$(1-x^2)^4(1+x^2)^4(1-x)^{-4}$

$\cancel{(1-x)^4}(1+x)^4 \; (1+x^2)^4 \; \cancel{(1-x)^{-4}}$

$\binom{4}{1}x \quad \binom{4}{2}x^8 \to 4$

$\binom{4}{3}x^3 \quad \binom{4}{3}x^6 \to 16$

Name: ____________ Student No: ____________

P1 [15pts] How many positive integers n less than 600 satisfy $\gcd(n, 600) = 1$? Show your calculation.

Note that $600 = 2^3 \cdot 3^1 \cdot 5^2$. So, if n is not div. by 2, 3 or 5 $\gcd(n,600)=1$

Then, let c_1: div by 2 c_2: div by 3 c_3: div by 5. Then using I.E. with c_i being our cond,

$\bar{N} = S_0 - S_1 + S_2 - S_3$ is our answer. $S_0 = 600$ $S_1 = \frac{600}{2} + \frac{600}{3} + \frac{600}{5}$ $S_2 = \frac{600}{6} + \frac{600}{10} + \frac{600}{15}$

$S_3 = \frac{600}{30}$ $\bar{N} = 600 - 620 + 200 - 20 = 160$ // 300 200 120 100 60 40

P2 [15pts] In how many ways can Alice select 9 balls from a bag that contains 3 red, 3 blue, 3 green and 3 white balls. Show your calculation and clearly write your final formula.

Alternative sol.

$x_1 + x_2 + x_3 + x_4 = 9 \quad 0 \le x_i \le 3 \quad c_i: x_i \ge 4$

$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4 = \binom{12}{3} - \binom{4}{1}\binom{8}{3} + \binom{4}{2}\binom{4}{3}$ $220 - 224 + 24 = 20$

$x_1 + x_2 + x_3 + x_4 = 3 \quad 0 \le x_i \le 3$ $\binom{6}{3}$ $\frac{12 \cdot 11 \cdot 10}{6} - 4 \cdot \frac{8 \cdot 7 \cdot 6}{6} + 6 \cdot 4$

$(1 + x + x^2 + x^3)^4 \quad [x^9] = ?$

$(1-x^4)^4 (1-x)^{-4}$

$\binom{4}{0} \; 1 \quad -x^9 \to \binom{-4}{9} \quad \binom{12}{9}$

$\binom{4}{1} -x^4 \quad -x^5 \quad \binom{-4}{5} \quad \binom{8}{5}$

$\binom{4}{2} x^8 \quad -x \quad \binom{-4}{1} \quad \binom{4}{1}$

P3 [15pts] Alice, Bob, and Charlie each create a random permutation of the numbers from 1 to 20. Then, they examine these permutations and find that there are no indices containing the same number in any pair of permutations. (For example, there is no match like the number three in the second index in [15,3,11,...] and [17,3,5,...]).

(a)Using the derangement formula, find a very simple upper bound for the probability of this occurring. Explain your calculation and reasoning.

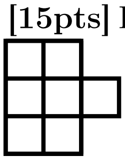
$P(A \times B) = \frac{1}{e}$ $P(A \times C) = \frac{1}{e}$ $P(B \times C) = \frac{1}{e} \Rightarrow \frac{1}{e^3}$

A	B	C
1	5	4
2	17	5
3	10	⋮
4	9	

(b) Why is this not an exact solution but just an upper bound?

Bec. once we determine $P(A \times B)$ and $P(A \times C)$, $P(B \times C)$ will be considerably less than $1/e$. Therefore the answer should be considerably less than $1/e^3$

P4 [15pts] Find the rook polynomial for the board below. Clearly write it in the format $1+ax+bx^2+\cdots$.

$R($  $, x) = x \cdot R(\; , x) + R(\; , x) = x(1+4x+2x^2) + (1+6x+6x^2)$

$= x + 4x^2 + 2x^3 + 1 + 6x + 6x^2$

$= 1 + 7x + 10x^2 + 2x^3$

Important notice
If you want to attempt counting one by one, you'll get 0. Write SKIP instead.

