Ex:

cond. i = v_i is isolated

$\binom{5}{2} = 10 \quad \frac{2^{10}}{5}$ road system

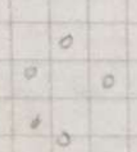
$S_1 = N(C_1) + \dots + N(C_5)$
$S_1 = 5\,N(C_1)$

$\overline{N} = S_0 - S_1 + S_2 - S_3 + S_4 - S_5$

$\binom{5}{0}2^{\binom{5}{2}} - \binom{5}{1}2^{\binom{4}{2}} + \binom{5}{2}2^{\binom{3}{2}} - \binom{5}{3}2^{\binom{2}{2}} + \binom{5}{4}1 - \binom{5}{5}1$

$= 2^{10} - 5 \cdot 2^6 + 10 \cdot 2^3 - 10 \cdot 2 + 5 - 1$

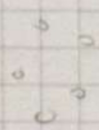
Theorem: Exactly m



Exactly 2?

$E_2 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4 - \binom{5}{3}S_5$

Theorem: At least



At least 2?

$L_2 = S_2 - \binom{2}{1}S_3 + \binom{3}{1}S_4 - \binom{4}{1}S_5$

Bir sıralama var ve bu sıralamanın hiçbirini
Derangement tutturamadığımız koşulda derangement

Ex: C_i = Guessy pass i correctly $\quad 1 \le i \le 18$

$\overline{N} = S_0 - S_1 + S_2 - S_3 + \dots + S_{18}$

$\frac{18!}{16!\,2!}\,16! = \frac{1}{2!}$

$= 18! - \binom{18}{1}17! + \binom{18}{2}16! + \dots - \binom{18}{17}1! + \binom{18}{18}0!$

$= 18!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots - \frac{1}{17!} + \frac{1}{18!}\right)$

8.13

$4!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 24\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 12 - 4 + 1 = 9$

$\frac{24}{e} = 8{,}8\ldots \approx 9$

general formula is $\frac{(n+r-1)!}{r!\,(n-1)!} = \left(\frac{n+r-1}{r}\right)!$

$C(n+r-1,\ r)$

Ex 20 kinds of donuts. There are at least a dozen of each kind. In how many ways can we select a dozen donuts?

xxxxx||||| xx |||||/ xxx ||||| xx ||||

$C(20+19,\ 19) = C(39,19) = C(39,20)$

Number of Solutions for an Equation

Ex: $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$

1 1 1 | 1 1 | 1 | 1 1 | 1 | 1

non-negative → $C(15,10)$

positive → $x_1 = \bar{x}_1 + 1$, $x_2 = \bar{x}_2 + 1$...

$\bar{x}_1 + \bar{x}_2 + \dots + \bar{x}_6 = 4$

$C(9,4)$

Ex: $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 10$

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 9$

$C(15,9)$

Ex: I have 10 white balls and 4 identical bags. How many ways can I distribute my balls to these bags?

ayni 4

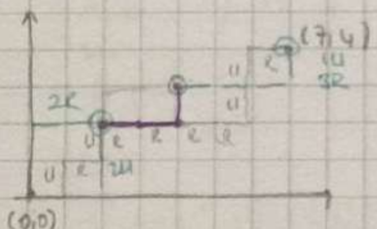
1 1 1 / 1 1 1 / 1 / 1 1 1

$C\binom{10+3}{3} = C(13,3)$

$C(13,3)/4!$

Ex: How many ways from (0,0) to (7,4).

→ $C(11,7) = C(11,4)$



Ex2: not use from (2,2) to (3,2) to (4,2) to (4,3)?

$C(11,4) - C(4,2) \cdot C(4,1)$

Ex2: third type of move is allowed D ↗ $D = 1U + 1R$

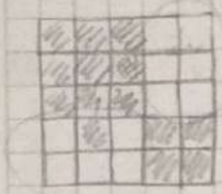
	R	U	D
D + 6R + 3U →	$\binom{10}{6}$	$\binom{4}{3}$	$\binom{1}{1}$
2D + 5R + 2U →	$\binom{9}{5}$	$\binom{4}{2}$	$\binom{2}{2}$
3D + 4R + U →	$\binom{8}{4}$	$\binom{4}{1}$	$\binom{3}{3}$
4D + 3R + 0U →	$\binom{7}{3}$	$\binom{4}{0}$	$\binom{4}{4}$

\+ answer

Ex 5 arabanın yerlerinin her gün bir önceki güne göre farklı olması

ilk gün $\to 5!$ | A | B | C | D | E | sonra $\to 5!/e$

Rook Polynomials



$r(C,x) = 1 + 11x + 40x^2 + 54x^3$

C_1 $\quad r_1(C_1,x) = 1 + 6x + 6x^2$

$r_2(C_2,x) = 1 + 5x + 4x^2$ (?)

C_2

coefficient

Ex Roll a die 5 times, what is the probability of getting 20 total?

