Friday 25/10/2019 Midterm 1 Duration: 90 minutes

Name: Solutions Student No:

Note: Unless otherwise stated, leave your answers with combinations, without calculating the numbers.

P1 [15 points] Consider the problem of choosing two soccer teams each having 11 players from a group of 22 people. (a) In how many ways can we choose these two teams? (b) What if two players insist on playing in the same team? How many options do we have? (c) What if Messi and Ronaldo are in the group and they are not allowed to be on the same team? How many options do we have?

(a) $\binom{22}{11}/2$

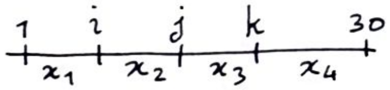
(b) $\binom{20}{9}$ (we select the teammates of these two players)

(c) $\binom{20}{10}$ (we just choose teammates of -say- Messi.)

P2 [20 points]

(a) How many printf statements will be called by this code:

```
for(int i=1; i<=30; i++) {
  for(int j=i+2; j<=30; j++) {
    for(int k=j+3; k<=30; k++) {
      printf("%d %d %d\n",i,j,k);
    }
  }
}
```



$x_1+x_2+x_3+x_4=29$

$x_1, x_4 \geq 0$

$x_2 \geq 2$

$x_3 \geq 3$

$y_2 = x_2 - 2$

$y_3 = x_3 - 3$

$x_1+y_2+y_3+x_4=24$

$\Downarrow$

$\binom{27}{3}$.

(b) How many non-negative integer solutions are there to the following equality/inequality system: $x_1 + x_2 + x_3 + \ldots + x_8 \leq 37$, $x_1 + x_2 + x_3 = 6$?

$x_1+x_2+x_3=6$ $\quad$ $x_4+x_5+\dots+x_8 \leq 31$

$x_4+x_5+\dots+x_8+x_9=31$

$x_9 \geq 0$.

$\downarrow$ $\quad$ $\downarrow$

$\binom{6+2}{2}$ $\quad$ $\binom{31+5}{5}$

So the answer is $\binom{8}{2}\cdot\binom{36}{5}$

P3 [20 points]

(a) Establish the validity of the argument:

$p \to q$

$(q \wedge r) \to s$

r

$\therefore p \to s$

1. $p \to q$ Premise.
2. $(q \wedge r) \to s$ Premise.
3. r Premise.
4. $\neg p \vee q$ St.1, equivalent.
5. $\neg(q \wedge r) \vee s$ St.2, equivalent.
6. $\neg q \vee \neg r \vee s$ St.5, DeMorgan
7. $(\neg q \vee s) \vee \neg r$ St.6. Associativity
8. $\neg q \vee s$ St.3,7 Disj.Syl.
9. $q \to s$ St.8. equiv.
10. $p \to s$ St. 1, 9. Syllogism. $\square$

(b) Let $p(x, y) : y - x = y + x^2$ where $x, y \in \mathbb{Z}$. Determine the truth value of the statements:

(a) $p(0,0)$ T F $\quad 0=0$

(b) $p(1,1)$ T F $\quad 0 \neq 2$

(c) $p(0,1)$ T F $\quad 1=1$

(d) $\forall y\ p(0,y)$ T F $\quad \forall y\ y=y$ ✓

(e) $\exists y\ p(1,y)$ T F $\quad y-1 \neq y+1$

(f) $\forall x\ \exists y\ p(x,y)$ T F $\quad$ Say $x=1$, for ex...

(g) $\exists y\ \forall x\ p(x,y)$ T F $\quad$ Impossible for $x \neq 0$.

(h) $\forall y\ \exists x\ p(x,y)$ T F $\quad \forall y$ give 0 to x.

P4 [30 points]

(a) Fibonacci numbers are defined as follows:
$F_0 = 0, F_1 = 1$
$F_n = F_{n-1} + F_{n-2}$ for $n \in Z^+$ with $n \geq 2$
Prove that:

$$F_0 + F_1 + F_2 + \ldots + F_n = \sum_{i=0}^{n} F_i = F_{n+2} - 1$$

using induction.

Proof.- For $n=0$, $F_0 = F_2 - 1$ $\quad 0 = 1-1$ ✓

- For $n=k$, assume that

$$F_0 + F_1 + \cdots + F_k = F_{k+2} - 1$$

Then for $n=k+1$, By Definition By Theorem

$$\underbrace{F_0 + F_1 + \cdots + F_k}_{\text{using assumption}} + F_{k+1} \stackrel{?}{=} F_{k+3} - 1$$

$$F_{k+2} - 1 \quad + F_{k+1} = F_{k+3} - 1$$

$$F_{k+3} \quad - 1 \quad = F_{k+3} - 1 \checkmark$$

So, we are done. □

(b) Prove the following statement by using mathematical induction: $\forall n \in \mathbb{Z}, n \geq 14$:

$$\exists i, j, k \in \mathbb{N}\ [n = 5i + 7j + 11k]$$

(For all integers $n \geq 14$, there exists natural numbers i, j and k such that n can be written as $5i + 7j + 11k$. $\mathbb{N} = \{0, 1, 2, \ldots\}$)

Proof. For $n=14$, $i=0$ $j=2$ $k=0$. gives
$14 = 5 \cdot 0 + 7 \cdot 2 + 11 \cdot 0$ ✓

For $n=k$, assume $k = 5a + 7b + 11c$

Then, for $n=k+1$, we can use one of these formulas:

1. $k+1 = 5(a-2) + 7b + 11(c+1)$
2. $k+1 = 5(a+3) + 7(b-2) + 11c$
3. $k+1 = 5(a+1) + 7(b+1) + 11(c-1)$

For all formulas to be useless, we must have $a<2 \wedge b<2 \wedge c<1$ but the largest integer we can get would be $5 \cdot 1 + 7 \cdot 1 + 11 \cdot 0 = 12$. However, we deal with $n \geq 14$, so we can always use atleast one of three.
Thus, $s(k) \to s(k+1)$ and we are done. □

P5 [20(+10bonus) points]

a) Prove that $\forall n \in \mathbb{Z}^+$, $gcd(5n+3, 7n+4) = 1$.

$gcd(5n+3, 7n+4)$

$= gcd(5n+3, 7n+4-5n-3)$

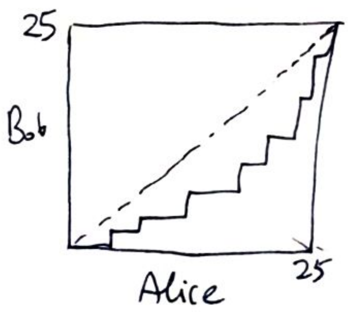
$= gcd(5n+3, 2n+1)$

$= gcd(n+1, 2n+1)$

$= gcd(n+1, n)$

$= 1.$ □

b) Suppose that Freddy has 50 dollars, in 1 dollar coins. So, he has fifty \$1 coins. He wants to give these dollars to his children Alice and Bob. He will do this day by day. Every day, he will give one of the coins to either Alice or Bob randomly. He wants to give 25 to each at the end of day 50. But since Alice is a little envious, he wants to keep Alice always having more money. So, at any time, the total money given to Bob shouldn't be more than the total money given to Alice. In how many ways can Freddy do this?



answer:

$$\binom{50}{25} - \binom{50}{26}$$

or equivalently

$$\frac{1}{26} \cdot \binom{50}{25}.$$