Differential Equations

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1 Introduction to Differential Equations

So far, we have looked at derivatives, integrals, and how they relate to each other. You have also most likely heard the saying "Calculus is the study of change" at some point. From being able to tell how long it would take to fill up a bucket by knowing how a water's rate of flow changes over time, to calculating the volume of rotated solids by knowing how the radius changes over time, we know this to be true (and very cool!)

However, our knowledge from previous units is not enough to solve problems similar to this.

$$\frac{dy}{dx} = 10y$$

What is important to follow here is that y is still a function of x. So, what this problem is asking us to do in this case is to find such a function of x that ten times that function is equal to its derivative with respect to x. Why is this important you ask? Imagine the following scenario.

Imagine that you are a scientist trying to figure out what the population of an invasive new species will be in ten years. You know that births per year are equal to the current population size divided by a hundred. How would you go about solving this?

When thinking about this problem, you may come to the realization that what we just said can be written in mathematical form. If $p(t)$ is the population at any given time t (in years), we can write down what we just said as

$$\frac{dp}{dt} = \frac{p}{100}.$$

So, let's see if we can find a solution to this problem.

When we had problems such as the following,

$$\frac{dy}{dx} = 3x^2 + 5$$

it was enough to solve it by integrating the right side with respect to x. However, in this case, we don't know what y is equal to as a function of x. In fact, that is exactly what we are trying to find out.

2 Verifying Differential Equations

Let us consider our first example,

$$\frac{dy}{dx} = 10y$$

How would we know if an equation satisfies this condition? Now, out of thin air, what if I told you that the answer to this problem was $y = Ce^{10x}$ where C is a constant that, for now, can be any number. How would you make sure that I am not lying to you?

Well, let's see what I just gave you the answer to. The statement $\frac{dy}{dx} = 10y$ states that the derivative our whatever our function $y(x)$ is, should be equal to ten times that function. So, is this true?

Well, we know how to differentiate Ce^{10x}. We know that C is just some random constant, so we know (through our differentiation rules) that

$$\frac{d}{dx}Ce^{10x} = 10Ce^{10x}$$

Now does our statement hold true?

If we look back at our original statement, that

$$\frac{dy}{dx} = 10y$$

and substitute what we found for $\frac{dy}{dx}$ and what I said that y should be equal to, we will get

$$10Ce^{10x} = 10Ce^{10x}$$

So I wasn't lying to you. The derivative of that function really was equal to ten times the function itself!

Try some more verification exercises, this time, on your own. I will tell you that the answer to the following equation is some function, and you will write down whether I am right.

Exercise 1

Consider the given differential equations. Determine if the given function for y satisfies the corresponding differential equation.

$$\frac{dy}{dx} = 20y + 2yx$$

I say: $y = Ce^{20x+x^2}$

$$\frac{dy}{dx} = y$$

I say: $y = \frac{x^2}{2} + C$

$$\frac{dy}{dx} = \frac{1}{y}$$

I say: $y = \sqrt{2x + C}$

3 Slope Fields

Before we see how to solve these equations, let's think a little deeper about what they represent. Consider the following differential equation:

$$\frac{dy}{dx} = xy$$

What does this tell us?

We know that the derivative is the slope of the tangent line to a graph at a given point. So, this equation tells us the instantaneous rate of change (or slope) of the function y(x) at any point (x, y).

This is a good point to talk about that pesky C we keep seeing. You may have noticed that a differential equation does not have a single solution. Let's go back to our first example:

$$\frac{dy}{dx} = 10y$$

You may remember that I said that the solution to this was $y = Ce^{10x}$. This means that $y = 2e^{10x}$ is a solution, and so is $5e^{10x}$, and so is $6.28\pi e^{10x}$.

In the context of slope fields, this is important. Not every function that satisfies a differential equation will go through a random point (x_0, y_0). However, if you do have a function that satisfies that differential equation, and does go through that random point, the differential equation will give you its slope at that point.

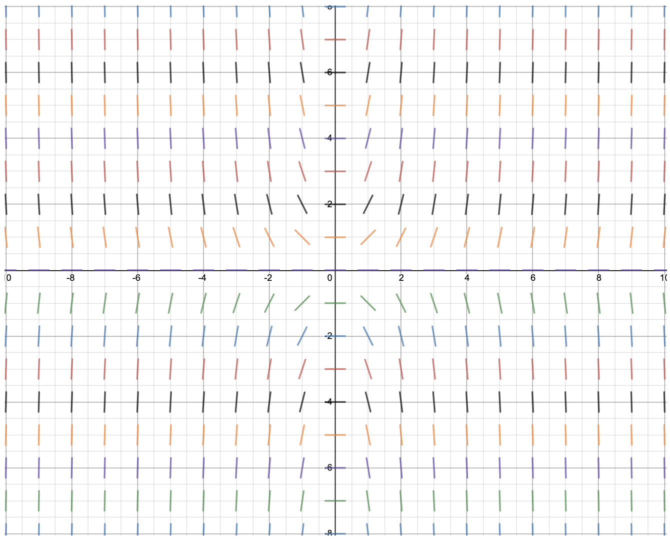
A neat way to represent the slope a function that satisfies our differential equation would have is through *slope fields*. A slope field consists of small line segments at different points with slopes representing the slope a function satisfying the differential equation would have at that point.

