler circuits are cheaper to build, consume less power, and run faster than complex circuits.
- With this in mind, we always want to reduce our Boolean functions to their simplest form.
- There are a number of Boolean identities that help us to do this.

Boolean Algebra - Affect on implementation



Reduces to F = X'Y + XZ





Boolean Algebra – Laws

Most Boolean identities have an AND (product) form as well as an OR (sum) form. We give our identities using both forms. Our first group is rather intuitive:

Identity	AND	OR
Name	Form	Form
Identity Law Null Law Idempotent Law Inverse Law	$1x = x$ $0x = 0$ $xx = x$ $x\overline{x} = 0$	$0 + x = x$ $1 + x = 1$ $x + x = x$ $x + \overline{x} = 1$

Boolean Algebra – Laws (Cont.)

Our second group of Boolean identities should be familiar to you from your study of algebra:

Identity	AND	OR
Name	Form	Form
Commutative Law Associative Law Distributive Law	xy = yx $(xy)z = x(yz)$ $x+yz = (x+y)(x+z)$	x+y = y+x $(x+y)+z = x + (y+z)$ $x(y+z) = xy+xz$

■ Using distributive law x+x'y=(x+x')(x+y)

Boolean Algebra – Laws (Cont.)

- Our last group of Boolean identities are perhaps the most useful.
- If you have studied set theory or formal logic, these laws are also familiar to you.

Identity Name	AND Form	OR Form
Absorption Law DeMorgan's Law	$x(x+y) = x$ $(\overline{xy}) = \overline{x} + \overline{y}$	$\mathbf{x} + \mathbf{x}\mathbf{y} = \mathbf{x}$ $\overline{(\mathbf{x}+\mathbf{y})} = \overline{\mathbf{x}}\overline{\mathbf{y}}$
Double Complement Law	(<u>x</u>)	= x

Boolean Algebra – DeMorgan's Law

- Sometimes it is more economical to build a circuit using the complement of a function (and complementing its result) than it is to implement the function directly.
- DeMorgan's law provides an easy way of finding the complement of a Boolean function.
- Recall DeMorgan's law states:

$$\overline{(xy)} = \overline{x} + \overline{y}$$
 and $\overline{(x+y)} = \overline{x}\overline{y}$

Boolean Algebra – Proof: DeMorgan

Theorem 3.7: DeMorgan's Law

For each pair of elements x, y in a Boolean algebra:

$$\overline{(x+y)} = \overline{x} \cdot \overline{y}$$

$$\overline{(x \cdot y)} = \overline{x} + \overline{y}$$

Proof:

$$(x + y) + \overline{x} \cdot \overline{y} = (x + y + \overline{x}) \cdot (x + y + \overline{y})$$

$$= (x + \overline{x} + y) \cdot (y + \overline{y} + x)$$

$$= (1 + y) \cdot (1 + x)$$

$$= 1 \cdot 1$$

Note: Using Associativity of each operation (+), (\cdot)

Boolean Algebra – DeMorgan's Law (Cont.)

- DeMorgan's law can be extended to any number of variables.
- Replace each variable by its complement and change all ANDs to ORs and all ORs to ANDs.
- Thus, we find the complement of:

$$F(X,Y,Z) = (XY) + (\overline{X}Z) + (Y\overline{Z})$$
is:
$$\overline{F}(X,Y,Z) = \overline{(XY) + (\overline{X}Z) + (Y\overline{Z})}$$

$$= (\overline{XY})(\overline{XZ})(\overline{YZ})$$

$$= (\overline{X}+\overline{Y})(X+\overline{Z})(\overline{Y}+Z)$$

Truth Tables for the Laws of Boolean

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	Boolean Expression	Description	Equivalent Switching Circuit	Boolean Algebra Law or Rule			
	A + 1 = 1	A in parallel with closed = "CLOSED"	A	Annulment			
	A + 0 = A	A in parallel with open = "A"	A	Identity			
	A . 1 = A	A in series with closed = "A"	A	Identity			
	A . 0 = 0	A in series with open = "OPEN"	A .	Annulment			
	A + A = A	A in parallel with A = "A"	A	Idempotent			
	A . A = A	A in series with A = "A"	AAA	Idempotent			