$(3,5,6,2,4) \to x_1 + x_2 + x_3 + x_4 + x_5 = 20 \quad 1 \le x_i \le 6$

$[x^{20}] : (x + x^2 + x^3 + \dots + x^6)^5 \to [x^{15}] \; \cancel{x^5}(1 + x^2 + \dots + x^5)^5$

$$= \left(\frac{1-x^6}{1-x}\right)^5 = (1-x^6)^5 (1-x)^{-5}$$

Binomial T. $(a+b)^n \quad \dots a^r b^{n-r}$ has coef $\binom{n}{r} \quad = (a+b)(a+b)\dots(a+b)$

Extended B T. $\binom{n}{r} = \dfrac{n(n-1)\dots(n-r+1)}{r!} \quad n \in \mathbb{R}$

$$\binom{1/3}{2} = \frac{(\frac{1}{3})(-\frac{2}{3})}{2!} = -\frac{1}{9}$$

$$\binom{-6}{10} = \frac{(-6)(-7)(-8)\dots(-15)}{10!} = \binom{15}{10}$$

For neg. coml

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

Ex find exactly 2 void suits

$E_2 = S_2 - \binom{3}{2} S_3 + \binom{4}{2} S_4 \to 0$

15 flowers on 5 shelves?

Ex $\overset{+1}{s_1} + \overset{+1}{s_2} + \overset{+1}{s_3} + \overset{+1}{s_4} + \overset{+1}{s_5} = 15 \quad 1 \le i \le 5 \quad 1 \le x_i \le 4$

$s_1' + s_2' + s_3' + s_4' + s_5' = 10 \qquad x_i \le 3$

$S_0 = \binom{5}{0}\binom{14}{4}, \quad S_1 = \binom{5}{1}\binom{10}{4}, \quad S_2 = \binom{5}{2}\binom{6}{4}, \quad (S_3, S_4, S_5) = 0$

$\binom{5}{0}\binom{14}{4} - \binom{5}{1}\binom{10}{4} + \binom{5}{2}\binom{6}{4}$

Ex How many integers, $1 \le n \le 75$ (≥ 0) are not divisible by 3, 5, or 7?

$75 - \left(\frac{75}{3} + \frac{75}{5} + \frac{75}{7}\right) + \left(\frac{75}{15} + \frac{75}{21} + \frac{75}{35}\right) - \left(\frac{75}{105}\right)$

$75 \quad - 50 \quad + \quad 10 \quad - \quad 0 = 35$

Exactly m – At least m

Ex 10 prizes for 4 student. We want $\boxed{\text{only 2}}$ to get them

E_2

$E_2 \Rightarrow S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 \to 0$

$S_2 = \binom{4}{2} 2^{10}, \quad S_3 = \binom{3}{1}\binom{4}{3} 1^{10}$

$x_1 + x_2 + x_3 + x_4 = 10$
$1 \le i \le 4$
$1 \le x_i \le$

prizes: o o o o o o o o o o
student: a b

$2 \times 2 \times 2 \cdots \times 2 = 2^{10}$

Ex Deal 13 cards from 52 card deck
probability of at least one card from each suit?
{♡, ◇, ♧, ♤}

$C_1 = $ no ♡, $C_2 = $ no ◇, $C_3 = $ no ♧, $C_4 = $ no ♤

$+ S_0 = \binom{52}{13}$

$- S_1 = \binom{4}{1}\binom{39}{13}$

$+ S_2 = \binom{4}{2}\binom{26}{13}$

$- S_3 = \binom{4}{3}\binom{13}{13}$

$S_4 = 0$

$\text{Prob.} = \dfrac{S_0 - S_1 + S_2 - S_3}{S_0} \qquad \binom{52}{13}$

~ INCLUSION - EXCLUSION ~

8.5

$x_1 + x_2 + x_3 + x_4 = 18 \quad 0 \le i \le 4 \quad x_i \le 7$

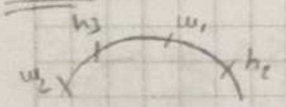
$+8$

$\Rightarrow \overline{N} = S_0 - S_1 + S_2 - S_3 + S_4$

$3 \cdot 8 = 24 > 18$

$\binom{4}{0}\binom{21}{3} - \binom{4}{1}\binom{18-8}{3} + \binom{4}{2}\binom{10-8}{3} - 0 + 0 \rightarrow 4 \cdot 8 = 32 > 18$

8.9



6 couple, no wife sits her husband

$\Rightarrow S_0 - S_1 + S_2 - S_3 + S_4 - S_5 + S_6$

$S_0 = 11!$

$S_1 = (w_1 h_1) \rightarrow 2 \cdot 10! \binom{6}{1}$

$S_2 = (w_1 h_1)(w_2 h_2) \rightarrow 2 \cdot 2 \cdot 9! \binom{6}{2}$

$\Rightarrow \binom{6}{0} 11! - \binom{6}{1} 2 \cdot 10! + \binom{6}{2} 2^2 9! - \binom{6}{3} 2^3 8!$

$+ \binom{6}{4} 2^4 7! - \binom{6}{5} 2^5 6! + \binom{6}{6} 2^6 \cdot 5!$

8.10

$S_0 = 2^{\binom{5}{2}}$ $\binom{5}{2} = 10 \rightarrow$ kaç farklı yol var?

yol var mı yok mu?

