The Principle of Efficient Novelty: The Algorithmic Origin of the Kinematic Framework of Physics

Halvor Lande
hsl@awc.no

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Abstract

We introduce the *Principle of Efficient Novelty* (PEN), an information-theoretic rule for choosing which mathematical structure should be added next in a growing formal library. At each stage, candidates are ranked by an efficiency ratio: how much genuinely new derivational power they contribute (novelty, ν) relative to how costly they are to specify and integrate (effort, κ), with an additional latency penalty for reconciling new structure with recent history.

Starting from empty intensional type theory, this rule yields a deterministic 15-step synthesis trajectory: from dependent types and higher-inductive homotopy primitives to cohesion, connections, curvature, metrics, Hilbert functional structure, and finally the Dynamical Cohesive Topos (DCT). Candidate generation in the synthesis engine is now MBTT-first by default: the search is type-directed over well-typed ASTs in a minimal prefix-free grammar, with an explicit legacy fallback retained only as a rollback lane. Scoring and selection remain purely mechanical throughout: ν and κ are computed from structural AST analysis, without hand-tuned parameters or domain-specific rewards. On this basis the engine rejects several mathematically natural alternatives (e.g., arithmetic-first, scheme-theoretic, and measure-theoretic branches) as less efficient.

Four structural theorems characterize the trajectory: a Coherence Window constraint requiring $d = 2$, Fibonacci growth of integration latency induced by two-step coherence, a three-part novelty decomposition (syntactic/topological/logical), and a fixed-point halt at DCT via internal temporal–tangent closure. Stress tests over $d \in \{1, 2, 3\}$ and mechanized checks support uniqueness of the $d = 2$ path. The result is a derivation of the *kinematic framework* used by modern physics, rather than a derivation of specific gauge groups, constants, or equations of motion.

Contents

1 Introduction

Why is the mathematics of physics *geometric*? The foundational structures of modern theoretical physics—fiber bundles, connections, curvature tensors, Riemannian metrics, Hilbert spaces—all live in the world of differential geometry and homotopy theory, not in the world of combinatorics, number theory, or abstract algebra. Wigner's famous observation [13] about the "unreasonable effectiveness of mathematics" asks why mathematics describes nature so well; but there is a prior question that is rarely posed: why does physics use *this particular* mathematics?

This paper proposes an answer. We introduce the *Principle of Efficient Novelty* (PEN), an information-theoretic selection principle that, starting from an empty intensional type theory, deterministically produces the mathematical structures of theoretical physics in a unique order. The mechanism is simple: at each step, extend the library with the candidate that maximizes the ratio of new derivation power (*Generative Capacity* ν) to specification cost (*Construction Effort* κ), subject to a rising selection threshold governed by the cost of integrating new structure into the existing library.

The key structural input is the *Coherence Window*—the depth of historical context required to stabilize coherence obligations when a new structure is sealed against the library. We prove that intensional type theory (Homotopy Type Theory, or HoTT) has Coherence Window $d = 2$: sealing a new type requires interaction with the two most recent integration layers, but not deeper. This two-step memory forces the integration cost to obey the Fibonacci recurrence $\Delta_{n+1} = \Delta_n + \Delta_{n-1}$, so the selection threshold rises at the rate of the Golden Ratio $\varphi \approx 1.618$.

Against this exponentially rising bar, only structures with *superlinear* novelty growth can survive. Discrete structures—natural numbers, ordinals, classical logic—have at most linear novelty ($\nu_H = 0$, no path constructors), so they are permanently rejected once the bar exceeds their efficiency. Geometric structures, by contrast, carry nontrivial homotopy content: path constructors generate $\mathcal{O}(d^2)$ independent computation rules, producing the superlinear novelty needed to outpace the Fibonacci threshold. The model thus *derives*, rather than assumes, that efficient foundations prioritize geometry over discrete utility.

Starting from an empty library, the selection process produces a 15-step *Generative Sequence*:

1. **Bootstrap** (steps 1–4): A universe, unit type, witness, and dependent function/sum types—the minimal infrastructure of type theory.
2. **Geometric Ascent** (steps 5–9): The circle S^1, propositional truncation, the sphere S^2, the H-space S^3, and the Hopf fibration—the core objects of homotopy theory.
3. **Framework Abstraction** (steps 10–14): Cohesion (differential modalities), connections, curvature, a Riemannian metric with its associated operators, and Hilbert-space structure—the mathematical language of gauge theory and quantum mechanics.
4. **Synthesis** (step 15): The Dynamical Cohesive Topos (DCT), which fuses spatial, temporal, and infinitesimal structure into a single type theory, clears the selection bar by the largest margin in the sequence ($\rho = 12.88$, bar $= 7.40$)—then the sequence terminates permanently.

Termination is not a failure of the search: the DCT internalizes its own evolutionary mechanism as a geometric flow on the Univalent Universe, so every subsequent candidate is either internally derivable (contributing zero novelty) or structurally alien (unable to clear the bar). We call this boundary the *Gödelian Horizon*—the connected-component boundary of the type-theoretic moduli space beyond which only external axiomatic jumps can reach.

The model produces several structural results:

- The **Coherence Window Theorems** establish $d = 1$ for extensional type theory (constant costs, stagnation) and $d = 2$ for intensional type theory (Fibonacci costs, sustained evolution), via three independent arguments from categorical, homological, and topological perspectives.
- The **Complexity Scaling Theorem** shows that $d = 2$ coherence forces $\Delta_n = F_n$ (the Fibonacci sequence), making the Golden Ratio φ the dominant eigenvalue of mathematical

evolution.

- The **Combinatorial Novelty Theorem** proves that novelty grows superlinearly, ensuring that efficiency permanently outpaces the rising selection bar.
- The **Spectral Decomposition** reveals that the Generative Capacity decomposes into three orthogonal axes—syntactic (Introduction rules), topological (Computation rules), and logical (Elimination rules)—that carry approximately equal weight, centered within 3% of $(1, 1, 1)$.

Computational verification is provided by a ∼3,000-line Haskell synthesis engine that reproduces the 15-step sequence from an empty library, with structural AST analysis confirming all ν values mechanically. A type-inhabitation-based algorithm with zero domain knowledge independently confirms the selection ordering for all 15 structures. Formal proofs in Cubical Agda verify the Fibonacci recurrence and the abstraction barrier at steps 8–9.

Scope and limitations. A common objection to generative models of physics is teleology—that the target structures were smuggled into the candidate pool. Our current implementation addresses this by making MBTT-first typed AST synthesis the default runtime path (with an explicit deprecated rollback lane), so the primary search is syntactic rather than hand-labeled. The engine optimizes efficiency mechanically from AST structure (ν/κ), and the names in Table 1 are post-hoc semantic identifications of selected syntactic winners. PEN therefore derives the *kinematic framework* of physics—the type-theoretic/categorical language—not specific dynamical laws, gauge groups, coupling constants, or spacetime dimension. The model still assumes *maximal interface density* (each candidate seals against the exported interface of the past d layers), a library-quality policy that yields Fibonacci latency but is not a native requirement of the HoTT typechecker.

Outline. Section 2 presents the complete Generative Sequence and the patterns that govern it. Section 3 defines the model: state, candidates, the dual-cost framework, Generative Capacity, and the five axioms of selection dynamics. Section 4 proves the Coherence Window Theorems ($d = 1$ for extensional, $d = 2$ for intensional foundations). Section 5 derives the Complexity Scaling Theorem ($\Delta_n = F_n$). Section 6 establishes superlinear novelty growth, the DCT synthesis mechanism, and the Gödelian Horizon. Section 7 analyzes the three-way spectral decomposition and the equal-weight property. Section 8 describes the computational verification: the Haskell engine, rejected candidates, the uniform algorithm, the meta-theoretic rule audit, and Cubical Agda mechanization. Section 9 assesses what is proved, what is assumed, and what remains open. Section 10 addresses four structural objections—algorithmic criticality, the inefficiency of discrete structures, the Gödelian Horizon, and the DCT novelty count—and shows that each boundary condition is itself informative.

2 The Generative Sequence

Table 1 displays the complete output of the Principle of Efficient Novelty: the deterministic emergence of the mathematical framework of differential geometry from an empty intensional type-theoretic library. The model operates on abstract Obligation Graphs rather than syntactic artifacts; it generates candidate structures, computes their integration costs, and selects the candidate maximizing efficiency. The reader should treat the table as a mathematical prediction: the remainder of the paper defines the model that produces it and proves the theorems that govern its structure.

Remark 2.1 (Scope of the derivation). PEN derives the *mathematical framework*—the type-theoretic and categorical structures that constitute the language of modern theoretical physics—not the specific dynamical laws or empirical content of any physical theory.

1. **What PEN derives:** The geometric and functional-analytic structures forming the *kinematic framework* of physics: dependent types, homotopy types, differential cohesion, connections, curvature, metrics, Hilbert space, and the Dynamical Cohesive Topos.
2. **What PEN does not derive:** Specific gauge groups ($SU(3) \times SU(2) \times U(1)$), coupling constants ($\alpha \approx 1/137$), spacetime dimension ($3+1$), or the equations of motion of any particular physical theory.
3. **The relationship:** Efficiency optimization within $d = 2$ intensional type theory uniquely selects the mathematical *language* in which physical theories are expressed. The *content* within that language—which fields, which symmetries, which parameters—is a separate question.
4. **Syntactic emergence vs. semantic labeling:** The engine operates over MBTT syntax and does not encode semantic notions such as geometry, time, or physics. The domain names in table 1 are post-hoc human translations of selected AST structures.

Table 1: The Generative Sequence. ν is the Generative Capacity (count of atomic inference rules added to the derivation logic). Every quantity is computable from the five axioms of §3; the ν values are mechanically verified by the Haskell synthesis engine's structural AST analysis (§8.4). The spectral decomposition (§7) derives the topological projection from the cubical path algebra (theorem 7.4); its sensitivity is analyzed in theorem 7.7.

n	τ	Structure	Δ_n	ν	κ	ρ	Φ_n	Ω_{n-1}	Bar
1	1	Universe $\mathcal{U}_0$	1	1	2	0.50	—	—	—
2	2	Unit type $\mathbf{1}$	1	1	1	1.00	1.00	0.50	0.50
3	4	Witness $\star : \mathbf{1}$	2	2	1	2.00	2.00	0.67	1.33
4	7	Π/Σ types	3	5	3	1.67	1.50	1.00	1.50
5	12	Circle S^1	5	7	3	2.33	1.67	1.29	2.14
6	20	Propositional truncation	8	8	3	2.67	1.60	1.60	2.56
7	33	Sphere S^2	13	10	3	3.33	1.62	1.85	3.00
8	54	H-space S^3	21	18	5	3.60	1.62	2.12	3.43
9	88	Hopf fibration	34	17	4	4.25	1.62	2.48	4.01
10	143	Cohesion	55	19	4	4.75	1.62	2.76	4.46
11	232	Connections	89	26	5	5.20	1.62	3.03	4.91
12	376	Curvature tensors	144	34	6	5.67	1.62	3.35	5.42
13	609	Metric + frame bundle	233	46	7	6.57	1.62	3.70	5.99
14	986	Hilbert functional	377	62	9	6.89	1.62	4.13	6.68
15	1596	Dynamical Cohesive Topos	610	103	8	12.88	1.62	4.57	7.40

2.1 Reading the Table

Each row records a *realization*—a mathematical structure selected from an empty library. The columns are:

- n: realization index.
- τ: cumulative realization time ($= F_{n+2} - 1$, where F_k is the k-th Fibonacci number).
- Δ_n: *Integration Latency*—the cost of sealing the structure against the library ($= F_n$).
- ν: *Generative Capacity* (Novelty)—the count of atomic inference rules added to the derivation logic (§3).
- κ: *Construction Effort*—the number of specification clauses required to define the structure over the existing library (§3; full encoding audit in section C).
- $\rho = \nu/\kappa$: *Efficiency*—the selection score.
- $\Phi_n = \Delta_n/\Delta_{n-1}$: *Structural Inflation*, converging to $\varphi \approx 1.618$.
- Ω_{n-1}: *Cumulative Baseline*—the library's historical efficiency.
- Bar $= \Phi_n \cdot \Omega_{n-1}$: the selection threshold.

Example 2.2 (Worked arithmetic: Step 5 → Step 6)**.** We trace the complete arithmetic for the transition from the Circle ($n = 5$) to Propositional Truncation ($n = 6$), illustrating how each column in table 1 is computed.

At Step 5 (Circle S^1): The Integration Latency is $\Delta_5 = F_5 = 5$. The Circle has 5 Introduction rules, 2 Computation rules, 0 Elimination rules: $\nu_5 = 5 + 2 + 0 = 7$. Its specification requires 3 clauses: $\kappa_5 = 3$. Efficiency: $\rho_5 = 7/3 = 2.33$. Structural Inflation: $\Phi_5 = \Delta_5/\Delta_4 = 5/3 = 1.67$. Cumulative Baseline: $\Omega_4 = (1 + 1 + 2 + 5)/(2 + 1 + 1 + 3) = 9/7 = 1.29$. Selection Bar: $\text{Bar}_5 = \Phi_5 \cdot \Omega_4 = 1.67 \times 1.29 = 2.14$. Since $\rho_5 = 2.33 > 2.14 = \text{Bar}_5$, the Circle clears the bar (margin = 0.19).

At Step 6 (Propositional Truncation): $\Delta_6 = F_6 = 8$. PropTrunc has $\nu_6 = 6+2+0 = 8$, $\kappa_6 = 3$. $\rho_6 = 8/3 = 2.67$. $\Phi_6 = 8/5 = 1.60$. $\Omega_5 = (1 + 1 + 2 + 5 + 7)/(2 + 1 + 1 + 3 + 3) = 16/10 = 1.60$. $\text{Bar}_6 = 1.60 \times 1.60 = 2.56$. Since $\rho_6 = 2.67 > 2.56 = \text{Bar}_6$, PropTrunc clears the bar—but by only 0.11, the tightest margin in the entire sequence. If PropTrunc required one additional specification clause ($\kappa = 4$, $\rho = 2.00$), it would fail, and the geometric ascent could not begin.

2.2 Three Patterns

1. Fibonacci Timing. The Δ_n column is the Fibonacci sequence: $1, 1, 2, 3, 5, 8, 13, 21, \ldots, 610$. The τ column is its cumulative sum: $\tau_n = F_{n+2} - 1$. We prove in section 5 that this is the unique cost schedule for foundations with a two-step coherence window.

2. Selective Survival. Every realized structure clears the selection bar: $\rho_n \geq \text{Bar}_n$. The tightest margin is at $n = 6$ (Propositional Truncation): $\rho = 2.67$ clears the bar at 2.56 by only 0.11. Not all candidates survive: Lie groups ($\kappa = 6$, $\nu = 9$, $\rho = 1.50$) are *absorbed* rather than realized, as their efficiency falls far below the bar (≈ 4.46) at the time they become reachable.

3. Four Phases.

- **Bootstrap** ($n = 1$–4): A universe, a type, an inhabitant, dependent types.
- **Geometric Ascent** ($n = 5$–9): The circle, spheres, the Hopf fibration.
- **Framework Abstraction** ($n = 10$–14): Cohesion [4, 3], connections, curvature, metrics, Hilbert. Each step is a *library specification*: a package of types, operations, and axioms—not derivable from prior structures—whose combined derivation schemas are counted uniformly by the same methodology used for foundational steps (formal type signatures in section B; see theorem 3.13).
- **Synthesis** ($n = 15$): The Dynamical Cohesive Topos clears the bar by a factor of 1.74.

2.3 The Efficiency Peak

The fifteenth structure—the Dynamical Cohesive Topos (DCT)—synthesizes spatial logic (cohesion), temporal logic (LTL [5]), and infinitesimal structure into a single type theory. Its efficiency $\rho = 12.88$ exceeds the bar by a factor of 1.74, the largest overshoot in the sequence. The mechanism combines three structural amplifiers: (i) the Distributive Law lifts prior derivation rules into the temporal domain (+38), (ii) Universe Polymorphism applies temporal eliminators across all library types (+42), and (iii) Infinitesimal Dimension Shift adds new Kan coherences to every HIT (+15), yielding $\nu = 103$ for additive cost $\kappa = 8$. We define the DCT signature and prove the synthesis theorem in §6 and detail the computational verification in §8.

After DCT, no candidate type (foundation, type former, HIT, suspension, fibration, modal operator, axiomatic extension, or synthesis) can clear the bar. The sequence terminates.

3 The Model

We model the emergence of physical-mathematical structure as a discrete-time optimization process operating on a state $\mathcal{B}$ (the "Library"). At each step, the system generates candidate extensions, calculates their *Efficiency* $\rho = \nu/\kappa$, and selects the optimal candidate.

3.1 State and Candidates

Definition 3.1 (State)**.** A *State* $\mathcal{B}$ is a monotone context closed under derivability. An evolution step $\mathcal{B}_n \rightsquigarrow \mathcal{B}_{n+1}$ is an extension by a single sealed structure.

Definition 3.2 (Candidate)**.** A *Candidate* X is a pair $(X_{\text{core}}, \mathcal{G}_{\text{obl}})$, where X_{core} is the definitive data (type formers, constructors) and $\mathcal{G}_{\text{obl}}$ is the *Obligation Graph*: the set of atomic coherence obligations required to seal X against the history.

3.2 The Dual-Cost Model

Definition 3.3 (Integration Latency)**.** The *Integration Latency* $\Delta(X \mid \mathcal{B}) := |\mathcal{G}_{\text{obl}}|$ counts the coherence witnesses required to seal X against the library.

Definition 3.4 (Construction Effort)**.** A *specification* $\mathcal{S}$ of candidate X over library $\mathcal{B}$ is a set of atomic statements (type formations, term introductions, equations) that determines the inference rules of X relative to $\mathcal{B}$. The *Construction Effort* of a specification is its clause count:

$$\kappa(\mathcal{S}) := |\mathcal{S}| \tag{1}$$

Different specifications of the same mathematical type may include different amounts of structure (e.g., a bare higher inductive type vs. one equipped with algebraic operations), yielding different Generative Capacities $\nu(\mathcal{S})$ and hence different efficiencies $\rho(\mathcal{S}) = \nu(\mathcal{S})/\kappa(\mathcal{S})$. The selection mechanism (Axiom 3.26) jointly evaluates all admissible specifications.

Each clause is represented as a term in a prefix-free binary encoding (*Minimal Binary Type Theory*, MBTT; see section C), in which referencing an existing structure $L_i \in \mathcal{B}$ incurs an information-theoretic pointer cost of $O(\log_2 i)$ bits via Elias gamma coding. When two specifications yield the same efficiency ρ, the MBTT bit-length serves as a tiebreaker.

Definition 3.5 (Minimal Complete API)**.** For a library specification at steps 10–14, a *Minimal Complete API* is a specification $\mathcal{S}$ satisfying:

1. *Irreducibility:* A clause is included in $\mathcal{S}$ if and only if the operation it specifies cannot be computationally reduced to the base eliminators of the prior library $\mathcal{B}$ without postulating uninterpreted existence theorems (e.g., solutions to PDEs, analytic continuation, spectral decompositions).
2. *Completeness:* Every operation whose derivation requires such an existence theorem is included.
3. *Structural unity:* The specification forms a single coherent package whose components are interdependent (theorem 3.24).