Boolean Expression	Description	Equivalent Switching Circuit	Boolean Algebra Law or Rule
$NOT \overline{A} = A$	NOT NOT A (double negative) = "A"		Double Negation
$A + \overline{A} = 1$	A in parallel with NOT A = "CLOSED"	Ā	Complement
$A.\overline{A}=0$	A in series with NOT A = "OPEN"	A Ā	Complement
A+B=B+A	A in parallel with B = B in parallel with A	ABB	Commutative
A.B = B.A	A in series with B = B in series with A	AB	Commutative
$\overline{(A+B)}=\overline{A}.\overline{B}$	invert and replace OR with AND		de Morgan's Theorem
$\overline{(A.B)} = \overline{A} + \overline{B}$	invert and replace AND with OR		de Morgan's Theorem

Boolean Algebra – Complement of Functions

EXAMPLE 2.2

Find the complement of the functions $F_1 = x'yz' + x'y'z$ and $F_2 = x(y'z' + yz)$. By applying DeMorgan's theorems as many times as necessary, the complements are obtained as follows:

$$F'_1 = (x'yz' + x'y'z)' = (x'yz')'(x'y'z)' = (x + y' + z)(x + y + z')$$

$$F'_2 = [x(y'z' + yz)]' = x' + (y'z' + yz)' = x' + (y'z')'(yz)'$$

$$= x' + (y + z)(y' + z')$$

$$= x' + yz' + y'z$$

Boolean Algebra – Principle of Duality

- What is meant by the dual of a function?
 - The dual of a function is obtained by interchanging OR and AND operations and replacing 1s and 0s with 0s and 1s.
- Shortcut to getting function complement
 - ► Having F=X'YZ'+X'Y'Z
 - \blacksquare Generate the dual F=(X'+Y+Z')(X'+Y'+Z)
 - Complement each literal to get:
 - ► F'=(X+Y'+Z)(X+Y+Z')
- Each postulate consists of two expressions s.t. one expression is transformed into the other by interchanging the operations (+) and (·) as well as the identity elements 0 and 1.
- Such expressions are known as duals of each other.
- If some equivalence is proved, then its dual is also immediately true.
- E.g. If we prove: $(x \cdot x) + (\overline{x} \cdot \overline{x}) = 1$, then we have by duality: $(x + x) \cdot (\overline{x} + \overline{x}) = 0$

Boolean Algebra - Principle of Duality (Cont.)

- There are useful identities of Boolean expressions that can help us to transform an expression A into an equivalent expression B
- We can derive additional identities with the help of the dual of a Boolean expression.
- ■The dual of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0s and 1s.

Boolean Algebra – Duality

Examples:

The dual of
$$x(y + z)$$
 is $x + yz$
The dual of $x \cdot 1 + (y + z)$ is $(x + 0)(yz)$

The **dual** of a Boolean function F represented by a Boolean expression is the function represented by the dual of this expression.

This dual function, denoted by Fd, does not depend on the particular Boolean expression used to represent F.

Boolean Algebra – Duality

- Therefore, an identity between functions represented by Boolean expressions remains valid when the duals of both sides of the identity are taken.
- We can use this fact, called the **duality** principle, to derive new identities.
- For example, consider the absorption law x(x + y) = x
- By taking the duals of both sides of this identity, we obtain the equation x + xy = x, which is also an identity (and also called an absorption law).

Boolean Algebra – Example

We can use Boolean identities to simplify the

```
function: F(X,Y,Z) = (X + Y) (X + \overline{Y}) (\overline{XZ})
as follows:
```

```
(X + Y) (X + \overline{Y}) (X\overline{Z})
                                  Idempotent Law (Rewriting)
 (X + Y) (X + \overline{Y}) (\overline{X} + Z)
                                  DeMorgan's Law
 (XX + XY + XY + YY) (X + Z)
                                 Distributive Law
((X + YY) + X(Y + Y))(X + Z)
                                  Commutative & Distributive Laws
((X + 0) + X(1))(\overline{X} + Z)
                                  Inverse Law
  X(X + Z)
                                  Idempotent Law
  XX + XZ
                                  Distributive Law
  0 + XZ
                                  Inverse Law
                                  Idempotent Law
     XZ
```