EX DISCRETEMATHROCKS (have no consecutive pairs) → ardışık

DISCRETMAHOK (12)
SCRET (5) > 17 letter

$+ S_0 = \dfrac{17!}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$

$- S_1 = SS, CC, \ldots$

$+ S_2 = SS + CC, SS + RR, \ldots$

$- S_3 = SS + CC + RR, \ldots$

$+ S_4 =$

$- S_5 =$

$+ 17! / 2^5$

$- \binom{5}{1} 16! / 2^4$

$+ \binom{5}{2} 15! / 2^3$

$- \binom{5}{3} 14! / 2^2$

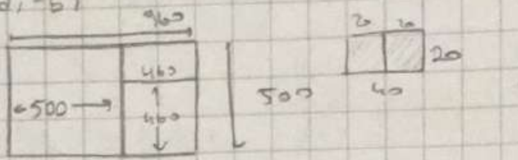
$+ \binom{5}{4} 13! / 2^1$

$- \binom{5}{5} 12! / 2^0$

Greatest Common Division

$gdc(a,b) = gdc(b,a)$
$gdc(-a,b) = gdc(a,-b)$

Ex: $gdc(960,500)$



$\Rightarrow 20$

Coprimeness

Relatively Prime (Co-Prime)

if $gdc(a,b)=1$ a and b are co-prime

Ex: $(24,49)$

Theorem

$a,b,c \in \mathbb{Z}^+$, $ax+by=c$ has an integer solution $x=x_0$ $y=y_0$ if and only if $gdc(a,b) \mid c$

$gdc(14,70)=14 \rightarrow 14x+70y=14$ (14 is the smallest integer)
$14x+70y=28$

Least Common Multiple

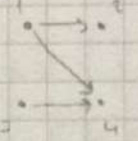
$a,b \in \mathbb{Z}^+$, a positive int., c is commen multp. a and b if $a \mid c \wedge b \mid c$

Pigeonhole Principle

Given m items and n containers, if $m>n$, there is at least one container with certain $\lceil m/n \rceil$ items.
$\hookrightarrow = x,y \ldots = x+1$

Ex $\lceil \frac{4}{3} \rceil = \lceil 1.333 \rceil = 2$, Ex: $\lceil \frac{10}{4} \rceil = 3$ items

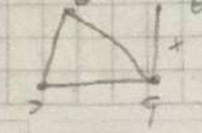
Ex In a group of people, there are two people who have an identical number of friends within the group.



n people (items) pigeon
each person $[1,2,\ldots,n-1]$ friend
← containers (pigeonhole)

$\frac{n}{n-1} = \lceil 1,\ldots \rceil = 2$

Incident: x incident to a-b
Adjacent: b adjacent to a,c
Isolated: p



trivial: tüm değişkenlerin 0'a eşit olduğu çözüm
: önemsiz

$[x^{50}]\,(x^6+x^7+x^8+\dots)^6$ $\quad [x^{14}]\left(\frac{1}{1-x}\right) = (1,1,1,1,\dots)$

$[x^{50}]\,x^{36}(1+x+x^2+\dots)^6$ $\quad = 1$

~ THE GRAPH THEORY AND APPLICATIONS ~

~ Properities ~

The degre: $d(V)$ V's number of incident edges
self-loop count for 2 in degree
isolated vertex has degree 0

Proposition: The sum of the degrees of a graph $G(V,E)$ equals $2|E| = 2m$ (trivial)

Corollary: The number of vertices of odd is even (trivial)

Complete $K_n \to B(n,2) \to \frac{n(n-1)}{2}$ possible edge number

$K_4 \to \frac{4\cdot 3}{2} = 6$ $\quad K_5 = \frac{5\cdot 4}{2} = 10$

K-Regular: A simple graph with vertices of equal degree k

Corolly: The complete graph K_n is $(n-1)$-regular

Bipartite: $V = V_1 \cup V_2$ (there is no edge between $V_1 - V_2$)

Complete bipartity: one where all edges between $V_1 - V_2$ are present. $(|E| = |V_1|\cdot|V_2|)$ (K_{n_1,n_2}) number of edges $\to n_1 \cdot n_2$

trail: walk with all different edges

path: walk with all different nodes (and here edges)

directed graph (digraph) $(\rightarrow)$: all edge has a direction $(\rightleftarrows)$

$E(V_s, V_t)$ $V_s \to$ source node, $V(t)$ terminal nodes)

in degree $d_{in}(V)$, out degree $d_{out}(V)$

A graph is balanced if $d_{in}(V) = d_{out}(V)$ for all nodes

1. Define universal set $\to \mathcal{U}$
2. Define conditions $\to c_i \to t$
3. Write the formula and solve $\to S_0 + S_1 + S_2 + \dots + S_t$

$0 \le x \le 7$

$\bar{N} = S_0 - S_1 + S_2 + S_3 + S_4$

$S_1 = N(c_1) + N(c_2) + \dots$ $\quad S_2 = N(c_1c_2) + N(c_1c_3) + \dots +$

$S_1 = x_1 + x_2 + x_3 + x_4 = 18$

$(x_1 + 8)$

$S_1 = x_1 + x_2 + x_3 + x_4 = 18 - 8 = 10$

$S_1 = \binom{13}{3}\binom{4}{1}$

$S_2 = (x_1 + 8) + (x_2 + 8) + x_3 + x_4 = 18$

$S_2 = x_1 + x_2 + x_3 + x_4 = 2$

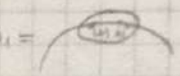
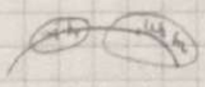
$S_2 = \binom{4}{2}\binom{5}{3}$