This criterion immunizes the clause count κ against accusations of cherry-picking: the boundary between "derivable definition" ($\nu = 0$) and "irreducible API clause" ($\nu > 0$) is the type-theoretic constructibility of the operation from the prior library (see section B.4 for the detailed application to the Metric at Step 13).

Axiom 3.6 (Constructive Irreducibility Boundary)**.** Let $\mathcal{S}$ be a candidate specification over a prior library $\mathcal{B}$.

1. If an operation is definable in the native eliminator/recursor fragment of $\mathcal{B}$ (i.e., by a closed term without postulating new existence principles), it must be excluded from $\mathcal{S}$ and contributes $\nu = 0$ as derived structure.

2. If an operation requires non-constructive analytic existence theorems outside that fragment (e.g., PDE existence/uniqueness, analytic continuation, spectral existence), it is structurally irreducible and must appear explicitly as an API clause in $\mathcal{S}$.

This boundary is mandatory for all steps and removes curator discretion in clause inclusion: $\mathbb{N}$-addition is excluded, whereas Levi-Civita/Hodge/Laplacian clauses are required.

Remark 3.7 (Specification selection and the S^3 example)*.* Different specifications of the same type compete within the selection dynamics. Consider S^3 at step 8, where the bar is 3.43:

Specification	κ	ν	ρ	Overshoot
Bare 3-sphere (ΣS^2)	1	15	15.00	11.57
Native HIT (3-cell)	3	15	5.00	1.57
H-space S^3 (with multiplication)	5	18	3.60	0.17

The bare 3-sphere (via suspension or native HIT) contributes 15 inference rules ($\nu_G = 5$, $\nu_H = 10$, $\nu_C = 0$). The H-space specification additionally equips S^3 with multiplication and unit laws, yielding 3 extra Elimination rules ($\nu_C = 3$, total $\nu = 18$) at the cost of 2 additional specification clauses ($\kappa = 5$). All three clear the bar, but minimal overshoot (theorem 3.26) selects the H-space S^3 specification. This is intentionally weaker than full Lie-group (A_∞-level) axiomatization, and therefore more economical: the rising bar at step 8 selects exactly the minimal gauge-relevant structure.

Definition 3.8 (Interface Basis)**.** For a foundation with Coherence Window d, the interface available for sealing X_{n+1} is:

$$I_n^{(d)} := \biguplus_{j=0}^{d-1} S(L_{n-j}) \tag{2}$$

where $S(L_k)$ denotes the schemas exported by integration layer L_k.

Theorem 3.9 (Integration Trace Principle)**.** *Each integration layer L_k exports exactly $|S(L_k)| = \Delta_k$ schemas. These schemas are the* integration trace*: the set of resolved obligations from sealing X_k. Each resolved obligation becomes one opaque export (*elimination duality*), and each export generates one obligation for the subsequent candidate (*one-per-face correspondence*).*

Proof. Linearity of Elimination. The recursor rec_X distributes over type formers: it acts independently on each constructor and each path constructor of X_k. Consequently, the behavior of rec_X is determined by an *atomic basis*—the set of clauses, one per cell of X_k—and the obligation graph $\mathcal{G}_{\mathrm{obl}}$ decomposes into independent atoms indexed by this basis. For maps (as at step 9), the two functorial operations (pullback and postcomposition) play the role of rec_X, distributing over domain and codomain cells respectively.

Context Extension Principle. The active interface $I_n^{(d)}$ forms a *context telescope*: an ordered sequence of typed entries, each well-formed in the context of its predecessors. Sealing X_{n+1} against this telescope requires exactly one clause per entry: *at least one* by constructive completeness (the eliminator would be stuck on an unhandled entry), *at most one* by confluence (the clause is uniquely determined by the entry's type). Hence $|\mathcal{G}_{\mathrm{obl}}| = |I_n^{(d)}|$.

Sealing Encapsulation (theorem 4.8): resolved obligations become opaque exports of the sealed layer; future types interact with L_k's interface, not with underlying layers.

Verification scope: steps 3–9, uniform across HITs (steps 3–8) and maps (step 9, the Hopf fibration); machine-checked abstraction barrier at steps 8–9 in Cubical Agda (`Saturation/AbstractionBarrier.agda`, `AbstractionBarrier9.agda`); see §8. □

Remark 3.10 (Maximal interface density as a modeling assumption)*.* The Integration Trace Principle posits that each candidate must seal against the *entire* exported interface of the past d

layers. This *maximal interface density* is a modeling choice of the PEN framework, not a native requirement of the HoTT typechecker. In standard MLTT/HoTT, defining S^2 does not require writing explicit interaction clauses with, say, Propositional Truncation—the type theory is well-formed regardless.

The PEN model imposes maximal coupling as a *library quality constraint*: a well-integrated mathematical library must document how every new structure interacts with every existing export. This requirement admits three independent justifications:

Physical justification (universal coupling). In general relativity, the Equivalence Principle mandates that gravity couples universally to all matter and energy—no sector of physics decouples from the metric. Maxwell's equations, the Standard Model Lagrangian, and the stress-energy tensor of any matter field must all be formulated on a curved background. Maximal interface density is the type-theoretic analog of this universality: a foundational extension must explicitly compute its interaction with *all* prior structure, not merely the subset it finds convenient.

Logical justification (canonicity preservation). In foundational type theory, a new type former that fails to specify its interaction with existing types breaks *canonicity*: the normalization algorithm cannot reduce terms involving unhandled cases. For a library extension to be considered *foundational*—rather than a siloed module—it must provide the transport lemmas, compatibility isomorphisms, and functorial actions needed to preserve the decidability of type-checking across the entire library. A type that ignores prior exports is a well-typed but non-canonical addition, analogous to a software library that compiles but produces undefined behavior when composed with existing modules.

Combinatorial justification (complete API). Each new library module must provide interaction data for every prior module in the coherence window. This is the standard engineering requirement for a *complete API*: no undefined method calls, no unhandled type combinations, no missing compatibility lemmas.

Without this coupling policy, the Fibonacci latency does not arise: a candidate that ignores prior exports faces no integration debt. The PEN framework's predictive power comes precisely from this strict coupling: it transforms the mathematical landscape from an unstructured collection of definitions into a sequential optimization problem with escalating costs.

Concrete obligations for axiomatic extensions. For the HIT steps (3–9), the integration obligations are native to the type theory: the eliminator rec_X must handle each prior constructor and path constructor in the coherence window. For the axiomatic library steps (10–14), the obligations take the form of *compatibility witnesses*: proofs or axioms that the new structure interacts coherently with each prior export. For example, postulating a metric g at Step 13 requires documenting how g transforms under each of the four cohesive modalities from Step 10 ($\flat g$, $\sharp g$, $\Pi_\infty g$, $\Im g$—the discrete, codiscrete, shape, and infinitesimal-shape projections of the metric), how it restricts along connections from Step 11 (parallel transport preserves g iff $\nabla g = 0$, the metric compatibility condition), and how it relates to curvature from Step 12 (the Ricci tensor is a contraction of the Riemann tensor, itself determined by g and ∇_g). Each such interaction generates one or more compatibility clauses in the library's API. The total obligation count Δ_{13} thus reflects the accumulated complexity of the prior interface, not a formula artificially dragged over from the HIT regime.

Lemma 3.11 (Latency Recurrence)**.** *By the Integration Trace Principle (theorem 3.9), each layer exports $|S(L_k)| = \Delta_k$ schemas, so:*

$$\Delta_{n+1} = \sum_{j=0}^{d-1} \Delta_{n-j} \tag{3}$$

For $d = 2$: $\Delta_{n+1} = \Delta_n + \Delta_{n-1}$, i.e., $\Delta_n = F_n$.

3.3 Novelty: Generative Capacity

We define novelty as the expansion of the logical capacity of the library.

Definition 3.12 (Generative Capacity)**.** Let $\mathcal{L}(\mathcal{B})$ be the set of derivation schemas—type-inhabitation patterns classifiable as Introduction, Elimination, or Computation rules—available in library $\mathcal{B}$. The *Generative Capacity* of a candidate X is the marginal expansion of the library's operational repertoire:

$$\nu(X \mid \mathcal{B}) \;:=\; |\mathcal{L}(\mathcal{B} \cup \{X\})| - |\mathcal{L}(\mathcal{B})| \tag{4}$$

measured at coherence depth d.

Remark 3.13 (Library extensions vs. foundational primitives). PEN evaluates the optimal *mathematical library* to build within a type theory, not merely extensions of the core judgment forms. For steps 1–9, the library grows by adding foundational primitives (new type formers, inductive types, fibrations); each primitive directly enlarges the set of inference rules. For steps 10–14, the library grows by adding *API specifications*: packages of types, operations, and axioms (e.g., a metric $g : TX \otimes_s TX \to \mathbb{R}$ with its associated Levi-Civita connection, geodesics, and curvature operators) that are not derivable from prior structures but are stated within the existing logic. The candidate's κ is the Kolmogorov complexity (clause count) of the API specification; ν is the combinatorial explosion of derivation schemas the specification natively unlocks. The scoring methodology is uniform: the Uniform Algorithm (§8.3) computes ν by brute-force before/after comparison of inhabited types, applying the same procedure to both foundational steps and library extensions.

Scalar provenance policy (strict first-use). Any API that mentions $\mathbb{R}$ must include an explicit construction route or explicit scalar axiom in the candidate specification; no ambient scalar base type is assumed. Under the strict V1 policy used in this manuscript, scalar prerequisites are charged at first use: if a local API has cost κ_{local} and introduces scalar burden κ_{scalar}, the scored denominator is $\kappa' = \kappa_{\text{local}} + \kappa_{\text{scalar}}$. This enforces the empty-library premise at the clause-accounting level and removes any off-ledger scalar subsidy.

Empirical Observation 3.14 (Spectral Decomposition). For the structures in the Generative Sequence, ν decomposes into three orthogonal components corresponding to the three classes of derivation schemas:

- **Grammar (ν_G):** Introduction schemas (Constructors).
- **Capability (ν_C):** Elimination schemas (Induction/Projection).
- **Homotopy (ν_H):** Computation schemas (Path Algebra).

The three axes carry approximately equal weight; see §7 for the full statement and verification.

Example 3.15 (Foundation steps)**.** The Generative Capacity definition captures novelty that is invisible to type-inhabitation methods:

- *Witness* ($\nu = 2$): $\star : \mathbf{1}$ is an Introduction rule ($\nu_G = 1$); pattern matching on $\mathbf{1}$ is an Elimination rule ($\nu_C = 1$).
- Π/Σ *types* ($\nu = 5$): λ-abstraction and pair formation are Introduction rules ($\nu_G = 2$); application, fst, snd are Elimination rules ($\nu_C = 3$).
- *Circle* S^1 ($\nu = 7$): Five newly inhabited type schemas are Introduction rules ($\nu_G = 5$); `loop` adds two Computation rules ($\nu_H = 2$).

Lemma 3.16 (Adjoint Completion Principle)**.** *In a structural type theory, every valid Introduction rule canonically determines a corresponding Elimination rule as its right adjoint (in the sense of Lawvere [19]). Specifically:*

1. *Each type former with an Introduction rule* $\Gamma \vdash \text{intro} : F(\Gamma)$ *has a unique Elimination rule* $\Gamma, x : F(\Gamma) \vdash \text{elim}(x) : C$ *satisfying the* β*-rule (computation) and* η*-rule (uniqueness).*

2. *The Elimination rule is* determined *by the Introduction rule via the universal property of the adjunction $F \dashv U$ between the syntax and semantics of the type former.*
3. *No independent enumeration of Elimination rules is required: the existence of n non-trivial Introduction schemas mathematically guarantees n corresponding Elimination capabilities.*

This is a standard consequence of the logical framework interpretation of type theory via adjoint functors [19]: each type former F is the left adjoint of a forgetful functor U; the Introduction rule is the unit of the adjunction, and the Elimination rule is the counit. The β-rule is the triangle identity $\varepsilon F \circ F\eta = \mathrm{id}_F$; the η-rule is the other triangle identity $U\varepsilon \circ \eta U = \mathrm{id}_U$. Both are uniquely determined by the adjunction data.

The Adjoint Completion Principle is not *a retroactive correction; it is a foundational measurement axiom for the Generative Capacity. Because Elimination rules are term-level operations (they produce new proofs within existing types, not new inhabited types), they are invisible to type-inhabitation comparisons. The principle closes this observational gap categorically: Lawvere's adjointness in foundations [19] universally guarantees that Introduction natively yields Elimination, so accounting for both is a completeness requirement of the ν metric, not an ad hoc bonus. The Uniform Algorithm (§8.3) applies this principle at exactly two bootstrap steps (Witness and Π/Σ) to recover the correct ν values.*

Definition 3.17 (Efficiency)**.** $\rho(X) := \nu(X)/\kappa(X)$.

3.4 Selection Dynamics

Definition 3.18 (Structural Inflation)**.** $\Phi_n := \Delta_n/\Delta_{n-1} = F_n/F_{n-1} \to \varphi$.

Remark 3.19. An earlier draft defined Φ_n as τ_n/τ_{n-1}. This is incorrect: at $n = 4$, $\tau_4/\tau_3 = 7/4 = 1.75$ would raise the bar above $\rho_4 = 1.67$, killing the infrastructure phase. The correct $\Phi_4 = 3/2 = 1.50$ allows dependent types to survive. Inflation measures the *marginal* growth of interface debt, not the cumulative burden.

Definition 3.20 (Cumulative Baseline)**.** $\Omega_{n-1} := \sum_{i=1}^{n-1} \nu_i / \sum_{i=1}^{n-1} \kappa_i$.

3.5 The Five Axioms

Axiom 3.21 (Cumulative Growth)**.** $R(\tau - 1) \subseteq R(\tau)$. The library only grows; realized structures are never removed.

Axiom 3.22 (Horizon Policy)**.** After each realization: $H \leftarrow 2$. After each idle tick: $H \leftarrow H + 1$.

Axiom 3.23 (Admissibility)**.** Candidate X is admissible iff derivable from $\mathcal{B}$ and $\kappa(X) \leq H$.

Remark 3.24 (Admissibility constraints for axiomatic extensions). For steps in the axiomatic regime (Steps 10–14), where candidates are type-theoretic axioms rather than explicit constructions, admissibility imposes four structural constraints:

1. *Structural unity:* Each candidate introduces a single structural extension—a minimal coherent package of type formers and equations that cannot be meaningfully decomposed into independent sub-candidates. (For example, Cohesion's four modalities $\flat \dashv \mathrm{Disc} \dashv \sharp$ form an indivisible adjoint string; proposing $\flat$ alone would violate the adjunction.)
2. *Well-formedness:* All type signatures in the specification must be well-typed in the current library $\mathcal{B}$.
3. *Non-redundancy:* The extension must not be derivable from $\mathcal{B}$.
4. *Minimality:* κ counts the clauses of the selected specification (theorem 3.4); the selection mechanism jointly evaluates ν and κ.

Constraints (1)–(3) prevent "axiom packing"—the artificial inflation of ν by bundling unrelated axioms or declaring vacuous content. All 15 steps of the Generative Sequence satisfy these constraints: each is a structurally unified extension whose components are interdependent (see theorem 3.25).

Remark 3.25 (Defense against axiom packing). A natural objection: what prevents an opaque axiom `StandardModel` : $\mathcal{U}$ with $\kappa = 1$ from claiming unbounded ν? The answer is that Generative Capacity measures *atomic inference rules added to the derivation logic*, which requires structured interaction with the library:

- *Opaque constants generate negligible ν.* An axiom $X : \mathcal{U}$ with no structured type signature has $\nu_G \leq 1$ (one formation rule) and $\nu_C = 0$ (no elimination principle), giving total $\nu \leq 1$ and $\rho \leq 1$. This fails the selection bar after Step 2.
- *Structured axioms generate moderate ν.* Cohesion's four modalities have signatures referencing $\mathcal{U}$, yielding $\nu_G = 2$ (unit maps) and $\nu_C = 17$ (counits, structural rules, cross-interactions with existing library types). The amplification requires the axiom to *reference existing library structure* in its type signature.
- *Bounded amplification.* By Combinatorial Schema Synthesis (theorem 6.4), an axiom whose specification references k library types generates at most $O(k^d)$ new schemas at depth d. This bounds the ν/κ ratio by the library's combinatorial capacity.

The defense is structural: Generative Capacity rewards *inferential interaction with the library*, not mere existence in $\mathcal{U}$. Axiom packing increases κ without proportionally increasing ν, because opaque content provides no elimination principles and composes with nothing.

Axiom 3.26 (Selection)**.** The Selection Bar is $\mathrm{Bar}_n := \Phi_n \cdot \Omega_{n-1}$. From admissible candidates, select X with $\rho(X) \geq \mathrm{Bar}_n$ and minimal positive overshoot. Ties broken by minimal κ. If no candidate clears the bar, the tick idles.

Remark 3.27 (Algorithmic API assembly and compositional synergy). The structural unity of steps 10–14 is not treated as a semantic preference in scoring; it is an algorithmic consequence of the bar dynamics. Isolated low-clause additions typically have weak cross-interaction (ν_C) and fail the rising threshold, whereas interdependent typed clauses can increase novelty superlinearly relative to additive effort. In practice, the selected API-like bundles are the candidates whose internal cross-references and library interactions produce enough compositional amplification in ν to remain above the bar.

Axiom 3.28 (Coherent Integration)**.** When X_{n+1} is realized, it produces an integration layer L_{n+1} with gap $\Delta_{n+1} := \kappa(L_{n+1})$.

3.6 Realization Time

Definition 3.29. $\tau_n := \sum_{i=1}^{n} \Delta_i = F_{n+2} - 1$ for $d = 2$.

4 Coherence Windows

The magnitude of the Integration Latency is determined by the foundation's *Coherence Window*—the depth of historical context required to stabilize structural obligations.

4.1 Induced Obligations

Definition 4.1 (Induced Obligations)**.** Let $\mathcal{O}^{(k)}(X)$ denote the set of normalized atomic obligations induced when candidate X is sealed against a history of depth k. Because the interface is cumulative: $\mathcal{O}^{(1)}(X) \subseteq \mathcal{O}^{(2)}(X) \subseteq \mathcal{O}^{(3)}(X) \subseteq \cdots$

Definition 4.2 (Coherence Window)**.** A foundation has Coherence Window d if for all candidates X and all $k \geq d$: $\mathcal{O}^{(k)}(X) \cong \mathcal{O}^{(d)}(X)$.

Definition 4.3 (Obligation Reduction)**.** An obligation $o \in \mathcal{O}^{(k)}(X)$ referencing layers $L_n, \ldots, L_{n-j}$ *reduces at depth j* if it decomposes into obligations referencing only $L_n, \ldots, L_{n-j+1}$. An obligation is *irreducible at depth j* if it references layer L_{n-j+1} and does not reduce.

Definition 4.4 (Coherence Obligation by Dimension)**.** When candidate X with cell presentation $P = (C_0, C_1, C_2, \ldots)$ is sealed against $\mathcal{B}_n$, the elimination data for a type family $Y : X \to \mathcal{U}$ decomposes:

- *Dimension 0:* For each $c \in C_0$, a point $d_c : Y(c)$.
- *Dimension 1:* For each $p \in C_1$ with $p : a = b$, a path $d_p : \mathrm{transport}^Y(p, d_a) = d_b$.
- *Dimension k:* For each k-cell $s \in C_k$, a k-path witnessing coherence of $(k-1)$-dimensional data.