Boolean Algebra Simplification Example

$$Q = (A + B).(A + C)$$

$$A.A + A.C + A.B + B.C - Distributive law$$

$$A + A.C + A.B + B.C - Idempotent AND law (A.A = A)$$

$$A(1 + C) + A.B + B.C - Distributive law$$

$$A.1 + A.B + B.C - Identity OR law (1 + C = 1)$$

$$A(1 + B) + B.C - Distributive law$$

$$A.1 + B.C - Identity OR law (1 + B = 1)$$

$$Q = A + (B.C) - Identity AND law (A.1 = A)$$

Boolean Algebra – Summary

Table 2.1Postulates and Theorems of Boolean Algebra

			<u> </u>		
	Postulate 2	(a)	x + 0 = x	(b)	$x \cdot 1 = x$
	Postulate 5	(a)	x + x' = 1	(b)	$x \cdot x' = 0$
	Theorem 1	(a)	x + x = x	(b)	$x \cdot x = x$
	Theorem 2	(a)	x + 1 = 1	(b)	$x \cdot 0 = 0$
	Theorem 3, involution		(x')' = x		
	Postulate 3, commutative	(a)	x + y = y + x	(b)	xy = yx
\	Theorem 4, associative	(a)	x + (y + z) = (x + y) + z	(b)	x(yz) = (xy)z
N	Postulate 4, distributive	(a)	x(y+z)=xy+xz	(b)	x + yz = (x + y)(x + z)
	Theorem 5, DeMorgan	(a)	(x + y)' = x'y'	(b)	(xy)' = x' + y'
	Theorem 6, absorption	(a)	x + xy = x	(b)	x(x + y) = x

Simplify the following Boolean functions to a minimum number of literals.

1.
$$x(x' + y) = xx' + xy = 0 + xy = xy$$
.

2.
$$x + x'y = (x + x')(x + y) = 1(x + y) = x + y$$
.

3.
$$(x + y)(x + y') = x + xy + xy' + yy' = x(1 + y + y') = x$$
.

4.
$$xy + x'z + yz = xy + x'z + yz(x + x')$$

= $xy + x'z + xyz + x'yz$
= $xy(1 + z) + x'z(1 + y)$
= $xy + x'z$.

5.
$$(x + y)(x' + z)(y + z) = (x + y)(x' + z)$$
, by duality from function 4.

- Find the simplified equivalent for the following equations
- a(a+b')
 - a.a+a.b'=a+a.b'=a.(1+b)=a.1=a
- a+(a'+b)'
 - a+(a')'.b'=a+ab'=a
- x.y+x'.z+y.z+y'z
 - xy+z
- xyz+x'yz+y'z
 - **Z**

- A.(A'+B)=?
- (A+B).(A+C)=\$
- ► F=A'.B+A+A.B=?
- A.B+A'C+B.C

A.B+A'.C

■ A'B'C' + A'B'C + ABC' + AB'C'

```
F (A,B,C) = A' B' C' + A' B' C + A B' C' + A B C' + A B C' + A B C

=A'B'(C'+C)+AB'(C'+C)+AB(C'+C)

=A'B'+AB'+AB

= A'B'+AB'+AB+AB'

=B'(A'+A)+A(B+B')

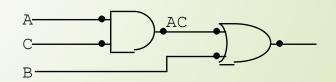
=B'+A
```

$$\rightarrow$$
 AB + A(B + C) + B(B + C)

$$= AB + AB + AC + BB + BC$$

$$= AB + AC + B + BC$$

$$= AB + AC + B$$



Boolean Algebra – Logically Equivalent Expressions

- Through our exercises in simplifying Boolean expressions, we see that there are numerous ways of stating the same Boolean expression.
 - ■These "synonymous" forms are logically equivalent.
 - Logically equivalent expressions have identical truth tables.
- In order to eliminate as much confusion as possible, designers express Boolean functions in *standardized* or *canonical* form.

Boolean Algebra – Sum of Products and Product of Sums

- There are two canonical forms for Boolean expressions: sumof-products and product-of-sums.
 - Recall the Boolean product is the AND operation and the Boolean sum is the OR operation.
- In the sum-of-products form, ANDed variables are ORed together.
 - For example: F(x,y,z) = xy + xz + yz
- In the product-of-sums form, ORed variables are ANDed together:
 - For example: F(x,y,z) = (x+y)(x+z)(y+z)

Boolean Algebra – Normal Forms

Consider the function:

$$f(w, x, y, z) = \overline{x} + w\overline{y} + \overline{wy}z$$

- A literal is an occurrence of a complemented or uncomplemented variable in a formula.
- A product term is either a literal or a product (conjunction) of literals.
- Disjunctive normal form: A Boolean formula written as a single product term or as a sum (disjunction) of product terms.