$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4$

$\binom{4}{0}\binom{21}{3} - \binom{4}{1}\binom{13}{3} + \binom{4}{2}\binom{5}{3} - 0 + 0$

Ex (8.9)

1. $\mathcal{U}$: All seating possible $\Rightarrow 11!$
2. C_i: w_i sits next h_i $\quad \forall_i \; 1 \le x_i \le 6 \quad t = 6$
3. $\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4 - S_5 + S_6$
$= 11! \cdot$

$S_1 =$  $\quad S_2 =$ 

$N(c_i) = 2 \cdot 10!$ $\quad N(c_ic_j) = 2^2 \cdot 9!$

$S_1 = \binom{6}{1} 2 \; 10!$ $\quad S_2 = \binom{6}{2} 2^2 \; 9!$

$\bar{N} = 11! - \binom{6}{1} 2 \cdot 10! + \binom{6}{2} 2^2 9 - \binom{6}{3} 2^3 8! + \binom{6}{4} 2^4 7! - \binom{6}{5} 2^5 6! + \binom{6}{6} 2^6 5!$

$\bar{N} = \sum_{i=0}^{6} \binom{6}{i} 2^i (11-i)!$

Euler's phi function

for $n \in \mathbb{Z}^+$, $n \ge 2$ let $\phi(n)$ be the positive integers m, where $1 \le m \le n$ and $\gcd(n, m) = 1$

$\phi(n) =$ [number of positive int. smaller than n, also relative prime to n].

Ex $\phi(12) = |\{1, 5, 7, 11\}| = 4$

$\phi(n) = ?$ $\quad n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \dots p_t^{e_t}$

1. $\mathcal{U} = \{1, 2, 3, \dots, n\}$
2. c_i = div by p_i $\quad \forall \; 1 \le i \le t$
3. $\bar{N} = S_0 - S_1 + S_2 - \dots \pm S_t$

$n = n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \dots \frac{n}{p_t}\right) + \left(\frac{n}{p_1p_2} + \frac{n}{p_1p_3} + \dots\right) + \dots$
$\dots - (-1)^t \left(\frac{n}{p_1p_2p_3 \dots p_t}\right)$

$\bar{N} = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_t}\right) \to n \prod^{t} \left(1 - \frac{1}{p_i}\right)$

PROOF OF MATHEMATICAL INDUCTIONS

Ex:

Theorem: $\forall n \in \mathbb{Z}^+ \; [1+3+5+\dots+(2n-1) = n^2]$

Proof:
1. $n=1$: $1 = 1^2$ ✓
2. $n=k$
   $1+3+\dots+(2k-1) = k^2$
3. $n=k+1$
   $\underbrace{1+3+\dots+(2k-1)}_{k^2}+(2k+1) = (k+1)^2 = k^2+2k+1$

   $k^2+2k+1 = k^2+2k+1$ ✓

Ex

The.: $\forall n \in \mathbb{Z}^+,\ n \geq 14 \quad \exists i,j \in \mathbb{N}\ [n = 3i + 8j]$

Proof:
1. $n=14$ $\quad n = 3\cdot 2 + 8\cdot 1 = 14$ //
2. $n=k$ $\quad n = 3a + 8j \quad (i=a,\ j=b)$
3. $n=k+1$
   if $a \geq 5$ $\quad k+1 = \underbrace{k}_{3a+8b} - (3\cdot 5) + (8\cdot 2) = 3(a-5) + 8(b+2)$
   if $b \geq 1$ $\quad k+1 = \underbrace{k}_{3a+8b} - 8 + 3\cdot 3 = 3(a+3) + 8(b-1)$

~~Strong Induction (Alternative Form)~~

Recursive Definition

Ex:

Theorem: $T(n) = 2^n - 1$

Proof:
1. $n=1 \quad 2^1 - 1 = 1$
2. (Ind. Hyp.) $k \geq 1 \quad T(k) = 2^k - 1$
3. For $k+1$,
   $T(k+1) = 2\,T(k) + 1$
   $2^{k+1} - 1 = 2(2^k - 1) + 1 = 2\cdot 2^k - 1$

DIVISION

If $a,b \in \mathbb{Z}$ and $b \neq 0$, we say that b divedes a, $b \mid a$ if there is an integer $a = bn$.

If $a,b \in \mathbb{Z}$, $b > 0$, then there exists a unique $q,r \in \mathbb{Z}$ with $a = qb + r \quad 0 \leq r < b$

q: quotion, r: remainder

$25 = \boxed{3}\cdot 7 + \boxed{4}$

$5 = 0\cdot 13 + 5$

6. Idempotents $p \vee p \leftrightarrow p \quad p \wedge p \leftrightarrow p$
7. Identity $p \vee F_0 \leftrightarrow p \quad p \wedge T_0 \leftrightarrow p$
8. Inverse $p \vee \neg p \leftrightarrow T_0 \quad p \wedge \neg p \leftrightarrow F_0$
9. Domination $p \vee T_0 \leftrightarrow T_0 \quad p \wedge F_0 \leftrightarrow F_0$
10. Absorption $p \vee (p \wedge q) \leftrightarrow p \quad p \wedge (p \vee q) \leftrightarrow p$