For example, considering the equation,

$$\frac{dy}{dx} = xy$$

We would draw a line segment with a slope of 1 at $(1, 1)$, a slope of 2 at $(2, 1)$, a slope of 3 at $(1, 3)$, a slope of 4 at $(2, 2)$…

Eventually, we would end up with a graph that looks like this:

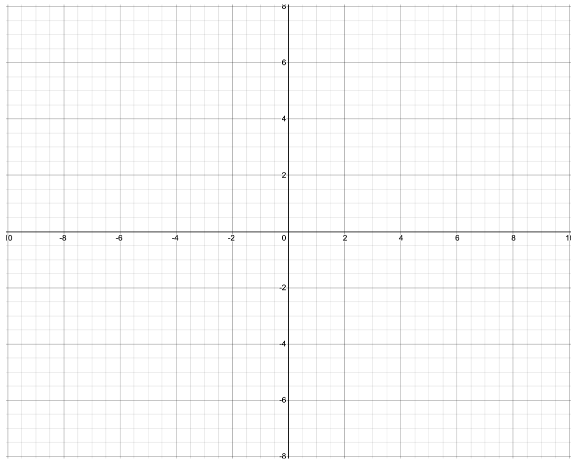


Now, let's get some practice with drawing slope fields!

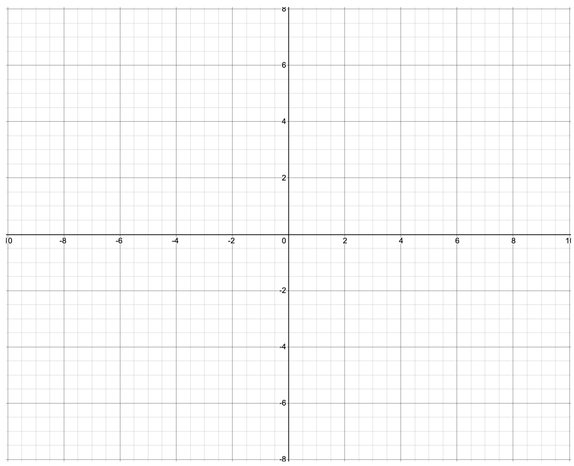
Exercise 2

Draw a slope field of the given function in the provided graph.

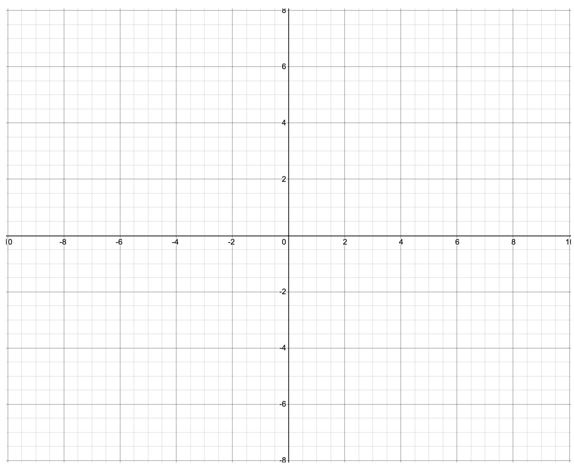
$\frac{dy}{dx} = x^2 y$



$\frac{dy}{dx} = \sin \frac{\pi x}{2}$



$\frac{dy}{dx} = 2x + y$



4 Solving Differential Equations

We have been looking at how to verify solutions to differential equations, how to draw slope fields of them, but now, time has come for us to solve them. We will be looking at a specific type of differential equation called a *separable differential equation*. This is also the only type of differential equation you will need to be able to solve throughout AP Calculus.

So, what is a separable differential equation? A separable differential equation is any differential equation

$$\frac{dy}{dx} = f(x, y(x))$$

that we can write in the form

$$\frac{dy}{dx} = g(x) \cdot h(y(x))$$

In other words, we should be able to factor whatever $\frac{dy}{dx}$ is equal to into one factor consisting of only **x**s (and constants), and another with only **y**s (and constants).

