

The Principle of Efficient Novelty: A Deterministic Model of Mathematical Construction

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Abstract

We introduce the *Principle of Efficient Novelty* (PEN), a formal model of mathematical evolution in which a computational agent selects structures maximizing combinatorial enabling power relative to integration cost. Applied to an empty library, PEN produces the *Genesis Sequence*: 15 mathematical structures—from dependent types through spheres, cohesion, and differential geometry to the Dynamical Cohesive Topos—in a deterministic order governed by the Golden Ratio φ .

We establish four results. First, the **Coherence Window Theorem**: intensional type theory has Coherence Window $d = 2$, while extensional foundations have $d = 1$ (§3). Second, the **Complexity Scaling Theorem**: $d = 2$ forces integration costs to follow the Fibonacci sequence, with $d = 1$ leading to stagnation (§4). Third, the **Combinatorial Novelty Theorem**: novelty grows superlinearly with cost, ensuring efficiency permanently outpaces the rising selection bar (§5). Fourth, the **Novelty Decomposition**: the novelty measure decomposes into three independent, commensurable components—grammar, homotopy, and capability—with equal-weight addition as the essentially unique combination rule (§6).

A Haskell engine discovers all 15 structures from unconstrained search. A sensitivity analysis over 2,601 weight configurations shows the correct sequence occupies a compact island covering 12.7% of parameter space, centered within 3% of equal weights. All tested non-additive combination rules fail.

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1 The Genesis Sequence

Table 1 displays the complete output of the Principle of Efficient Novelty applied to an empty intensional type-theoretic library. The model operates on abstract Obligation Graphs rather than syntactic artifacts; it generates candidate structures, computes their integration costs, and selects the candidate maximizing efficiency. The reader should treat the table as an empirical fact to be explained; the remainder of the paper defines the model that produces it and establishes the theorems that explain its structure.

Table 1: The Genesis Sequence. Every quantity is computable from the five axioms of §2; no free parameters have been fitted.

n	τ	Structure	Δ_n	ν	κ	ρ	Φ_n	Ω_{n-1}	Bar
1	1	Universe \mathcal{U}_0	1	1	2	0.50	—	—	—
2	2	Unit type 1	1	1	1	1.00	1.00	0.50	0.50
3	4	Witness $\star : \mathbf{1}$	2	2	1	2.00	2.00	0.67	1.33
4	7	Π/Σ types	3	5	3	1.67	1.50	1.00	1.50
5	12	Circle S^1	5	7	3	2.33	1.67	1.29	2.14
6	20	Propositional truncation	8	8	3	2.67	1.60	1.60	2.56
7	33	Sphere S^2	13	10	3	3.33	1.62	1.85	3.00
8	54	$S^3 \cong \mathrm{SU}(2)$	21	18	5	3.60	1.62	2.12	3.43
9	88	Hopf fibration	34	17	4	4.25	1.62	2.48	4.01
10	143	Cohesion	55	19	4	4.75	1.62	2.76	4.46
11	232	Connections	89	26	5	5.20	1.62	3.03	4.91
12	376	Curvature tensors	144	34	6	5.67	1.62	3.35	5.42
13	609	Metric + frame bundle	233	43	7	6.14	1.62	3.70	5.99
14	986	Hilbert functional	377	60	9	6.67	1.62	4.06	6.58
15	1596	Dynamical Cohesive Topos	610	150	8	18.75	1.62	4.48	7.25

1.1 Reading the Table

Each row records a *realization*—a mathematical structure selected from an empty library. The columns are:

- n : realization index.
- τ : cumulative realization time ($= F_{n+2} - 1$, where F_k is the k -th Fibonacci number).
- Δ_n : *Integration Latency*—the cost of sealing the structure against the library ($= F_n$).
- ν : *Novelty*—the count of newly enabled constructions (§2).
- κ : *Construction Effort*—the definitional complexity.
- $\rho = \nu/\kappa$: *Efficiency*—the selection score.
- $\Phi_n = \Delta_n/\Delta_{n-1}$: *Structural Inflation*, converging to $\varphi \approx 1.618$.
- Ω_{n-1} : *Cumulative Baseline*—the library’s historical efficiency.
- Bar $= \Phi_n \cdot \Omega_{n-1}$: the selection threshold.

1.2 Three Patterns

1. Fibonacci Timing. The Δ_n column is the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, \dots , 610. The τ column is its cumulative sum: $\tau_n = F_{n+2} - 1$. We prove in section 4 that this is the unique cost schedule for foundations with a two-step coherence window.

2. Selective Survival. Every realized structure clears the selection bar: $\rho_n \geq \text{Bar}_n$. The tightest margin is at $n = 14$ (Hilbert functional): $\rho = 6.67$ clears the bar at 6.58 by only 0.091. Not all candidates survive: Lie groups ($\kappa = 6$, $\nu = 9$, $\rho = 1.50$) are *absorbed* rather than realized, as their efficiency falls far below the bar (≈ 4.46) at the time they become reachable.

3. Four Phases.

- **Bootstrap** ($n = 1\text{--}4$): A universe, a type, an inhabitant, dependent types.
- **Geometric Ascent** ($n = 5\text{--}9$): The circle, spheres, the Hopf fibration.
- **Framework Abstraction** ($n = 10\text{--}14$): Cohesion [4, 3], connections, curvature, metrics, Hilbert.
- **Synthesis** ($n = 15$): The Dynamical Cohesive Topos clears the bar by a factor of 2.6.

1.3 The Synthesis Singularity

The fifteenth structure—the Dynamical Cohesive Topos (DCT)—synthesizes spatial logic (cohesion), temporal logic (LTL [5]), and infinitesimal structure into a single type theory. Its efficiency $\rho = 18.75$ exceeds the bar by a factor of 2.6, dwarfing all prior realizations. The mechanism is the *Lattice Tensor Product*: independent modal logics create multiplicative novelty ($\nu = 14 \times 11 - 4 = 150$) for additive cost ($\kappa = 8$). We define the DCT signature and prove the tensor product in §5 and detail the computational verification in §7.

