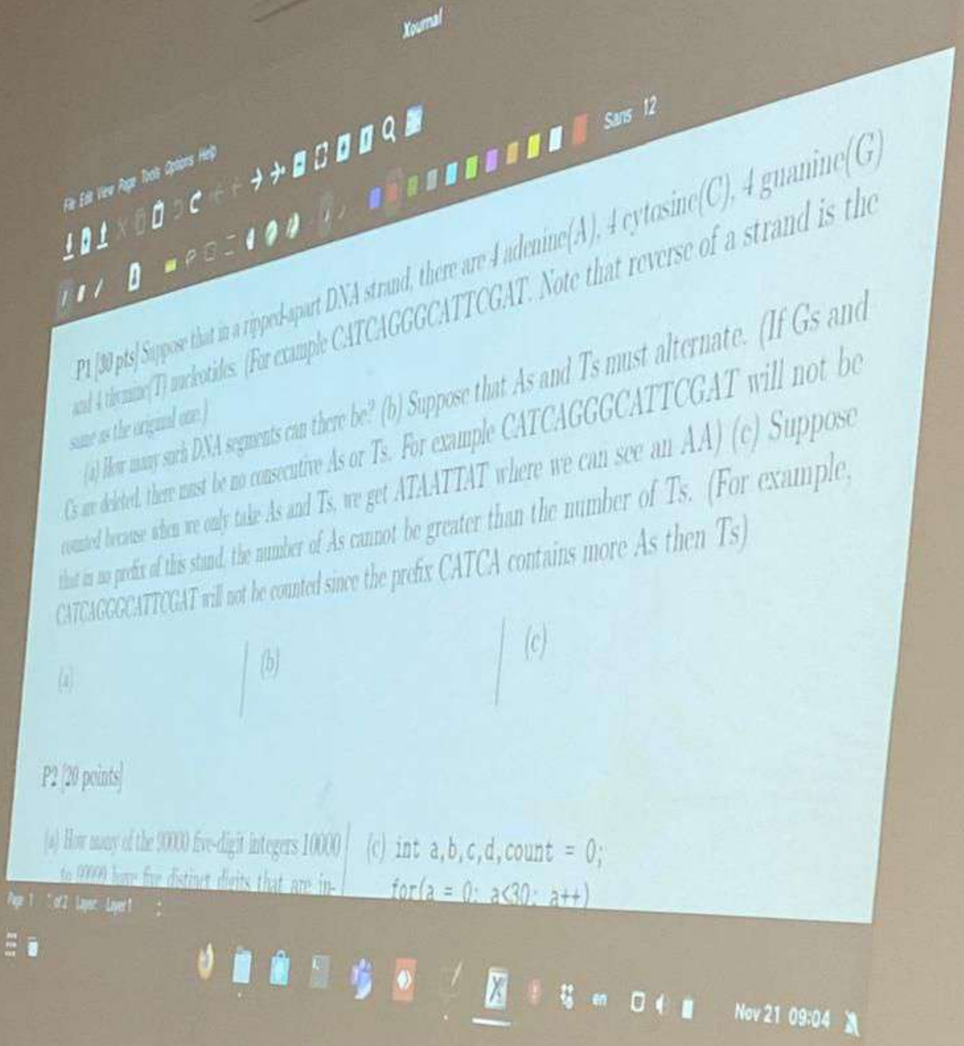


P1 [30 pts] Suppose that in a ripped-apart DNA strand, there are 4 adenine(A), 4 cytosine(C), 4 guanine(G) and 4 thymine(T) nucleotides. (For example CATCAGGGCATTCGAT. Note that reverse of a strand is the same as the original one.)

(a) How many such DNA segments can there be? (b) Suppose that As and Ts must alternate. (If Gs and Cs are deleted, there must be no consecutive As or Ts. For example CATCAGGGCATTCGAT will not be counted because when we only take As and Ts, we get ATAATTAT where we can see an AA) (c) Suppose that in no prefix of this stand, the number of As cannot be greater than the number of Ts. (For example, CATCAGGGCATTCGAT will not be counted since the prefix CATCA contains more As then Ts)

(a)

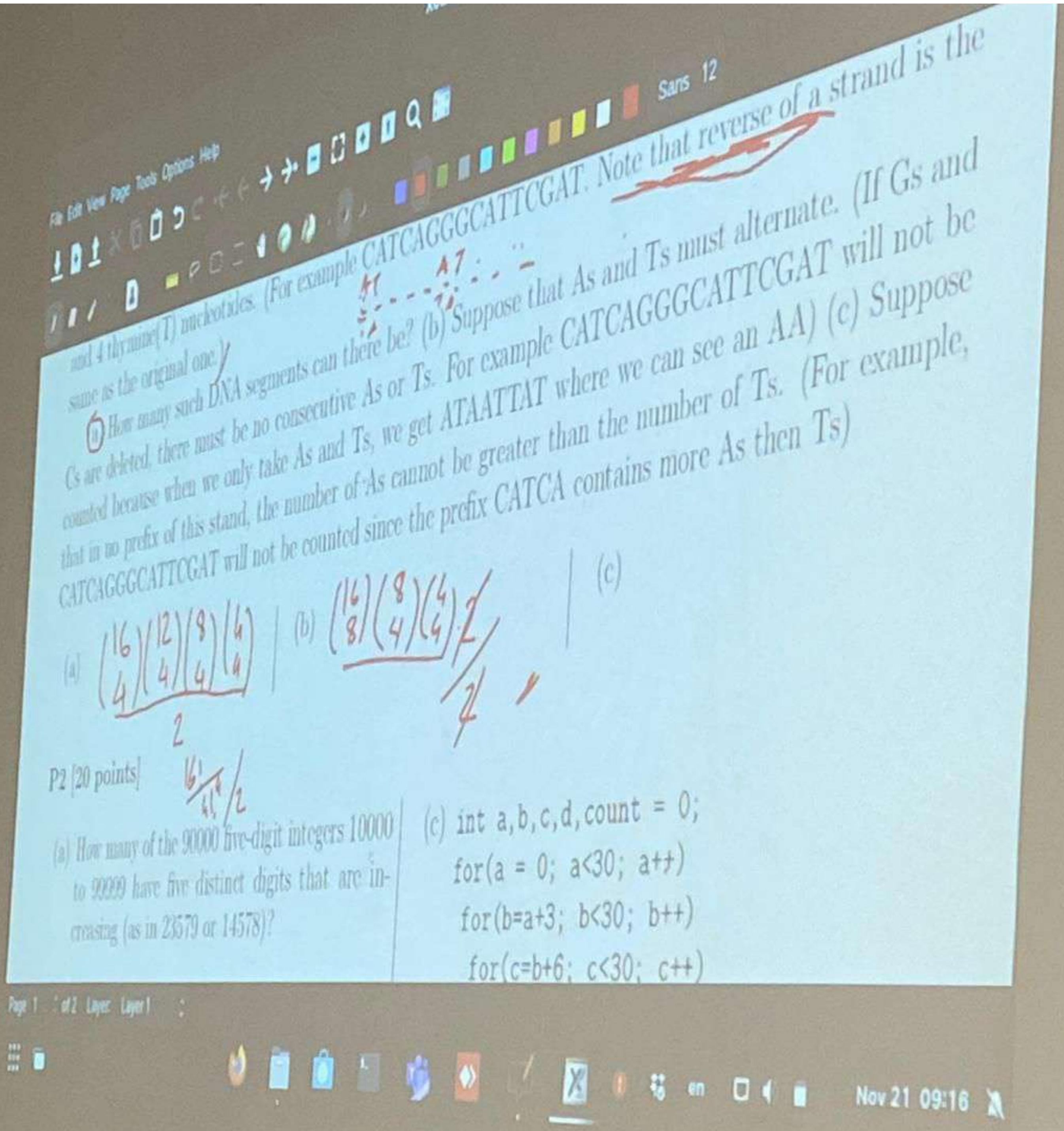
(b)

(c)

P2 [20 points]

(a) How many of the 90000 five-digit integers 10000 to 99999 have five distinct digits that are in-

(c)
```
int a,b,c,d,count = 0;
  for(a = 0; a<30; a++)
```



and 4 thymine(T) nucleotides. (For example CATCAGGGCATTCGAT. Note that reverse of a strand is the same as the original one.)

(a) How many such DNA segments can there be? (b) Suppose that As and Ts must alternate. (If Gs and Cs are deleted, there must be no consecutive As or Ts. For example CATCAGGGCATTCGAT will not be counted because when we only take As and Ts, we get ATAATTAT where we can see an AA) (c) Suppose that in no prefix of this stand, the number of As cannot be greater than the number of Ts. (For example, CATCAGGCATTCGAT will not be counted since the prefix CATCA contains more As then Ts)