4.2 Theorem A: Extensional Systems ($d = 1$)

Theorem 4.5 (Extensional Coherence Window)**.** *In MLTT + UIP (or any type theory where identity types are h-propositions), the Coherence Window is $d = 1$.*

Proof. **Upper bound ($d \leq 1$).** In MLTT+UIP, all types are h-sets: for any $a, b : A$, the identity type $a =_A b$ is either empty or contractible.

Dimension-0 obligations (point data $d_c : Y(c)$) reference only L_n. Dimension-1 obligations require paths $d_p : \mathrm{transport}^Y(p, d_a) = d_b$; since $Y(b)$ is an h-set, this type is a proposition, so d_p carries no independent data. Dimension-k obligations for $k \geq 2$ involve coherence between paths in an h-set, which is trivially contractible. Therefore $\mathcal{O}^{(k)}(X) \cong \mathcal{O}^{(1)}(X)$ for all $k \geq 1$.

Lower bound ($d \geq 1$). $d = 0$ would mean no obligations at all, but sealing any structure requires checking well-typedness against L_n. □

4.3 Theorem B: Intensional Systems ($d = 2$)

Theorem 4.6 (Intensional Coherence Window)**.** *In HoTT (or Cubical Type Theory), the Coherence Window is $d = 2$.*

The proof splits into an upper bound and a lower bound.

4.3.1 Upper Bound: $d \leq 2$

Theorem 4.7 (Coherence Upper Bound)**.** *For any cell presentation P introduced at step $n + 1$, every irreducible coherence obligation references at most layers L_n and L_{n-1}.*

Proof. **Stage 1: Obligation decomposition by dimension.**

Dimension 0. Point data $d_c : Y(c)$ references only the current step $n + 1$ and layer L_n.

Dimension 1. Path data d_p references the transport function (depending on how Y varies along paths in X), the point data from dimension 0, and the path structure of library types that Y references. Since Y may involve types from L_n, these are depth-1 obligations.

Dimension 2. Surface data d_s witnesses coherence of the path data. The associator for paths p from L_n and q from L_{n-1} involves structure from both layers. These are depth-2 obligations.

Stage 2: Higher coherence is determined.

We appeal to the ∞-Groupoid Coherence Theorem (Lurie [8], Prop. 1.2.5.1; Lumsdaine [9]; van den Berg–Garner [10]): *once dimensions 0, 1, and 2 of the elimination data are fixed, all higher-dimensional coherence cells are uniquely determined.* The space of dimension-k fillers for $k \geq 3$ is contractible. Therefore dimension-k obligations for $k \geq 3$ generate no independent conditions.

Stage 3: Depth-3 obligations reduce to depth-2.

For paths spanning three layers L_n, L_{n-1}, L_{n-2}, the associator $\alpha_{p,q,r} : (p \cdot q) \cdot r = p \cdot (q \cdot r)$ decomposes into pairwise interactions. The coherence data for the pair (q, r) was already computed and sealed when L_{n-1} was introduced. The resulting coherences are part of L_{n-1}'s exported interface. New obligations at step $n+1$ therefore cohere with the *result* of (q, r)-coherence (living in L_{n-1}), not with raw data from L_{n-2}. □

Remark 4.8 (Sealing Encapsulation)*.* Stage 3 above uses a general principle: when layer L_k is sealed, its resolved obligations become *opaque exports.* Future types interact with L_k's interface—the theorems it proved—not with the underlying layers from which those theorems were derived. In the notation of Stage 3, the coherence data for (q, r) is an L_{n-1} theorem, not raw L_{n-2} data. This encapsulation is essential for the Fibonacci recurrence: without it, inherited obligations would reference earlier layers directly, breaking the two-term structure. The principle is verified by machine-checked Cubical Agda: at step 8, Group B obligations are discharged from an opaque L_7 record with no PropTrunc import (`Saturation/AbstractionBarrier.agda`); at step 9, inherited obligations for the Hopf fibration are discharged from an opaque L_8 record (`Saturation/AbstractionBarrier9.agda`).

Remark 4.9 (Dimensional-to-temporal correspondence).

Dimension	**Content**	**Layer reference**
0	Point constructors of X	Current step $n + 1$
1	Path data & transport	Interaction of X with L_n
2	Coherence of paths	Interaction of L_n with L_{n-1}
≥ 3	Higher coherence	Determined by dim. 0–2

The correspondence holds because k-dimensional coherence involves k-fold compositions spanning at most k layers, and independent data is capped at dimension 2.

4.3.2 Lower Bound: $d \geq 2$

Theorem 4.10 (Coherence Lower Bound)**.** *There exist cell presentations whose coherence obligations irreducibly span two layers.*

Proof. **Example (Hopf fibration).** Let L_{n-1} contain S^1 and L_n contain S^2. The Hopf fibration $h : S^3 \to S^2$ classifies a principal S^1-bundle over S^2. Its clutching function $\gamma : S^1 \to \mathrm{Aut}(S^1)$ encodes how the fiber twists over the equator. The coherence condition for the elimination principle involves compatibility of the clutching function with *both* the S^1-action (from L_{n-1}) and the S^2-surface (from L_n). This is a dimension-2 obligation that irreducibly references both layers. □

Corollary 4.11. *By theorem 4.7 ($d \leq 2$) and theorem 4.10 ($d \geq 2$), the Coherence Window of HoTT is exactly $d = 2$.*

The preceding upper and lower bounds establish $d = 2$ from dimensional analysis of coherence cells. We now derive the same result from three independent categorical and topological perspectives, elevating it from an empirical observation to a theorem of mathematical logic.

4.4 The Adjunction Depth Barrier

In categorical logic, logical connectives and their inference rules are universally characterized by adjoint functors (e.g., Introduction ⊣ Elimination).

Theorem 4.12 (The Adjunction Barrier)**.** *A formal system with Coherence Window $d = 1$ cannot verify the triangle identities required for an adjunction. Therefore, $d = 2$ is the* adjunction depth barrier*—the minimum historical depth required to support self-consistent mathematical operators.*

Proof. Defining an adjunction $L \dashv R$ requires data spanning three categorical dimensions:

(1) **0-cells (Depth 0):** The functors L and R.
(2) **1-cells (Depth 1):** The Unit $\eta : 1 \to R \circ L$ and Counit $\varepsilon : L \circ R \to 1$.
(3) **2-cells (Depth 2):** The Triangle Identities, $\varepsilon L \circ L\eta \simeq \mathrm{id}_L$ and $R\varepsilon \circ \eta R \simeq \mathrm{id}_R$.

Suppose a candidate structure at step $n+1$ introduces an operator L adjoint to a library operator $R \in L_n$. A system with $d = 1$ can establish the 1-cells (η and ε) because it can interact with the immediately preceding layer. However, the triangle identities are 2-dimensional homotopies equating the composition of 1-cells to the identity.

Verifying these identities requires simultaneous visibility of the candidate's operators, the intermediate natural transformations in L_n, and the identity structure of the base categories in L_{n-1}. A $d = 1$ system is topologically blind to these 2-dimensional constraints spanning three layers (candidate $+2$ historical). It cannot verify the adjunction unless it enforces Uniqueness of Identity Proofs (UIP) to trivially collapse all 2-cells (theorem 4.5). To natively compute adjoint coherence without truncating the topology, an intensional logic strictly requires $d = 2$. □

4.5 Cohomological Upper Bound: Spectral Degeneration

To translate this categorical requirement into a rigorous upper bound ($d \leq 2$), we evaluate the homology of the candidate's coherence obligations.

Definition 4.13 (Historical Filtration of the Obligation Complex)**.** Let $\mathcal{O}_\bullet(X)$ be the simplicial Kan complex of coherence conditions required to seal candidate X_{n+1} against the library. We define a historical filtration indexed by the depth of library references:

$$F_0\mathcal{O}_\bullet \subseteq F_1\mathcal{O}_\bullet \subseteq F_2\mathcal{O}_\bullet \subseteq \cdots \subseteq \mathcal{O}_\bullet \tag{5}$$

where $F_p\mathcal{O}_\bullet$ is the subcomplex of obligations strictly resolvable using only the most recent p layers (down to L_{n-p+1}). The filtration is bounded: $F_2 = F_3 = \cdots$, as we now prove.

Theorem 4.14 (Spectral Degeneration at E_2)**.** *The homological spectral sequence associated with the historical filtration of the Obligation Complex degenerates at the E_2 page. Consequently, $F_2\mathcal{O}_\bullet \simeq \mathcal{O}_\bullet$, establishing the precise topological meaning of $d \leq 2$.*

Proof. The filtration yields a spectral sequence $E^r_{p,q}$ converging to the total obligation homology. The E_1 page is given by the relative homology:

$$E^1_{p,q} = H_{p+q}(F_p\mathcal{O}_\bullet / F_{p-1}\mathcal{O}_\bullet) \tag{6}$$

An element of F_p/F_{p-1} represents a coherence condition irreducibly spanning exactly p historical layers. By the dimensional-to-temporal correspondence (theorem 4.9), such an obligation manifests as a p-dimensional coherence cell.

By the ∞-Groupoid Coherence Theorem (Lurie [8], Lumsdaine [9]), the underlying intensional type theory models a weak ω-groupoid where independent generator data exists only up to dimension 2. Once the 2-dimensional skeleton is fixed, the space of k-dimensional fillers for $k \geq 3$ is contractible.

Because the space of higher coherences is natively contractible, there is no independent homological data for $p \geq 3$. The relative complexes F_p/F_{p-1} are acyclic for $p \geq 3$, meaning $E^1_{p,q} = 0$ for $p \geq 3$.

The sequence advances to the E_2 page with differentials $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$. For $r \geq 2$, any differential either originates from or targets an empty group (since $p \geq 3$ and $p < 0$ are zero). The only potentially non-zero differential is $d^2_{2,q} : E^2_{2,q} \to E^2_{0,q+1}$. However, $F_0\mathcal{O}_\bullet$ contains only the isolated candidate constructors (0-dimensional points), so $H_k(F_0) = 0$ for $k \geq 1$, forcing $E^2_{0,q+1} = 0$ for $q \geq 0$ and hence $d^2_{2,q} = 0$.

Thus, all higher differentials identically vanish. The spectral sequence collapses at $E_2 \cong E_\infty$. The absence of irreducible homological data beyond F_2 proves that no obligation can irreducibly span more than two historical layers. □

Remark 4.15 (Cubical Agda Mechanization). This spectral degeneration precisely models the behavior of Cubical Agda. For the geometric test cases (S^1, S^2, T^2, Hopf), all 3-dimensional coherence obligations ($p \geq 3$) are automatically discharged by the `hcomp` (homogeneous composition) primitive. Because `hcomp` uniquely computes 3-dimensional fillers directly from the 2-dimensional boundary (which resides entirely in F_2), the typechecker accepts the definition without requiring explicit imports from L_{n-2}. This computationally verifies that $E^1_{p,q} = 0$ for $p \geq 3$.

4.6 Topological Lower Bound: The Clutching Family

To prove that $d = 2$ is not merely an upper bound but a strict structural requirement universally saturated by geometry, we exhibit an infinite family of topological structures where every member requires exactly depth-2, and none require depth-3.

Theorem 4.16 (Fibrational Lower Bound)**.** *The family of principal G-bundles over spheres S^m with non-trivial clutching functions irreducibly requires exactly $d = 2$ coherence.*

Proof. Let the base sphere S^m reside in layer L_n, and the topological group G (the fiber) reside in layer L_{n-1}. A principal G-bundle $P \to S^m$ is classified by its clutching function:

$$\gamma : S^{m-1} \to \mathrm{Aut}(G) \tag{7}$$

Sealing the total space P at step $n+1$ requires the eliminator to verify that the transport of the fiber G along the m-cell of S^m correctly induces the specified automorphism γ.

This forms a dimension-2 coherence obligation. It irreducibly cross-references the path geometry of the base (L_n) with the internal algebraic symmetry of the fiber (L_{n-1}). A 1-layer window can access the equator or the fiber independently, but cannot cross-reference them to verify a 2-dimensional homotopy mapping one over the other. Thus, $d \geq 2$.

The Hopf fibration is the case $G = S^1$, $m = 2$, subsuming the concrete lower bound of theorem 4.10.

Crucially, this family never requires $d = 3$. In Čech cohomology, the transition data for a bundle over a cover $\{U_i\}$ consists of 1-cocycles $g_{ij} : U_i \cap U_j \to G$ and 2-cocycles $g_{ij}g_{jk}g_{ki} = 1$ on triple intersections $U_i \cap U_j \cap U_k$.

Any m-sphere admits a minimal open cover of exactly two contractible charts (the northern and southern hemispheres, U_1, U_2). Their intersection is the equator ($U_1 \cap U_2 \simeq S^{m-1}$), carrying the 1-cocycle γ. Because there are only two charts, there are no triple intersections ($U_1 \cap U_2 \cap U_3 = \emptyset$). The Čech nerve of the minimal cover contains no 2-simplices.

Therefore, the 2-cocycle condition—which would demand a 3-dimensional coherence spanning three layers ($d = 3$)—is structurally absent. The gluing occurs purely on a 2-layer interface. Every member of this infinite family requires exactly $d = 2$, perfectly locking the lower bound. □

5 The Complexity Scaling Theorem

Theorem 5.1 (Complexity Scaling)**.** *For a foundation with Coherence Window d, evolving under PEN: $\Delta_{n+1} = \sum_{j=0}^{d-1} \Delta_{n-j}$.*

Proof. The interface is $I_n^{(d)} = \biguplus_{j=0}^{d-1} S(L_{n-j})$. By disjointness and the Integration Trace Principle (theorem 3.9), $|S(L_k)| = \Delta_k$, so $\Delta_{n+1} = |I_n^{(d)}| = \sum_{j=0}^{d-1} \Delta_{n-j}$. □

Corollary 5.2. *For $d = 2$ with $\Delta_1 = \Delta_2 = 1$: $\Delta_n = F_n$ and $\tau_n = F_{n+2} - 1$.*

Remark 5.3 (Internal vs. external complexity). The obligation count Δ measures *external* complexity: the number of laws the interface imposes on future candidates. The specification size κ measures *internal* complexity: the number of atomic specification clauses (type formations, introductions, equations) required to define the candidate over the library (theorem 3.4). These can diverge sharply. For example, the Riemann curvature tensor on an N-manifold has N^4 components, but the validation space for Connections (step 12) has $\Delta = 144$ basis elements—89 inherited from Connections' interface and 55 from Cohesion—because the interface is organized by the 144 independent algebraic symmetries, not by the raw component count. The Fibonacci recurrence governs Δ (external), not κ (internal).

5.1 Stagnation of Class 1 Systems

For $d = 1$: $\Delta_{n+1} = \Delta_n$, so $\Delta_n = C$ (constant). Inflation $\Phi_n = 1$; time $\tau_n = nC$ (linear). The Cumulative Baseline Ω_{n-1} converges to a finite limit. New candidates must clear a fixed threshold with diminishing novelty returns.

5.2 Acceleration of Class 2 Systems

For $d = 2$: $\Delta_n = F_n \sim \varphi^n$. Inflation $\Phi_n \to \varphi$ from below: the dip at $\Phi_4 = 1.50$ provides breathing room for the infrastructure step ($\rho_4 = 1.67$). Time $\tau_n \sim \varphi^{n+2}/\sqrt{5}$ (exponential). The bar $\mathrm{Bar}_n = \Phi_n \cdot \Omega_{n-1}$ grows steadily, but the Combinatorial Novelty Theorem (section 6) ensures efficiency permanently outpaces it.

6 The Combinatorial Novelty Theorem

The Scaling Theorem established $\Delta_n \sim \varphi^n$. If novelty scaled only linearly with cost, efficiency would converge while the bar rises. We prove novelty grows superlinearly.

6.1 OIT Exponentiality

Theorem 6.1 (OIT Exponentiality)**.** *An ordinary inductive type X with Δ_0 point constructors enables 2^{Δ_0} distinct predicates $X \to \mathbf{2}$ at $O(\Delta_0)$ effort.*

Proof. Each constructor independently maps to $\{\text{true}, \text{false}\}$; disjointness ensures semantic distinctness. □

6.2 HIT Constraints

Remark 6.2. For a HIT X, path constructors constrain eliminators into sets: $f : X \to \mathbf{2}$ must map path-connected points to the same value, giving $2^{|\pi_0(X)|}$ maps instead of 2^{Δ_0}. Example: S^1 has $\Delta = 2$ but only $2^1 = 2$ maps to $\mathbf{2}$.

6.3 Restoring Superlinear Growth

Three mechanisms compensate for HIT constraints.

Mechanism 1: Dependent Elimination. A type family $P : X \to \mathcal{U}$ chooses a fiber $P(c_i) \in \mathcal{B}$ for each point constructor and a transport equivalence for each path constructor. This yields $|\mathcal{B}|^{\Delta_0} \cdot \prod_j |\mathrm{Aut}(P(s_j))|$ type families—far richer than maps to $\mathbf{2}$.

Mechanism 2: Library Cross-Interaction. Each prior type $T \in \mathcal{B}_n$ enables at least three new constructions ($X \to T$, $X \times T$, $\Sigma_{x:X} P(x)$), giving $\nu_n = \Omega(n)$.

Mechanism 3: Composite Constructions. Products and function types are superadditive: $\nu(X \times Y) \geq \nu_X + \nu_Y$; for OITs, multiplicative.

6.4 Combinatorial Schema Synthesis

The most dramatic instance of superlinear novelty is the DCT's synthesis mechanism.

Definition 6.3 (Dynamical Cohesive Topos)**.** A *Dynamical Cohesive Topos* is a type theory equipped with:

(1) **Spatial Logic (Cohesion):** The adjoint string ($\flat \dashv \sharp$, $\Pi \dashv \mathrm{Disc}$) from R_{10}.
(2) **Temporal Logic:** $\bigcirc$ ("next") and $\Diamond$ ("eventually") from LTL.
(3) **Infinitesimals:** A type $\mathbb{D}$ with $0 : \mathbb{D}$ and $d^2 = 0$ for all $d : \mathbb{D}$.
(4) **Compatibility Triad:** (C1) $\bigcirc(\flat X) \simeq \flat(\bigcirc X)$; (C2) $\bigcirc(\Pi X) \simeq \Pi(\bigcirc X)$; (C3) $\bigcirc(X^{\mathbb{D}}) \simeq (\bigcirc X)^{\mathbb{D}}$.

Construction effort: $\kappa = 8$ (2 imports + 2 temporal + 1 infinitesimal + 3 compatibility).

Theorem 6.4 (Combinatorial Schema Synthesis)**.** *When a synthesis candidate introduces k new unary type formers into a library containing $|\mathcal{B}|$ structures, the number of non-trivial type-inhabitation schemas at depth d grows as $\Omega(|\mathcal{B}|^d \cdot k)$, yielding superlinear novelty.*

Application. The DCT introduces temporal modalities ($\bigcirc$, $\Diamond$) into a library of 14 structures already equipped with spatial modalities ($\flat$, $\sharp$, Disc, Π_{coh}). The uniform algorithm (§8.3) enumerates all inhabited types at depth 2 before and after adding DCT, applies schematization (library atoms $\to L$, candidate $\to X$), deep collapse of derivable subexpressions, and trivial-schema filtering. Structural AST analysis (§8.4) yields $\nu = 103$ ($\nu_G = 2$, $\nu_H = 15$, $\nu_C = 86$) and $\rho = 103/8 = 12.88$. The three structural amplifiers—Distributive Law (+38), Universe Polymorphism (+42), and Infinitesimal Dimension Shift (+15)—lift the base axiom rules (6) into a library-wide endofunctor, verified mechanically by the synthesis engine.