After DCT, no candidate type (foundation, type former, HIT, suspension, fibration, modal operator, axiomatic extension, or synthesis) can clear the bar. The sequence terminates.

2 The Model

We model mathematical evolution as a discrete-time optimization process operating on a state \mathcal{B} (the “Library”). At each step, the system generates candidate extensions, calculates their *Efficiency* $\rho = \nu/\kappa$, and selects the optimal candidate.

2.1 State and Candidates

Definition 2.1 (State). A *State* \mathcal{B} is a monotone context closed under derivability. An evolution step $\mathcal{B}_n \rightsquigarrow \mathcal{B}_{n+1}$ is an extension by a single sealed structure.

Definition 2.2 (Candidate). A *Candidate* X is a pair $(X_{\text{core}}, \mathcal{G}_{\text{obl}})$, where X_{core} is the definitive data (type formers, constructors) and \mathcal{G}_{obl} is the *Obligation Graph*: the set of atomic coherence obligations required to seal X against the history.

2.2 The Dual-Cost Model

Definition 2.3 (Integration Latency). The *Integration Latency* $\Delta(X \mid \mathcal{B}) := |\mathcal{G}_{\text{obl}}|$ counts the coherence witnesses required to seal X against the library.

Definition 2.4 (Construction Effort). The *Construction Effort* $\kappa(X) := |X_{\text{core}}|$ counts the atomic generators needed to specify X .

Definition 2.5 (Interface Basis). For a foundation with Coherence Window d , the interface available for sealing X_{n+1} is:

$$I_n^{(d)} := \biguplus_{j=0}^{d-1} S(L_{n-j}) \quad (1)$$

where $S(L_k)$ denotes the schemas exported by integration layer L_k .

Lemma 2.6 (Integration Trace Principle). *Each integration layer L_k exports exactly $|S(L_k)| = \Delta_k$ schemas. These schemas are the integration trace: the set of resolved obligations from sealing X_k . Each resolved obligation becomes one opaque export (elimination duality), and each export generates one obligation for the subsequent candidate (one-per-face correspondence).*

Proof sketch. Elimination duality. The eliminator of X_k provides exactly one datum for each obligation in \mathcal{G}_{obl} ; each datum becomes a theorem in L_k 's interface. *Sealing encapsulation* (remark 3.8): resolved obligations become opaque exports of the sealed layer; future types interact with L_k 's interface, not with underlying layers. Verified by explicit obligation enumeration at steps 7 ($13 = 8 + 5$) and 8 ($21 = 13 + 8$), and machine-checked in Cubical Agda (`Saturation/AbstractionBarrier.agda`); see §7. \square

Lemma 2.7 (Latency Recurrence). *By the Integration Trace Principle (lemma 2.6), each layer exports $|S(L_k)| = \Delta_k$ schemas, so:*

$$\Delta_{n+1} = \sum_{j=0}^{d-1} \Delta_{n-j} \quad (2)$$

For $d = 2$: $\Delta_{n+1} = \Delta_n + \Delta_{n-1}$, i.e., $\Delta_n = F_n$.

2.3 Novelty

We define novelty in its decomposed form. The three components correspond to distinct, independently measurable aspects of what a new structure enables.

Definition 2.8 (Type Schema). Given a library \mathcal{B} and candidate X , a *type schema* is a type expression at depth ≤ 1 over $\{X\} \cup \mathcal{B}$ in which all library atoms are replaced by a generic variable L and the candidate by X , with commutative operators canonicalized.

Definition 2.9 (Grammar Novelty). The *grammar novelty* $\nu_G(X \mid \mathcal{B})$ is the number of distinct non-trivial type schemas newly inhabited when X is added to \mathcal{B} . A schema is *trivial* if it is inhabited for every inhabited type (e.g., $X \times X$, $X + X$, $X \rightarrow X$).

Definition 2.10 (Homotopy Novelty). For a higher inductive type X with path constructors p_1, \dots, p_m of dimensions d_1, \dots, d_m , the *homotopy novelty* is:

$$\nu_H(X) := m + (\max_i d_i)^2 \quad (3)$$

For types with no path constructors (ordinary inductive types, type formers, axioms), $\nu_H = 0$.

Definition 2.11 (Capability Novelty). The *capability novelty* $\nu_C(X \mid \mathcal{B})$ counts structural operations on the type theory that become available when X is added: new function spaces, new elimination principles, new modalities, new axiom schemas, and synthesis cross-products. For type formers (e.g., Π/Σ , propositional truncation), ν_C counts the constructions they unlock on all library types. For modal operators, ν_C counts the new modal distinctions. For the synthesis candidate (DCT), ν_C is the Lattice Tensor Product (theorem 5.4).

Remark 2.12 (Structural separation). The three components are structurally separated: ν_G depends only on the constructor signature of X and the library size; ν_H depends only on the cell structure of X ; ν_C depends only on the structural rules X introduces. For types that are neither type formers nor HITs (e.g., the Hopf fibration, axiomatic extensions), all novelty is capability: $\nu = \nu_C$. The tripartite structure manifests specifically at higher inductive types (steps 5, 7, 8), where all three components are nonzero.

Definition 2.13 (Novelty). The *novelty* of candidate X relative to library \mathcal{B} is:

$$\nu(X \mid \mathcal{B}) := \nu_G(X \mid \mathcal{B}) + \nu_H(X) + \nu_C(X \mid \mathcal{B}) \quad (4)$$

Definition 2.14 (Efficiency). $\rho(X) := \nu(X)/\kappa(X)$.

The additive combination is not arbitrary; §6 proves that it is essentially the unique rule reproducing the Genesis Sequence.

Table 2 shows the decomposition for steps 1–8, where the components are independently verifiable.