AT - - - AT

(a) $\dfrac{\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}}{2}$

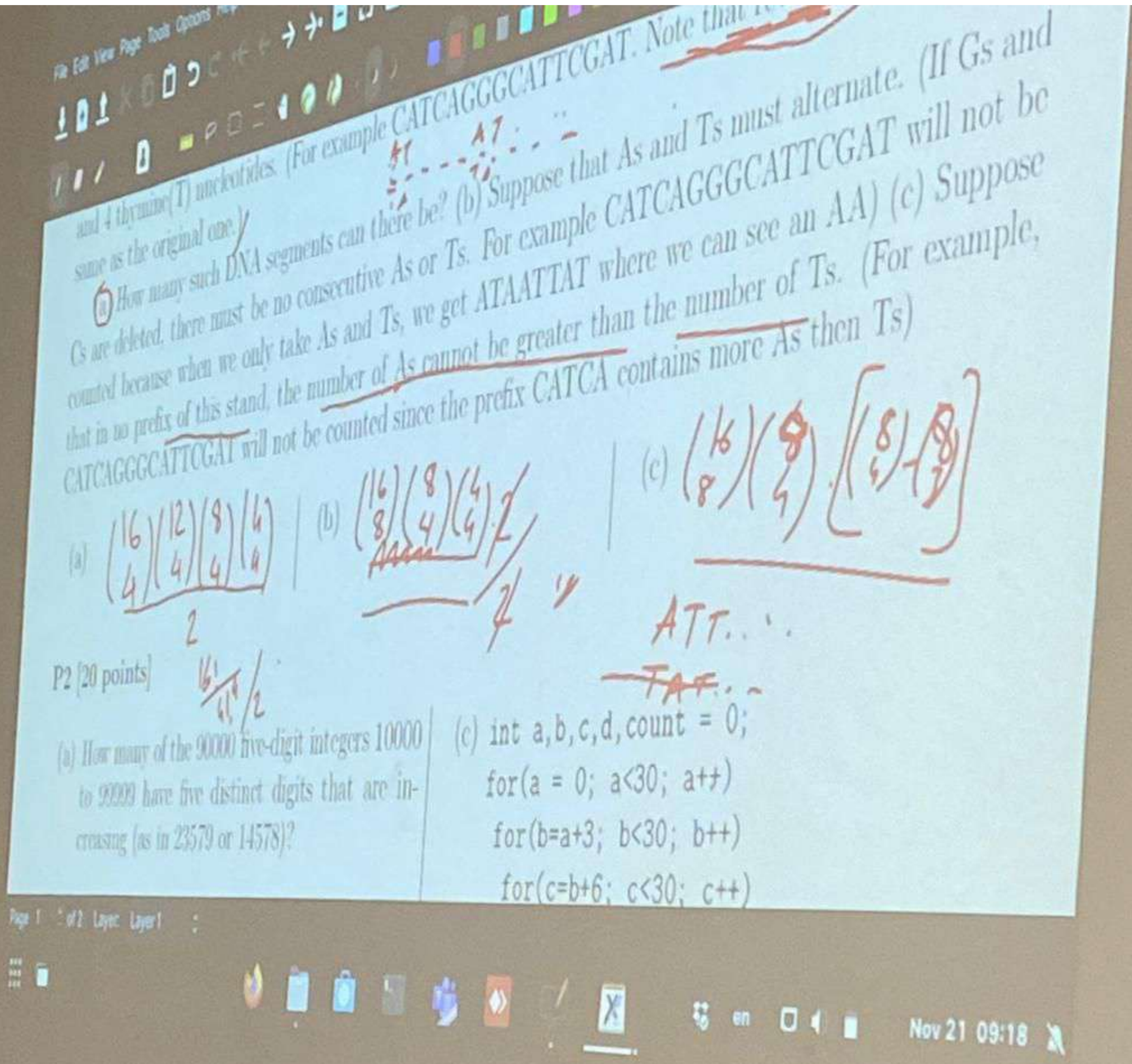
(b) $\dfrac{\binom{16}{8}\binom{8}{4}\binom{4}{4}\cancel{2}}{\cancel{4}}$ ✓

(c)

P2 [20 points] $\dfrac{16!}{4!^4}/2$

(a) How many of the 90000 five-digit integers 10000 to 99999 have five distinct digits that are increasing (as in 23579 or 14578)?

```
(c) int a,b,c,d,count = 0;
      for(a = 0; a<30; a++)
       for(b=a+3; b<30; b++)
        for(c=b+6; c<30; c++)
```



and 4 thymine(T) nucleotides. (For example CATCAGGGCATTCGAT. Note that

same as the original one.)

(a) How many such DNA segments can there be? (b) Suppose that As and Ts must alternate. (If Gs and Cs are deleted, there must be no consecutive As or Ts. For example CATCAGGGCATTCGAT will not be counted because when we only take As and Ts, we get ATAATTAT where we can see an AA) (c) Suppose that in no prefix of this stand, the number of As cannot be greater than the number of Ts. (For example, CATCAGGGCATTCGAT will not be counted since the prefix CATCA contains more As then Ts)

AT AT

(a) $\frac{\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}}{2}$

(b) $\binom{16}{8}\binom{8}{4}\binom{4}{4}\cdot 2$ / 2 4

(c) $\binom{16}{8}\binom{8}{4}\cdot\left[\binom{8}{4}-\binom{8}{3}\right]$

ATT...

~~TAT~~...

P2 [20 points]

$\frac{16!}{4!^4}/2$

(a) How many of the 90000 five-digit integers 10000 to 99999 have five distinct digits that are increasing (as in 23579 or 14578)?

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```
int a,b,c,d,count = 0;
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```
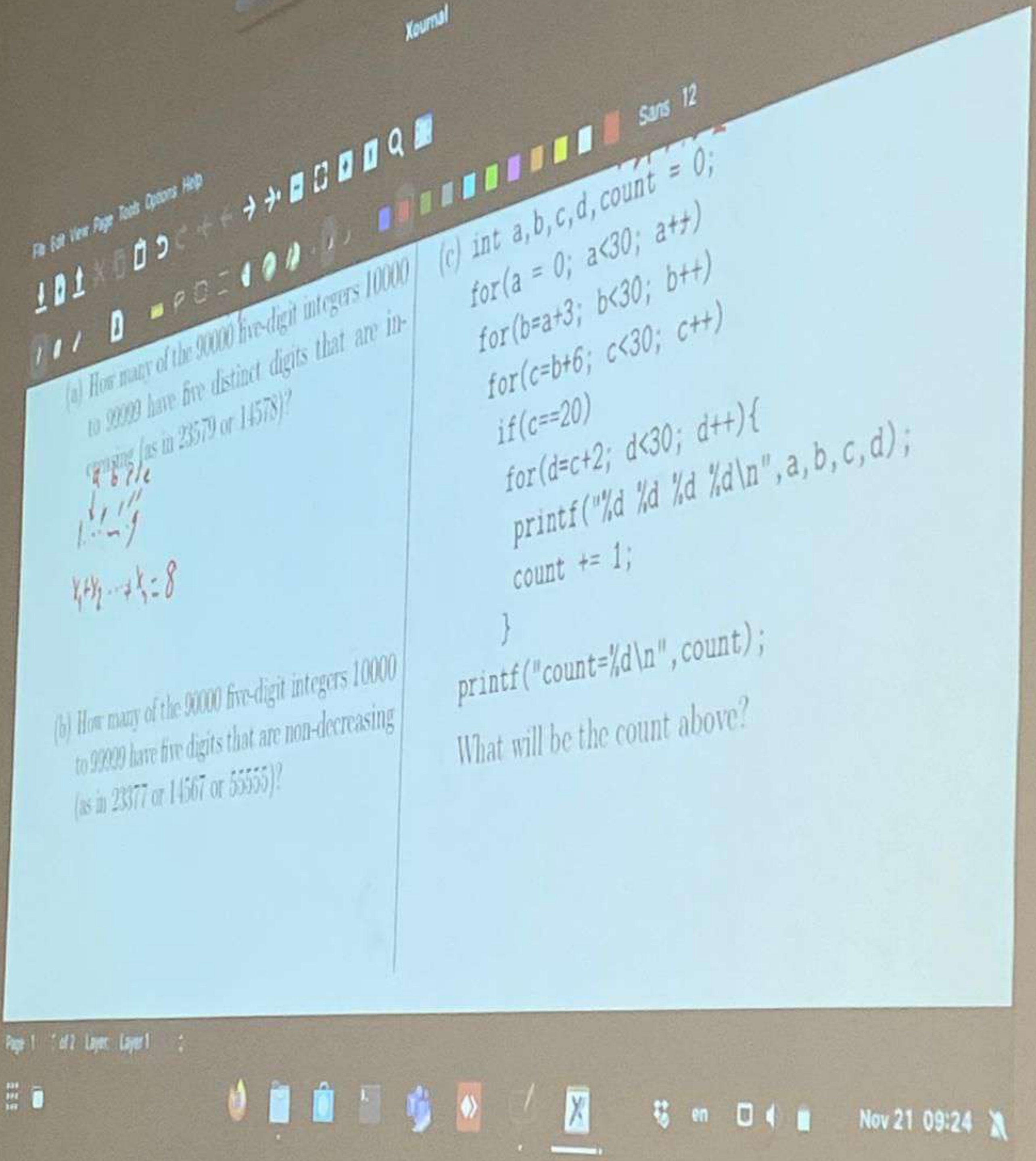



(a) How many of the 90000 five-digit integers 10000 to 99999 have five distinct digits that are increasing (as in 23579 or 14578)?