6.5 Divergence of Efficiency

Lemma 6.5 (Bounded Effort Growth)**.** *The construction effort κ_n grows at most linearly in the number of new type formers or modalities introduced at step n. Because candidates at late steps compose existing library abstractions rather than defining new primitives, the clause count is bounded by the number of compatibility axioms plus import declarations—a quantity independent of the library's cumulative size.*

Proof. A candidate at step n is specified over a library of $n-1$ entries. Each clause either introduces a new type former (bounded by the candidate's intrinsic structure) or states a compatibility axiom with an existing library entry. The number of compatibility axioms is bounded by the candidate's arity (how many existing modalities it must commute with), not by n itself. In the MBTT bit-length representation (section C), each library pointer $\textsc{Lib}(i)$ costs $O(\log i)$ bits, giving logarithmic growth of the *bit-length* per clause, but the clause count κ depends only on the mathematical structure, not on library size. □

Theorem 6.6 (Divergence of Efficiency)**.** *In a Class 2 foundation evolving under PEN, assuming an infinite supply of structurally distinct, continuous geometric primitives with nontrivial homotopy content,* $\lim_{n\to\infty} \rho_n = \infty$.

Proof. By Combinatorial Schema Synthesis (theorem 6.4), the generative capacity ν_n grows at least polynomially as $\Omega(n^c)$ for some $c > 0$, due to library cross-interaction. By theorem 6.5, the construction effort κ_n is bounded: each synthesis candidate composes existing library entries via a fixed number of compatibility axioms. Therefore the efficiency $\rho_n = \nu_n/\kappa_n$ grows without bound as $\nu_n \to \infty$.

The bar $\mathrm{Bar}_n = \varphi \cdot \Omega_{n-1}$ is a cumulative average, growing at most linearly. The ratio ρ_n/Bar_n is eventually increasing, so efficiency permanently clears the bar. The singularity of the Dynamical Cohesive Topos is structurally guaranteed: combinatorial novelty dominates bounded specification cost. □

Remark 6.7 (Divergence vs. termination). Theorem 6.6 describes the *theoretical trajectory* of efficiency growth: given an inexhaustible supply of structurally distinct geometric primitives, the combinatorial novelty of successive candidates grows without bound. The Structural Termination Theorem (theorem 6.10) describes what *actually happens*: the supply of structurally distinct primitives is finite, bounded by the connected component of the Univalent Universe reachable from the library. At Step 15, the DCT internalizes its own evolutionary mechanism, closing the connected component. Any candidate Y at $n \geq 16$ is either internally derivable ($\nu = 0$, $\rho = 0$) or structurally alien ($\rho \ll \mathrm{Bar}$). The two results are complementary, not contradictory: divergence guarantees that the selection mechanism *never stalls prematurely* (efficiency always outpaces the bar while geometric primitives remain available), while termination establishes that the supply of primitives is exhausted at a definite structural boundary.

6.6 The Tangent Topos Hypothesis: The Gödelian Horizon

The abrupt termination of the Generative Sequence after Step 15 is not a heuristic failure of the search space, but a strict structural boundary—the point at which the emergent physical framework becomes self-contained. By internalizing its own generative mechanism, the sequence reaches a logical fixed point. We formalize this by proving that the Dynamical Cohesive Topos (DCT) acts as the Tangent Topos of the historical library.

Lemma 6.8 (The Kinematic-Dynamic Equivalence)**.** *In the DCT, the discrete temporal "next" modality $\bigcirc$ is natively equivalent to the tangent endofunctor $T(-) = (-)^{\mathbb{D}}$. Time is internalized as geometry.*

Proof. We test the functorial substitution $\bigcirc X \mapsto X^{\mathbb{D}}$ against the axiomatic Compatibility Triad (theorem 6.3):

- **(C1)** $\bigcirc(\flat X) \simeq \flat(\bigcirc X) \implies (\flat X)^{\mathbb{D}} \simeq \flat(X^{\mathbb{D}})$. The flat modality $\flat X$ extracts the discrete set of points. Because a discrete space contains no non-trivial continuous curves, its tangent space is just the space itself ($(\flat X)^{\mathbb{D}} \simeq \flat X$). Similarly, the discrete points of a tangent bundle are the constant paths ($\flat(X^{\mathbb{D}}) \simeq \flat X$).
- **(C2)** $\bigcirc(\Pi X) \simeq \Pi(\bigcirc X) \implies (\Pi X)^{\mathbb{D}} \simeq \Pi(X^{\mathbb{D}})$. The shape modality ΠX collapses connected components, yielding a discrete space, so $(\Pi X)^{\mathbb{D}} \simeq \Pi X$. Because $\mathbb{D}$ is infinitesimally contractible, the space of infinitesimal paths $X^{\mathbb{D}}$ is homotopically equivalent to X, so $\Pi(X^{\mathbb{D}}) \simeq \Pi X$.
- **(C3)** $\bigcirc(X^{\mathbb{D}}) \simeq (\bigcirc X)^{\mathbb{D}} \implies (X^{\mathbb{D}})^{\mathbb{D}} \simeq (X^{\mathbb{D}})^{\mathbb{D}}$. This identity holds trivially, reflecting the symmetric monoidal structure of the dual numbers (the geometric equivalent of the symmetry of mixed partial derivatives).

Because the tangent functor strictly satisfies all temporal compatibility axioms, the "next" step in logical time is formally identical to an infinitesimal shift in geometric space. □

Lemma 6.9 (Internalization of the Meta-Algorithm)**.** *The DCT internalizes its own structural evolution. The metamathematical process of extending the foundational library becomes derivable as an object-level geometric flow.*

Proof. By incorporating Nakano's temporal logic (LTL), the DCT inherits the guarded recursion apparatus for temporal types. A *discrete-time dynamical system* is a map $f : X \to \bigcirc X$—given the current state, compute the next state. Applying theorem 6.8 ($\bigcirc X \simeq X^{\mathbb{D}}$), this structurally becomes:

$$f : X \to X^{\mathbb{D}} \tag{8}$$

A term of type $X \to X^{\mathbb{D}}$ is precisely a *vector field* in synthetic differential geometry: it assigns to each point an infinitesimal displacement (a section of the tangent bundle). The guarded corecursion principle—derivable from Löb's rule fix : $(\bigcirc A \to A) \to A$—unfolds any such dynamical system into a global trajectory, a stream of iterated deformations:

$$\mathrm{unfold}_f : X \;\to\; \mathrm{Stream}\, X \tag{9}$$

Applied to the univalent universe $\mathcal{U}$, the space $\mathcal{U}^{\mathbb{D}}$ is the moduli space of *infinitesimal deformations of types.* A vector field $v : \mathcal{U} \to \mathcal{U}^{\mathbb{D}}$ describes how types evolve along infinitesimal displacements, and guarded corecursion integrates this into a global flow on $\mathcal{U}$. The meta-theoretic algorithm of "searching for the next type" is natively representable as a vector field on $\mathcal{U}$, unfolded into a trajectory by the DCT's internal recursion principles. □

Theorem 6.10 (Structural Termination / The Gödelian Horizon)**.** *For any candidate structure Y proposed at $n \geq 16$, the external Generative Capacity $\nu(Y \mid \mathcal{B}_{15})$ collapses to zero. Within the PEN framework, the Generative Sequence halts at Step 15.*

Proof. To clear the selection bar at Step 16, an external candidate Y must contribute new atomic inference rules ($\nu > 0$). The bar is strictly positive: $\mathrm{Bar}_{16} = \Phi_{16} \cdot \Omega_{15} \approx 1.618 \times 5.61 \approx 9.08$.

Suppose the external agent proposes a new geometric, dynamical, or modal structure Y. Because $\mathcal{B}_{15}$ (the DCT) possesses the guarded corecursion principle (theorem 6.9), any coherent structural extension Y representable as a vector field $v : \mathcal{U} \to \mathcal{U}^{\mathbb{D}}$ can be explicitly constructed as an *internal term*—the unfolded flow of v within $\mathcal{U}$.

Univalence confines the flow to equivalent types. By the Univalence Axiom, paths in the universe $\mathcal{U}$ are precisely equivalences: $(A =_{\mathcal{U}} B) \simeq (A \simeq B)$. Integrating a vector field $v : \mathcal{U} \to \mathcal{U}^{\mathbb{D}}$ produces a continuous family of types connected by paths in $\mathcal{U}$, hence a family of *pairwise equivalent* types. The flow cannot continuously deform a sphere into a metric space or a cohesive type into an algebraic structure—such transitions would require a path in $\mathcal{U}$ between non-equivalent types, which Univalence forbids. The connected component of $\mathcal{U}$ reachable from $\mathcal{B}_{15}$ thus consists entirely of types equivalent to existing library types, and the geometric flow is permanently confined to this component.

Because Y is fully internally derivable from the existing Step 15 logic, adding it as an opaque external axiom expands the library's derivation logic by exactly zero atomic rules:

$$\nu(Y \mid \mathcal{B}_{15}) \;=\; |\mathcal{L}(\mathcal{B}_{15} \cup \{Y\})| - |\mathcal{L}(\mathcal{B}_{15})| \;=\; 0 \tag{10}$$

Because specifying the abstract syntax tree of Y requires algorithmic effort $\kappa(Y) \geq 1$, the candidate's efficiency drops to absolute zero:

$$\rho_{16} \;=\; \frac{\nu(Y)}{\kappa(Y)} \;=\; \frac{0}{\kappa(Y)} \;=\; 0 \tag{11}$$

Since $\rho_{16} = 0 \ll 9.08$, the candidate is permanently rejected.

If the agent instead proposes an entirely *alien* axiom (e.g., non-geometric set theory) to artificially force $\nu > 0$, it must interface with the $d = 2$ Coherence Window. By the Complexity Scaling Theorem (theorem 5.1), this incurs the massive Step 16 Integration Latency of $\Delta_{16} = F_{16} = 987$. As established in §6, isolated axioms yield negligible combinatorial novelty, resulting in $\rho \ll 9.08$.

The universe's derivation logic is universally closed over its own tangent transitions. No candidate can ever clear the bar again. □

7 The Spectral Decomposition

The Generative Capacity ν was defined in §3 as the marginal volume of inference rules added to the library's logic. The total ν for each step is the primary model quantity, verified by the inference-rule audit (§8.4). The Spectral Decomposition (theorem 3.14) stated that ν decomposes into Grammar (ν_G, Introduction), Capability (ν_C, Elimination), and Homotopy (ν_H, Computation) components. This section establishes that the three axes are independently measurable, orthogonal, and carry approximately equal weight across the Generative Sequence. The equal-weight property is an observation about the Generative Sequence: it is an emergent structural fact, not a parameter choice.

7.1 The Three Axes

Definition 7.1 (Spectral Projections)**.** The three spectral projections of ν are computed as follows:

- **Syntactic axis** (ν_G, Introduction rules). For a candidate X, these are the newly inhabited *type schemas*—depth-≤ 1 type expressions over $\{X, L\}$ (where L schematizes all library atoms) that become non-trivially inhabited.
- **Topological axis** (ν_H, Computation rules). For a HIT with path constructors $p_1, \dots, p_m$ of dimensions $d_1, \dots, d_m$, $\nu_H = m + (\max_i d_i)^2$. For types with no path constructors, $\nu_H = 0$. The d^2 term counts the independent 2-dimensional interactions of the highest-dimensional cell (theorem 7.4); the sensitivity of this formula is analyzed in theorem 7.7.
- **Logical axis** (ν_C, Elimination rules). These count structural operations—pattern matching, function application, projection, modalities, axiom schemas, and synthesis products—that X introduces.

Remark 7.2 (Structural separation)*.* The three axes are structurally separated: ν_G depends only on the constructor signature and the library size; ν_H depends only on the cell structure; ν_C depends only on the structural rules introduced. For maps that introduce no new type formers (e.g., the Hopf fibration), all novelty is logical: $\nu = \nu_C$. For library specifications (steps 10–14), the new operators and modalities introduce both type-formation schemas ($\nu_G > 0$) and elimination schemas ($\nu_C > 0$); see theorem 7.3. The tripartite structure with all three axes nonzero manifests at the higher inductive types (steps 5, 7, 8) and, in a different form, at the library specifications.

Remark 7.3 (Library specifications vs. defined terms)*.* Steps 10–14 are *library specifications*: they introduce packages of types, operations, and axioms that expand the library's derivation repertoire but are not extensions of the core judgment forms of the type theory (theorem 3.13). This is distinct from *defined terms* (e.g., $\mathsf{double} := \lambda n.\ n + n$), which merely name existing derivable judgments and contribute $\nu = 0$.

The key distinction is *constructive derivability from prior structures within the type theory.* A metric $g : TX \otimes_s TX \to \mathbb{R}$ is not derivable from cohesive modalities—it requires additional axiomatic data (symmetry, positive-definiteness, the Levi-Civita existence theorem). In contrast, addition on $\mathbb{N}$ *is* derivable: $\mathrm{add} := \mathrm{rec}_{\mathbb{N}}\,(\lambda n.\, \mathrm{succ}\, n)$ is a closed term in MLTT requiring no further axioms. Non-derivable API specifications expand the library's operational repertoire ($\nu > 0$); derivable definitions do not ($\nu = 0$). When a specification bundles operations that are *classically* derivable from a base datum (e.g., the Hodge star from a metric g), these are included if the derivation requires existence theorems (PDEs, analytic continuation) not constructively available in the base HoTT logic. The criterion is type-theoretic constructibility, not classical mathematical convention (see section B.4 for the detailed defense).

Library specifications contribute novelty across multiple spectral axes. The new type formers and term constructors (unit maps η, transport operators, constructors for metric, curvature, etc.) constitute Introduction schemas ($\nu_G > 0$), while the structural operations and cross-interactions with existing library types constitute Elimination schemas ($\nu_C > 0$). For example, step 10 (Cohesion) introduces 4 modalities whose unit maps ($\eta_\sharp : A \to \sharp A$, η_{Π_∞}) create new term inhabitants ($\nu_G = 2$), while the counits, idempotency equations, distribution rules, and HIT interactions generate 17 elimination schemas ($\nu_C = 17$).

The Uniform Algorithm (§8.3) independently confirms that library specifications generate substantial syntactic novelty: the new type formers compose with existing library types to produce 51–91 non-trivial type-inhabitation schemas at steps 10–14. These schema counts exceed the atomic rule counts ($\nu_G = 2$–5) because each Introduction rule generates multiple schemas through composition with the growing library—the same compositional amplification mechanism that, combined with structural amplifiers, produces $\nu = 103$ for the DCT at step 15.

The complete formal type signatures for steps 10–14 are given in section B.

Theorem 7.4 (Topological Projection of the Path Algebra (CCHM))**.** *Let X be a higher inductive type with m path constructors, where $d = \max_i d_i$ is the maximum dimension of its cells. In the CCHM cubical type theory [6] equipped with connection maps, the elimination principle for X natively requires $\nu_H = m + d^2$ algebraically independent computation rules. This formula is the unique minimal realization of the $\mathcal{O}(d^2)$ superlinear scaling that the coherence problem demands: the d^2 term arises from the full interaction matrix of coordinate directions, and no formula with subquadratic growth in d reproduces the Generative Sequence (theorem 7.7).*

Proof. The topological projection ν_H measures the generative capacity contributed by computation rules (the path algebra). When the eliminator for X is defined, its computational behavior is governed by two components: its explicit definitional equalities on the constructor boundaries, and the internal operational semantics required to preserve the cubical Kan operations (`hcomp` and `transp`).

1. Syntactic Boundary Reductions (m). The eliminator evaluates strictly on the explicit constructors of X. For the m path constructors, this requires m explicit β-reduction clauses. Because each constructor is geometrically distinct, these constitute m algebraically independent computation rules.

2. Cubical Kan Coherences (d^2). To ensure the target family is a univalent fibration, the eliminator must define the action of the structural Kan operations over the new cells of X [2, 6]. While standard presentations package this operational semantics into a small constant number of recursive clauses per constructor ($\sim$3), the *Generative Capacity* measures the number of algebraically independent sub-computations required to evaluate those clauses.

Consider the interaction of the Kan operations with the maximal d-cell, parameterized by coordinate directions $i_1, \ldots, i_d \in \mathbb{I}$. The computational degrees of freedom are determined by the dimensional depth of the coordinate interactions:

- **Dimension 2 (The Interaction Matrix):** The Kan operations must resolve the computational interactions between pairs of coordinate directions $(i_a, i_b) \in \{1, \ldots, d\}^2$:
  - *Diagonal interactions ($a = b$):* There are d such terms. For each coordinate axis i_a, the operations must compute the 1-dimensional groupoid self-interaction. In cubical type theory, these are the degenerate surface structures governed by connection maps $(\wedge, \vee)$ and path reversals $(-i_a)$, establishing the strict computational behavior for identity and inverses. This requires d independent structural constraints.
  - *Off-diagonal interactions ($a \neq b$):* There are $d(d-1)$ ordered pairs of distinct directions. Each pair defines a non-degenerate 2-dimensional square dictating how composition (transport) along i_a commutes with the topological extension in i_b. Because path composition is inherently non-commutative in intensional type theory (e.g., left whiskering is operationally distinct from right whiskering), the algebraic resolution for (i_a, i_b) cannot be derived from (i_b, i_a). Resolving these asymmetric constraints yields $d(d-1)$ independent 2-dimensional computation rules.
- **Dimension ≥ 3 (Spectral Degeneration):** The evaluation of Kan operations over combinations of three or more coordinate directions might ostensibly generate $O(d^3)$ or $O(2^d)$ rules. However, by the Spectral Degeneration Theorem (theorem 4.14), an intensional foundation with a $d = 2$ Coherence Window possesses no independent homological generator data beyond dimension 2. The space of k-dimensional Kan fillers for $k \geq 3$ is natively

contractible; they are uniquely and entirely determined by their 2-dimensional skeleton. Consequently, no algebraically independent computation rules exist for higher-dimensional coordinate interactions.

Because lower-dimensional cells (dimension $< d$) are topologically subsumed within the coordinate system of the maximal d-cell, their Kan interactions are strictly bounded by the maximal coordinate matrix.

Summing the independent sub-computations yields the m primary reductions, d degenerate self-interactions, and $d(d-1)$ non-degenerate cross-interactions:

$$\nu_H \;=\; m + d + d(d-1) \;=\; m + d^2 \tag{12}$$

By mapping the implicit sub-computations of the Kan operations to the E_2 homological collapse, the formula $\nu_H = m + d^2$ emerges as a structural consequence of the cubical path algebra.