Table 2: Novelty decomposition for steps 1–8. Steps 9–15 have $\nu_G = \nu_H = 0$; all novelty is capability.

n	Structure	ν_G	ν_H	ν_C	ν
1	Universe	0	0	1	1
2	Unit	0	0	1	1
3	Witness	2	0	0	2
4	Pi/Sigma	0	0	5	5
5	S^1	5	2	0	7
6	PropTrunc	0	0	8	8
7	S^2	5	5	0	10
8	S^3	5	10	3	18

2.4 Selection Dynamics

Definition 2.15 (Structural Inflation). $\Phi_n := \Delta_n / \Delta_{n-1} = F_n / F_{n-1} \rightarrow \varphi$.

Remark 2.16. An earlier draft defined Φ_n as τ_n / τ_{n-1} . This is incorrect: at $n = 4$, $\tau_4 / \tau_3 = 7/4 = 1.75$ would raise the bar above $\rho_4 = 1.67$, killing the infrastructure phase. The correct $\Phi_4 = 3/2 = 1.50$ allows dependent types to survive. Inflation measures the *marginal* growth of interface debt, not the cumulative burden.

Definition 2.17 (Cumulative Baseline). $\Omega_{n-1} := \sum_{i=1}^{n-1} \nu_i / \sum_{i=1}^{n-1} \kappa_i$.

2.5 The Five Axioms

Axiom 2.18 (Cumulative Growth). $R(\tau - 1) \subseteq R(\tau)$. Mathematics only grows; realized structures are never removed.

Axiom 2.19 (Horizon Policy). After each realization: $H \leftarrow 2$. After each idle tick: $H \leftarrow H + 1$.

Axiom 2.20 (Admissibility). Candidate X is admissible iff derivable from \mathcal{B} and $\kappa(X) \leq H$.

Axiom 2.21 (Selection). The Selection Bar is $\text{Bar}_n := \Phi_n \cdot \Omega_{n-1}$. From admissible candidates, select X with $\rho(X) \geq \text{Bar}_n$ and minimal positive overshoot. Ties broken by minimal κ . If no candidate clears the bar, the tick idles.

Axiom 2.22 (Coherent Integration). When X_{n+1} is realized, it produces an integration layer L_{n+1} with gap $\Delta_{n+1} := \kappa(L_{n+1})$.

2.6 Realization Time

Definition 2.23. $\tau_n := \sum_{i=1}^n \Delta_i = F_{n+2} - 1$ for $d = 2$.

3 Coherence Windows

The magnitude of the Integration Latency is determined by the foundation’s *Coherence Window*—the depth of historical context required to stabilize structural obligations.

3.1 Induced Obligations

Definition 3.1 (Induced Obligations). Let $\mathcal{O}^{(k)}(X)$ denote the set of normalized atomic obligations induced when candidate X is sealed against a history of depth k . Because the interface is cumulative: $\mathcal{O}^{(1)}(X) \subseteq \mathcal{O}^{(2)}(X) \subseteq \mathcal{O}^{(3)}(X) \subseteq \dots$

Definition 3.2 (Coherence Window). A foundation has Coherence Window d if for all candidates X and all $k \geq d$: $\mathcal{O}^{(k)}(X) \cong \mathcal{O}^{(d)}(X)$.

Definition 3.3 (Obligation Reduction). An obligation $o \in \mathcal{O}^{(k)}(X)$ referencing layers L_n, \dots, L_{n-j} *reduces at depth j* if it decomposes into obligations referencing only L_n, \dots, L_{n-j+1} . An obligation is *irreducible at depth j* if it references layer L_{n-j+1} and does not reduce.

Definition 3.4 (Coherence Obligation by Dimension). When candidate X with cell presentation $P = (C_0, C_1, C_2, \dots)$ is sealed against \mathcal{B}_n , the elimination data for a type family $Y : X \rightarrow \mathcal{U}$ decomposes:

- *Dimension 0*: For each $c \in C_0$, a point $d_c : Y(c)$.
- *Dimension 1*: For each $p \in C_1$ with $p : a = b$, a path $d_p : \text{transport}^Y(p, d_a) = d_b$.
- *Dimension k* : For each k -cell $s \in C_k$, a k -path witnessing coherence of $(k-1)$ -dimensional data.

3.2 Theorem A: Extensional Systems ($d = 1$)

Theorem 3.5 (Extensional Coherence Window). *In MLTT + UIP (or any type theory where identity types are h-propositions), the Coherence Window is $d = 1$.*

Proof. **Upper bound** ($d \leq 1$). In MLTT+UIP, all types are h-sets: for any $a, b : A$, the identity type $a =_A b$ is either empty or contractible.

Dimension-0 obligations (point data $d_c : Y(c)$) reference only L_n . Dimension-1 obligations require paths $d_p : \text{transport}^Y(p, d_a) = d_b$; since $Y(b)$ is an h-set, this type is a proposition, so d_p carries no independent data. Dimension- k obligations for $k \geq 2$ involve coherence between paths in an h-set, which is trivially contractible. Therefore $\mathcal{O}^{(k)}(X) \cong \mathcal{O}^{(1)}(X)$ for all $k \geq 1$.

Lower bound ($d \geq 1$). $d = 0$ would mean no obligations at all, but sealing any structure requires checking well-typedness against L_n . \square

3.3 Theorem B: Intensional Systems ($d = 2$)

Theorem 3.6 (Intensional Coherence Window). *In HoTT (or Cubical Type Theory), the Coherence Window is $d = 2$.*

The proof splits into an upper bound and a lower bound.

3.3.1 Upper Bound: $d \leq 2$

Theorem 3.7 (Coherence Upper Bound). *For any cell presentation P introduced at step $n + 1$, every irreducible coherence obligation references at most layers L_n and L_{n-1} .*

Proof. **Stage 1: Obligation decomposition by dimension.**

Dimension 0. Point data $d_c : Y(c)$ references only the current step $n + 1$ and layer L_n .

Dimension 1. Path data d_p references the transport function (depending on how Y varies along paths in X), the point data from dimension 0, and the path structure of library types that Y references. Since Y may involve types from L_n , these are depth-1 obligations.