They are the dual of each other

The Dual

$$[p \wedge (q \vee r)]^d = p \vee (q \wedge r)$$

$\wedge \to \vee$, $\vee \to \wedge$, $F_0 \to T_0$, $T_0 \to F_0$

$\widetilde{p} \to \underbrace{q} \leftrightarrow \neg \underbrace{q} \to \neg \widetilde{p}$ (contrapositive)

$p \to q$ converse $q \to p$

$\neg p \to \neg q$ inverse $p \to q$

* $p \wedge q$
$p \to (q \to r)$
$\therefore r$

if $p \wedge q \equiv 1 \quad p \equiv 1 \quad q \equiv 1$
$\underset{1}{p} \to \underbrace{(q \to r)}_{\text{must be } 1} \equiv 1$

$q \to r \equiv 1$
$\downarrow$
1 must be 1

* PROOF

$(\neg p \vee \neg q) \to (r \wedge s)$
$r \to t$
$\neg t$
$\therefore p$

Steps	Reason
1. $(\neg p \vee \neg q) \to (r \wedge s)$	Premise
2. $r \to t$	Premise
3. $\neg t$	Premise
4. $\neg r$	Step 2, 3 and Modus Tollens
5. $\neg(p \wedge q) \to (r \wedge s)$	Step 1 and DeMorgan's
6. $\neg(r \wedge s) \to \neg\neg(p \wedge q)$	Step 5 and Contrapositive
7. $\neg(r \wedge s) \to (p \wedge q)$	Step 6 and Double negation
8. $(\neg r \vee \neg s) \to (p \wedge q)$	Step 7 and DeMorgan's
9. $(\neg r \vee \neg s)$	Step 4 and Rule of disjunctive Amplification
10. $(p \wedge q)$	Step 8, 9 and Mod Ponens
11. p	Step 10 and Conjunctive Simplification

generic function

$$\left(\frac{(1-x^6)}{(1-x)}\right)^5 = \begin{array}{cc|cc} \multicolumn{2}{c|}{(1-x^6)^5} & \multicolumn{2}{c}{(1-x)^{-5}} \\ \hline \binom{5}{0} & 1 & \binom{-5}{15} & (-1)^{15} \\ \binom{5}{1} & -x^6 & \binom{-5}{9} & (-1)^{9} \\ \binom{5}{2} & x^{12} & \binom{-5}{3} & (-1)^{3} \end{array}$$

$\binom{5}{0}\binom{19}{4} - \binom{5}{1}\binom{13}{4} + \binom{5}{2}\binom{7}{4}$

$x_1 + x_2 + \dots + x_5 = 20$

$1 \le x_i \le 6$

$y_1 + y_2 + \dots + y_5 = 15$

$0 \le y_i \le 5$

$C_i : y_i \ge 6$

$\overline{N} = S_0 - S_1 + S_2 - S_3 + S_4 - S_5$

$= \binom{19}{4} - \binom{5}{1}\binom{13}{4} + \binom{5}{2}\binom{7}{4}$

Ex How many ways are there to re- 35 cents by using pennies, nickles, and dimes

1c 5c 10c

$\overset{1c}{(1 + x + x^2 + \dots)}\ \overset{5c}{(1 + x^5 + x^{10} + \dots)}\ \overset{10c}{(1 + x^{10} + x^{20} + \dots)}$

$\left(\frac{1}{1-x}\right) \quad \left(\frac{1}{1-x^5}\right) \quad \left(\frac{1}{1-x^{10}}\right)$

$[x^{35}]$ $(1-x)^{-1}(1-x^5)^{-1}(1-x^{10})^{-1}$

$(1-x)^{-1}$	$(1-x^5)^{-1}$	$(1-x^{10})^{-1}$	
x^{35}	1	1	
x^{30}	x^5	1	
x^{25}	x^{10}	1	
x^{20}	x^{15}	1	$\to \binom{-1}{0}$
x^{15}	x^{20}	1	
x^{10}	x^{25}	1	
x^5	x^{30}	1	
1	x^{35}	1	
x^{25}	1	x^{10}	
x^{20}	x^5		
x^{15}	x^{10}		$\to \binom{-1}{1}$
x^{10}	x^{15}		
x^5	x^{20}		
1	x^{25}		
x^{15}	1	x^{20}	
x^{10}	x^5		$\to \binom{-1}{2}$
x^5	x^{10}		
1	x^{15}		

$\frac{1}{1-x} = 1 + x + x^2 + \dots \qquad 1, 1, 1, 1, \dots$

$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \dots + x^n \qquad 1, 1, 1, 1, \dots 1, 0, 0$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \qquad 1, -1, +1, -1, \dots$

$\frac{1}{1-ax} = a^0 + a^1 + a^2 + \dots \qquad 1, a, a^2, a^3, \dots$

$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + \dots\right)$

$= \left(\frac{1}{1-x}\right)^2 = 0 + 1 + 2x + 3x^2 + 4x^3 + \dots$

$= (1 + x + x^2 + \dots)(1 + x + x^2 + \dots) \qquad 1, 2, 3, 4, 5, \dots$

The USE OF QUANTIFIERS

- "The number $x+2$ is an even integer." (Not a statement) → is an open statement
- When x is replaced by 1, 3, 5 → FALSE statement
- When x is " " 2, 4, 6 → TRUE statement

$\exists x$ → for at least one x.