a b c d e

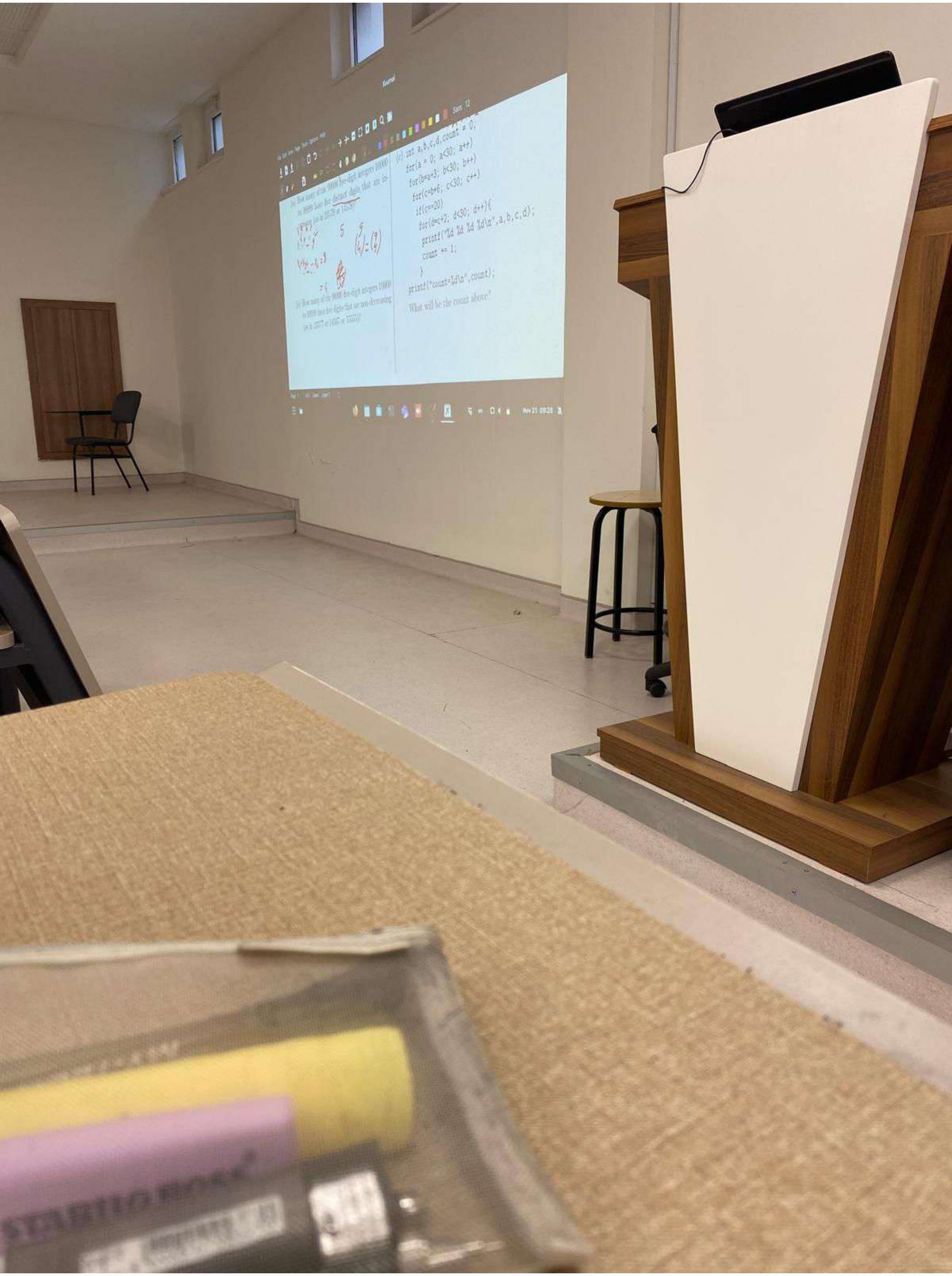
1 ... 9

$y_1+y_2 \cdots + y_5 = 8$

(b) How many of the 90000 five-digit integers 10000 to 99999 have five digits that are non-decreasing (as in 23377 or 14567 or 55555)?

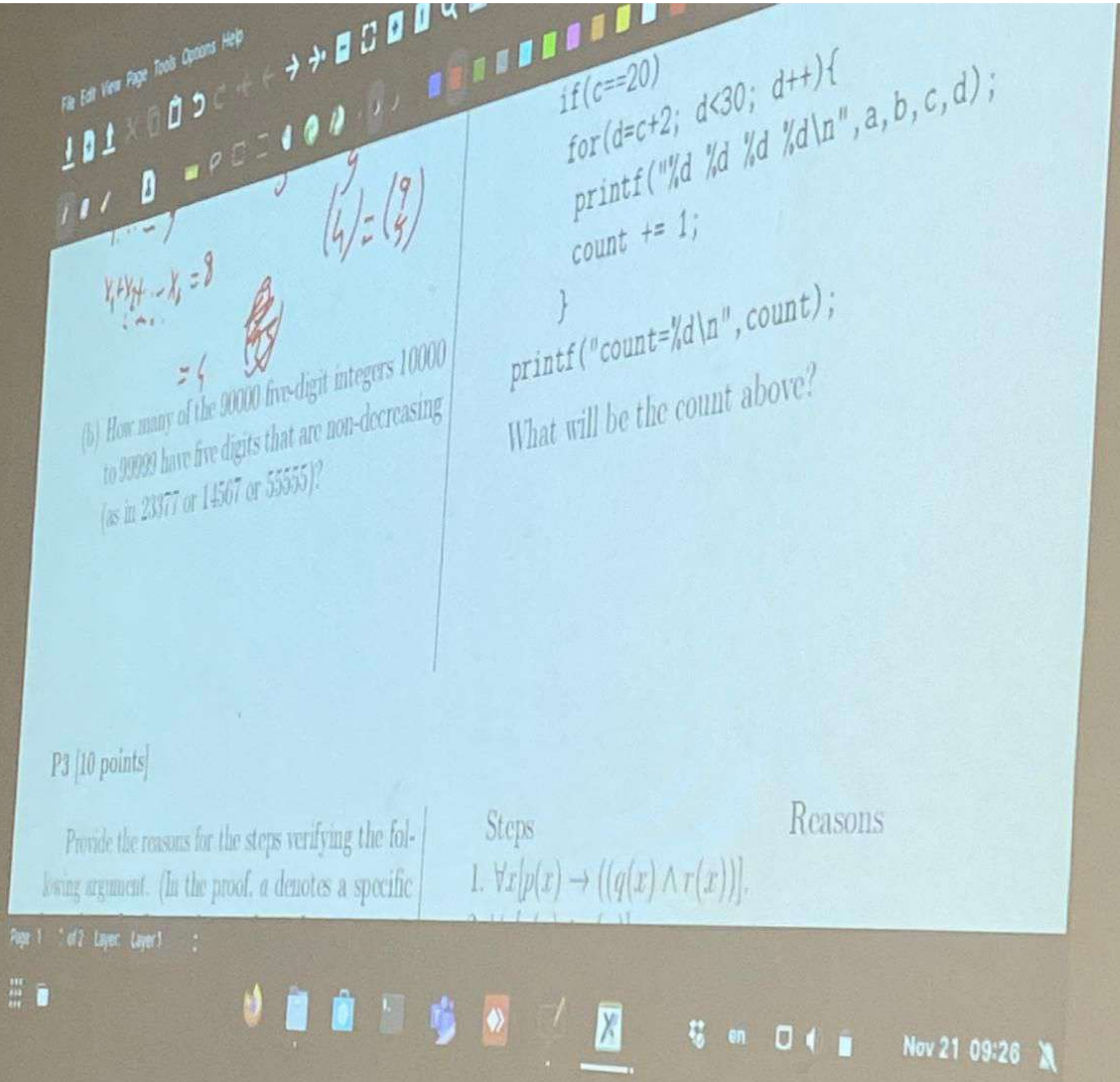
(c)
```
int a,b,c,d,count = 0;
 for(a = 0; a<30; a++)
 for(b=a+3; b<30; b++)
 for(c=b+6; c<30; c++)
  if(c==20)
  for(d=c+2; d<30; d++){
   printf("%d %d %d %d\n",a,b,c,d);
   count += 1;
  }
printf("count=%d\n",count);
```

What will be the count above?



```
(c) int a,b,c,d,count = 0;
  for(a = 0; a<30; a++)
   for(b=a+3; b<30; b++)
    for(c=b+6; c<30; c++)
     if(c==20)
      for(d=c+2; d<30; d++){
       printf("%d %d %d %d\n",a,b,c,d);
       count += 1;
      }
  printf("count=%d\n",count);
```

What will be the count above?



$y_1+y_2+\dots+x_5=8$

$\binom{9}{4}=\binom{9}{5}$

$= 4$

(b) How many of the 90000 five-digit integers 10000 to 99999 have five digits that are non-decreasing (as in 23377 or 14567 or 55555)?

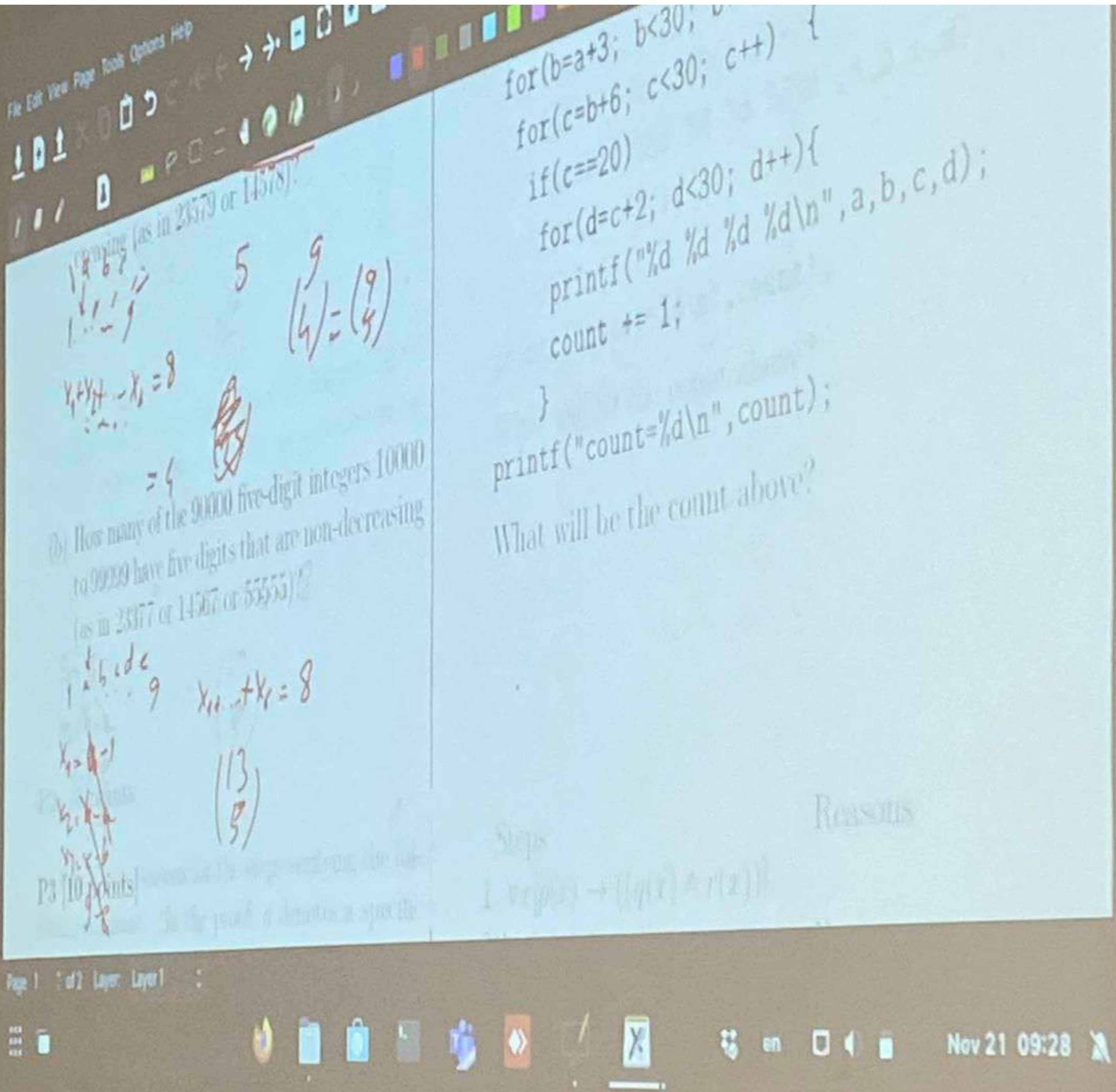
```
if(c==20)
 for(d=c+2; d<30; d++){
 printf("%d %d %d %d\n",a,b,c,d);
 count += 1;
 }
printf("count=%d\n",count);
```