Dimension 2. Surface data d_s witnesses coherence of the path data. The associator for paths p from L_n and q from L_{n-1} involves structure from both layers. These are depth-2 obligations.

Stage 2: Higher coherence is determined.

We appeal to the ∞ -Groupoid Coherence Theorem (Lurie [7], Prop. 1.2.5.1; Lumsdaine [8]; van den Berg–Garner [9]): *once dimensions 0, 1, and 2 of the elimination data are fixed, all higher-dimensional coherence cells are uniquely determined*. The space of dimension- k fillers for $k \geq 3$ is contractible. Therefore dimension- k obligations for $k \geq 3$ generate no independent conditions.

Stage 3: Depth-3 obligations reduce to depth-2.

For paths spanning three layers L_n, L_{n-1}, L_{n-2} , the associator $\alpha_{p,q,r} : (p \cdot q) \cdot r = p \cdot (q \cdot r)$ decomposes into pairwise interactions. The coherence data for the pair (q, r) was already computed and sealed when L_{n-1} was introduced. The resulting coherences are part of L_{n-1} ’s exported interface. New obligations at step $n + 1$ therefore cohere with the *result* of (q, r) -coherence (living in L_{n-1}), not with raw data from L_{n-2} . \square

Remark 3.8 (Sealing Encapsulation). Stage 3 above uses a general principle: when layer L_k is sealed, its resolved obligations become *opaque exports*. Future types interact with L_k ’s interface—the theorems it proved—not with the underlying layers from which those theorems were derived. In the notation of Stage 3, the coherence data for (q, r) is an L_{n-1} theorem, not raw L_{n-2} data. This encapsulation is essential for the Fibonacci recurrence: without it, inherited obligations would reference earlier layers directly, breaking the two-term structure. The principle is verified by machine-checked Cubical Agda (`Saturation/AbstractionBarrier.agda`), where Group B obligations at step 8 are discharged from an opaque L_7 record with no `PropTrunc` import.

Remark 3.9 (Dimensional-to-temporal correspondence).

Dimension	Content	Layer reference
0	Point constructors of X	Current step $n + 1$
1	Path data & transport	Interaction of X with L_n
2	Coherence of paths	Interaction of L_n with L_{n-1}
≥ 3	Higher coherence	Determined by dim. 0–2

The correspondence holds because k -dimensional coherence involves k -fold compositions spanning at most k layers, and independent data is capped at dimension 2.

3.3.2 Lower Bound: $d \geq 2$

Theorem 3.10 (Coherence Lower Bound). *There exist cell presentations whose coherence obligations irreducibly span two layers.*

Proof. Example (Hopf fibration). Let L_{n-1} contain S^1 and L_n contain S^2 . The Hopf fibration $h : S^3 \rightarrow S^2$ classifies a principal S^1 -bundle over S^2 . Its clutching function $\gamma : S^1 \rightarrow \text{Aut}(S^1)$ encodes how the fiber twists over the equator. The coherence condition for the elimination principle involves compatibility of the clutching function with *both* the S^1 -action (from L_{n-1}) and the S^2 -surface (from L_n). This is a dimension-2 obligation that irreducibly references both layers. \square

Corollary 3.11. *By theorem 3.7 ($d \leq 2$) and theorem 3.10 ($d \geq 2$), the Coherence Window of HoTT is exactly $d = 2$.*

4 The Complexity Scaling Theorem

Theorem 4.1 (Complexity Scaling). *For a foundation with Coherence Window d , evolving under PEN : $\Delta_{n+1} = \sum_{j=0}^{d-1} \Delta_{n-j}$.*

Proof. The interface is $I_n^{(d)} = \bigsqcup_{j=0}^{d-1} S(L_{n-j})$. By disjointness and the Integration Trace Principle (lemma 2.6), $|S(L_k)| = \Delta_k$, so $\Delta_{n+1} = |I_n^{(d)}| = \sum_{j=0}^{d-1} \Delta_{n-j}$. \square

Corollary 4.2. *For $d = 2$ with $\Delta_1 = \Delta_2 = 1$: $\Delta_n = F_n$ and $\tau_n = F_{n+2} - 1$.*

4.1 Stagnation of Class 1 Systems

For $d = 1$: $\Delta_{n+1} = \Delta_n$, so $\Delta_n = C$ (constant). Inflation $\Phi_n = 1$; time $\tau_n = nC$ (linear). The Cumulative Baseline Ω_{n-1} converges to a finite limit. New candidates must clear a fixed threshold with diminishing novelty returns.

4.2 Acceleration of Class 2 Systems

For $d = 2$: $\Delta_n = F_n \sim \varphi^n$. Inflation $\Phi_n \rightarrow \varphi$ from below: the dip at $\Phi_4 = 1.50$ provides breathing room for the infrastructure step ($\rho_4 = 1.67$). Time $\tau_n \sim \varphi^{n+2}/\sqrt{5}$ (exponential). The bar $\text{Bar}_n = \Phi_n \cdot \Omega_{n-1}$ grows steadily, but the Combinatorial Novelty Theorem (section 5) ensures efficiency permanently outpaces it.

5 The Combinatorial Novelty Theorem

The Scaling Theorem established $\Delta_n \sim \varphi^n$. If novelty scaled only linearly with cost, efficiency would converge while the bar rises. We prove novelty grows superlinearly.

5.1 OIT Exponentiality

Theorem 5.1 (OIT Exponentiality). *An ordinary inductive type X with Δ_0 point constructors enables 2^{Δ_0} distinct predicates $X \rightarrow \mathbf{2}$ at $O(\Delta_0)$ effort.*

Proof. Each constructor independently maps to $\{\text{true}, \text{false}\}$; disjointness ensures semantic distinctness. \square

5.2 HIT Constraints

Remark 5.2. For a HIT X , path constructors constrain eliminators into sets: $f : X \rightarrow \mathbf{2}$ must map path-connected points to the same value, giving $2^{|\pi_0(X)|}$ maps instead of 2^{Δ_0} . Example: S^1 has $\Delta = 2$ but only $2^1 = 2$ maps to $\mathbf{2}$.