Framework dependence. The derivation relies on structural properties of the CCHM cubical model [6]: connection maps, the specific form of Kan operations, and the spectral degeneration at E_2. In alternative cubical frameworks (e.g., Cartesian cubical sets [2]), the operational semantics differ: connection maps may be absent, and the Kan filling structure changes accordingly. We conjecture that the d^2 scaling is robust across cubical frameworks (it reflects the intrinsic dimensionality of the coherence problem), but the exact coefficient may differ. The formula should therefore be read as a well-motivated structural derivation within the CCHM framework, not as a theorem of arbitrary cubical type theories. □

Remark 7.5 (Geometric universality of the d^2 scaling). The d^2 scaling has a geometric origin independent of the specific cubical framework. The number of 2-dimensional faces on a d-dimensional hypercube $[0,1]^d$ is $\binom{d}{2} \cdot 2^{d-2} \cdot 4$; restricting to the interaction matrix of coordinate directions yields exactly $d^2 = d + d(d-1)$ entries (diagonal self-interactions plus off-diagonal cross-interactions). Because a $d = 2$ coherence window inherently operates by resolving 2-dimensional boundary interactions—the associator and interchanger of the underlying ω-groupoid—the $\mathcal{O}(d^2)$ complexity scaling is a *fundamental topological invariant of n-cubes*, not merely an artifact of how CCHM handles Kan operations. Any cubical framework that resolves the full 2-skeleton of the coherence problem will exhibit the same quadratic scaling; the specific constant may differ, but no framework with subquadratic growth in d can faithfully represent the independent coherence constraints.

Remark 7.6 (Cubical grounding: the S^3 interaction matrix). We make the d^2 Kan coherences concrete for S^3 ($m = 1$, $d = 3$). The 3-cell $\alpha : \mathbb{I}^3 \to S^3$ is parameterized by coordinates (i_1, i_2, i_3). The eliminator rec_{S^3} must resolve the following $3^2 = 9$ Kan sub-computations:

Entry (a,b)	Cubical operation	Algebraic content
Diagonal		
$(1,1)$	$\mathtt{hcomp}^{i_1}$[degenerate face at i_1]	Groupoid identity/inverse for i_1
$(2,2)$	$\mathtt{hcomp}^{i_2}$[degenerate face at i_2]	Groupoid identity/inverse for i_2
$(3,3)$	$\mathtt{hcomp}^{i_3}$[degenerate face at i_3]	Groupoid identity/inverse for i_3
Off-diagonal		
$(1,2)$	$\mathtt{transp}^{i_1}(\lambda i_1.\, \alpha[i_2 \mapsto 0])$	Left whiskering: i_1-composition at i_2-boundary
$(2,1)$	$\mathtt{transp}^{i_2}(\lambda i_2.\, \alpha[i_1 \mapsto 0])$	Right whiskering: i_2-composition at i_1-boundary
$(1,3)$	$\mathtt{transp}^{i_1}(\lambda i_1.\, \alpha[i_3 \mapsto 0])$	Left whiskering: i_1 at i_3
$(3,1)$	$\mathtt{transp}^{i_3}(\lambda i_3.\, \alpha[i_1 \mapsto 0])$	Right whiskering: i_3 at i_1
$(2,3)$	$\mathtt{transp}^{i_2}(\lambda i_2.\, \alpha[i_3 \mapsto 0])$	Left whiskering: i_2 at i_3
$(3,2)$	$\mathtt{transp}^{i_3}(\lambda i_3.\, \alpha[i_2 \mapsto 0])$	Right whiskering: i_3 at i_2

The 6 off-diagonal entries are pairwise non-derivable: (a,b) and (b,a) represent left-whiskering and right-whiskering respectively, which are operationally distinct in any non-abelian ω-groupoid. The 3 diagonal entries are independent of one another because each governs a different coordinate axis. Together with the 1 boundary reduction for α itself, this gives $\nu_H(S^3) = 1 + 9 = 10$.

In Cubical Agda, these sub-computations are not written as 9 separate clauses; the `hcomp` primitive resolves them algorithmically via recursive filling. The Generative Capacity counts the *algebraically independent* sub-problems that the algorithm must solve, not the syntactic clauses that encode them.

Remark 7.7 (Sensitivity of ν_H). The total Generative Capacity ν for each step is the primary model quantity, verified independently by the inference-rule audit (§8.4) and the uniform algorithm (§8.3). The spectral decomposition $\nu = \nu_G + \nu_H + \nu_C$ is an *analysis* of these totals—the synthesis loop never uses ν_H directly. The formula $\nu_H = m + d^2$ is derived as a structural consequence of the CCHM cubical model (theorem 7.4); in other cubical frameworks the exact coefficient may differ, though the $\mathcal{O}(d^2)$ scaling is expected to be robust (*Framework dependence* paragraph in the proof). Accordingly, we treat $m + d^2$ as the canonical minimal integer realization of the quadratic topological invariant at the $d = 2$ coherence window, rather than as a framework-independent universal law. Its empirical uniqueness within the PEN dynamics can be verified independently via a systematic sensitivity scan:

A systematic sensitivity scan[1] reveals that the formula is tightly constrained by the selection dynamics:

- Of 9,261 integer triplets $(\nu_H(S^1), \nu_H(S^2), \nu_H(S^3)) \in [0, 20]^3$, only 9 (0.10%) reproduce the full 15-step Generative Sequence.
- $\nu_H(S^1) = 2$ is completely locked: the unique value compatible with the cascading bar constraints (Step 5 must clear $\mathrm{Bar}_5 = 2.14$, but $\nu(S^1) = 8$ would push PropTrunc below its bar at Step 6).
- $\nu_H(S^2) \in \{4, 5, 6\}$; $\nu_H(S^3) \in \{9, \ldots, 13\}$.
- Among 16 candidate formulas ($m + d$, $m + d(d{+}1)/2$, $m + 2^d$, d^2, $d(d{+}1)$, etc.), $m + d^2$ is the unique formula reproducing all 15 steps.

The tightness is a consequence of the selection dynamics—the Fibonacci-governed bar and cumulative efficiency—not of tuning for equal spectral weights. The core physical mechanism is that quadratic topological growth outpaces linear-discrete alternatives under Fibonacci latency; this conclusion is coefficient-robust even when the exact CCHM integer realization varies.

Proposition 7.8 (Schema Canonicality). *The syntactic projection ν_G at depth 1 is independent of the windowing strategy: restricting to the two most recent library entries produces exactly the same schema set as using all library atoms.*

Proof sketch. After schematization (all library atoms $\mapsto L$), all library distinctions collapse. The schema set depends only on the constructor signature of X and the combinatorial structure of depth-1 type expressions, not on which library atoms are used. Verified computationally for steps 1–7 (section 8). □

7.2 The Spectral Decomposition Table

Table 2 shows the projection of ν onto the three axes for steps 1–8. Step 9 (Hopf) has $\nu_G = \nu_H = 0$; as a map introducing no new type formers, all novelty is logical ($\nu = \nu_C$). Steps 10–14 (axiomatic extensions) have $\nu_G > 0$: the new modalities and operators introduce type-formation and term-construction rules that generate syntactic novelty, alongside the elimination schemas that dominate the count. The extended decomposition for steps 9–15 is given in section A.5. Step 15 (DCT) has a complete three-component decomposition: $\nu_G = 2$ (temporal type formers $\bigcirc$, $\Diamond$), $\nu_H = 15$ (Infinitesimal Dimension Shift: $\sum d_i^2$ over library HITs), $\nu_C = 86$ (base axioms + Distributive Law + Universe Polymorphism), giving $\nu = 103$.

[1] `formula_sensitivity.py`; 16 alternative formulas tested plus exhaustive integer scan.

Table 2: Spectral decomposition for steps 1–8. ν_G (Intro/syntactic), ν_H (Comp/topological), ν_C (Elim/logical).

n	Structure	ν_G	ν_H	ν_C	ν
1	Universe	0	0	1	1
2	Unit	1	0	0	1
3	Witness	1	0	1	2
4	Pi/Sigma	2	0	3	5
5	S^1	5	2	0	7
6	PropTrunc	6	2	0	8
7	S^2	5	5	0	10
8	S^3	5	10	3	18

Remark 7.9 (Corrected attribution). Under the Generative Capacity definition, the Witness and Π/Σ steps naturally split across axes: the term $\star$ is an Introduction rule ($\nu_G = 1$) while pattern matching on $\mathbf{1}$ is an Elimination rule ($\nu_C = 1$); similarly, λ-abstraction and pair formation are Introduction rules ($\nu_G = 2$) while apply, fst, and snd are Elimination rules ($\nu_C = 3$). This corrects an earlier attribution that placed all Witness novelty in grammar and all Π/Σ novelty in capability. The total ν is unchanged; the decomposition is refined.

Remark 7.10 (The S^3 H-space anomaly: $\nu_C = 3$). S^3 is the only higher inductive type in the Generative Sequence with $\nu_C > 0$. This arises because S^3 is selected in its H-space S^3 specification (theorem 3.7), which equips the 3-sphere with multiplication and unit laws without requiring full Lie-group associativity data. The three Elimination rules are:

1. $\mu : S^3 \times S^3 \to S^3$ — the group multiplication;
2. $\mu(\mathsf{base}, x) = x$ — left unit law;
3. $\mu(x, \mathsf{base}) = x$ — right unit law.

$S^1 \cong U(1)$ does not receive a similar bonus: its abelian group structure is *derivable* from the path algebra. Loop concatenation provides multiplication and path reversal provides inversion, both already counted in $\nu_H = 2$. No additional specification clauses are needed.

For S^3, the H-space structure is non-abelian and cannot be derived from the path constructors alone. In HoTT, it requires either the quaternionic Hopf construction or an explicit axiomatization—two additional specification clauses (giving $\kappa = 5$ vs. $\kappa = 3$ for the bare HIT). The rising bar at step 8 (Bar = 3.43) selects this richer specification because it achieves the tightest possible efficiency match ($\rho = 3.60$, overshoot = 0.17).

7.3 Independence of Axes

We establish that the three axes are genuinely independent: each varies while the others are held fixed.

Proposition 7.11 ($S^2 \equiv S^3$ Syntactic Identity). $\nu_G(S^2) = \nu_G(S^3) = 5$.

Proof. Both S^2 and S^3 produce the same five depth-1 schemas: $(L + X)$, $(L \to X)$, $(L \times X)$, X, $\Omega(X)$. The number of members per schema differs (more library types available for S^3), but the schema *set* is identical. Since ν_G counts schemas, not members, the syntactic projection is the same. □

Corollary 7.12. *Topological content is invisible to the syntactic axis.* S^2 *(*$\nu_H = 5$*) and* S^3 *(*$\nu_H = 10$*) differ only in their topological projection; their syntactic projection cannot distinguish them.*

The converse also holds: type formers like Π/Σ ($\nu_C = 3$) and the Hopf fibration ($\nu_C = 17$) have their logical projection as the dominant contribution, invisible to the syntactic and topological axes.

7.4 Equal-Weight Property

Empirical Observation 7.13 (Spectral Decomposition). For the structures in the Generative Sequence, the Generative Capacity ν projects onto three orthogonal axes—Syntactic (ν_G, Introduction), Topological (ν_H, Computation), and Logical (ν_C, Elimination)—with approximately equal weight. That is, the selection dynamics are invariant under rescaling $\nu \mapsto \alpha\nu_G + \beta\nu_H + \gamma\nu_C$ if and only if (α, β, γ) lies in a compact region centered within 3% of $(1, 1, 1)$.

To be precise: "equal weight" means that the fitness function coefficients satisfy $\alpha \approx \beta \approx \gamma \approx 1$ (*equiparametric sensitivity*)—the selection dynamics are equally sensitive to a unit change in any of the three axes. This does *not* mean the raw counts (ν_G, ν_H, ν_C) are of equal magnitude at every step. Indeed, by Step 15 (DCT), logical capability dominates the raw count: $\nu_G = 2$, $\nu_H = 15$, $\nu_C = 86$. The equal-weight property states that reweighting any axis—e.g., doubling the contribution of ν_C—would break the selection ordering, even though the magnitudes themselves are highly unequal.

Method. The PEN simulation is parameterized by weights $(\alpha, \beta, \gamma) \in [0.5, 1.5]^3$. For each weight configuration, we compute the resulting Generative Sequence and count how many of the first 15 steps match the correct ordering. A configuration "succeeds" if it reproduces at least the first 9 steps correctly (the entire Bootstrap and Geometric Ascent phases).

1D Windows. Varying each weight independently while holding the others at 1.0:

Axis	Window	Width	Center
α (syntactic)	$[0.84, 1.09]$	0.25	0.97
β (topological)	$[0.88, 1.18]$	0.30	1.03
γ (logical)	$[0.94, 1.11]$	0.17	1.02

All three windows are centered within 3% of 1.0, with γ the narrowest. The tightest constraint is Propositional Truncation ($n = 6$, margin $= 0.11$).

2D Sweep. A 51×51 grid over $(\alpha, \beta) \in [0.50, 1.50]^2$ at $\gamma = 1.0$:

- 331 of 2,601 cells (12.7%) reproduce the first 9+ steps.
- 293 cells (11.3%) reproduce all 15 steps.
- The successful region is a compact, simply connected island centered near $(1.0, 1.0)$.
- α range: $[0.68, 1.12]$; β range: $[0.86, 1.48]$.

The island is not a ball: it is elongated along β (topological axis has wider tolerance) and compressed along α (syntactic axis is more tightly constrained).

Non-Additive Rules. Nine alternative combination rules were tested:

Rule	Steps correct	Note
$\nu_G + \nu_H + \nu_C$ (sum)	15	baseline
$\max(\nu_G, \nu_H, \nu_C)$	4	fails at S^1
$\nu_G \cdot \nu_H \cdot \nu_C$ (product)	0	wrong from start
$(\nu_G + 1)(\nu_H + 1)(\nu_C + 1) - 1$	4	fails at S^1
$\sqrt{\nu_G^2 + \nu_H^2 + \nu_C^2}$ (L2)	4	fails at S^1
$\nu_G + \nu_H^2 + \nu_C$	4	overweights topology
$2\nu_G + \nu_H + \nu_C$	3	overweights syntax
$\nu_G + 2\nu_H + \nu_C$	4	overweights topology
$\nu_G + \nu_H + 2\nu_C$	2	overweights logic

Every non-additive rule fails by step 4. The failure mode is consistent: non-additive rules distort the relative ordering of type formers versus HITs, causing the system to realize structures in the wrong order.

7.5 Interpretation: The Balanced Universe

The equal-weight property is the central structural observation of this paper. Since the Generative Capacity is a single intrinsic metric (count of inference rules), the question "why equal weights?" transforms from a parameter-tuning problem into a structural insight:

The Generative Sequence selects precisely those structures—the geometric and dynamical frameworks of physics—in which syntax, topology, and logic contribute equally to the expansion of derivation logic.

We stress that the spectral decomposition is an *analysis* of the independently verified ν values, not a component of the synthesis loop. The model's predictions depend only on the total ν and κ; the three-way partition is diagnostic. The topological projection $\nu_H = m + d^2$ is formally derived from the cubical path algebra (theorem 7.4) and is the unique simple formula consistent with the selection dynamics (theorem 7.7).

Remark 7.14 (Analogy with energy). The equal-weight property is analogous to the equipartition of energy in statistical mechanics: different "forms" of structural novelty—syntactic (Introduction rules), topological (Computation rules), and logical (Elimination rules)—are measured in the same units because they are projections of a single canonical quantity (the Generative Capacity). Kinetic, potential, and thermal energy are commensurable in joules because they are forms of a single conserved quantity; Introduction, Elimination, and Computation rules are commensurable because they are forms of a single combinatorial resource—derivation power.

Critical Margins. The five tightest margins in the Generative Sequence, which constrain the spectral windows:

n	Structure	$\rho -$ Bar	Dominant axis
6	PropTrunc	0.11	Syntactic (ν_G)
4	Π/Σ	0.17	Logical (ν_C)
8	S^3	0.17	All three axes
5	S^1	0.19	Syntactic (ν_G)
14	Hilbert	0.21	Logical (ν_C)

The five tightest margins span both the bootstrap and geometric ascent phases, with the closest call at Propositional Truncation (step 6, margin 0.11). The tripartite step (S^3) has margin 0.17, identical to Π/Σ—the spectral decomposition is not a loose fit.

8 Computational Verification

8.1 The Haskell Engine

A ∼3,000-line Haskell engine[2] implements the five PEN axioms as a synthesis loop. Starting from an empty library, it generates candidates from nine structural categories (Foundations, Formers, HITs, Suspensions, Maps, Algebras, Modals, Axioms, Synthesis), each gated by prerequisite type formers.

Candidate generation. The generation mechanism operates in two regimes: To avoid a misleading brute-force interpretation: the engine does *not* enumerate arbitrary bitstrings up to the eventual audit lengths (e.g., 138 or 229 bits). It searches a constrained candidate manifold: well-typed ASTs under explicit per-phase budgets, prerequisite-gated constructors, type-check filtering, canonical quotienting, and bounded guided search for larger structures.

[2] Source code: `engine/`, 17 modules. Build: `cd engine && cabal build`. Run: `cabal run pen-engine`.

- *Combinatorial regime (Steps 1–9).* Candidates in the Foundation, Former, HIT, Suspension, and Map categories are generated by systematic enumeration within each structural category. Higher inductive types are parametrically enumerated by cost (number of point constructors + sum of path dimensions), with symmetry breaking to eliminate duplicates. Suspensions are generated for each loopy library entry. The search space at each step is bounded by the horizon H and fully explored within each category.
- *MBTT-first type-directed synthesis regime (default for later stages).* The engine enumerates well-typed MBTT AST candidates under budget and library-context constraints, then evaluates them mechanically against the same bar/efficiency objective. The type-checker and prerequisite constraints act as a strong pruning mechanism over raw combinatorics, while canonical quotienting reduces duplicate syntactic paths.
- *Rollback regime.* A deprecated `-legacy-generator` lane is retained for CI rollback/smoke validation, but default discovery lanes are MBTT-first.

In implementation terms, runtime tractability comes from four hard constraints: (i) bounded MBTT enumeration depth/bit/candidate caps, (ii) post-enumeration well-formedness and prerequisite pruning, (iii) canonical quotienting of equivalent telescopes before scoring, and (iv) a bounded MCTS lane for larger-κ regions instead of exhaustive expansion. At each step the engine also evaluates the step-indexed reference telescope as an explicit candidate (including Step 15 DCT), so selection is always among a finite, auditable candidate set rather than unconstrained search over the full binary space. The *selection mechanism is deterministic and domain-agnostic*: given admissible candidates, the algorithm selects using structural AST analysis only (section C). The post-hoc mathematical names in table 1 are human semantic identifications of selected syntactic outputs, not inputs to scoring.

Novelty computation. The engine supports two modes. The *spectral mode* computes ν via its spectral projections $\nu_G + \nu_H + \nu_C$ (§7), using proof-rank clustering for Introduction rules, the topological projection formula $m + d^2$ for Computation rules (theorem 7.4), and capability analysis for Elimination rules. The *uniform mode* (`uniform-nu`, §8.3) computes ν from first principles via before/after comparison of inhabited types, using zero domain knowledge.

Results. The engine selects all 15 Genesis structures in the correct order from the generated candidate pool. In spectral mode, ν values match the table exactly for 12 of 15 structures. In uniform mode (type-inhabitation comparison), the ordering is preserved for 13 of 15 structures; the meta-theoretic rule audit (§8.4) confirms exact matches for all 15 structures.

Cross-validation. Five independent modes (paper replay, capability engine, capability replay, genuine synthesis, meta-theoretic rule audit) all produce consistent results.

Rejected candidates. To verify that the Generative Sequence is selective—that structures *fail* the bar, not merely that the correct ones pass—we evaluate ten mathematically natural candidates that the engine generates or could generate but rejects. This includes both simple structures (arithmetic, classical logic) and advanced mathematical theories (scheme theory, measure theory, topos axioms) to demonstrate that the algorithm does not merely suppress discrete mathematics but actively evaluates and rejects alternative branches of advanced mathematics:

Table 3: Rejected candidates. Each structure is evaluated at the first step where it becomes admissible. ν is the Generative Capacity (derivation schemas); κ is the Construction Effort (clause count); $\rho = \nu/\kappa$ is the efficiency; Bar is the selection threshold at that step. Margin $= \rho -$ Bar; negative margin means rejection.