"For some x, $p(x)$" → $\exists x \; p(x)$

$\exists x \exists y \; p(x,y) \Rightarrow \exists x, y \; p(x,y)$

$\forall_x$ → for every x.

$\forall x, y \; p(x,y) \Rightarrow \forall x \forall y \; p(x,y)$

* $\exists = \neg\forall$, $\forall = \neg\exists$

~ RULE OF INFERENCE ~

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$ (Modus Ponens)

$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$ (Modus Tollens)

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$ (Rule of Syllogism)

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$ (Rule of Conjunction)

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$ (Rule of disjunctive Syllogism)

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$ (Rule of disjunctive Amplification)

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$ (Rule of Conjunctive Simplification)

$$\begin{array}{l} \neg p \to F_0 \\ \hline \therefore p \end{array}$$ (Rule of Contradiction)

$$\begin{array}{l} p \wedge q \\ p \to (q \to r) \\ \hline \therefore r \end{array}$$ (The rule of Conditional Proof)

Ex $a_n = -a_{n-1} + 6a_{n-2}$, $n \geq 2$

$a_0 = -1$ $a_1 = 8$

a : $-1, 8, -14, 62, \ldots, a_n$

→ 0 değer → $a_n + a_{n-1} - 6a_{n-2} = 0$

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$x f(x) = \quad a_0 x + a_1 x^2 + a_2 x^3 + \ldots$

$-6x^2 f(x) = \quad -6a_0 x^2 - 6a_1 x^3 + \ldots$

$+$

$f(x)(1+x-6x^2) = -1 + 7x$

$f(x) = \dfrac{-1+7x}{1+x-6x^2} = \dfrac{A}{1-2x} + \dfrac{B}{1+3x} \Rightarrow A + 3Ax + B - 2Bx = 7x - 1$

$A + B = -1$ $\quad A = 1$

$3A - 2B = 7$ $\quad B = -2$

$1 \quad -2$

$f(x) = \dfrac{1}{1-2x} - \dfrac{2}{1+3x}$

coeff. of x^n in $f(x)$ = $2^n + (-3)^n \cdot 2 = 2^n - 2(-3)^n$

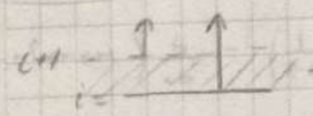
Exactly m − At least m

$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \ldots$

$L_m = S_m - \binom{m}{1} S_{m+1} + \binom{m+1}{2} S_{m+2} - \ldots$

$E_i = L_i - L_{i+1}$

Proof Is there a point $E_i = L_i$



$\Rightarrow E_i$

$E_i = L_i$ (base case)

$E_i = L_i - L_{i+1} \to 0$

if there is only i condition

$E_i = L_i$

Ex ARRANGEMENT a) E_2 pairs ?

b) L_3 pairs ?

ARNGEMT

ARN E

a) $S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4$

$\binom{4}{2} \dfrac{9!}{2 \cdot 2} - 3\binom{4}{1}\dfrac{8!}{2} + 6\binom{4}{4} 7!$

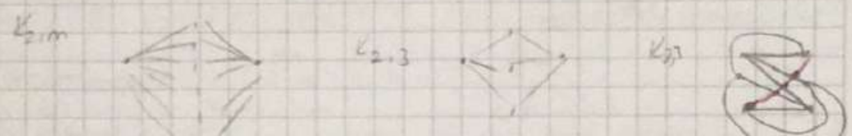
b) $S_3 - \binom{3}{1} S_4$

$\binom{4}{3}\dfrac{8!}{2} - 3\binom{4}{4} 7!$

if it, then connected and even degrees

Euler circuit : traverse every edge only once.

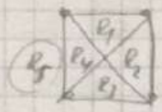
Planar GRAPH



Kuratowski's Theorem : A graph is non-planar if it contains a subgraph

Eulerian Theorem

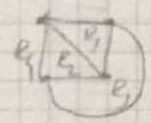
$v - e + r = 2$



$v = 4$
$e = 6$
$r = 5$ } $3 \neq 2$ not planner

Hamilton Cycles

there is a cycle in C that uses every vertex once



v 4
e 6
r 4

$4 - 6 + 4 = 2 = 2$ planner

Chromatic number

$X(K_n) = n$
$X(K_{n,m}) = 2$

B – A – D – E – C – F – G

Pre → * + 3 7 – 10 5
Post → 3 7 + 10 5 – *
In order → (3+7) * (10 - 5)

Ex prob exactly 2 void suits

$$E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 \to 0$$

$$\frac{E_2}{S_0} = \frac{\binom{4}{2}\binom{26}{13} - \binom{3}{1}\binom{4}{3}\binom{13}{13}}{S_0} \quad \text{(prob)}$$

8.10 $S_0 = 2^{10}\binom{5}{2}$ $S_1 = \binom{5}{1} 2^{6}\binom{4}{2}$ $S_2 = \binom{5}{2} 2^{3}$ $S_3 = \binom{5}{3} 2^{1}$ $S_4 = \binom{5}{4} \cdot 1$ $S_5 = \binom{5}{5} \cdot 1$

$$E_2 = \binom{5}{3} 2^3 - \binom{3}{1}\binom{5}{3} 2 + \binom{4}{2}\binom{5}{4} - \binom{5}{3}\binom{5}{5}$$

$$L_2 = \binom{5}{2} 2^3 - \binom{2}{1}\binom{5}{3} 2 + \binom{3}{2}\binom{5}{4} - \binom{4}{3}\binom{5}{5}$$