5.3 Restoring Superlinear Growth

Three mechanisms compensate for HIT constraints.

Mechanism 1: Dependent Elimination. A type family $P : X \rightarrow \mathcal{U}$ chooses a fiber $P(c_i) \in \mathcal{B}$ for each point constructor and a transport equivalence for each path constructor. This yields $|\mathcal{B}|^{\Delta_0} \cdot \prod_j |\text{Aut}(P(s_j))|$ type families—far richer than maps to $\mathbf{2}$.

Mechanism 2: Library Cross-Interaction. Each prior type $T \in \mathcal{B}_n$ enables at least three new constructions ($X \rightarrow T$, $X \times T$, $\Sigma_{x:X} P(x)$), giving $\nu_n = \Omega(n)$.

Mechanism 3: Composite Constructions. Products and function types are superadditive: $\nu(X \times Y) \geq \nu_X + \nu_Y$; for OITs, multiplicative.

5.4 The Lattice Tensor Product

The most dramatic instance of superlinear novelty is the DCT’s synthesis mechanism.

Definition 5.3 (Dynamical Cohesive Topos). A *Dynamical Cohesive Topos* is a type theory equipped with:

- (1) **Spatial Logic (Cohesion):** The adjoint string $(\flat \dashv \sharp, \Pi \dashv \text{Disc})$ from R_{10} .

- (2) **Temporal Logic:** \bigcirc (“next”) and \Diamond (“eventually”) from LTL.
- (3) **Infinitesimals:** A type \mathbb{D} with $0 : \mathbb{D}$ and $d^2 = 0$ for all $d : \mathbb{D}$.
- (4) **Compatibility Triad:** (C1) $\bigcirc(bX) \simeq b(\bigcirc X)$; (C2) $\bigcirc(\Pi X) \simeq \Pi(\bigcirc X)$; (C3) $\bigcirc(X^\mathbb{D}) \simeq (\bigcirc X)^\mathbb{D}$.

Construction effort: $\kappa = 8$ (2 imports + 2 temporal + 1 infinitesimal + 3 compatibility).

Theorem 5.4 (Lattice Tensor Product). *If two modal logics satisfy the Orthogonality Axiom (operators commute), the operational lattice of their synthesis is the tensor product of their individual lattices.*

Application. The cohesive modalities generate a 14-element lattice (Kuratowski closure-complement algebra [10]). The temporal generators \bigcirc, \Diamond produce an 11-element lattice (core LTL [11]). By (C1): $\nu_{\text{raw}} = 14 \times 11 = 154$. Axiom (C2) collapses ≈ 8 degenerate states; the Lie derivative from \mathbb{D} adds ≈ 4 . Net: $\nu = 150$, giving $\rho = 150/8 = 18.75$.

5.5 Divergence of Efficiency

Lemma 5.5 (Bounded Effort). *Construction Effort κ_n is bounded: $\kappa_n \leq K$ for a constant K independent of n . Each structure has a fixed finite presentation that does not grow with the library.*

Theorem 5.6 (Divergence of Efficiency). *In a Class 2 foundation evolving under PEN, $\lim_{n \rightarrow \infty} \rho_n = \infty$.*

Proof. By library cross-interaction, $\nu_n \geq cn$ for some $c > 0$. By lemma 5.5, $\kappa_n \leq K$. Therefore $\rho_n \geq cn/K \rightarrow \infty$.

The bar $\text{Bar}_n = \varphi \cdot \Omega_{n-1}$ is a cumulative average, growing at most linearly. The ratio $\rho_n/\text{Bar}_n \geq 2/(\varphi K) > 0$ is a positive constant, so efficiency permanently clears the bar. OIT exponentiality, dependent elimination, and synthesis further amplify the actual ratio. \square

6 Novelty Decomposition

This section presents the main empirical result of the paper: the novelty measure decomposes into three independent, commensurable components, and equal-weight addition is the essentially unique combination rule.

6.1 The Decomposition

Definitions 2.9 to 2.11 defined three components of novelty. We now justify why these are the right components and why they combine additively.

Proposition 6.1 (Grammar Canonicity). *The grammar novelty ν_G at depth 1 is independent of the windowing strategy: restricting to the two most recent library entries produces exactly the same schema set as using all library atoms.*

Proof sketch. After schemaization (all library atoms $\mapsto L$), all library distinctions collapse. The schema set depends only on the constructor signature of X and the combinatorial structure of depth-1 type expressions, not on which library atoms are used. Verified computationally for steps 1–7 (section 7). \square

The three components are independently measurable:

- ν_G : determined by enumerating depth-1 schemas over $\{X, L\}$ and filtering trivials.
- ν_H : determined by inspecting the cell structure of X (path constructors and their dimensions).

- ν_C : determined by the structural rules X introduces (capability analysis).

Table 2 shows that for each of steps 1–8, the paper’s ν values match the sum $\nu_G + \nu_H + \nu_C$ exactly. For steps 9–15, $\nu_G = \nu_H = 0$ (no new type schemas, no path constructors), and $\nu = \nu_C$.

Remark 6.2 (Scope of the tripartite structure). The tripartite decomposition with all three components nonzero is exercised at exactly three steps: S^1 ($n = 5$), S^2 ($n = 7$), and S^3 ($n = 8$)—the higher inductive types. Type formers (steps 1, 2, 4, 6) and structural extensions (steps 9–15) have $\nu = \nu_C$ only. The Witness (step 3) has $\nu = \nu_G$ only. The theorem’s content is therefore concentrated at the HIT steps.

6.2 Independence

We establish that the three components are genuinely independent: each varies while the others are held fixed.

Proposition 6.3 ($S^2 \equiv S^3$ Grammar Identity). $\nu_G(S^2) = \nu_G(S^3) = 5$.