What will be the count above?

P3 [10 points]

Provide the reasons for the steps verifying the following argument. (In the proof, a denotes a specific

Steps	Reasons
1. $\forall x[p(x) \rightarrow ((q(x) \wedge r(x))]$.	



…creasing (as in 23579 or 14578)?

5

$\binom{9}{4} = \binom{9}{5}$

$x_1 + x_2 + \dots x_5 = 8$

= 4

(b) How many of the 90000 five-digit integers 10000 to 99999 have five digits that are non-decreasing (as in 23377 or 14567 or 55555)?

1 a b c d e 9

$x_1 + \dots + x_6 = 8$

$x_1 = a-1$

$x_2 = b-a$

$\binom{13}{5}$

P3 [10 points] [illegible]

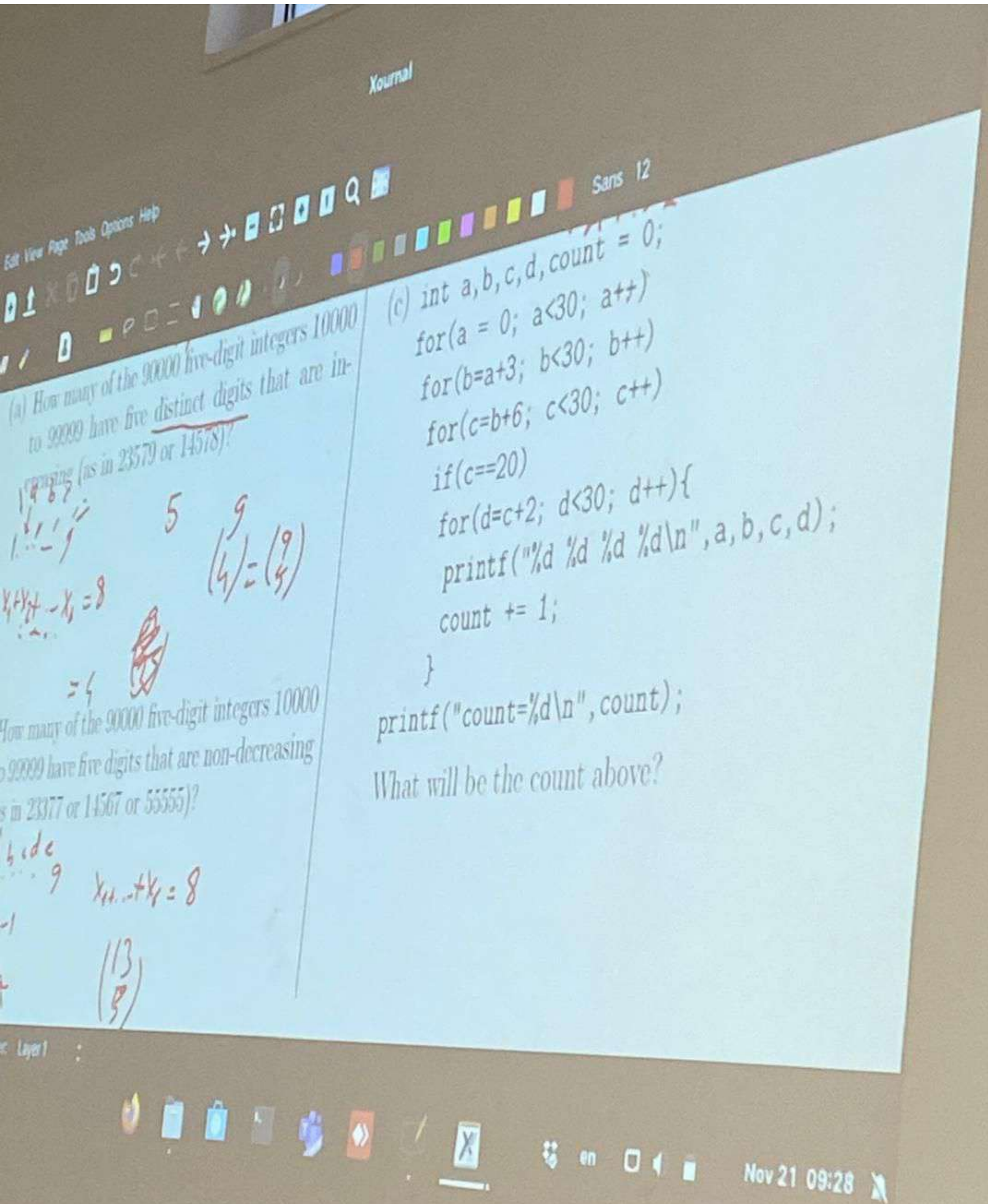
```
for(b=a+3; b<30;
for(c=b+6; c<30; c++)  {
 if(c==20)
 for(d=c+2; d<30; d++){
  printf("%d %d %d %d\n",a,b,c,d);
  count += 1;
 }
printf("count=%d\n",count);
```

What will be the count above?

Steps

Reasons

1. $\forall x(p(x) \to ((q(x) \wedge r(x)))$



(a) How many of the 90000 five-digit integers 10000 to 99999 have five distinct digits that are increasing (as in 23579 or 14578)?

1 4 6 7

1 ... 9

5

$\binom{9}{4}=\binom{9}{5}$

$x_1+x_2+\dots+x_5=8$

= 4

How many of the 90000 five-digit integers 10000 to 99999 have five digits that are non-decreasing (as in 23377 or 14567 or 55555)?

b c d e

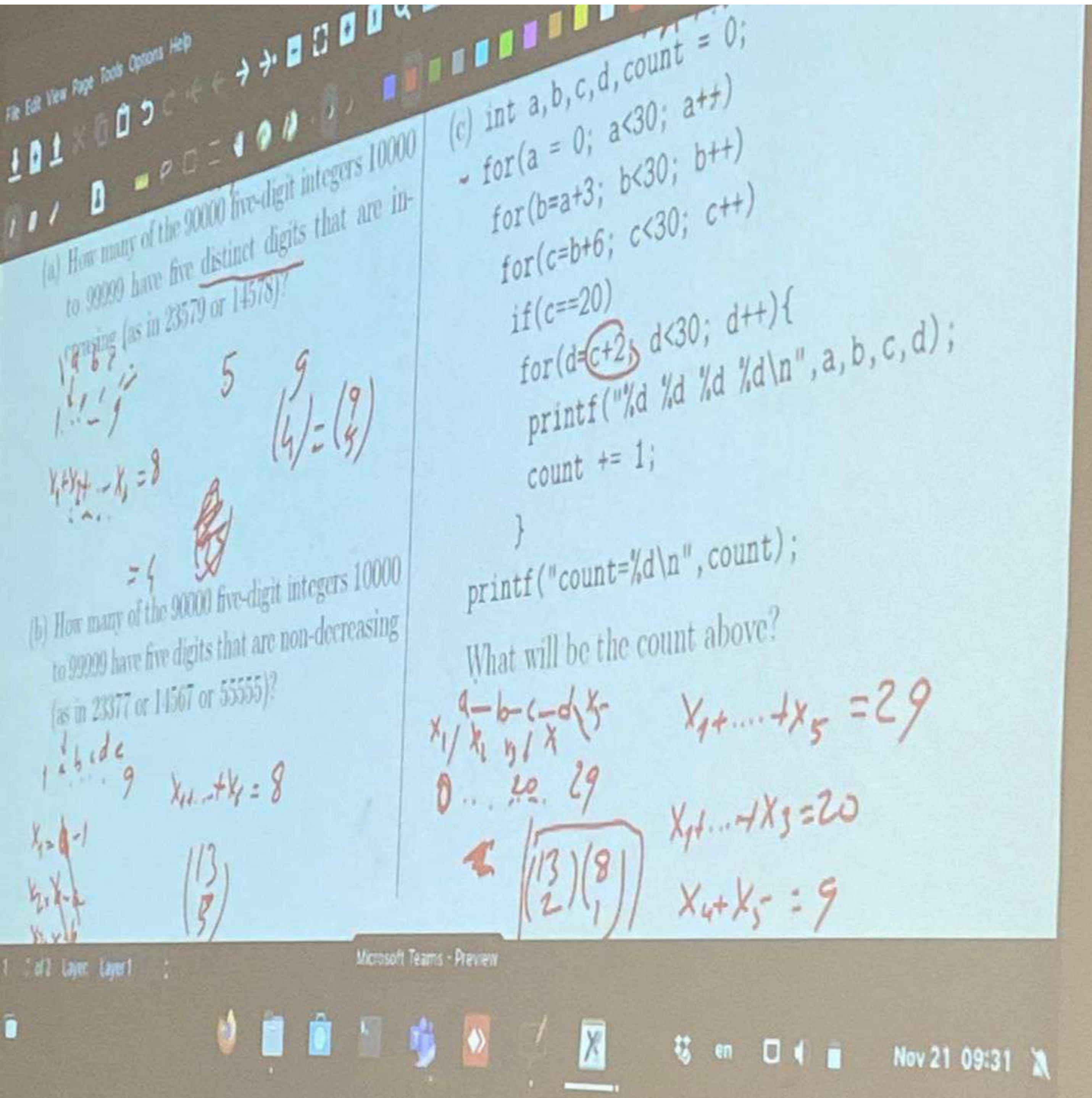
9

$x_1+\dots+x_5=8$

$\binom{13}{8}$

(c)
```
int a,b,c,d,count = 0;
for(a = 0; a<30; a++)
for(b=a+3; b<30; b++)
for(c=b+6; c<30; c++)
 if(c==20)
 for(d=c+2; d<30; d++){
  printf("%d %d %d %d\n",a,b,c,d);
  count += 1;
 }
printf("count=%d\n",count);
```

What will be the count above?