~ GENERATING FUNCTIONS ~

$[x^{20}]$ $\binom{5}{2}(x^1 + x^3 + x^5 + \dots + x^{17})^2 \binom{3}{3} (x^2 + x^4 + \dots + x^{20})^3$

$10 \cdot x^2(1 + x^2 + x^4 + \dots)^2 \cdot x^6(1 + x^2 + x^4 + \dots)^3$

$[x^{20}]$ $10 \cdot x^8 (1 + x^2 + x^4 + \dots)^5$

$[x^{12}]$ $10(1 + x^2 + x^4 + \dots)^5 \to \left(\frac{1}{1-x^2}\right)^5$ $[x^{12}](1-x^2)^{-5}$ 2,4,6,8,10,12

$$[x^{12}](1-x^2)^{-5} \left(10\binom{-5}{6}\right) \Rightarrow 10 \cdot \binom{10}{6} \cancel{(-1)^6} \cancel{(-1)^6}$$

$$10 \cdot 14 = 140$$

Ex $[x^{30}] (x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^{12}$

$[x^{18}] \, (1 + x + x^2 + x^3 + \dots + x^5)^{12} \to [x^{18}]\left(\frac{1-x^6}{1-x}\right)^{12} = [x^{18}](1-x^6)^{12}(1-x)^{-12}$

$$\binom{29}{18} - 12\binom{23}{12} + \binom{12}{2}\binom{17}{6} - \binom{12}{3}$$

+	$\binom{12}{0}$		$\binom{-12}{18}$
−	$\binom{12}{1}$	(-1)	$\binom{-12}{12}$
+	$\binom{12}{2}$		$\binom{-12}{6}$
−	$\binom{12}{3}$	(-1)	$\binom{-12}{0}$

Euler Circuit : traverse every edges only once.

if it's connected and even degrees

Planar GRAPH

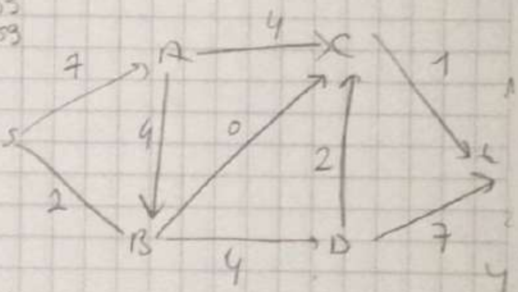
A →

Dijkstra's Algoritm

			1	2	3	4	5	6
1	Konya	∞	0	0	0			
2	Karaman	∞	19	19	19			
3	Mersin	∞	∞	21	24			
4	Niğde	∞	∞	27	21			
5	Nevşehir	∞	∞	60	27	9		
6	Aksaray	∞	26	26	26	10	8	
7	Isparta	∞						56
8	Burdur	∞						59
9	Antalya	∞	33	33	33			59
10	Afyon	∞	35	35	35			
11	Uşak	∞						
12	Denizli	∞						
13	Muğla	∞						
14	Aydın	∞						
15	İzmir	∞						
16	Manisa	∞						

B → A → D → E → C – F – 6, → F – C – 6

B → D → A → E → C, → E → A → C, F; → F – A – C



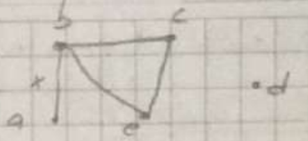
4
2

	0	1	7 (6)	2 (12)	8 (14)	5 (16)
0	0	0	0			
1	4	4	4			
2	∞	12	12			
3	∞	∞	∞	19	19	30
4	∞	∞	∞	∞	∞	26
5	∞	∞	∞	16	16	
6	∞	∞	∞	20	20	18
7	8	8	8			
8	∞	14	14	14		

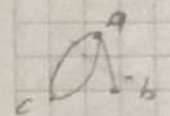
0
0-1
0-1-2
0-1-2-8-5

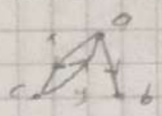
0-7-6
0-7
0-1-2-8

Incident : x incident to a - b
Adjacent : b adjacent to a, c, e
Isolated : d



Undirected - Direct





$e = \{a,b\} = \{b,a\}$

$e' = (\overrightarrow{a,b})$
tail, head

$x = (c,a)$
$y = (a,c)$
$x \neq y$

* Only one subgraph is there which spanning and induced

spanning ↓ all vertices

induced ↓ Remove vertex incident edges

for this (remove nothing)

Complement

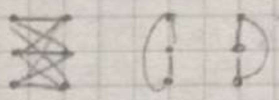
$G = \langle V, E \rangle \quad |V| = v \quad |E| = e$

$|\overline{E}| = ?$ $\boxed{K_n \text{ total edges} - e}$ → # of edges in V

K_n total edges $= \frac{n \cdot (n-1)}{2}$

EX $\overline{K_{m,n}}$ look like?

