Fall 2024: CSE 221 | **Discrete Mathematics I** | **Akdeniz University**

15/11/2024 | Midterm | Duration: 90 minutes

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Important: *Give concise and readable answers. Don't calculate the final integers, leave as combinatorial formulas. Do not count things one by one, use the methods we discussed.*

P1 [18 pts (6+6+6)] Suppose that in a DNA strand, there are 3 adenine(A), 3 cytosine(C), 3 guanine(G) and 3 thymine(T) nucleotides. (Note that you can turn it around since it is a 3D object. So, GCTGACGTACTA is the same as ATCATGCAGTCG!)

(a) How many such DNA segments can there be? (b) Suppose that within the DNA segment, A's and T's must alternate (there must be no consecutive A's or T's) when G's and C's are ignored. For example CATAGTCGTGCA will not be counted because when we see only A's and T's, we get ATATTA where we can see a TT) (c) Suppose that in addition to A's and T's, now, similarly G's and C's must also alternate among themselves.)

(a) pl. A's, pl. T, pl. C's, Pl. G

$\dfrac{\underbrace{\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}}}{2}$ ← reverses are the same.

(b) $\binom{12}{3}\cdot\binom{9}{3}\cdot 2/2$

$\underbrace{\binom{12}{6}\cdot 2}_{\text{placing As Ts}}\ \underbrace{\binom{6}{3}}_{\text{plac Gs}}\underbrace{\binom{3}{3}}_{\text{pl Cs}}/2$

(A)

(c) $\underbrace{\binom{12}{6}}_{1}\binom{6}{6}\cdot 2$

placing As Ts

P2 [25 points (10+15)]

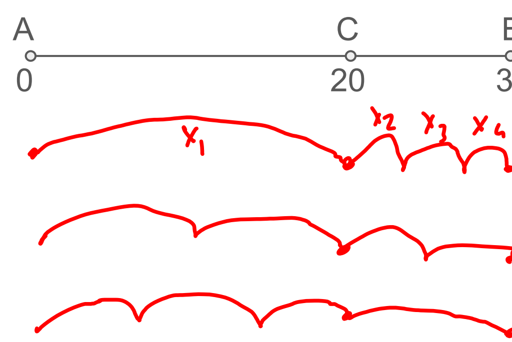
(a) You will throw a 20-sided die (sides numbered from 1 to 20) 5 times. In how many different ways can you end up with a total of 20? (For example, [12,6,18,3,10] adds up to 49, so it shouldn't be counted. Whereas [6,4,2,4,4] or [1,16,1,1,1] add up to 20, so they must be counted.)

$x_1+x_2+x_3+x_4+x_5=20$ $\quad x_i \geq 1$

$y_1+\dots\ y_5=15$ $\quad y_i = x_i - 1$

$\rightarrow \binom{19}{4}$

(b) Suppose that a frog jumps $h \in \mathbb{Z}^+$ centimeters whenever it jumps, always to the right. It starts from point A and, by making exactly 4 jumps, arrives at point B, which is 30 cm to the right of A. Let C be the point 20 cm to the right of A. What is the probability that the frog visits C (i.e., lands exactly on point C at some point during its journey)?



$x_1=20$ $\quad x_i \geq 1$

$x_2+x_3+x_4=10$

$x_1+x_2=20$

$x_3+x_4=10$

$x_1+x_2+x_3=20$

$x_4=10$

$$\frac{1\cdot\binom{9}{2}+\binom{19}{1}\binom{9}{1}+\binom{19}{2}\cdot 1}{\binom{29}{3}}$$

$x_1+\dots=30$ $\quad x_i \geq 1$

P3 [12 points (12x1)] Fill in the blanks so that the following is a valid proof.

Steps	Reasons
1. $\forall x[p(x) \to q(x)]$	Premise.
2. $\forall x[r(x) \to \neg q(x)]$	Premise.
3. $\exists x[r(x)]$	Premise.
4. $r(c)$	R. of Exis. Spec., Step 3.
5. $r(c) \to \neg q(c)$	R. of Univ. Spec., Step 2.
6. $p(c) \to q(c)$	R. of Univ. Spec., Step 1.
7. $\neg q(c)$	Modus Ponens, Steps 4, 5.
8. $\neg p(c)$	Modus Tollens, Steps 6, 7.
9. $\exists x \neg p(x)$	R. of Exis. Gen., Step 8.