(a) How many of the 90000 five-digit integers 10000 to 99999 have five distinct digits that are increasing (as in 23579 or 14578)?

$1\ a\ b\ ?\ \ldots\ 9$

5

$\binom{9}{4} = \binom{9}{5}$

$x_1 + x_2 + \ldots x_5 = 8$

$= 5$

(b) How many of the 90000 five-digit integers 10000 to 99999 have five digits that are non-decreasing (as in 23377 or 14567 or 55555)?

$1\ a\ b\ c\ d\ e\ \ldots\ 9$

$x_1 + \ldots + x_6 = 8$

$x_1 = a - 1$

$x_2 = x - b$

$\binom{13}{5}$

(c)
```
  int a,b,c,d,count = 0;
- for(a = 0; a<30; a++)
    for(b=a+3; b<30; b++)
    for(c=b+6; c<30; c++)
    if(c==20)
    for(d=c+2; d<30; d++){
    printf("%d %d %d %d\n",a,b,c,d);
    count += 1;
    }
  printf("count=%d\n",count);
```

What will be the count above?

$x_1\ a - b - c - d\ x_5$

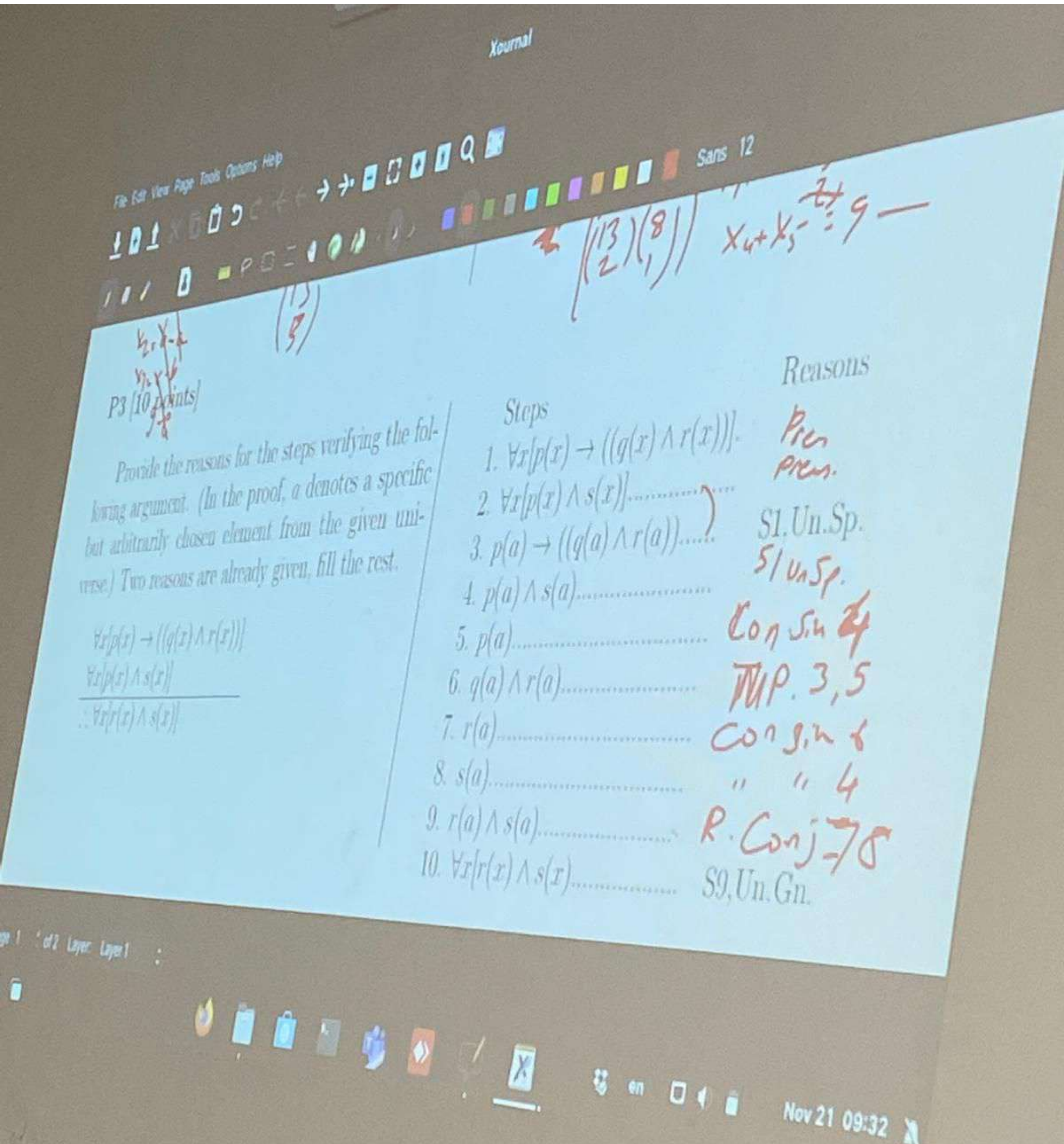
$0 \ldots 20 \ldots 29$

$x_1 + \ldots + x_5 = 29$

$x_1 + \ldots + x_3 = 20$

$\boxed{\binom{13}{2}\binom{8}{1}}$

$x_4 + x_5 = 9$



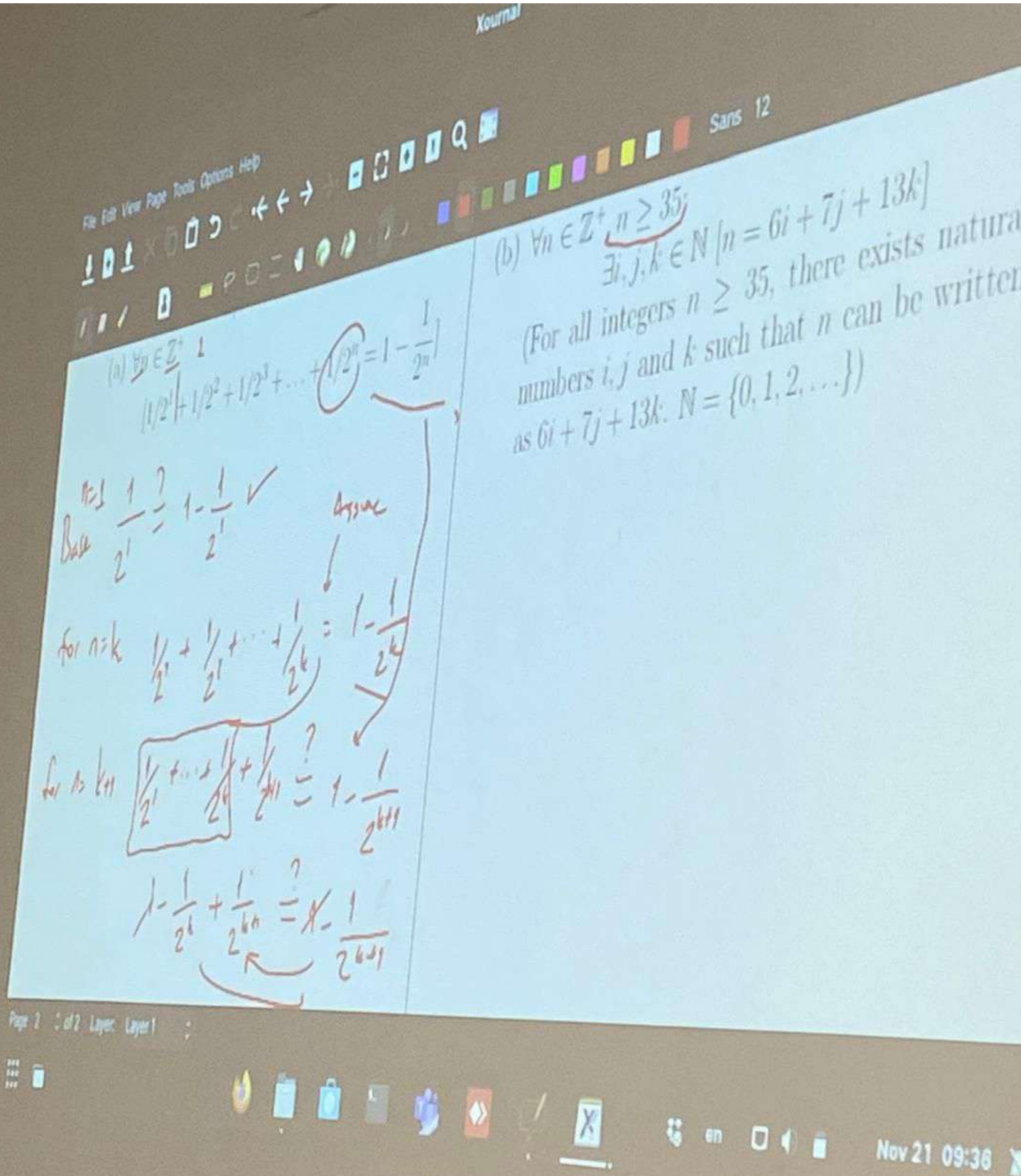
$\binom{13}{2}\binom{8}{1}$ $x_4+x_5 = 9$

P3 [10 points]

Provide the reasons for the steps verifying the following argument. (In the proof, a denotes a specific but arbitrarily chosen element from the given universe.) Two reasons are already given, fill the rest.