Boolean Algebra – Normal Forms

Consider the function:

$$f(w, x, y, z) = z(x + \overline{y})(w + \overline{x} + \overline{y})$$

- → A sum term is either a literal or a sum (disjunction) of literals.
- Conjunctive normal form: A Boolean formula written as a single sum term or as a product (conjunction) of sum terms.

Boolean Algebra – Canonical Formulas

■ How to obtain a Boolean formula given a truth table?

X	Y	Z	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

Boolean Algebra – Minterm Canonical Formula

Χ	Y	Z	f	
0	0	0	0	$\overline{x} \overline{y} z$
0	0	1	1	
0	1	0	0	$\overline{x} y z$
0	1	1	1	
1	0	0	1	
1	0	1	0	$x \overline{y} \overline{z}$
1	1	0	0	
1	1	1	0	

$$f(x,y,z) = \overline{x}\,\overline{y}\,z + \overline{x}y\,z + x\,\overline{y}\,\overline{z}$$

Boolean Algebra – m-Notation

X	Y	Z	f	
0	0	0	0	$\overline{x} \overline{y} z$
0	0	1	1 -	
0	1	0	0	$\overline{x} y z$
0	1	1	1	
1	0	0	1	
1	0	1	0	$x \overline{y} \overline{z}$
1	1	0	0	
1	1	1	0	

- ightharpoonup f(x,y,z) can be written as $f(x,y,z)=m_1+m_3+m_4$
- $f(x, y, z) = \Sigma m(1,3,4)$

Boolean Algebra – Maxterm Canonical Formula

X	Y	Z	f	
0	0	0	0	x + y + z
0	0	1	1	
0	1	0	0	$x + \overline{y} + z$
0	1	1	1	
1	0	0	1	$\overline{x} + y + \overline{z}$
1	0	1	0	
1	1	0	0	$\overline{x} + \overline{y} + z$
1	1	1	0	
				$\overline{x} + \overline{y} + \overline{z}$
(())		_	\	

$$f(x,y,z) = (x+y+z)(x+\overline{y}+z)$$
$$(\overline{x}+y+\overline{z})(\overline{x}+\overline{y}+z)(\overline{x}+\overline{y}+\overline{z})$$

Boolean Algebra – Sum of Products

- It is easy to convert a function to sum-of-products form using its truth table.
- We are interested in the values of the variables that make the function true (=1).
- Using the truth table, we list the values of the variables that result in a true function value.
- Each group of variables is then ORed together.

ਜ	(x	V	z.)	=	$x\bar{z}+y$
E /	(A)	У,	4	_	A Z T y

x	У	z	xz+y
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Boolean Algebra – Sum of Products

The sum-of-products form for our function is:

$$F(x,y,z) = \overline{x}y\overline{z} + \overline{x}yz + x\overline{y}\overline{z} + xy\overline{z} + xy\overline{z}$$

We note that this function is not in simplest terms. Our aim is only to rewrite our function in canonical sum-of-products form.

$$F(x,y,z) = x\overline{z} + y$$

x	У	z	xz+y
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

CNF and DNF

Find the f function using both DNF and CNF for the following truth tables

X	Y	f
0	0	0
0	1	1
1	0	1
1	1	0

X	Y	\int
0	0	1
0	1	1
1	0	1
1	1	1

X	Y	Z	f
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Logic Gates

- We have looked at Boolean functions in abstract terms.
- In this section, we see that Boolean functions are implemented in digital computer circuits called gates.
- A gate is an electronic device that produces a result based on two or more input values.
 - In reality, gates consist of one to six transistors, but digital designers think of them as a single unit.
 - Integrated circuits contain collections of gates suited to a particular purpose.

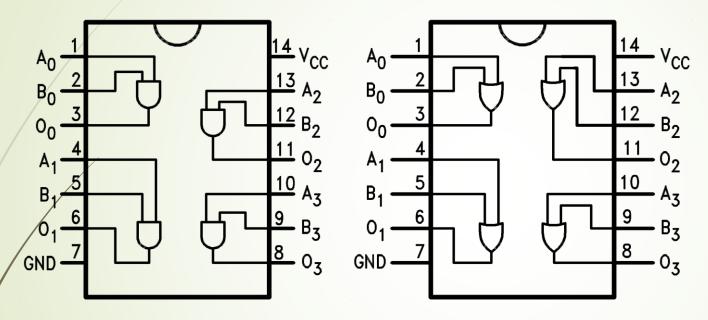
Logic Gates – AND OR NOT

The three simplest gates are the AND, OR, and NOT gates.