P5 [20 points] A florist needs to prepare a bouquet that contains 20 flowers with

roses, tulips, daisies, orch lil, jas.

$(x^4+x^5+\dots x^{17})(x^2+x^4+x^6+\dots)(x+x^3+x^5+\dots)(1+x)^2(x^5+x^{10}+\dots)$

- at least 4 roses at most 17
- positive even number of tulips
- odd number of daisies
- zero or one orchid
- zero or one lily
- a positive multiple of five of jasmines

$x^4(1+x\dots)\, x^2(1+x^2+x^4)\, x(1+x^2+x^4+\dots)(1+x)^2\, x^5(1+x^5+\dots)$

$x^{12}(1+x+x^2+\dots)(1+x^2+\dots)^2(1+x)^2(1+x^5+x^{10}+\dots)\ [x^{2}$

$\to (1-x)^{-1}(1-x^2)^{-2}\cancel{(1+x)^2}(1-x^5)^{-1} = (1-x)^{-3}(1+x^5+x^{10}+\dots)$

$(1-x)^{-2}\cancel{(1+x)^{-2}}$

$[x^8]=?$

$\binom{-3}{8}(-x)^8$

$\binom{-3}{3}(-x)^3$

1, x^5 → no need for GF because it makes it harder

$(1-x^5)^{-1}$

$\binom{10}{8}+\binom{5}{3}$ //

In how many ways can the florist do this? Make your calculations in the space above ↑↑, write your final answer clearly.

P6 [20 points] Let the series $\{a_n\}$ be defined with the recursive definition: $a_0 = 0, a_1 = 1$ and $\forall n \geq 2 : a_n = 5a_{n-1} - 6a_{n-2}$. By using generating functions, find a closed formula for a_n (A formula that depends only on n, so that if we need a_{10000} we can just substitute n with 10000 and calculate the result.) Make your calculations below and clearly write the final result.

Note that $\forall n \geq 2$ $\boxed{a_n - 5a_{n-1} + 6a_{n-2}} = 0$. Let $G(x)$ be the G.F. for the series a_n.

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$
$$-5x\,G(x) = -5a_0x - 5a_1x^2 - 5a_2x^3 + \dots$$
$$+\,6x^2G(x) = +6a_0x^2 + 6a_1x^3 + \dots$$

0

$$G(x)(1-5x+6x^2) = x$$

$$G(x) = \frac{x}{(1-3x)(1-2x)} = \frac{A}{(1-3x)} + \frac{B}{(1-2x)}$$

$A - 2Ax + B - 3Bx = x$

$A+B=0$ $-2A-3B=1$

$-B=1$ $B=-1$

$A=1$

$$G(x) = \frac{1}{1-3x} - \frac{1}{1-2x}$$

For arbitrary $k \geq 0$, the term in (a_n) with index k → $a_k = 3^k - 2^k = \boxed{3^k - 2^k}$

$\boxed{a_n = 0, 1, 5, 19, \dots}$ (a_0, a_1, a_2, a_3) //

$0 + 1x + 5x^2 + 19x^3$ I.E.

$a_0 = 0$ $1-1=0$
$a_1 = 1$ $3-2=1$
$a_2 = 5$ $9-4=5$
$a_3 = 19$ $27-8=19$
⋮

Table 1: Some generating functions that can be useful. For all $m, n \in \mathbb{Z}^+$, $a \in \mathbb{R}$

1) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$

2) $(1+ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}a^2x^2 + \cdots + \binom{n}{n}a^nx^n$

3) $(1+x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \cdots + \binom{n}{n}x^{nm}$

4) $(1-x^{n+1})/(1-x) = 1 + x + x^2 + x^3 + \cdots + x^n$

5) $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$

6) $1/(1-ax) = 1 + ax + a^2x^2 + a^3x^3 + \cdots$ 3 3^k