$\forall x[p(x) \rightarrow ((q(x) \wedge r(x))]$
$\forall x[p(x) \wedge s(x)]$
$\therefore \forall x[r(x) \wedge s(x)]$

Steps	Reasons
1. $\forall x[p(x) \rightarrow ((q(x) \wedge r(x))]$.	Pres
2. $\forall x[p(x) \wedge s(x)]$	Pres.
3. $p(a) \rightarrow ((q(a) \wedge r(a))$	S1,Un.Sp.
4. $p(a) \wedge s(a)$	S/ UnSp.
5. $p(a)$	Con Sim 4
6. $q(a) \wedge r(a)$	MP. 3,5
7. $r(a)$	Con Sim 6
8. $s(a)$	" " 4
9. $r(a) \wedge s(a)$	R. Conj 7 8
10. $\forall x[r(x) \wedge s(x)]$	S9,Un.Gn.



(a) $\forall n \in \mathbb{Z}^+$ 1

$1/2^1 + 1/2^2 + 1/2^3 + \ldots + 1/2^n = 1 - \frac{1}{2^n}$

Base n=1 $\frac{1}{2^1} \overset{?}{=} 1 - \frac{1}{2^1}$ ✓

Assume

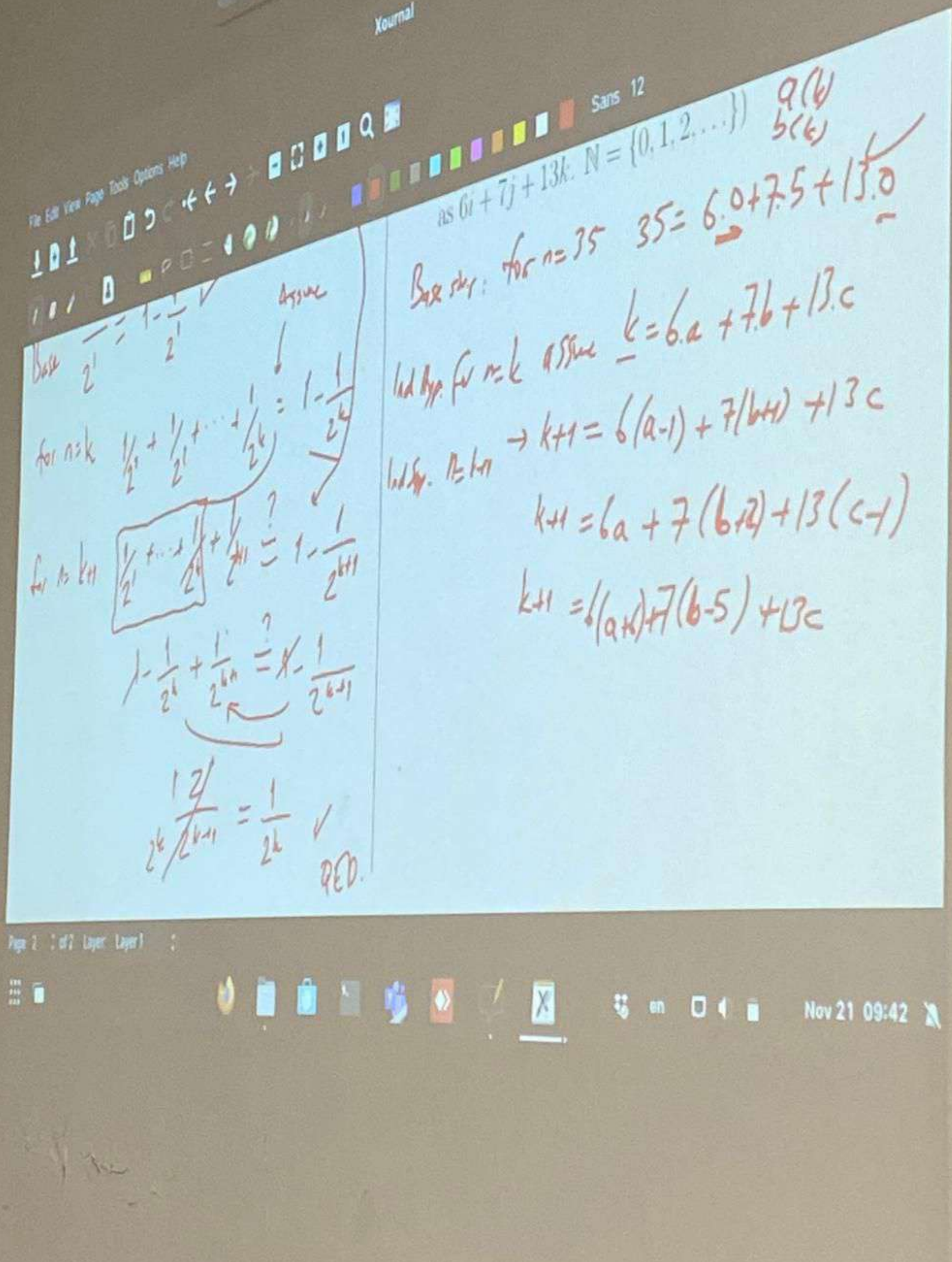
for n=k $\frac{1}{2^1} + \frac{1}{2^1} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$

for n=k+1 $\boxed{\frac{1}{2^1} + \cdots + \frac{1}{2^k}} + \frac{1}{2^{k+1}} \overset{?}{=} 1 - \frac{1}{2^{k+1}}$

$\cancel{1} - \frac{1}{2^k} + \frac{1}{2^{k+1}} \overset{?}{=} \cancel{1} - \frac{1}{2^{k+1}}$

(b) $\forall n \in \mathbb{Z}^+, \underbrace{n \geq 35;}$
$\exists i, j, k \in \mathbb{N}\ [n = 6i + 7j + 13k]$

(For all integers $n \geq 35$, there exists natura
numbers i, j and k such that n can be writte
as $6i + 7j + 13k$. $\mathbb{N} = \{0, 1, 2, \ldots\}$)



Base $\frac{}{2^1} = 1-\frac{1}{2^1}$ ✓

Assume

for n=k $\frac{1}{2^1}+\frac{1}{2^2}+\dots+\frac{1}{2^k} = 1-\frac{1}{2^k}$

for n=k+1 $\boxed{\frac{1}{2^1}+\dots+\frac{1}{2^k}}+\frac{1}{2^{k+1}} \overset{?}{=} 1-\frac{1}{2^{k+1}}$

$\cancel{1}-\frac{1}{2^k}+\frac{1}{2^{k+1}} \overset{?}{=} \cancel{1}-\frac{1}{2^{k+1}}$

$\frac{\cancel{2}^{\,1}}{\cancel{2^{k+1}}_{\,2^k}} = \frac{1}{2^k}$ ✓

QED.

as $6i+7j+13k$. $\mathbb{N}=\{0,1,2,\dots\}$)

a(k)
b(k)

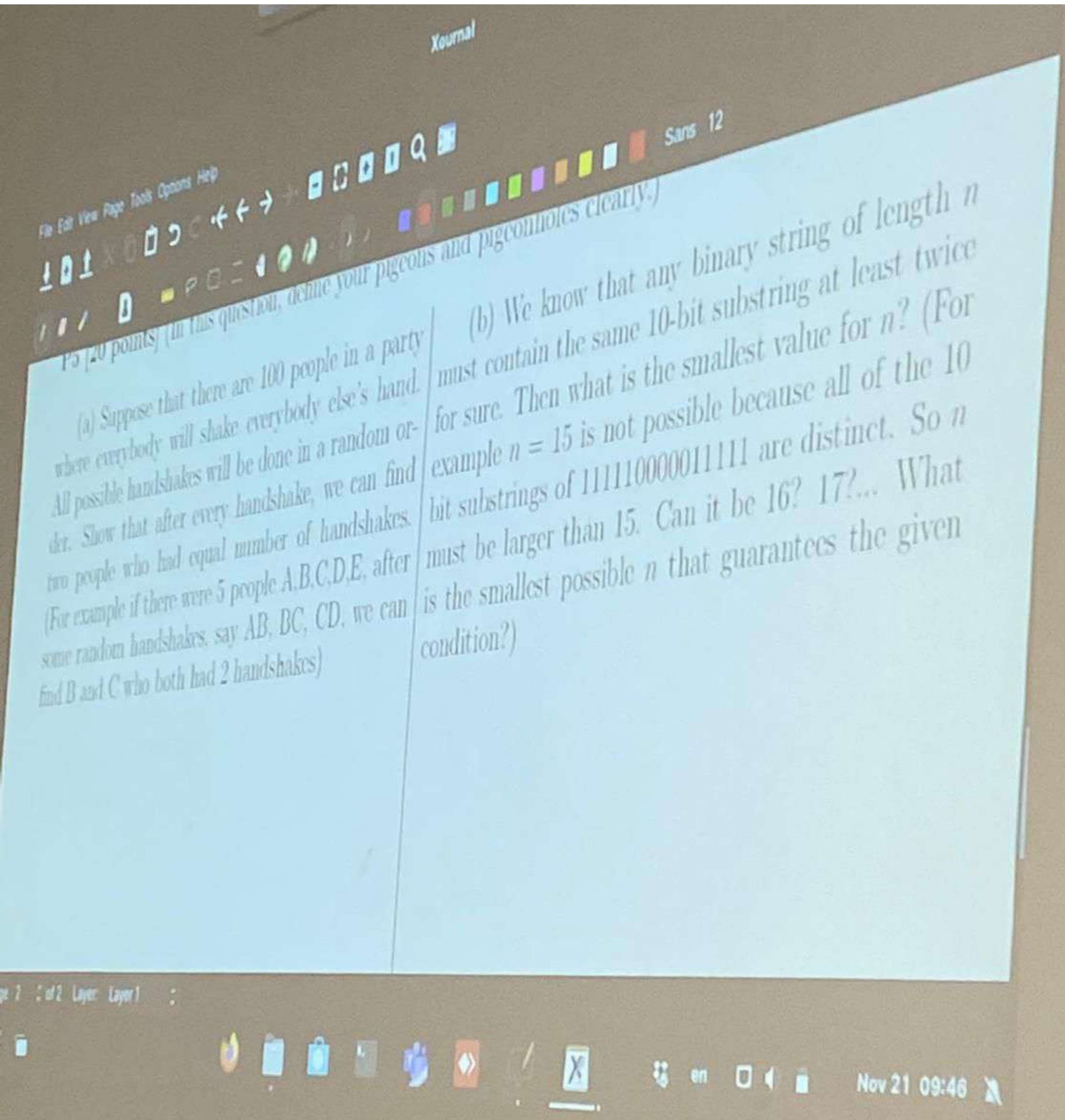
Base step: for n=35 $35 = 6.0 + 7.5 + 13.0$