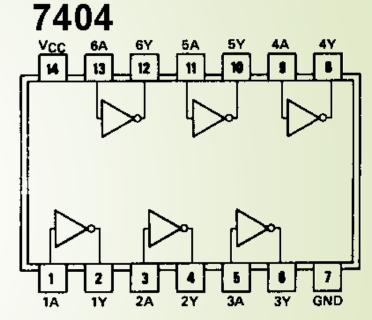
	х — ч —		<u> </u>	Y	х — ч —		X+2	X		>>— [₹]	×
	X	AN	DΥ		3	х оғ	7 Y		NO	ТХ	
ı	Х	Y	XY		X	Y	X+Y		Х	$\overline{\mathbf{x}}$	1
	0 0 1	0 1 0	0 0 0		0 0 1	0 1 0	0 1 1		0 1	1 0	
	1	1	1		1	1	1				

They correspond directly to their respective Boolean operations, as you can see by their truth tables.

Logic Gates – AND OR NOT



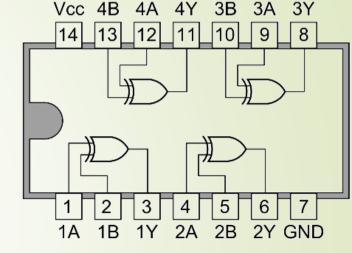
74AC08, 74ACT08 Quad 2-Input AND Gate 74AC32, 74ACT32 Quad 2-Input OR Gate



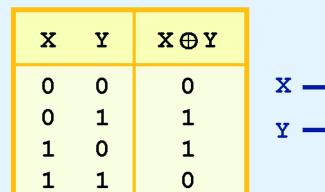
7404 6-Input NOT Gate

Logic Gates – XOR

- Another very useful gate is the exclusive OR (XOR) gate.
- The output of the XOR operation is true only when the values of the inputs differ.



74HCT86 Quad 2-input EXCLUSIVE-OR gate



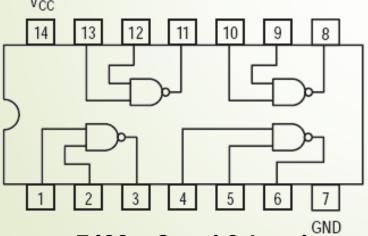
X XOR Y



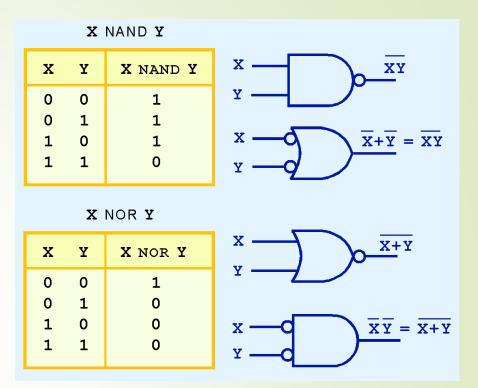
Note the special symbol ⊕ for the XOR operation.

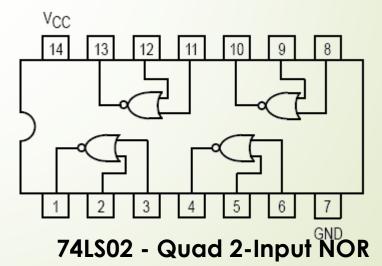
Logic Gates – NAND NOR

- NAND and NOR are two very important gates.
- Their symbols and truth tables are shown at the right.