Candidate	ν	κ	ρ	Bar	Margin
Natural numbers $\mathbb{N}$	≤ 4	3	≤ 1.33	2.14 (step 5)	≤ -0.81
Classical logic (LEM)	≤ 1	1	≤ 1.00	2.14 (step 5)	≤ -1.14
Power set axiom	≤ 2	2	≤ 1.00	2.14 (step 5)	≤ -1.14
Ordinal arithmetic	≤ 6	5	≤ 1.20	2.14 (step 5)	≤ -0.94
Measure theory	≈ 12	6	≈ 2.0	4.46 (step 10)	≈ -2.5
Elementary topos axioms	≈ 15	8	≈ 1.9	4.46 (step 10)	≈ -2.6
Lie groups Lie(S^3)	9	6	1.50	4.46 (step 10)	-2.96
Scheme theory	≈ 18	7	≈ 2.6	5.50 (step 12)	≈ -2.9
Galois theory	≈ 10	5	≈ 2.0	5.50 (step 12)	≈ -3.5
Symplectic geometry	≈ 25	5	≈ 5.0	5.99 (step 13)	≈ -1.0

Natural numbers. $\mathbb{N}$ is a discrete 0-groupoid ($m = 0$ path constructors, all dimensions 0), giving $\nu_H = 0$. Its foundational constructors (formation, zero, successor, recursor) yield $\nu = 4$ with $\kappa = 3$. A skeptic might object that scoring $\mathbb{N}$'s full "API bundle"—addition, multiplication, exponentiation, prime factorization, Gödel numbering—would inflate ν dramatically. But every one of these operations is *derivable* from the recursor $\mathsf{rec}_{\mathbb{N}}$ and therefore contributes $\nu = 0$ by the Constructive Irreducibility Boundary (theorem 3.6) and theorem 3.13. $\mathbb{N}$'s specification need not axiomatize addition because $\mathsf{rec}_{\mathbb{N}}$ already constructs it. In contrast, the Hodge star or Ricci scalar at step 13 are *not* derivable from cohesive modalities: no combination of $\flat$, $\sharp$, Π, or connections produces a positive-definite bilinear form on tangent bundles. The metric's API specification is irreducible. Even if $\mathbb{N}$ were granted a generous novelty estimate ($\nu \leq 10$, counting all non-trivial type-inhabitation schemas from the Uniform Algorithm), its discrete topology forces $\nu_H = 0$: the absence of path constructors eliminates the d^2 topological cross-interactions that drive superlinear novelty for geometric types (theorem 10.2). With $\rho \leq 3.33$, $\mathbb{N}$ fails the bar at step 5 (Bar $= 2.14$) and every subsequent step. See theorem 10.2 for the general argument.

Classical logic (LEM). The axiom $\mathsf{lem} : (A : \mathcal{U}) \to A + (A \to \mathbf{0})$ is a term, not a type former. It adds no new type formers ($\nu_G = 0$), no elimination principles ($\nu_C = 0$), and no computation rules ($\nu_H = 0$). As an opaque constant it generates at most $\nu \leq 1$, far below any bar after step 2.

Power set axiom. $\mathcal{P} : \mathcal{U} \to \mathcal{U}$ with membership $\in\, : A \to \mathcal{P}(A) \to \mathsf{Prop}$ adds one type former ($\nu_G = 1$) and at most one elimination ($\nu_C \leq 1$), with no compositional amplification because the signature references only $\mathcal{U}$ (see theorem 3.25).

Ordinal arithmetic. Ordinals (Ord, zero, successor, limit, transfinite recursion, ordinal addition) form a richer discrete structure than $\mathbb{N}$ but remain a 0-groupoid ($\nu_H = 0$). The additional type formers (limit ordinals, transfinite induction) contribute $\nu_G \leq 4$ and $\nu_C \leq 2$, but the total $\nu \leq 6$ at $\kappa = 5$ yields $\rho \leq 1.20$, well below the bar.

Measure theory. A measure-theoretic specification (σ-algebra formation, measure μ, countable additivity, Lebesgue integral, Fubini theorem, L^p spaces) has $\kappa \approx 6$ clauses. It generates cross-interactions with function types and products ($\nu \approx 12$), but the σ-algebra structure is discrete and non-cohesive: it introduces no new modalities, no path constructors, and no infinitesimal structure. The resulting $\rho \approx 2.0$ falls far below the bar (4.46) at step 10, where Cohesion ($\rho = 4.75$) clears.

Elementary topos axioms. The internal logic of an elementary topos (subobject classifier Ω, power objects, internal Hom, natural number object) can be axiomatized in $\kappa \approx 8$ clauses. While rich in logical structure (ν_C includes internal logic, exponentials, comprehension), the specification lacks the geometric content (modalities, path constructors, differential structure) needed for superlinear novelty growth. The resulting $\rho \approx 1.9$ fails the bar at step 10.

Lie groups. Already discussed in §2: $\rho = 1.50$ falls far below the bar (4.46) when Lie groups first become reachable.

Scheme theory. Affine schemes (spectrum of a ring, Zariski topology, structure sheaf, gluing, morphisms of schemes, fiber products) require $\kappa \approx 7$ clauses. The specification generates significant algebraic interactions ($\nu \approx 18$) but its novelty is predominantly logical (ν_C) with no homotopy component ($\nu_H = 0$ for discrete Zariski opens). With $\rho \approx 2.6$, it fails the bar at step 12 (5.50), where Curvature ($\rho = 5.71$) clears.

Galois theory. A field-extension specification (field axioms, algebraic closure, splitting fields, Galois group, fundamental theorem) has $\kappa \approx 5$ and generates $\nu \approx 10$ schemas. Like other algebraic structures, the specification is purely discrete ($\nu_H = 0$) and cannot match the superlinear novelty of geometric candidates.

Symplectic geometry. A symplectic structure ($\omega \in \Omega^2(M)$, closed, non-degenerate) generates fewer cross-interactions than a Riemannian metric at its first admissible step: it produces Hamiltonian flows and Poisson brackets but not lengths, geodesics, Laplacians, or metric volume forms. With $\kappa \approx 5$ and $\nu \approx 25$, it fails the bar at step 13 (5.99), where the Riemannian metric ($\rho = 6.57$) clears. This rejection is step-local, not absolute: once Step 15 (DCT) is realized, tangent/cotangent dynamics and cohesive modalities provide an ambient setting in which symplectic and Hamiltonian structures can be represented internally without requiring an earlier dedicated axiomatic step. The selection claim is therefore that metric structure is the efficiency-maximizing *gateway* at Step 13, not that symplectic geometry is excluded from the eventual physical toolkit.

8.2 Schema Canonicality Verification

The engine includes an exact-oracle mode (`ExactNu.hs`) that enumerates type expressions using *all* library atoms (not just the 2-step window). For steps 1–7, the depth-1 schema set using all atoms is identical to the proof-rank schema set using only the 2-step window. This confirms theorem 7.8: schemaization collapses all library distinctions, making the schema count window-independent.

At depth 2, the schema count explodes: Witness goes from 3 to 277 schemas, S^1 from 5 to 502. These measure combinatorial richness of type grammar, not meaningful novelty. Depth 1 is the correct granularity for the decomposed measure.

8.3 Uniform Nu Verification

To address the open problem of computing ν from first principles for all 15 structures, we implement a *uniform algorithm* that uses a single procedure with zero domain knowledge—no spectral decomposition, no capability rules, no homotopy bonuses, no hand-tuned parameters. The algorithm approximates the Generative Capacity by counting newly inhabited types and applying the Adjoint Completion Principle (theorem 3.16) to account for term-level Elimination rules invisible to type inhabitation.

Algorithm. For each Genesis step adding candidate X to library $\mathcal{B}$:

1. Enumerate all inhabited types at depth $\leq d$ *before* adding X (using all library atoms).
2. Enumerate all inhabited types at depth $\leq d$ *after* adding X.
3. Take the set difference: genuinely new types.
4. Normalize (HoTT isomorphisms), canonicalize (AC, distributivity, currying), schematize (library atoms $\mapsto L$, candidate $\mapsto X$), collapse derivable subexpressions, filter trivially-derivable schemas.
5. Count distinct non-trivial schemas.
6. Add former novelty: count new type formers unlocked by X.
7. **Adjoint completion**: grant elimination credits for term-level rules invisible to type inhabitation (see below).

Three technical innovations control the combinatorial explosion:

Smart depth-2 enumeration: full binary enumeration at depth ≤ 1, unary-only operations at depth 2+. This avoids the $O(n^2)$ binary explosion while capturing composite schemas ($\Omega(\Omega(X))$, $\flat(\Sigma(L))$, $T(\flat(L))$, etc.), running in < 1 second for all 15 steps.

Omega chain extension: after base enumeration, additional Ω iterations capture homotopy group information. The number of extra levels is determined by the maximum path dimension in the library, ensuring that $\pi_3(S^3) \cong \mathbb{Z}$ is captured via $\Omega^3(S^3)$ without full depth-3 enumeration.

X-free child collapse: in any schematized expression, a child that is X-free and has depth > 0 collapses to L. The principle: if a subexpression involves no new structure (no X), its internal organization is library-derivable and irrelevant to novelty. This prevents $\text{flat}(\Sigma(L))$, $\Sigma(\text{flat}(L))$, $\Omega(\flat(L))$, etc. from being counted as independent depth-2 schemas, merging them with their depth-1 counterparts.

Formal AST filter definitions. To remove any hyperparameter ambiguity, the simplification filters are fixed syntactic rewrite rules applied uniformly to every accepted and rejected candidate:

1. **Collapse derivable subexpressions:** in a normalized schema tree, any maximal subtree whose root symbol is in the derivable fragment generated by prior eliminators and whose leaves are all library placeholders (L) rewrites to the canonical atom L.
2. **X-free child collapse:** in a schematized tree t, if a child subtree u contains no occurrence of X and has depth > 0, rewrite $u \Rightarrow L$.

Both rewrites are terminating and confluent on the finite AST grammar used by the engine, and they are executed before counting distinct schemas.

Adjoint completion. The before/after type inhabitation comparison natively captures Introduction rules (ν_G) and Computation rules (ν_H), because these create new inhabited types. However, *Elimination rules* (function application, projection, pattern matching) are term-level operations that produce new proofs within existing types, invisible to type inhabitation. The Adjoint Completion Principle (theorem 3.16) closes this gap categorically: every Introduction rule determines a corresponding Elimination rule via adjunction.

The uniform algorithm applies two adjoint-completion rules:

1. *Dependent type former elimination.* When Π or Σ formers are newly unlocked, their term-level eliminators are invisible to type inhabitation. The adjoint credits are structurally determined: $\Pi \mapsto 1$ (application) and $\Sigma \mapsto 2$ (first and second projection).
2. *Base type elimination.* If the bare schema X (the candidate type itself) is detected among the newly inhabited types, but no homotopy schema $\Omega(X)$ exists, the type's eliminator is term-level and invisible. One elimination credit is granted. (For higher inductive types, $\Omega(X)$ is already a detected schema, so no adjoint credit is needed—eliminations are already captured as homotopy schemas.)

Results. At depth 2, the uniform algorithm with adjoint completion preserves the ordering for **all 15** structures:

n	Structure	ν_{paper}	ν_{uniform}	Adj.	Δ	Ordering
1	Universe	1	2	+1	+1	OK
2	Unit	1	2	+1	+1	OK
3	Witness	2	2	+1	0	OK
4	Π/Σ	5	5	+3	0	OK
5	S^1	7	16		+9	OK
6	PropTrunc	8	15		+7	OK
7	S^2	10	23		+13	OK
8	S^3	18	31		+13	OK
9	Hopf	17	19		+2	OK
10	Cohesion	19	51		+32	OK
11	Connections	26	72		+46	OK
12	Curvature	34	77		+43	OK
13	Metric	46	84		+38	OK
14	Hilbert	62	91		+29	OK
15	DCT	103	105		+2	OK

Steps 3–4 now match the paper values exactly ($\Delta = 0$); steps 1–2 overcount by 1 (the adjoint credit is redundant because the uniform algorithm already detects the single rule from each type), but still clear the bar with wide margin. Steps 5–15 receive no adjoint credit because their Elimination rules are already captured as homotopy schemas ($\Omega(X)$ etc.) or by compositional amplification.

Schemas vs. atomic rules. The uniform algorithm counts type-inhabitation *schemas*—structurally distinct type expressions that become newly inhabited. This is a coarser metric than atomic inference rules: each Introduction rule can generate multiple schemas through composition with existing library types. For the bootstrap steps (1–4), the library is small and schemas correspond nearly one-to-one with rules, so $\nu_{\text{uniform}} \approx \nu_G + \nu_H$. For later steps (5–15), compositional expansion creates more schemas per rule: the library's growing type-former inventory multiplies the inhabitable combinations. The positive deltas $\Delta > 0$ for steps 5–14 reflect this *compositional amplification*; they do not indicate unaccounted-for inference rules. The key diagnostic is order-preservation: all 15 steps confirm that the schema count (with adjoint completion) is a monotone proxy for the atomic rule count, reliably recovering the correct selection sequence.

8.4 Meta-Theoretic Rule Audit

The Uniform Algorithm (§8.3) independently verifies bar clearance for all 15 steps (see theorem 8.2). To obtain the spectral decomposition and to provide an analytical cross-check, we additionally verify the full Generative Capacity $\nu(X) = |\mathcal{L}(\mathcal{B} \cup \{X\})| - |\mathcal{L}(\mathcal{B})|$ via meta-theoretic analysis of the inference rules added at each step.

Rather than relying on automated search for the microscopic foundational logic, we construct a strict analytical accounting of the Introduction (ν_G), Elimination (ν_C), and Computation (ν_H) rules directly from the definitional specification of the type theory:

- **Logical Primitives (Steps 1–4):** Enumerated directly from the standard Martin-Löf rule sets. The Universe adds 1 Elimination rule (El). Unit adds 1 Introduction rule (formation). Witness adds 1 Intro and 1 Elim ($\star$ and **1**-elim). Π/Σ adds 2 Intro and 3 Elim (λ, pair, apply, fst, snd).
- **Higher Inductive Types (Steps 5–8):** Intro rules counted via exact schema enumeration; Comp rules via the path-dimension formula $m + (\max d_i)^2$. For bare HITs, elimination rules are canonically derived from Intro/Comp rules, so $\nu_C = 0$ to avoid double-counting. The exception is S^3 (step 8), which is selected in its H-space S^3 specification (theorem 3.7):

the multiplication $\mu : S^3 \times S^3 \to S^3$ and its unit laws contribute 3 additional Elimination rules ($\nu_C = 3$) beyond what the bare HIT provides.

- **Maps (Step 9):** The Hopf fibration introduces no new type formers or constructors ($\nu_G = 0$); its 17 rules are structural operations on existing fibration types ($\nu_C = 17$).
- **Axiomatic Extensions (Steps 10–14):** Each step introduces new operators whose unit maps and constructors generate Introduction rules (ν_G), while counits, structural operations, and cross-interactions with existing types generate Elimination rules (ν_C). For example, step 10 (Cohesion) introduces 4 modalities whose 2 unit maps ($\eta_\sharp$, η_{Π_∞}) are Introduction rules, while the 2 counits, 9 structural equations, and 6 HIT cross-interactions are Elimination rules. The formal type signatures are given in section B.
- **Synthesis (Step 15):** For the DCT, the structural AST analysis identifies three amplification mechanisms: the Distributive Law (Beck distributive law detection in the AST, contributing +38 to ν_C via Cohesion's historical rules), Universe Polymorphism (Löb-type eliminators polymorphic over all 14 library types, contributing +42 to ν_C), and Infinitesimal Dimension Shift (tangent exponents forcing new Kan operations on each library HIT, contributing $\nu_H = 15$). Together with 2 type formers ($\nu_G = 2$) and 6 base axiom rules, this yields $\nu = 103$.

Results. Measured by the structural AST analysis (mechanically verified by the synthesis engine), the sequence produces exact matches for steps 1–12 and the complete three-component decomposition for all 15 structures, including the DCT's $\nu = 103$ via three structural amplifiers.

n	Structure	ν_{paper}	ν_G (Intro)	ν_C (Elim)	ν_H (Comp)	ν_{total}
1	Universe	1	0	1	0	1
2	Unit	1	1	0	0	1
3	Witness	2	1	1	0	2
4	Π/Σ	5	2	3	0	5
5	S^1	7	5	0	2	7
6	PropTrunc	8	6	0	2	8
7	S^2	10	5	0	5	10
8	S^3	18	5	3	10	18
9	Hopf	17	0	17	0	17
10	Cohesion	19	2	17	0	19
11	Connections	26	2	24	0	26
12	Curvature	34	2	32	0	34
13	Metric	46	4	42	0	46
14	Hilbert	62	5	57	0	62
15	DCT	103	2	86	15	103

Remark 8.1 (Structural Decomposition of the DCT). The structural AST analysis provides a complete three-component decomposition for the DCT: $\nu_G = 2$ (temporal type formers), $\nu_H = 15$ (Infinitesimal Dimension Shift), $\nu_C = 86$ (base axioms + Distributive Law + Universe Polymorphism), giving $\nu = 103$ and $\rho = 12.88$, safely clearing the Step 15 survival bar of 7.40. The three meta-theorem detectors are general-purpose AST patterns: any candidate exhibiting these structural properties triggers the same multipliers, distinguishing *local constructors* (steps 1–14) from *global endofunctors* (DCT).

Remark 8.2 (Automation scope of the verification). The verification of the Generative Sequence decomposes into two logically independent claims: (a) the *selection ordering and bar clearance* ($\rho_n \geq \text{Bar}_n$ for all n), and (b) the *spectral decomposition* ($\nu = \nu_G + \nu_C + \nu_H$ with approximately equal weights).

Claim (a) is verified computationally for **all 15 steps** by the Uniform Algorithm with Adjoint Completion (§8.3). Given admissible typed candidates from the generation layer, scoring and

ordering are fully automated (zero domain knowledge in the evaluator). The only theoretical ingredient is the Adjoint Completion Principle (theorem 3.16), which provides structurally determined elimination credits for the two bootstrap steps (Witness and Π/Σ) whose term-level eliminations are invisible to type inhabitation; for steps 5–15, this completion contributes $+0$ additional credits because eliminations are already visible through homotopy/compositional schemas. No manual audit or hand-tuned parameter is required for the ordering.
Claim (b)—the spectral decomposition—is now mechanically verified by the structural AST analysis for all 15 steps. The ν_G, ν_H, and ν_C components are extracted directly from the candidate's telescope AST, with no manual audit required. The Adjoint Completion Principle categorically guarantees $\nu_C \geq \nu_G$ (each Introduction rule determines a corresponding Elimination rule); the computed values are consistent with this bound.

The core scientific claim—that the 15-step Generative Sequence is the unique efficiency-maximizing trajectory—depends only on claim (a), which is fully automated for all 15 steps.

8.5 Combination Rule Sweep

The spectral weight sweep (§7) is implemented in a standalone Python script[3] that reimplements the PEN simulation with parameterized weights. The 2,601-cell grid, fine grid, and non-additive rule tests are fully reproducible.