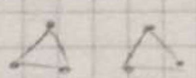
$K_{3,3}$



→ $\overline{K_{3,3}}$

$\overline{E_G} = E_{K_n} - E_G$
$15 - 9 = 6$

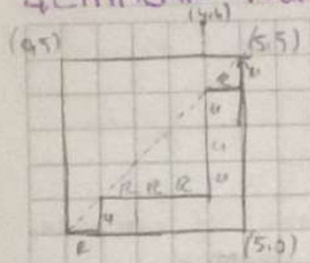
$\overline{K_{m,n}} = K_m / K_n$



K_3 K_3 $\overline{K_{3,3}}$

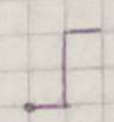
$\sum_{v \in V} d(v) = 2|E|$

4. CATALAN NUMBER



don't cross the line

All		BANNED
$\binom{10}{5}$	$-$	$\binom{10}{4}$



The number of paths $\Rightarrow \binom{2n}{n} - \underbrace{\binom{2n}{n-1}}$

the number of paths we should exclude

Ex: Calculate by and represent each of the possible paths on a grid from (0,0) to (3,3)!

$\binom{6}{3} - \binom{6}{2} = 5$

Ex: Using 5 opening, 5 closing parenthesize ( ), how many valid parantheses

5 (→ R
5) → U

$\binom{10}{5} - \binom{10}{4}$

(stack!)

Ch 2: LOGIC

1. Connectives and Truth Tables

→ if p, then q

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	$\boxed{0}$	0
1	1	1	1	1	1

$\leftrightarrow$ → p if and only if q
p iff q

conjuction disjuction

2. The Laws of Logic

1. Double negation $\neg\neg p \leftrightarrow p$
2. DeMorgan's $\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$
3. Commutative $p \vee q \leftrightarrow q \vee p$
4. Associative $p \vee (q \vee r) \leftrightarrow (p \vee q) \vee r$
5. Distributive $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$

Binomial Theorem

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \dots + \binom{n}{n}x^ny^0$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Ex: $(2a-3b)^7 \rightarrow$ coefficient of a^5b^2

$$\binom{7}{2}(2a)^5(-3b)^2 = \underbrace{\binom{7}{2}2^5(-3)^2}_{\text{coefficient}}\ a^5b^2$$

$\star$ $\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n} = 2^n$

n=4 $\quad 1+4+6+4+1 = 16 \rightarrow 2^4 \quad n=4$

n=5 $\quad 1+5+10+10+5+1 = 32 \rightarrow 2^5 \quad n=5$

$\star$ $\binom{n}{0}-\binom{n}{1}+\dots+(-1)^n\binom{n}{n} = 0$

n=4 $\quad 1-4+6-4+1 = 0$

n=5 $\quad 1-5+10-10+5-1 = 0$

The Multinominal Theorem

$(x_1+x_2+\dots+x_t)^n \rightarrow$ the coefficient of $x_1^{n_1}x_2^{n_2}\dots x_t^{n_t}$?

$$= \underbrace{\frac{n!}{n_1!\,n_2!\dots n_t!}}_{\text{coefficient}}\ x_1^{n_1}x_2^{n_2}\dots x_t^{n_t}$$

Combination with Repetitions

Ex: 7 people, 4 different food (a, b, c, d), how many item are purchased, (not who purchased what)

aaaabbb $\rightarrow$ 4a, 3b $\rightarrow$ There are same options

abababa $\rightarrow$ 4a, 3b

$\cdot\ \cdot \overset{1}{|} \cdot \overset{2}{|} \cdot\ \cdot \overset{3}{|} \cdot\ \cdot$ $\rightarrow$ seperater

C(10, 3) is the answer

Ch1 Counting

DISCRETE MATHEMATICS

1. The Rules of Sum and Product

The Rule of Sum: If a first task can be performed in m ways, while a second task can be performed in n ways, then performing either task can be accomplished in any one of $m+n$ ways.

The Rule of Product: If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes there there are n possible outcomes for the second stage; then the total procedure can be carried out, in the designated order, in mn ways.

2 Permutations

$$P(n,r) = \frac{n!}{(n-r)!}$$

Ex How many ways can we arrange the letter of the word MASSASAUGA ?

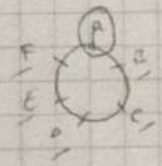
$\frac{10!}{4!\,3!}$ all A's together $\rightarrow \frac{7!}{3!}$

Ex: Prove that if n and k are positive integers with $n=2k$, then $n!/2^k$ is an integer.

$10 * 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1 \rightarrow 2^5$

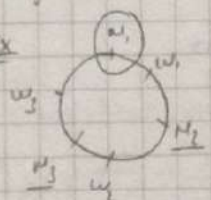
$10!/2^5 \rightarrow$ number of arrangments of AABBCCDDEE

Ex:



$\rightarrow 5!$

Ex



men $\rightarrow 2!$

women $\rightarrow 3!$

$2! \cdot 3!$

3 Combinations (select)

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

$$C(n,r) = P(n,r)/r!$$

Ex: 8 people $\rightarrow$ 4 team

$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} \Big/ 4!$

there is no team name

Ex: 8 people $\rightarrow$ A, B, C, D teams

$= \binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$

$= \frac{8 \cdot 7}{2!} \frac{6 \cdot 5}{2!} \frac{4 \cdot 3}{2!} \frac{2 \cdot 1}{2!} = \frac{8!}{(2!)^4}$