Let's look at some examples of separable and non-separable differential equations.

$$\frac{dy}{dx} = yx^2 + \pi yx + 8y$$

This equation is separable. We can factor it into

$$\frac{dy}{dx} = y(x^2 + \pi x + 8)$$

Let's look at another equation.

$$\frac{dy}{dx} = x^2y + xy^2$$

This equation is not separable. There is no way to get in into our aforementioned form.

Now that we know whether an equation is separable or not, how do we solve it if it is?

To solve a separable differential equation, we are going to treat $\frac{dy}{dx}$ somewhat like a fraction. Once we have our equation in the form of

$$\frac{dy}{dx} = g(x) \cdot (h(y(x)),$$

we are going to "multiply" both sides by dx (look into differential forms if you are curious about what we are actually doing here), and divide both sides by $h(y(x))$ to get

$$\frac{1}{h(y(x))}dy = g(x)dx$$

Then, we can integrate both sides to get

$$\int \frac{1}{h(y(x))}dy = \int g(x)dx$$

Once we solve this equation, we will be able to (with some manipulation) get a function of x that satisfies our original equation.

Let's try our original example.

$$\frac{dy}{dx} = 10y$$

We only have a function of y(x) on the right side, so we can divide both sides by y, and multiply both sides by dx to get

$$\frac{1}{y}dy = 10dx$$

Then, with integration, we can see that

$$\int \frac{1}{y}dy = \int 10dx \implies \ln y = 10x + C$$

Raising e to both sides, we get

$$e^{\ln y} = e^{10x+C} = e^{10x} \cdot e^{C}$$

Because C represents some constant without a predefined value, we can just write in C_1 for e^C. Although I have written C_1 here, you will later see that we will keep using the letter C when we have this situation to avoid clutter. I like to think of it as the C "absorbing" any constants it has around.

Finally, we have

$$y = C_1 e^{10x}$$

Now, let's do some practice with separable differential equations.

Exercise 3

Find a solution in the form of $y = f(x)$ for the given differential equations.

$$\frac{dy}{dx} = 3yx$$

$$\frac{dy}{dx} = y^2 \sin x$$

$$\frac{dy}{dx} = 3yx^{-1}$$

5 Initial Conditions and Applications of Differential Equations

By this point, we should be running out time and energy, so I have decided to combine these last two topics into this section.

Now, what are initial conditions? When we solve a differential equation, we find a kind of "family" of solutions that are all valid. That is where the C we have comes from. However, we usually want a specific function to model the situation we are observing. For example, if we know that the invasive species population has the differential equation

$$\frac{dp}{dt} = \frac{p}{100},$$

we now know how to solve for p(t). I will skip over the steps for solving for p, and tell you that $p(t) = Ce^{\frac{1}{100}x}$. However, if there are currently 100 of these species, the equation we have cannot tell us how many there will be in ten years... or can it?

We know that $p(t) = Ce^{\frac{1}{100}x}$. To get a function we can actually use, we need to determine a value for C. This is where the fact that we know what the population is right now comes in handy. If we consider the current time to be $t = 0$, then we know that $p(0) = 100$. When we plug in 0 for $p(x)$, we get

$$p(0) = Ce^{\frac{1}{100}\cdot 0} = C$$

Because we know that $p(0) = 100$, we have found the value of C to be 100. Finally, we can conclude that

$$p(x) = 100e^{\frac{1}{100}x}$$

Now to answer our original question of how many of these species there would be in 10 years,

$$p(x) = 100e^{\frac{1}{100}x} \implies p(10) = 100e^{\frac{1}{100}\cdot 10} = 100e^{\frac{1}{10}} \approx 111$$

So, at the end of 10 years, there will be just around 111 of our invasive species living here.

Now, let's try some exercises with initial value problems.

Exercise 4

The per-year rate of change of how many people live in a certain city in the U.S. can be determined by the differential equation

$$\frac{dP}{dt} = \frac{1}{100}P$$

If there were 120,000 people living in the city this year, determine how many people will be living in the city in 34 years to the nearest hundredth.

Your friend has told you that he will have $12000 in his college savings bank account in two years. If the rate of increase of the money in his bank account can be modelled by

$$\frac{dm}{dt} = 1.02m$$

find how long he will need to wait in order to gain enough money through interest to pay for a fifteen-thousand dollar tuition.

The per-year rate of change of a population of deer can be modelled by the differential equation

$$\frac{dP}{dt} = 0.05L(1000 - L)$$

If there are currently 400 deer at $t = 0$,

- Find how many deer there will be in two years
- Will this population keep growing forever? Is there a limit? If so, find it. If not, explain why there will be no limit.