7400 - Quad 2-Input NAND Gate

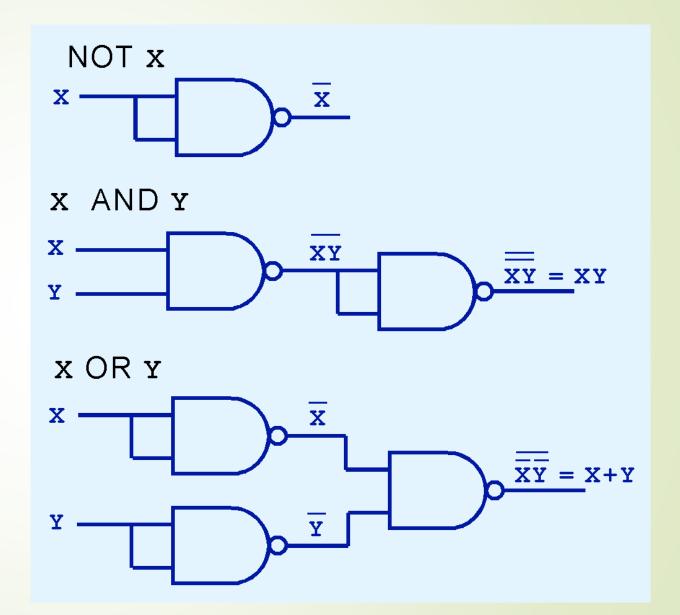




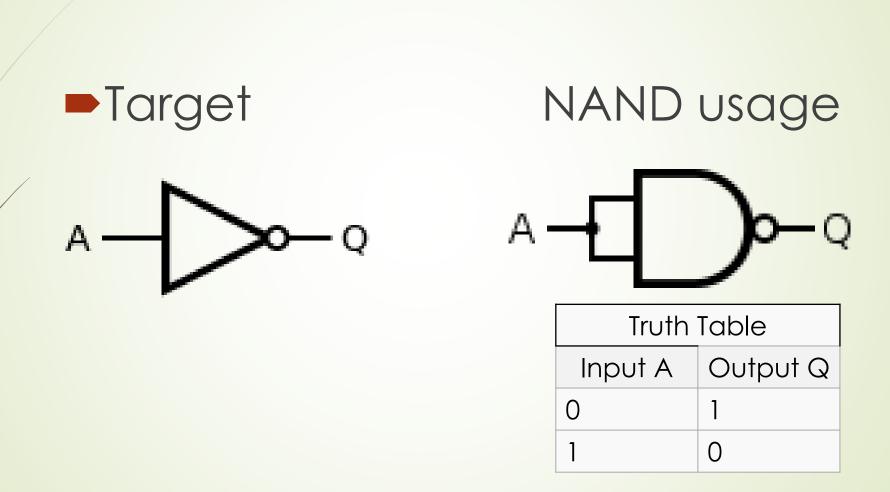
Gate

Logic Gates

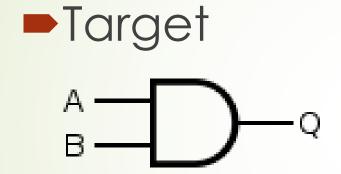
NAND and NOR are known as universal gates because they are inexpensive to manufacture and any Boolean function can be constructed using only NAND or only NOR gates.