P4 [20 points (10+10)] Prove the following statement by using mathematical induction:

(a) Fibonacci numbers are defined as follows:

Defn
$F_0 = 0, F_1 = 1$
$F_n = F_{n-1} + F_{n-2}$ for $n \in \mathbb{Z}^+$ with $n \geq 2$

Prove that for all $n \geq 1$:

$$F_n F_{n+1} = F_{n+2} F_{n-1} + (-1)^{n+1}$$

F_0 0
F_1 1
F_2 1
F_3 2
3
5
8
13

Base. $n=1 \Rightarrow F_1 F_2 = F_3 F_0 + (-1)^2$
$1 \qquad 0 + 1$ ✓

Ind. Hyp. for $n=k$ $\quad F_k F_{k+1} = F_{k+2} F_{k-1} + (-1)^{k+1}$

Ind. Step for $n = k+1$

$$F_{k+1} F_{k+2} \overset{?\,\text{Thm}}{=} F_{k+3} F_k + (-1)^{k+2}$$

defn $\qquad$ defn.

$$(F_k + F_{k-1}) F_{k+2} \overset{?}{=} (F_{k+2} + F_{k+1}) F_k + (-1)^k$$

$$\cancel{F_k F_{k+2}} + F_{k-1} F_{k+2} \overset{?}{=} \cancel{F_{k+2} F_k} + F_{k+1} F_k + (-1)^k$$

$(-1)^{k+1}$ ✓

QED.

(b) A new country has a currency named "Orh". They want to mint three types of coins so that all amounts 100 Orh or more (100,101,102,...) can be obtained with these coins. You suggest they can do this with 6 Orh, 9 Orh and 20 Orh coins. And they want you to prove it.

$\forall n, \exists i,j,k \in \mathbb{N} \; n = 6i + 9j + 20k$

- Base for $n=100$ $\quad 6.0 + 9.0 + 20.5 = 100$
- Ind. hyp. for $n = k$ $\quad k = 6a + 9b + 20.c$
- Ind. Step for $n = k+1$ $\quad k+1 = 6(a+2) + 9(b+1) + 20(c-1)$

$= 6(a+1) + 9(b-5) + 20(c+2)$
$= 6(a-8) + 9(b+1) + 20(c+2)$

$c \geq 1$ or $b \geq 5$ or $a \geq 8$

oth. largest $n = 20.0 + 9.4 + 6.7 = 42 + 36$
78

So at least one of the eq. can be used for $k+1$. QED.

Quad Erat Demonstrandum

P5 [25 points (10+15)] (In this question, define your pigeons and pigeonholes clearly.)

(a) You visit an alien planet where human-like creatures live. You see that they also have a friendship relation. You tell them you can find two of them who has the same number of friends but they don't believe you and say "How do you know? Maybe we all have a different number of friends?" So you forgive their lack of math knowledge and prove your argument:

Possible number of friends $\rightarrow$ $0 \ldots\ldots n-2$
Pigeonholes $\rightarrow$ $1 \ldots\ldots n-1$
$(n-1)$

pigeons: aliens. $\rightarrow n$ $\quad \left\lceil \frac{n}{n-1} \right\rceil = 2$ of them must have the same number of friends

QED.

(b) Let $k \in \mathbb{Z}^+$. Prove that there exists a positive integer n such that k divides n and the only digits in n can be only 0s and 5s.

Let's consider these numbers:

$\underset{a_1}{5}, \underset{a_2}{55}, \ldots, \underset{a_k}{\underbrace{55\ldots5}_{k}}$. Now, either $\nexists i,j$ s.t. $a_i = a_j \bmod k$

or $\exists i,j$ s.t. $a_j = a_i \bmod k$. In the first scenario, $\exists i : a_i = 0 \bmod k$ so, a_i is the number we are looking for. → consist of only 5s. In the second scenario, by P.h.p. $\left\lceil \frac{k}{k-1} \right\rceil = 2$ of the numbers a_i is equiv. mod k. Then assume $a_i > a_j$, $a_i - a_j$ is div by k and consists only 5s and 0s. QED.

pigeons: a_i numbers
p.h.: possible modulo k except 0.

✓ G A G T A C T C A G C T

X T C G A C T C A T G A G