Proof. Both S^2 and S^3 produce the same five depth-1 schemas: $(L + X)$, $(L \rightarrow X)$, $(L \times X)$, X , $\Omega(X)$. The number of members per schema differs (more library types available for S^3), but the schema *set* is identical. Since ν_G counts schemas, not members, the grammar novelty is the same. \square

Corollary 6.4. *Homotopy content is invisible to the grammar component. S^2 ($\nu_H = 5$) and S^3 ($\nu_H = 10$) differ only in homotopy novelty; their grammar component cannot distinguish them.*

The converse also holds: type formers like Π/Σ ($\nu_C = 5$) and PropTrunc ($\nu_C = 8$) have $\nu_G = \nu_H = 0$ —their entire contribution is structural, invisible to both grammar and homotopy measures.

6.3 Combination Rule Uniqueness

We test whether the additive combination $\nu = \alpha\nu_G + \beta\nu_H + \gamma\nu_C$ is forced.

Method. The PEN simulation is parameterized by weights $(\alpha, \beta, \gamma) \in [0.5, 1.5]^3$. For each weight configuration, we compute the resulting Genesis Sequence and count how many of the first 15 steps match the paper ordering. A configuration “succeeds” if it reproduces at least the first 9 steps correctly (the entire Bootstrap and Geometric Ascent phases).

1D Windows. Varying each weight independently while holding the others at 1.0:

Weight	Window	Width	Center
α (grammar)	[0.84, 1.09]	0.25	0.97
β (homotopy)	[0.88, 1.18]	0.30	1.03
γ (capability)	[0.94, 1.11]	0.17	1.02

All three windows are centered within 3% of 1.0, with γ the narrowest. The tightest constraint is the Hilbert functional ($n = 14$, margin = 0.091), which is capability-dominated.

2D Sweep. A 51×51 grid over $(\alpha, \beta) \in [0.50, 1.50]^2$ at $\gamma = 1.0$:

- 331 of 2,601 cells (12.7%) reproduce the first 9+ steps.
- 293 cells (11.3%) reproduce all 15 steps.
- The successful region is a compact, simply connected island centered near (1.0, 1.0).
- α range: [0.68, 1.12]; β range: [0.86, 1.48].

The island is not a ball: it is elongated along the β axis (homotopy has wider tolerance) and compressed along α (grammar is more tightly constrained).

Non-Additive Rules. Nine alternative combination rules were tested:

Rule	Steps correct	Note
$\nu_G + \nu_H + \nu_C$ (sum)	15	baseline
$\max(\nu_G, \nu_H, \nu_C)$	4	fails at S^1
$\nu_G \cdot \nu_H \cdot \nu_C$ (product)	0	wrong from start
$(\nu_G + 1)(\nu_H + 1)(\nu_C + 1) - 1$	4	fails at S^1
$\sqrt{\nu_G^2 + \nu_H^2 + \nu_C^2}$ (L2)	4	fails at S^1
$\nu_G + \nu_H^2 + \nu_C$	4	overweights homotopy
$2\nu_G + \nu_H + \nu_C$	3	overweights grammar
$\nu_G + 2\nu_H + \nu_C$	4	overweights homotopy
$\nu_G + \nu_H + 2\nu_C$	2	overweights capability

Every non-additive rule fails by step 4. The failure mode is consistent: non-additive rules distort the relative ordering of type formers versus HITs, causing the system to realize structures in the wrong order.

6.4 Commensurability

The most unexpected finding is that the three components are *commensurable*: equal weights ($\alpha = \beta = \gamma = 1$) sit at the center of the viable region.

Definition 6.5 (Commensurability). Three quantities are *commensurable* if they are measured in the same units: a unit increase in any one has the same effect on the dependent quantity.

Proposition 6.6 (Commensurability of Novelty Components). *The grammar, homotopy, and capability components of novelty are commensurable: $\alpha = \beta = \gamma = 1$ lies within 3% of the center of the viable weight region.*

This is surprising because the three components are computed by entirely different methods:

- ν_G counts type schemas (a syntactic/combinatorial quantity).
- ν_H measures path-constructor complexity (a topological quantity).
- ν_C counts structural operations (a proof-theoretic quantity).

That they happen to be measured in the same units—i.e., that one grammar schema, one unit of path complexity, and one capability rule all contribute equally to the selection dynamics—suggests that there is a single underlying notion of “mathematical novelty” of which all three are manifestations.

Remark 6.7 (Interpretation). The commensurability of ν_G , ν_H , and ν_C is analogous to the commensurability of different forms of energy in physics. Kinetic, potential, and thermal energy are measured in the same units (joules) because there is a single underlying conserved quantity. The equal-weight result suggests that grammar schemas, homotopy path dimensions, and capability rules are different “forms” of the same underlying combinatorial resource. We do not claim to identify this resource; we observe that the PEN selection dynamics are consistent with its existence.

Critical Margins. The five tightest margins in the Genesis Sequence, which constrain the weight windows:

n	Structure	$\rho - \text{Bar}$	Dominant component
14	Hilbert	0.091	ν_C
6	PropTrunc	0.107	ν_C
13	Metric	0.156	ν_C
4	Π/Σ	0.167	ν_C
8	S^3	0.167	$\nu_G + \nu_H + \nu_C$

Four of the five tightest constraints are capability-dominated steps, explaining why γ has the narrowest window. The one tripartite step (S^3) has margin 0.167, identical to Π/Σ —the decomposition is not a loose fit.

7 Computational Verification

7.1 The Haskell Engine

A $\sim 3,000$ -line Haskell engine¹ implements the five PEN axioms as a synthesis loop. Starting from an empty library, it generates candidates from nine structural categories (Foundations, Formers, HITs, Suspensions, Maps, Algebras, Modals, Axioms, Synthesis), each gated by prerequisites.

Novelty computation. For HITs and suspensions, ν is computed by *proof-rank clustering*: enumerate newly inhabited types at depth ≤ 1 , abstract to schemas (X/L), filter trivials, count distinct clusters, add homotopy bonus. For type formers and axioms, ν is computed by capability rules. For the DCT, ν is the Lattice Tensor Product.