Logic Gates NAND with NOT gate



Logic Gates NAND with AND

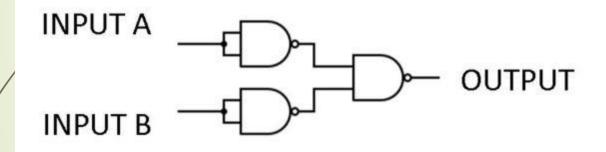




Truth Table					
Input A	Input B	Output Q			
0	0	0			
0	1	0			
1	0	0			
1	1	1			

Logic Gates NAND with OR

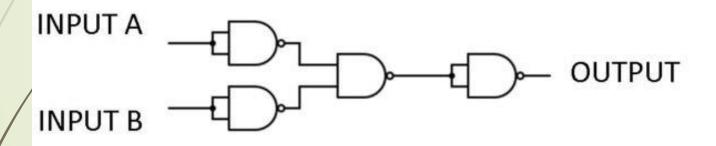
OR gate from NAND gates



NAND (Not AND)				
Α	В	S		
0	0	1		
0	1	1		
1	0	1		
1	1	0		

Logic Gates NAND with NOR Gate

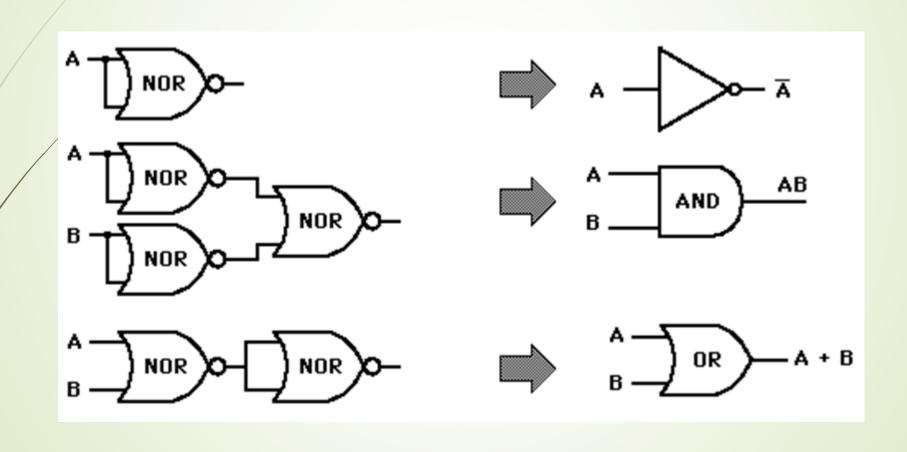
NOR gate from NAND gates



NAND (Not AND)			
Α	В	S	
0	0	1	
0	1	1	
1	0	1	
1	1	0	

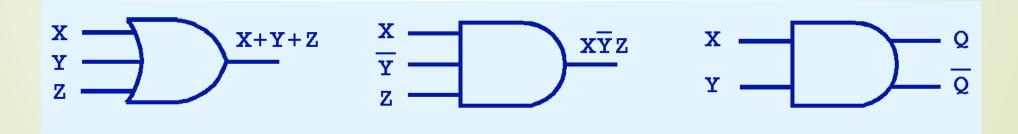
NOR			
Α	В	S	
0	0	1	
0	1	0	
1	0	0	
1	1	0	

Logic Gates – Creating other gates using NOR gate



Logic Gates

- Gates can have multiple inputs and more than one output.
 - A second output can be provided for the complement of the operation.
 - We'll see more of this later.



Logic Gates – Example 1

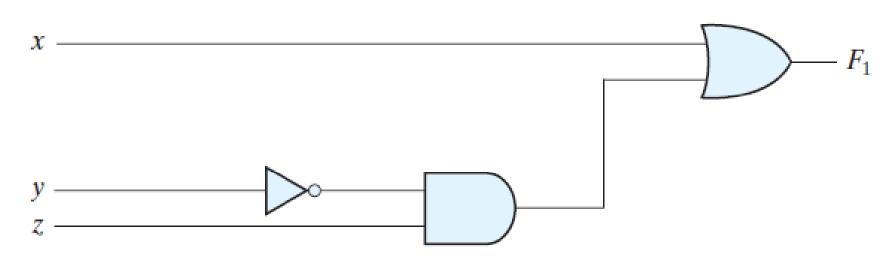
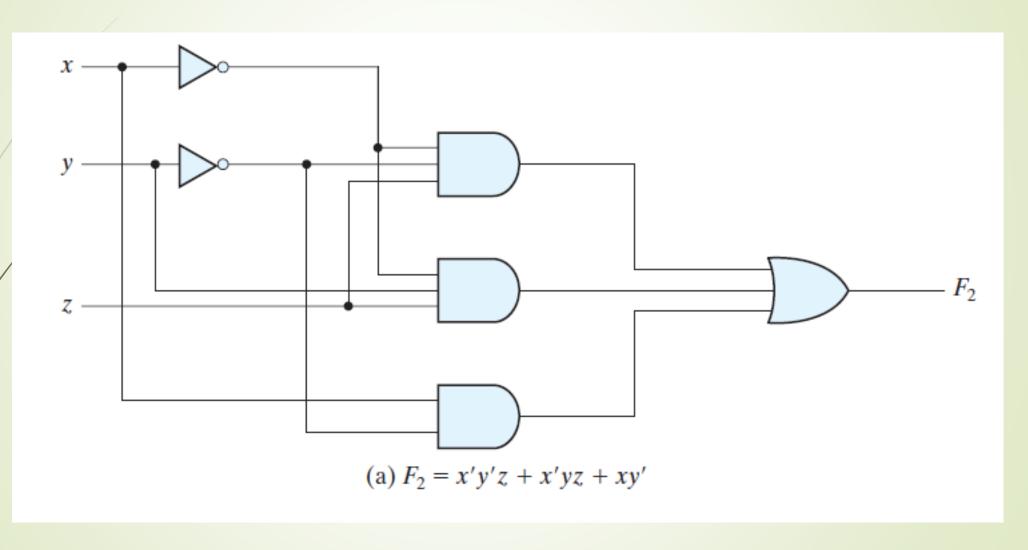


FIGURE 2.1

Gate implementation of $F_1 = x + y'z$

Logic Gates – Example 2



Logic Gates – Example 3