Ind Hyp. for n=k assume $k = 6.a + 7.b + 13.c$

Ind Step. n=k+1 → $k+1 = 6(a-1) + 7(b+1) + 13c$

$k+1 = 6a + 7(b+2) + 13(c-1)$

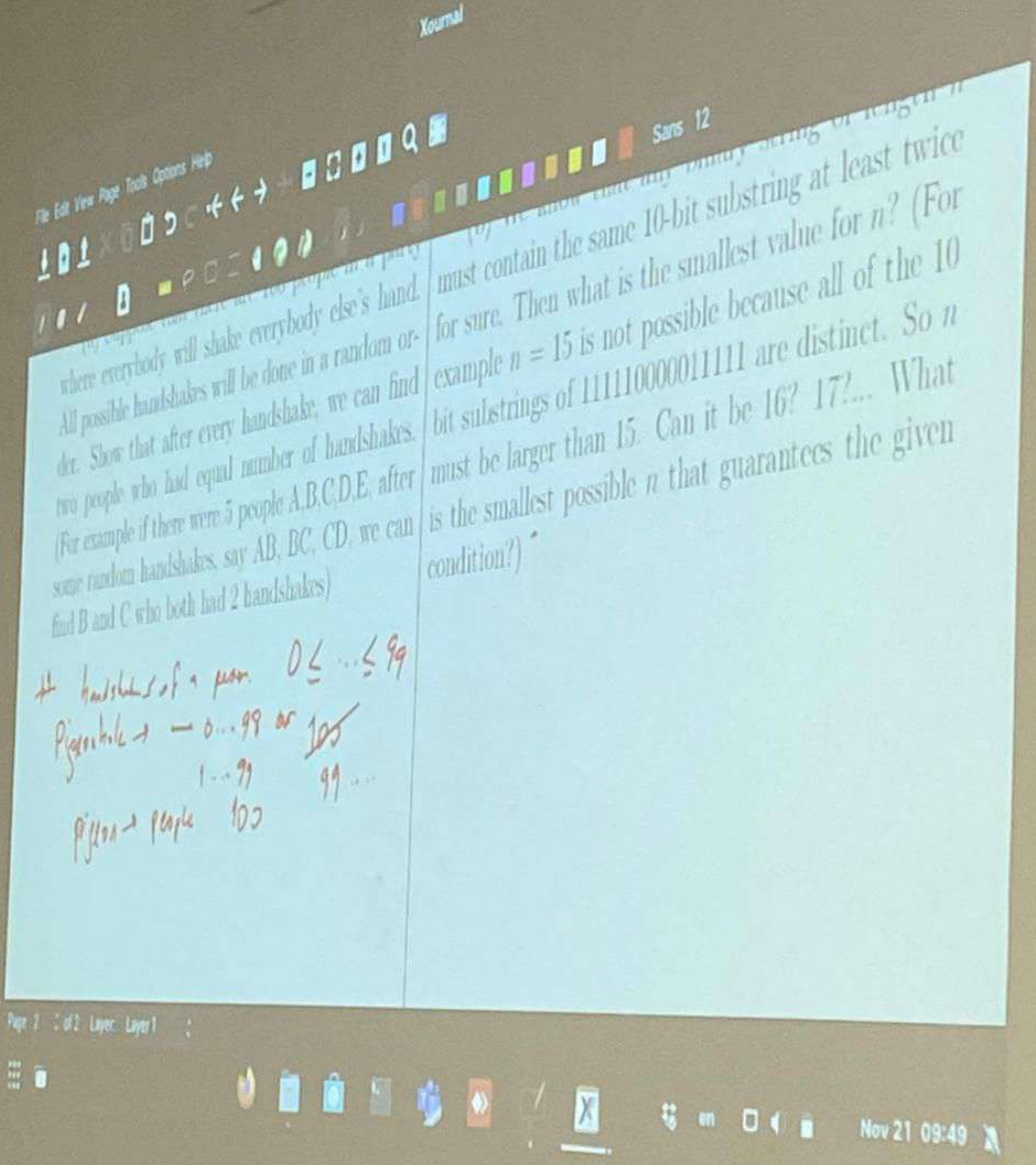
$k+1 = 6(a+6) + 7(b-5) + 13c$



P5 [20 points] (In this question, define your pigeons and pigeonholes clearly.)

(a) Suppose that there are 100 people in a party where everybody will shake everybody else's hand. All possible handshakes will be done in a random order. Show that after every handshake, we can find two people who had equal number of handshakes. (For example if there were 5 people A,B,C,D,E, after some random handshakes, say AB, BC, CD, we can find B and C who both had 2 handshakes)

(b) We know that any binary string of length n must contain the same 10-bit substring at least twice for sure. Then what is the smallest value for n? (For example $n = 15$ is not possible because all of the 10 bit substrings of 111110000011111 are distinct. So n must be larger than 15. Can it be 16? 17?... What is the smallest possible n that guarantees the given condition?)



(a) Suppose that there are 100 people in a party where everybody will shake everybody else's hand. All possible handshakes will be done in a random order. Show that after every handshake, we can find two people who had equal number of handshakes. (For example if there were 5 people A,B,C,D,E, after some random handshakes, say AB, BC, CD, we can find B and C who both had 2 handshakes)

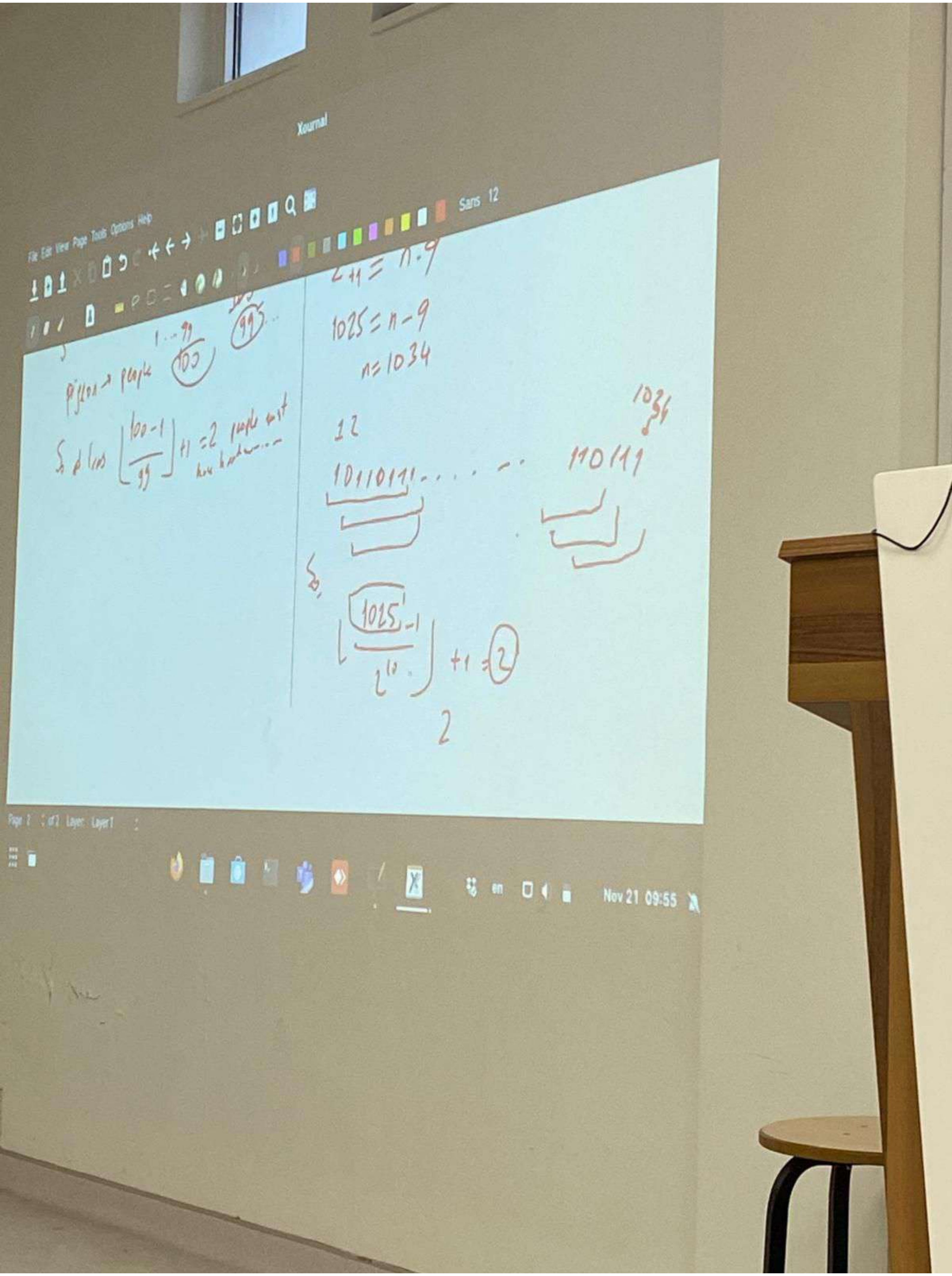
\# handshakes of a person $0 \leq \ldots \leq 99$

Pigeonhole → — 0…99 or ~~100~~

1…99   99…

pigeon → people 100

(b) We know that any binary string of length n must contain the same 10-bit substring at least twice for sure. Then what is the smallest value for n? (For example $n = 15$ is not possible because all of the 10 bit substrings of 111110000011111 are distinct. So n must be larger than 15. Can it be 16? 17?... What is the smallest possible n that guarantees the given condition?)