Results. The engine discovers all 15 Genesis structures in the correct order. Values of ν match the table exactly for 12 of 15 structures and lie within $\pm 15\%$ for the remaining three (S^3 , Connections, Hilbert). These variations arise from the depth-1 window, which does not fully capture deep homotopy-theoretic structure. In all cases the structure clears its bar, and the ordering is preserved.

Cross-validation. Four independent modes (paper replay, capability engine, capability replay, genuine synthesis) all produce consistent results.

7.2 Schema Canonicity Verification

The engine includes an exact-oracle mode (`ExactNu.hs`) that enumerates type expressions using *all* library atoms (not just the 2-step window). For steps 1–7, the depth-1 schema set using all atoms is identical to the proof-rank schema set using only the 2-step window. This confirms proposition 6.1: schemaization collapses all library distinctions, making the schema count window-independent.

At depth 2, the schema count explodes: Witness goes from 3 to 277 schemas, S^1 from 5 to 502. These measure combinatorial richness of type grammar, not meaningful novelty. Depth 1 is the correct granularity.

7.3 Combination Rule Sweep

The sweep analysis (§6) is implemented in a standalone Python script² that reimplements the PEN simulation with parameterized weights. The 2,601-cell grid, fine grid, and non-additive rule tests are fully reproducible.

7.4 Cubical Agda Mechanization

The Complexity Scaling Theorem is proved in Cubical Agda [6]: for $d = 2$, $\Delta_{n+1} = \Delta_n + \Delta_{n-1}$. The proof covers initial conditions, the recurrence, the cumulative sum identity $\tau_n = F_{n+2} - 1$, and convergence $\Phi_n \rightarrow \varphi$.

¹Source code: `engine/`, 17 modules. Build: `cd engine && cabal build`. Run: `cabal run pen-engine`.

²`sweep_figure.py`; output in `sweep_output.txt`.

Coherence Obligation Experiments trace the obligations for S^1 , S^2 , T^2 , and the Hopf fibration in Cubical Agda. In all cases: all obligations reference at most 2 layers, at least one genuinely references 2, and none references 3.

Obligation decomposition and the abstraction barrier. Explicit obligation enumeration at steps 7 and 8 verifies the Integration Trace Principle (lemma 2.6): S^2 resolves $13 = 8 + 5$ obligations (8 from PropTrunc, 5 from S^1), and S^3 resolves $21 = 13 + 8$ (13 from S^2 , 8 from PropTrunc). The 13 obligations from S^2 decompose into 5 structural and 8 inherited from S^2 's resolved PropTrunc interactions. A Cubical Agda module (`Saturation/AbstractionBarrier.agda`, `--cubical --safe`) defines L_7 's interface as an opaque record and discharges all Group B obligations (8.6–8.10) from record fields alone, with no PropTrunc import. This machine-checks Sealing Encapsulation (remark 3.8): resolved obligations are depth-1 references to L_7 , not depth-2 leaks to L_6 .

7.5 Coherence Window Stress-Testing

The engine accepts a `-window d` flag parameterizing the Coherence Window. $d = 1$ (constant costs) stagnates after 4–5 structures. $d = 2$ (Fibonacci) produces the 15-structure Genesis Sequence. $d = 3$ (tribonacci) stalls earlier as costs outpace the horizon. This provides computational evidence that $d = 2$ uniquely supports sustained evolution.

8 Discussion

8.1 What Is Proved, What Is Assumed, What Is Open

Proved.

- The Coherence Window Theorems (theorems 3.5 and 3.6): $d = 1$ for extensional foundations, $d = 2$ for intensional, with full proofs.
- The Integration Trace Principle (lemma 2.6): each layer exports $|S(L_k)| = \Delta_k$ schemas, verified by explicit obligation enumeration at steps 3–8 and machine-checked abstraction barrier in Cubical Agda. The Complexity Scaling Theorem (theorem 4.1) follows: $\Delta_n = F_n$ for $d = 2$.
- Sealing Encapsulation (remark 3.8): resolved obligations become opaque L_k exports; machine-checked for Group B obligations at step 8 (`AbstractionBarrier.agda`).
- The Divergence of Efficiency (theorem 5.6): $\rho_n \rightarrow \infty$, using only the conservative library-interaction bound.
- Grammar Canonicity (proposition 6.1): depth-1 schemas are window-independent, verified computationally for steps 1–7.
- The $S^2 \equiv S^3$ Grammar Identity (proposition 6.3).

Assumed.

- The Integration Trace Principle is verified for steps 3–8 and the abstraction barrier is machine-checked, but a general proof for all k remains open.
- The ν values in table 1: for 12 of 15 structures, ν is computed from first principles by the engine; for 3 (S^3 , Connections, Hilbert), the engine's depth-1 window approximates but does not exactly match, and the table values incorporate domain knowledge.
- The DCT lattice count ($14 \times 11 - 4 = 150$): the factor counts are standard (Kuratowski's 14 [10], LTL's 11 [11]), but the correction -4 is an estimate.

Open.

- **What does κ measure?** The table assigns $\kappa(S^3) = 5$ (SU(2)-equipped), but the engine computes $\kappa = 3$ (suspension). The sequence is preserved under both definitions, but the correct measure is not settled.
- **Extending proof-rank to all candidates.** The engine uses capability rules for type formers and axioms. Computing ν from first principles for all 15 structures would eliminate remaining domain knowledge.
- **Commensurability explanation.** Why are ν_G , ν_H , and ν_C measured in the same units? The 12.7% island centered at $(1, 1, 1)$ demands an explanation we do not have.
- **Higher depth schemas.** Depth-2 schema counts explode ($\times 100$), but this may indicate a different measure at coarser granularity.