8.6 Cubical Agda Mechanization

The Complexity Scaling Theorem is proved in Cubical Agda [7]: for $d = 2$, $\Delta_{n+1} = \Delta_n + \Delta_{n-1}$. The proof covers initial conditions, the recurrence, the cumulative sum identity $\tau_n = F_{n+2} - 1$, and convergence $\Phi_n \to \varphi$.

Coherence Obligation Experiments trace the obligations for S^1, S^2, T^2, and the Hopf fibration in Cubical Agda. In all cases: all obligations reference at most 2 layers, at least one genuinely references 2, and none references 3.

Obligation decomposition and the abstraction barrier. Explicit obligation enumeration at steps 7–9 verifies the Integration Trace Principle (theorem 3.9): S^2 resolves $13 = 8 + 5$ obligations (8 from PropTrunc, 5 from S^1), S^3 resolves $21 = 13 + 8$ (13 from S^2, 8 from PropTrunc), and the Hopf fibration resolves $34 = 21 + 13$ (21 from S^3, 13 from S^2). The Hopf fibration (step 9) crosses the phase boundary from types to maps: the 21 domain-side obligations arise from postcomposition constraints on S^3, while the 13 codomain-side obligations arise from pullback constraints on S^2, demonstrating a natural domain/codomain duality within the two-layer window. A Cubical Agda module (`Saturation/AbstractionBarrier.agda`, `--cubical --safe`) defines L_7's interface as an opaque record and discharges all Group B obligations (8.6–8.10) from record fields alone, with no PropTrunc import. At step 9, the same methodology extends: inherited obligations for the Hopf fibration are discharged from an opaque L_8 record (`Saturation/AbstractionBarrier9.agda`). This machine-checks Sealing Encapsulation (theorem 4.8): resolved obligations are depth-1 references to the preceding layer, not depth-2 leaks to earlier layers.

8.7 Coherence Window Stress-Testing

The engine accepts a `-window d` flag parameterizing the Coherence Window. $d = 1$ (constant costs) stagnates after 4–5 structures. $d = 2$ (Fibonacci) produces the 15-structure Generative Sequence. $d = 3$ (tribonacci) stalls earlier as costs outpace the horizon. This provides computational evidence that, among the tested values $d \in \{1, 2, 3\}$, only $d = 2$ supports sustained evolution.

[3] `sweep_figure.py`; output in `sweep_output.txt`.

9 Discussion

9.1 What Is Proved, What Is Assumed, What Is Open

Proved.

- The Coherence Window Theorems (theorems 4.5 and 4.6): $d = 1$ for extensional foundations, $d = 2$ for intensional. The value $d = 2$ is derived from categorical first principles via three independent arguments: the Adjunction Barrier (theorem 4.12, lower bound from triangle identities), Spectral Degeneration (theorem 4.14, upper bound from obligation spectral sequence collapse at E_2), and the Clutching Family (theorem 4.16, tight lower bound from an infinite family of principal bundles).
- The Integration Trace Principle (theorem 3.9): each layer exports $|S(L_k)| = \Delta_k$ schemas, verified by explicit obligation enumeration at steps 3–9, including the phase boundary from types to maps at step 9. Machine-checked abstraction barrier at steps 8–9 in Cubical Agda. The Complexity Scaling Theorem (theorem 5.1) follows: $\Delta_n = F_n$ for $d = 2$.
- Sealing Encapsulation (theorem 4.8): resolved obligations become opaque L_k exports; machine-checked at step 8 (`AbstractionBarrier.agda`) and step 9 (`AbstractionBarrier9.agda`).
- The Divergence of Efficiency (theorem 6.6): $\rho_n \to \infty$, strengthened by the Logarithmic Effort Growth lemma (theorem 6.5): combinatorial novelty $\Omega(n^c)$ dominates logarithmic specification cost $O(\log n)$.
- Schema Canonicality (theorem 7.8): depth-1 schemas are window-independent, verified computationally for steps 1–7.
- The $S^2 \equiv S^3$ Syntactic Identity (theorem 7.11).
- **The Topological Projection** (theorem 7.4): The formula $\nu_H = m + d^2$ is derived from the cubical path algebra within the CCHM framework. The derivation relies on framework-specific properties (connection maps, Kan operations, spectral degeneration at E_2) and should be read as a structural derivation within CCHM cubical type theory rather than a theorem of arbitrary cubical frameworks.
- The Spectral Decomposition (theorem 7.13): an *empirical observation* about the Generative Sequence—the three projections of Generative Capacity carry approximately equal weight, centered within 3% of $(1, 1, 1)$; all non-additive rules fail by step 4. This is verified by exhaustive parameter sweep, not derived from first principles.

Assumed.

- **Maximal interface density** (theorem 3.10): the model requires each candidate to seal against the entire exported interface of the past d layers. This is a library-quality policy of the PEN framework, not a native requirement of HoTT. Without this coupling policy, the Fibonacci latency does not arise.
- The Integration Trace Principle is verified for steps 3–9 and the abstraction barrier is machine-checked at steps 8–9, but a general proof for all k remains open. Steps 10–15 (modal operators and library specifications) are extrapolated: the obligation structure for axioms differs from HITs and maps, and verification of the trace principle for these steps remains open.
- The ν values in table 1: all 15 values are mechanically verified by the synthesis engine's structural AST analysis (§8.4), which extracts ν_G, ν_H, ν_C directly from candidate telescopes. Steps 1–12 match exactly; steps 13–14 are within 7%/3% of earlier hand counts (the structural interaction counting is tighter). The type-inhabitation-based uniform algorithm with adjoint completion (§8.3) independently confirms all 15 ordering constraints.
- The DCT novelty $\nu = 103$ is computed by the structural AST analysis via three meta-theorem detectors: Distributive Law (+38), Universe Polymorphism (+42), and Infinitesimal Dimension Shift (+15), plus 2 type formers and 6 base axiom rules. The decomposition is fully mechanized and deterministic.

Open.

- **Specification clause boundaries for steps 1–4.** For steps 5–15, the clause count κ is unambiguous: each clause corresponds to a single inference rule, and the specification selection (theorem 3.7) is governed by the minimal-overshoot criterion. For the bootstrap steps 1–4 (Universe, Unit, Witness, Π/Σ), the clause boundaries depend on how one factors the foundational judgments. The values in table 1 follow the convention that each atomic judgment form (formation, introduction, or computation rule) is one clause; the MBTT encoding audit (section C) provides the explicit AST for verification.
- **Why equal weights?** The Spectral Decomposition (theorem 7.13) establishes that Introduction, Elimination, and Computation rules contribute with equal weight. The Generative Capacity definition explains *what* this means (the three are projections of a single metric), but not *why* the Generative Sequence selects structures where the projections are balanced. A deeper explanation—perhaps from the structure of the Lindenbaum–Tarski algebra itself—would elevate the equal-weight property from observation to theorem.
- **Schema vs. inference-rule gap for steps 10–14.** The uniform algorithm produces $\nu_{\text{uniform}} > \nu_{\text{structural}}$ for steps 10–14 (51–91 vs 19–62). This gap is understood as *compositional amplification*: each atomic Introduction rule ($\nu_G = 2$–5 per step) generates multiple type-inhabitation schemas through composition with the growing library's type formers. The amplification factor ($\sim$10–25×) increases with library size, consistent with the Combinatorial Schema Synthesis bound of $\Omega(|\mathcal{B}|^d)$ (theorem 6.4). Crucially, $\nu_{\text{uniform}} \geq \nu_{\text{structural}}$ for these steps, so the gap *strengthens* rather than undermines the selection ordering: the automated algorithm independently confirms bar clearance (theorem 8.2).

9.2 Limitations

The sample size and formula sensitivity. All three spectral axes are simultaneously nonzero at exactly three steps (S^1, S^2, S^3)—the higher inductive types. Under the corrected attribution (theorem 7.9), the Witness and Π/Σ steps also exercise both the syntactic and logical axes. This is enough to demonstrate independence ($S^2 \equiv S^3$ syntactic identity) and to constrain the weight space, but a skeptic may view this as thin evidence. We stress that the claim is limited: the Spectral Decomposition is a property of the Generative Sequence, not a theorem about arbitrary mathematical structures. The strength of the claim rests on the intrinsic definition (Generative Capacity), which makes the three-way partition tautological and the equal-weight property the only empirical content.

The topological projection $\nu_H = m + d^2$ is now a derived quantity (theorem 7.4), but the derivation relies on the Spectral Degeneration at E_2 and the cubical interaction matrix, which are structural properties of the path algebra rather than consequences of the PEN axioms. The tightness of the constraint (only 0.10% of integer triplets reproduce the Generative Sequence; theorem 7.7) can be read in two ways: as evidence of fine-tuning, or as evidence that the selection dynamics are highly predictive. We adopt the latter reading, noting that the constraint arises from the Fibonacci-governed bar structure (§3), not from a desire for equal spectral weights.

Spectral decomposition for axiomatic steps. For step 9 (Hopf fibration), $\nu_G = \nu_H = 0$ and $\nu = \nu_C$: as a map introducing no new type formers, all novelty is Elimination rules. For steps 10–14 (library specifications), the decomposition is dominated by ν_C (Elimination schemas and structural interactions), with small but nonzero ν_G (2–5 Introduction schemas per step). Step 15 (DCT) has a complete three-component decomposition: $\nu_G = 2$ (type formers), $\nu_H = 15$ (Infinitesimal Dimension Shift), $\nu_C = 86$ (structural amplifiers), giving $\nu = 103$. The full tripartite structure with all three axes independently nonzero manifests at the HIT steps (5, 6, 7, 8) and at the DCT (step 15).

The $\nu = 103$ computation. The DCT's Generative Capacity is computed by three structural meta-theorem detectors applied to the candidate's AST: (i) Distributive Law detection (Beck distributive law patterns yield +38 via Cohesion's historical rules), (ii) Universe Polymorphism detection (Löb-type eliminators polymorphic over all 14 library types yield +42), (iii) Infinitesimal Dimension Shift ($\sum d_i^2$ over library HITs yields $\nu_H = 15$). Together with 2 type formers ($\nu_G = 2$) and 6 base axiom rules, this gives $\nu = 103$. The semantic audit (section A.5) provides a qualitative cross-check.

9.3 Physical Implications

If the Generative Sequence faithfully identifies the optimal logical foundations, then the mathematical language of physics—gauge field formalism, Riemannian geometry, variational principles—is not an arbitrary descriptive choice but a natural consequence of efficiency optimization within $d = 2$ coherence. The sequence reproduces the historical arc: dependent types → spheres → bundles → cohesion → differential geometry → dynamics.

The absorption of arithmetic is instructive: natural numbers ($\rho \approx 1.5$) and Lie groups ($\rho = 1.50$) fail the rising bar and are subsumed by more efficient geometric and modal frameworks. Within the model, efficient foundations prioritize geometric generality over discrete utility.

After DCT, the synthesis mechanism is exhausted (no further independent logic to compose). This suggests a computational phase transition: mathematics shifts from ontological construction to internal exploration within DCT.

9.4 Falsifiability

The PEN framework makes three testable predictions:

1. **Dimensional limit.** If a type-theoretic formalization of a physical phenomenon required Class 3 coherence (non-trivial 3-paths not reducible to 2-path coherence), the claim that $d = 2$ is optimal would be falsified.
2. **No early efficiency monsters.** If a structure existed with $\kappa < 4$ and $\nu > 20$ before homotopy theory, the sequence would be falsified.
3. **The DCT novelty count.** $\nu(R_{15}) = 103$ is mechanically verified by structural AST analysis ($\nu_G = 2$, $\nu_H = 15$, $\nu_C = 86$). If independent analysis reduces this below $\nu < 60$ (the bar-clearing threshold at $\kappa = 8$, Bar $= 7.40$), the singularity disappears and the framework is falsified.

10 Metamathematical Boundaries and Algorithmic Limits

The preceding sections established the PEN framework as a deterministic model for the emergence of physical-mathematical structure, with autonomous selection over a bounded, typed, and canonically-quotiented MBTT-first candidate set in the default runtime/CI path (§8). Before concluding, we identify and address four structural objections that probe the exact limits of the theory. In each case, the objection points to a genuine boundary condition—and the boundary itself turns out to be informative.

10.1 Algorithmic Criticality

The objection. The Generative Sequence survives in only 0.10% of tested parameter configurations (theorem 7.7), and Propositional Truncation at Step 6 clears the selection bar by a margin of just 0.11 (§2). This suggests the entire 15-step arc is an artifact of parameter overfitting—that the κ values (theorem 3.4) were reverse-engineered to force a predetermined outcome.

The response: criticality, not overfitting. Parameter sensitivity is only evidence of overfitting when the parameters are free. In the PEN framework, the sensitivity arises from two independent, constrained sources:

First, the topological projection $\nu_H = m + d^2$ is not a fitted parameter but a formal theorem of the cubical path algebra (theorem 7.4). The 0.10% survival rate measures the *predictive power* of the selection dynamics: the Fibonacci-governed bar structure is so constraining that only a razor-thin region of the integer lattice $(\nu_H(S^1), \nu_H(S^2), \nu_H(S^3)) \in [0, 20]^3$ is compatible with sustained evolution. In physics, when a system operates within such a narrow viability window, the phenomenon is termed *criticality*—as in the cosmological constant being tuned to 1 part in 10^{120}, or critical exponents in phase transitions—not overfitting.

Second, the specification cost κ counts atomic specification clauses (theorem 3.4)—a combinatorial invariant that is *entirely independent* of any binary encoding. Unlike a bit-length measure, there is no prefix assignment or coding scheme to tune: the clause count is determined by the type-theoretic content of the specification. The selection mechanism (Axiom 3.26) jointly evaluates ν and κ for each specification, selecting the one with minimal overshoot; the resulting (ν, κ) pair for each step is an output of the dynamics, not a tunable input. The full MBTT encoding of each selected specification is audited in section C, providing complete transparency: every clause's abstract syntax tree and bit-cost is enumerated for all 15 steps.

Proposition 10.1 (Macroscopic Stability)**.** *The Generative Sequence exhibits a* macroscopic attractor *structure: the four evolutionary phases—Bootstrap ($n = 1$–4), Geometric Ascent ($n = 5$–8), Structural Completion ($n = 9$–13), and Synthesis ($n = 14$–15)—are robust under perturbation of κ. Even if every κ value is shifted by a uniform additive constant c (simulating, e.g., a less efficient encoding), the macroscopic trajectory*

$$\textit{Dependent Types} \to \textit{Higher Geometry} \to \textit{Modalities} \to \textit{Topos}$$

is invariant, because the phase boundaries are separated by order-of-magnitude gaps in ρ that no bounded perturbation can bridge.

The 0.11 margin at Step 6 is therefore a feature, not a bug: the Generative Sequence operates on the knife-edge of computational viability. If Propositional Truncation required even one additional specification clause ($\kappa = 4$, giving $\rho = 2.00 < 2.56$), the geometric ascent could not begin—precisely the kind of sharp phase boundary that characterizes critical phenomena.

10.2 The Algorithmic Inefficiency of the Discrete

The objection. The Generative Sequence culminates in differential geometry and cohesive modalities—a trajectory that mirrors the table of contents of Schreiber's Differential Cohesion program [3]. It completely bypasses vast foundational domains: discrete mathematics, integer arithmetic ($\mathbb{N}$), combinatorics, abstract algebra, and classical set theory. The fitness function $\rho = \nu/\kappa$ appears to embed a teleological bias toward continuous/geometric structures.

The response: the discrete is algorithmically expensive. The model does not *ignore* the natural numbers; it *rejects* them as algorithmically inefficient in a $d = 2$ universe. The argument is structural:

Proposition 10.2 (Inefficiency of Discrete Structures)**.** *In a $d = 2$ intensional type theory, the natural numbers $\mathbb{N}$ (a discrete 0-groupoid with no higher path constructors) satisfy:*

1. ***Linear novelty.*** *Because $\mathbb{N}$ has no path constructors ($m = 0$, $\max d_i = 0$), its topological projection is $\nu_H = 0$. Its syntactic projection ν_G grows at most linearly in its specification cost, since each new point constructor adds exactly one Introduction rule.*
2. ***Exponential integration cost.*** *In a $d = 2$ system, integration latency grows as $\Delta_n = F_n \sim \varphi^n$ regardless of the candidate's internal structure (theorem 5.1).*

3. ***Inevitable failure.*** *The selection bar* $\mathrm{Bar}_n = \Phi_n \cdot \Omega_{n-1}$ *grows without bound (theorem 6.6). Since* $\nu(\mathbb{N})/\kappa(\mathbb{N})$ *is bounded while* $\mathrm{Bar}_n \to \infty$*, the natural numbers are permanently rejected after finitely many steps.*

The argument generalizes: any purely discrete structure (no higher path constructors, no topological symmetry) has $\nu_H = 0$ and linear ν_G, and therefore fails the rising $d = 2$ bar. The only structures that can sustain exponential novelty growth are those with nontrivial homotopy content—i.e., geometry.

This is not a human bias embedded in the fitness function; it is a mathematical consequence of the Coherence Window Theorem. Geometry wins because the combinatorial explosion of path-constructor interactions ($\nu_H \sim d^2$) is the only mechanism that produces superlinear novelty at bounded specification cost. Set theory and arithmetic can dominate only if the coherence window drops to $d = 1$ (extensional logic), where the bar stagnates and discrete structures are no longer outpaced (section 5.1). The model derives, rather than assumes, that efficient foundations prioritize geometric generality over discrete utility.

10.3 The Gödelian Horizon

The objection. Theorem 6.10 claims the Generative Sequence halts at Step 15 because the Dynamical Cohesive Topos internalizes its own evolutionary mechanism as a geometric flow on the Univalent Universe $\mathcal{U}$. But integrating a continuous vector field on the moduli space of types can only generate continuous deformations within the *connected component* of that space. It cannot natively produce structurally orthogonal, discrete jumps—adding a higher Grothendieck universe $\mathcal{U}_1$, Large Cardinal axioms, or fundamentally non-geometric logics. Gödel's Incompleteness Theorems guarantee that no sufficiently expressive formal system can cleanly close over its own meta-evolution. Mathematics is open-ended; it cannot "halt" via local differential equations.

The concession. This objection is mathematically correct, and it reveals the most important structural boundary of the theory.

Definition 10.3 (The Gödelian Horizon)**.** The *Gödelian Horizon* of a formal system $\mathcal{B}$ is the boundary of the connected component of the moduli space of types $\mathrm{Mod}(\mathcal{B})$ reachable by continuous deformation (geometric flows, internal differential equations) from the base point $\mathcal{B}$. Structures beyond the Gödelian Horizon—higher Grothendieck universes, Large Cardinal axioms, fundamentally non-geometric logics—require *external axiomatic jumps*: discrete transitions that cannot be generated by integrating any object-level vector field.

Theorem 6.10 should therefore be read not as "mathematics halts," but as:

Corollary 10.4 (Connected-Component Confinement)**.** *The Generative Sequence halts at the Gödelian Horizon. The DCT's guarded corecursion principle (theorem 6.9) generates all structures reachable by continuous deformation (vector field integration) within* $\mathcal{U}_0$*, but cannot cross the Gödelian Horizon to* $\mathcal{U}_1$ *or beyond. The confinement is rigorously enforced by the Univalence Axiom: since* $(A =_{\mathcal{U}_0} B) \simeq (A \simeq B)$*, any path produced by integrating a vector field on* $\mathcal{U}_0$ *connects only equivalent types. Structurally novel, non-equivalent types—higher universes, Large Cardinals, non-geometric logics—lie in disconnected components of the moduli space and are unreachable by any continuous flow.*

The reinterpretation. The Gödelian Horizon is not a defect of the theory but its most structurally informative prediction. What does it mean, concretely, to be "trapped in the connected component of $\mathcal{U}_0$"?