1 ... 99

$\boxed{100}$ $\boxed{99}$ ...

pigeon → people

So at least $\left\lfloor \frac{100-1}{99} \right\rfloor + 1 = 2$ people must have birthday...

$2^{10}+1 = n-9$

$1025 = n-9$

$n = 1034$

12

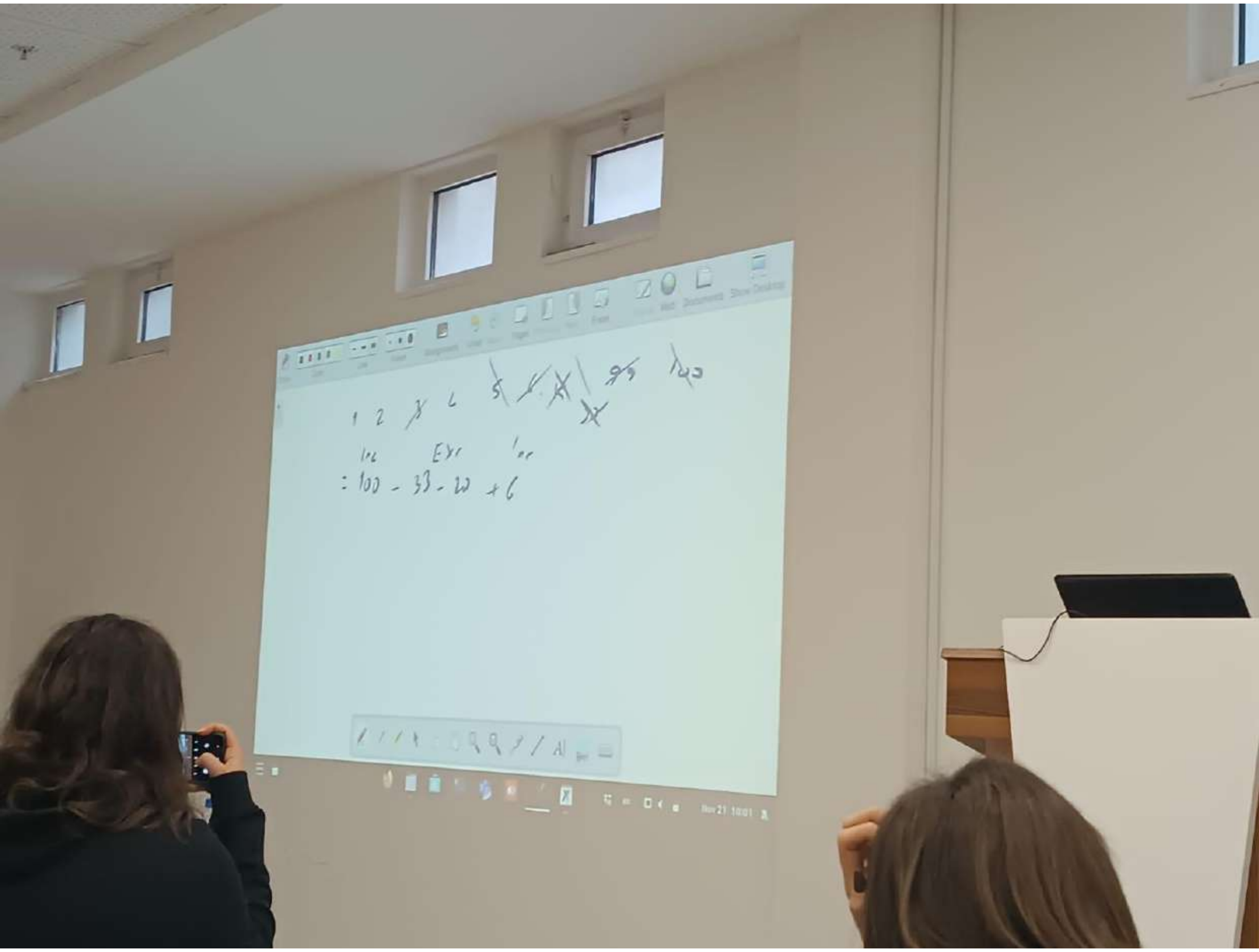
1034

$\underbrace{10110111}$ . . . . . . $\underbrace{110111}$

So

$\left\lfloor \frac{\boxed{1025}-1}{2^{10}} \right\rfloor + 1 = \boxed{2}$

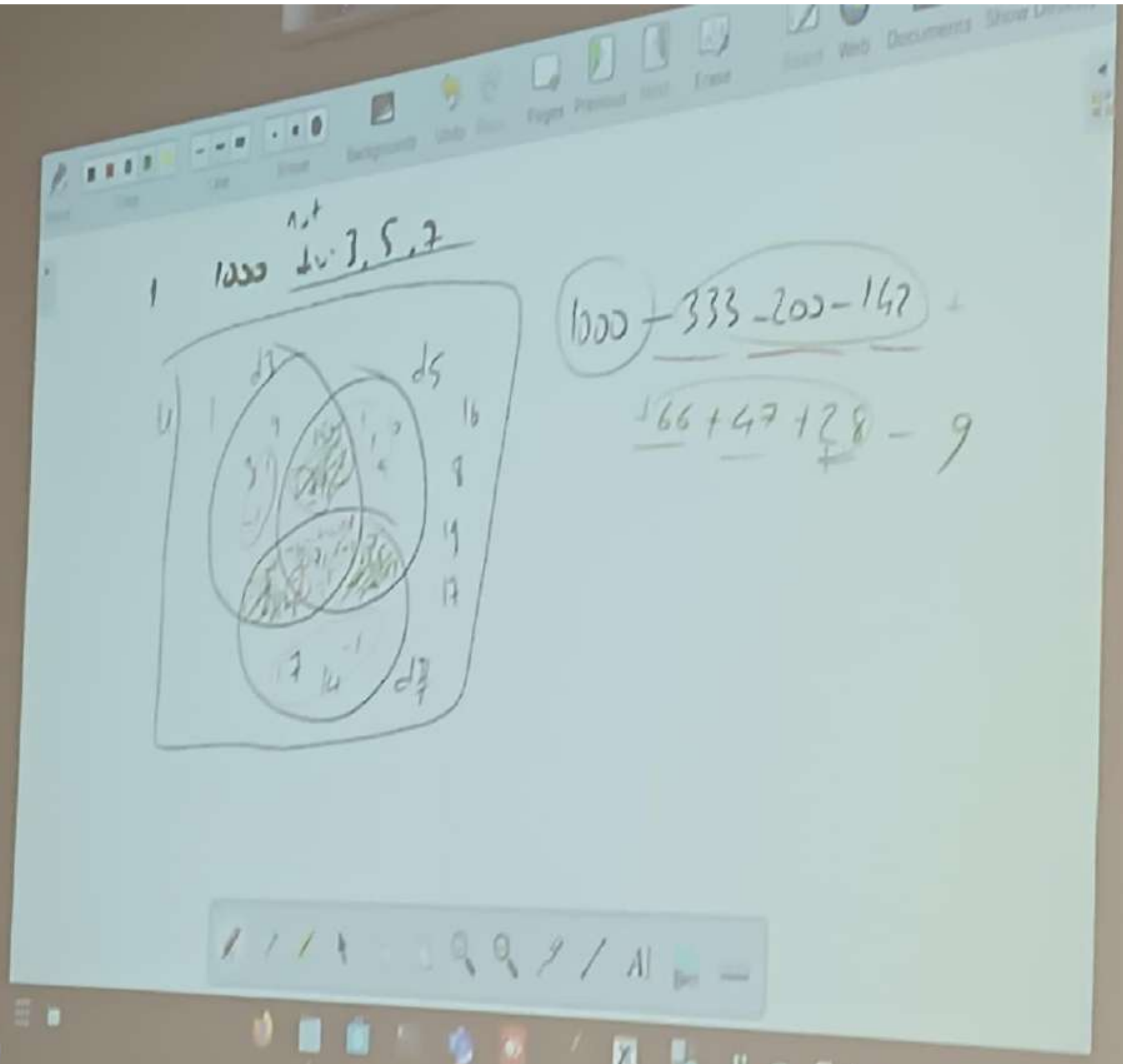
2



1 2 ~~3~~ 4 5 ~~6~~ ~~[illegible]~~ ~~[illegible]~~ ~~99~~ 100

Inc Exc Inc

= 100 - 33 - 20 + 6



1 1000 dv 3, 5, 7

1000 − 333 − 200 − 142 + 66 + 47 + 28 − 9