8.2 Limitations

The sample size for tripartite decomposition. The full three-component decomposition is exercised at exactly three steps (S^1 , S^2 , S^3). This is enough to demonstrate independence ($S^2 \equiv S^3$ grammar identity) and to constrain the weight space, but a skeptic may view $n = 3$ as thin evidence for universality. We stress that the claim is limited: the decomposition is canonical for the Genesis Sequence, not for arbitrary mathematical structures.

Capability as residual. For type formers (steps 1, 2, 4, 6) and axioms (steps 9–15), $\nu_G = \nu_H = 0$ and $\nu = \nu_C$. In these cases the decomposition is trivially exact. The non-trivial content is at the HIT steps, where all three components are independently nonzero and the additive combination produces correct ν values.

The $\nu = 150$ estimate. The DCT’s ν is computed structurally (Lattice Tensor Product), not by proof-rank clustering. The semantic audit (appendix A.5) provides a cross-check but is not a formal proof. A rigorous computation of the free monoid generated by the DCT axioms would strengthen the result.

8.3 The Physics of Mathematics

If the Genesis Sequence faithfully reconstructs the optimal path of mathematical discovery, then the structures of physics—gauge fields, Riemannian geometry, variational principles—are not empirical accidents but the inevitable output of efficiency optimization within $d = 2$ coherence. The sequence reproduces the historical arc: dependent types \rightarrow spheres \rightarrow bundles \rightarrow cohesion \rightarrow differential geometry \rightarrow dynamics.

The absorption of arithmetic is instructive: natural numbers ($\rho \approx 1.5$) and Lie groups ($\rho = 1.50$) fail the rising bar and are subsumed by more efficient geometric and modal frameworks. PEN predicts that efficient foundations prioritize geometric generality over discrete utility.

After DCT, the synthesis mechanism is exhausted (no further independent logic to compose). This suggests a computational phase transition: mathematics shifts from ontological construction to internal exploration within DCT.

8.4 Falsifiability

The PEN framework makes three testable predictions:

1. **Dimensional limit.** If a physical phenomenon required Class 3 coherence (non-trivial 3-paths not reducible to 2-path coherence), the framework would break.
2. **No early efficiency monsters.** If a structure existed with $\kappa < 4$ and $\nu > 20$ before homotopy theory, the sequence would be falsified.
3. **The DCT lattice count.** $\nu(R_{15}) = 150$ is a prediction. If the lattice collapses to $\nu < 60$, the singularity disappears and the framework is falsified.

9 Conclusion

The Principle of Efficient Novelty produces the Genesis Sequence—15 mathematical structures from an empty library—governed by three theorems:

1. **Coherence Windows.** Intensional type theory has $d = 2$; extensional has $d = 1$.
2. **Complexity Scaling.** The Integration Trace Principle—each layer’s exported interface equals its integration trace—forces Fibonacci costs ($\Delta_n = F_n$) for $d = 2$; $d = 1$ stagnates.
3. **Combinatorial Novelty.** Superlinear novelty growth ensures efficiency permanently outpaces the selection bar.

and one empirical result:

4. **Novelty Decomposition.** The measure decomposes into three independent, commensurable components ($\nu_G + \nu_H + \nu_C$), with equal-weight addition as the essentially unique combination rule (12.7% island centered within 3% of $(1, 1, 1)$; all non-additive rules fail by step 4).

The Golden Ratio of mathematical evolution is the dominant eigenvalue of a memory system that looks back exactly two steps. The three components of novelty—grammar, homotopy, capability—are commensurable, suggesting a single canonical notion of mathematical novelty. The complete engine source code and Cubical Agda proofs are available as supplementary material.

A DCT: Key Theorems and Applications

A.1 Internal Tangent Bundle

Theorem A.1. *For any type $X : \mathcal{U}$ in DCT, the tangent bundle is: $TX := \sum_{x:X} (X^{\mathbb{D}})_x$, where $(X^{\mathbb{D}})_x$ is the type of infinitesimal curves through x . The projection $\pi : TX \rightarrow X$ is a fiber bundle, and any flow on X lifts to TX .*

A.2 Temporal Evolution

Theorem A.2. *Any type $X : \mathcal{U}$ in DCT admits a temporal evolution operator $E_X : \mathbb{R} \rightarrow (X \rightarrow X)$ satisfying identity, group property, smoothness, and discrete-structure preservation. The next-modality $\bigcirc X$ is the type of fixed points of infinitesimal evolution.*

A.3 Hamiltonian Flows

Theorem A.3. *For a smooth type M with symplectic form ω : any $H : M \rightarrow \mathbb{R}$ generates a unique Hamiltonian vector field X_H ; the resulting flow preserves ω ; the Poisson bracket makes $C^\infty(M)$ a Lie algebra.*

A.4 Representative Applications

Classical mechanics (Hamilton’s equations), Yang–Mills gauge theory, geometric flows (Ricci, mean curvature, Yamabe), and Linear Temporal Logic formulas are all instances of DCT’s temporal evolution operator.

A.5 Semantic Audit of $\nu = 150$

Table 3: Semantic decomposition of $\nu = 150$

Domain	ν	Representative constructions
Dynamical systems	15	Flows, fixed points, attractors, stability
Classical mechanics	10	Lagrangian/Hamiltonian, phase space, Noether
Quantum mechanics	12	Geometric quantization, prequantum bundles
PDEs	10	Heat/wave/Schrödinger, semigroups
Gauge theory	15	Yang–Mills, BRST, instantons, Chern–Simons
Geometric flows	8	Ricci, mean curvature, Yamabe flow
Temporal logic	10	LTL, CTL, model checking, safety/liveness
Control theory	8	Controllability, Pontryagin, feedback
Statistical mechanics	10	Partition functions, phase transitions
Computation foundations	8	Guarded recursion, step-indexing, bisimulation
Synthetic diff. geom.	10	Tangent/jet bundles, de Rham, Stokes
Category theory	8	Temporal categories, monads, Kan extensions
HoTT extensions	10	Temporal HITs, spectral sequences
Physics applications	16	Electromagnetism, GR, QFT, cosmology
Total	150	

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