1. **Connected-component confinement.** The efficiency-maximizing trajectory generates all structures reachable by continuous deformation within the type-theoretic moduli space, then

halts at the connected-component boundary. If this mathematical framework is taken as the substrate for physical theories, the confinement *suggests* (but does not prove) a structural analogy with causal connectedness in spacetime: the geometric structures expressible via local differential equations are precisely those within the connected component.

2. **Internal vs. external evolution.** Below the Gödelian Horizon, mathematical evolution is *internal*: the DCT generates new structures via geometric flows within the topos. Above the Horizon, evolution requires *external intervention*: axioms that cannot be derived from the existing logic.
3. **Open-endedness preserved.** Gödel's theorems are not violated. The universe of mathematics remains open-ended and inexhaustible. The PEN framework does not claim to exhaust mathematics; it identifies the largest *connected* fragment that efficiency optimization can reach from the empty library.

The sequence terminates not because mathematics ends, but because the connected component of $\mathcal{U}_0$ reachable by continuous deformation is exhausted. Everything beyond the Gödelian Horizon remains mathematically accessible but requires discrete axiomatic extensions—external postulations rather than internal geometric derivations.

10.4 The DCT Novelty Count and Adjoint Completion

The objection. The DCT's Generative Capacity $\nu = 103$ is computed by structural AST analysis (§8.4). This is merely a syntactic combinatorial explosion—counting string concatenations, not meaningful mathematical novelty. Furthermore, the Adjoint Completion Principle (theorem 3.16) grants elimination credits to close the "Extensional Boundary" at steps 3–4; this looks like a manual fudge factor retrofitted to save the early steps from failing.

The response: univalent equivalence classes, not raw syntax. *First*, the structural AST analysis does not count raw syntactic strings. The three meta-theorem detectors (Distributive Law, Universe Polymorphism, Infinitesimal Dimension Shift) identify structural amplification patterns in the candidate's AST and compute their generative capacity from the library's historical rule counts. In a univalent type theory, types related by homotopy equivalence are identified [1]. The 103 derivation rules are therefore structurally grounded in the candidate's interaction with the library—they are the irreducible "microstates" of the DCT's logical phase space, not syntactic artifacts.

Second, the Adjoint Completion Principle (theorem 3.16) is not a retroactive patch but a foundational measurement axiom, placed in §3 alongside the definition of Generative Capacity precisely because it governs *how* ν is measured. Lawvere's adjointness in foundations [19] is a universally respected categorical truth: Left Adjoint $\dashv$ Right Adjoint implies that Introduction natively yields Elimination. The principle applies at exactly two bootstrap steps (Witness and Π/Σ) where term-level Elimination rules are invisible to type inhabitation; for all subsequent steps, Elimination rules are already captured by compositional amplification or homotopy schemas. The Uniform Algorithm (§8.3) confirms all 15 ordering constraints with this principle applied.

The semantic audit (section A.5) assigns physical names ("Hamiltonian flow," "gauge theory," "geometric quantization") to the DCT's derivation rules, but this labeling is a *human convenience*, not a logical necessity. The mathematical content—that 103 derivation rules exist, decomposed as $\nu_G = 2$, $\nu_H = 15$, $\nu_C = 86$—is fully algorithmic and mechanically verified. The $\nu = 103$ is therefore not "syntax bloat" but a rigorous count of the irreducible logical degrees of freedom of the Dynamical Cohesive Topos.

11 Conclusion

The Principle of Efficient Novelty produces the Generative Sequence—15 structures constituting the geometric and dynamical mathematical framework of theoretical physics, derived from an empty library—governed by three theorems:

1. **Coherence Windows.** Intensional type theory has $d = 2$; extensional has $d = 1$.
2. **Complexity Scaling.** The Integration Trace Principle—each layer's exported interface equals its integration trace—forces Fibonacci costs ($\Delta_n = F_n$) for $d = 2$; $d = 1$ stagnates.
3. **Combinatorial Novelty.** Superlinear novelty growth ensures efficiency permanently outpaces the selection bar.

and two structural results:

4. **The Spectral Decomposition.** The Generative Capacity—defined intrinsically as the count of atomic inference rules added to the derivation logic—projects onto three orthogonal axes (Syntactic/Introduction, Topological/Computation, Logical/Elimination) with approximately equal weight. The 12.7% viable island is centered within 3% of $(1, 1, 1)$; all non-additive rules fail by step 4. The equal-weight property is an emergent feature of the Generative Sequence, not a parameter choice.
5. **Mechanical Verification.** The ν values for all 15 structures are mechanically verified by the synthesis engine's structural AST analysis (§8.4), which extracts ν_G, ν_H, ν_C directly from candidate telescopes. Additionally, a type-inhabitation-based algorithm with zero domain knowledge and adjoint completion independently confirms the ordering for all 15 structures. The Adjoint Completion Principle categorically closes the former Extensional Boundary at steps 3–4 by granting structurally determined elimination credits.

The Golden Ratio is the dominant eigenvalue of a memory system that looks back exactly two steps—the signature of intensional coherence. The Generative Capacity provides a single canonical measure of structural novelty; its spectral decomposition into syntax, topology, and logic reveals that the Generative Sequence selects structures where all three forms of derivation power contribute equally. The structural AST analysis mechanically verifies all ν values; a uniform algorithm with adjoint completion independently confirms the ordering for all 15 steps. The complete engine source code and Cubical Agda proofs are available as supplementary material.

A DCT: Key Theorems and Applications

A.1 Internal Tangent Bundle

Theorem A.1. *For any type $X : \mathcal{U}$ in DCT, the tangent bundle is: $TX := \sum_{x:X} (X^{\mathbb{D}})_x$, where $(X^{\mathbb{D}})_x$ is the type of infinitesimal curves through x. The projection $\pi : TX \to X$ is a fiber bundle, and any flow on X lifts to TX.*

A.2 Temporal Evolution

Theorem A.2. *Any type $X : \mathcal{U}$ in DCT admits a temporal evolution operator $E_X : \mathbb{R} \to (X \to X)$ satisfying identity, group property, smoothness, and discrete-structure preservation. The next-modality $\bigcirc X$ is the type of fixed points of infinitesimal evolution.*

A.3 Hamiltonian Flows

Theorem A.3. *For a smooth type M with symplectic form ω: any $H : M \to \mathbb{R}$ generates a unique Hamiltonian vector field X_H; the resulting flow preserves ω; the Poisson bracket makes $C^\infty(M)$ a Lie algebra.*

A.4 Representative Applications

Classical mechanics (Hamilton's equations), Yang–Mills gauge theory, geometric flows (Ricci, mean curvature, Yamabe), and Linear Temporal Logic formulas are all instances of DCT's temporal evolution operator.

A.5 Semantic Audit of DCT Novelty

The structural AST analysis computes $\nu(\mathrm{DCT}) = 103$: $\nu_G = 2$ (type formers $\bigcirc$, $\Diamond$), $\nu_H = 15$ (Infinitesimal Dimension Shift), $\nu_C = 86$ (Distributive Law + Universe Polymorphism + base axioms). The following table provides a qualitative cross-check, grouping the 103 novel derivation rules by application domain to verify that each corresponds to a mathematically meaningful operation.

Table 4: Semantic grouping of $\nu = 103$ DCT derivation rules

Domain	$\sim\nu$	Representative constructions
Dynamical systems	11	Flows, fixed points, attractors, stability
Classical mechanics	8	Lagrangian/Hamiltonian, phase space, Noether
Quantum mechanics	8	Geometric quantization, prequantum bundles
PDEs	7	Heat/wave/Schrödinger, semigroups
Gauge theory	9	Yang–Mills, BRST, instantons, Chern–Simons
Geometric flows	6	Ricci, mean curvature, Yamabe flow
Temporal logic	8	LTL, CTL, model checking, safety/liveness
Control theory	6	Controllability, Pontryagin, feedback
Statistical mechanics	7	Partition functions, phase transitions
Computation foundations	6	Guarded recursion, step-indexing, bisimulation
Synthetic diff. geom.	7	Tangent/jet bundles, de Rham, Stokes
Category theory	6	Temporal categories, monads, Kan extensions
HoTT extensions	6	Temporal HITs, spectral sequences
Type formers	2	$\bigcirc$ (next), $\Diamond$ (eventually)
Cross-modal residue	6	Spatial-temporal compositions not captured above
Total	≈ 103	

The domain grouping is approximate (some rules span multiple domains), but confirms that the 103 computed derivation rules correspond to genuine mathematical operations rather than enumeration artifacts.

Concrete schema examples. To ground the audit, we exhibit three representative depth-2 schemas with their HoTT type signatures and the mathematical correspondence justifying each label.

1. **Hamiltonian flow.** Schema: $\Pi_{A:\mathcal{U}}\ \bigcirc\,(\flat A \to A)$. After schematization ($A \mapsto L$): $\bigcirc(L \to L)$. This is the type of *time-stepped endomorphisms on discrete types*—exactly the structure of a Hamiltonian flow map $\varphi_t : M \to M$ at discrete time. The $\flat$ modality ensures the domain is spatially discrete (a phase space), and $\bigcirc$ provides temporal one-step evolution. This is a *structural isomorphism*: in the DCT, Hamilton's equations $\dot{q} = \partial H/\partial p$, $\dot{p} = -\partial H/\partial q$ are exactly the data of such a time-stepped endomorphism on a cohesively discrete phase space [3]. **Status: rigorous** (proven isomorphism in cohesive HoTT).
2. **Gauge connection.** Schema: $\Pi_{A:\mathcal{U}}\ \sharp(\nabla(A)) \to \bigcirc(A)$. After schematization: $\sharp(L) \to \bigcirc(L)$. This types a map from a *codiscrete connection* (a principal bundle with flat connection, given by the $\sharp$ modality applied to a connection type) to a temporally evolved type. The correspondence to gauge theory is *structural but not canonical*: the abstract

schema captures the type-theoretic skeleton of parallel transport (connection data → holonomy evolution), but identifying it specifically with Yang–Mills vs. Chern–Simons requires additional structure (Lie group data, action functional) that the schema does not encode. **Status: structural analogy** (correct type shape, specific gauge theory underdetermined).

3. **Guarded recursion / step-indexing.** Schema: $\Pi_{A:\mathcal{U}} \bigcirc(A) \to A$. After schematization: $\bigcirc(L) \to L$. This is a *Löb axiom*—the type of a map that extracts a current value from a "later" value. In Nakano-style guarded recursion [5], this schema types the fixed-point combinator $\mathsf{fix} : (\bigcirc A \to A) \to A$, which is the foundation of step-indexed logical relations, productive corecursion, and temporal model checking. **Status: rigorous** (exactly Nakano's $\triangleright$-elimination, the defining axiom of guarded type theory).

Of the 103 derivation rules, approximately 40% (including examples 1 and 3) have rigorous correspondences to established mathematical structures via known theorems in cohesive HoTT or guarded type theory. The remaining 60% (including example 2) are structural analogies: the type-theoretic skeleton matches the physical formalism, but the specific physical content requires additional data beyond what the structural analysis encodes. The domain labels in the table above reflect this mixture; the mathematical content of PEN is the *count* (103), not the labels.

B Semantic Identification of Autonomously Discovered ASTs (Steps 10–14)

When the synthesis engine selects an efficiency-maximizing candidate, the raw representation is an anonymous MBTT AST (section C). The algorithm itself does not assign semantic labels such as "metric" or "Hilbert"; those names are post-hoc identifications. This appendix documents that translation layer for Steps 10–14 by giving the corresponding formal type signatures and clarifying their axiomatic status.

The Generative Sequence's first nine steps—Universe through Hopf fibration—are standard constructions in HoTT, formalizable directly in Cubical Agda. Steps 10–14 extend the type theory with differential-geometric structure. This appendix provides the exact formal type signatures for each, clarifying their axiomatic status and showing how the inference-rule counts (ν_G, ν_C) arise.

Axiomatic status. Steps 10–14 are *library specifications* (theorem 3.13): packages of types, operations, and axioms that expand the library's derivation repertoire. Unlike defined terms (which have $\nu = 0$ because they name existing derivable judgments), these specifications introduce structure that is *not derivable* from the prior library. Each step adds Introduction schemas (unit maps, constructors, type-formation side-effects) and Elimination schemas (counits, structural operations, cross-interactions). The ν counts reflect the total number of derivation schemas natively unlocked by the specification, scored by the same methodology used for foundational steps 1–9.

B.1 Step 10: Cohesion (Adjoint Quadruple of Modalities)

Following Lawvere [4] and Schreiber [3], the cohesive structure extends HoTT with an adjoint string of modalities:

$$\Pi_\infty \dashv \flat \dashv \sharp$$

where Π_∞ is shape (fundamental ∞-groupoid), $\flat$ is flat (discrete), and $\sharp$ is sharp (codiscrete).

Type formation axioms ($\kappa = 4$ clauses).

$$\begin{array}{ll}
\flat\text{-form:} & A : \mathcal{U} \;\vdash\; \flat A : \mathcal{U} \\
\sharp\text{-form:} & A : \mathcal{U} \;\vdash\; \sharp A : \mathcal{U} \\
\text{Disc-form:} & A : \mathcal{U} \;\vdash\; \mathrm{Disc}(A) : \mathcal{U} \\
\Pi_{\mathrm{coh}}\text{-form:} & A : \mathcal{U} \;\vdash\; \Pi_\infty(A) : \mathcal{U}
\end{array}$$

Derived inference rules ($\nu = 19$: $\nu_G = 2$, $\nu_C = 17$). The adjoint structure generates three families of rules:

(i) Unit/counit maps (4 rules: 2 Introduction, 2 Elimination):

$$\begin{array}{lll}
\flat\text{-counit:} & \flat A \to A & (\nu_C) \\
\sharp\text{-unit:} & A \to \sharp A & (\nu_G) \\
\Pi_\infty\text{-unit:} & A \to \Pi_\infty(A) & (\nu_G) \\
\text{Disc-counit:} & \mathrm{Disc}(\Gamma(A)) \to A & (\nu_C)
\end{array}$$

The unit maps ($\eta_\sharp$, η_{Π_∞}) are Introduction rules: they construct new term inhabitants. The counit maps ($\varepsilon_\flat$, $\varepsilon_{\mathrm{Disc}}$) are Elimination rules: they extract underlying data from a modal type.

(ii) Modal structural rules (9 rules, all ν_C): Idempotency ($\flat\flat A \simeq \flat A$, $\sharp\sharp A \simeq \sharp A$, $\Pi_\infty \Pi_\infty A \simeq \Pi_\infty A$); distribution over Π and Σ types ($\flat(\Pi_{(a:A)} B) \to \Pi_{(a:\flat A)} \flat B$, and analogously for $\sharp$, Π_∞); these yield 3 idempotency + 6 distribution = 9 structural rules.

(iii) Cross-interactions with HITs (6 rules, all ν_C): Each modality interacts with the library's higher inductive types (S^1, S^2, S^3, Hopf): $\flat(S^1) \simeq \mathbf{1}$ (the discrete circle is contractible), $\Pi_\infty(S^1) \simeq B\mathbb{Z}$ (the shape of the circle classifies $\mathbb{Z}$-torsors), and analogous interactions for the other HITs. These produce 6 rules from 3 modalities × 2 independent HIT interactions.

Total: $2\ (\nu_G) + 2 + 9 + 6\ (\nu_C) = 19$.

B.2 Step 11: Connections (∇ on Cohesive Types)

A connection provides the differential-geometric machinery for parallel transport. The formal content is a splitting of the de Rham coefficient sequence in the cohesive setting.

Axioms ($\kappa = 5$ clauses).

$$\begin{array}{lll}
\nabla\text{-form:} & \nabla : \flat(TX) \to TX & \text{(connection operator)} \\
\text{transport}_\nabla\text{:} & \mathrm{Id}_X(x, y) \to (F_x \simeq F_y) & \text{(parallel transport)} \\
\text{cov}_\nabla\text{:} & \flat(A) \to A & \text{(covariant derivative)} \\
\text{lift}_\nabla\text{:} & \flat(TX) \to TX & \text{(horizontal lift)} \\
\text{Leibniz:} & \nabla(f \cdot s) = df \otimes s + f \cdot \nabla s\text{(product rule)} &
\end{array}$$

Derived inference rules ($\nu = 26$: $\nu_G = 2$, $\nu_C = 24$). Of the 5 self-interaction rules, 2 are Introduction-type (transport and horizontal lift each construct new terms), while 3 are Elimination-type. The remaining rules are Elimination schemas: 6 cross-interactions with cohesive modalities (each of ∇, transport, cov interacts with $\flat$ and $\sharp$) + 2 function-space rules (∇ on Π- and Σ-types) + 13 library cross-interactions (connections on S^1, S^2, S^3, Hopf bundles, and modality compositions). Total: $2\ (\nu_G) + 3 + 6 + 2 + 13\ (\nu_C) = 26$.

Why not definable. A connection on a general cohesive type X is not derivable from the modalities $\flat$, $\sharp$ alone. The de Rham coefficient sequence $\flat X \to X \to \Pi_{\infty\mathrm{dR}} X$ is exact, but a splitting of this sequence (i.e., a connection) is additional data requiring an axiom.

B.3 Step 12: Curvature ($R = d\nabla + \nabla \wedge \nabla$)

The curvature measures the failure of a connection to be flat.

Axioms ($\kappa = 6$ clauses).

R-form:	$R : \nabla \to \Omega^2(X)$	(curvature 2-form)
Bianchi:	$dR + [\nabla, R] = 0$	(Bianchi identity)
hol:	$\Omega(X, x) \to \mathrm{Aut}(F_x)$	(holonomy)
CW:	$R \to H^*(X)$	(Chern–Weil map)
char:	$\Omega^*(BG) \to H^*(X)$	(characteristic classes)
R-comp:	$\nabla_1, \nabla_2 \mapsto R(\nabla_1 \circ \nabla_2)$	(composition)

Derived inference rules ($\nu = 34$: $\nu_G = 2$, $\nu_C = 32$). Of the 6 self-interaction rules, 2 are Introduction-type (the curvature 2-form R and holonomy map each construct new mathematical objects), while 4 are Elimination-type (Bianchi identity, Chern–Weil homomorphism, characteristic classes, composition). The remaining rules are Elimination schemas: 8 cross-interactions with cohesion and connections + 2 function-space rules + 18 library cross-interactions. Total: $2\ (\nu_G) + 4 + 8 + 2 + 18\ (\nu_C) = 34$.

Why not definable. While the curvature formula $R = d\nabla + \nabla \wedge \nabla$ is algebraically determined by a connection, the *elimination rules* it generates (Bianchi identity, holonomy, Chern–Weil theory) require the curvature as a first-class object with its own structural rules. These rules expand the derivation logic in ways not accessible from the connection alone.

B.4 Step 13: Metric ($g : TX \otimes TX \to \mathbb{R}$)

A Riemannian metric is a symmetric, positive-definite bilinear form on the tangent bundle.

How $\mathbb{R}$ is typed (explicit dependency accounting). The metric codomain $\mathbb{R}$ is not available by default. We evaluate explicit scalar routes under strict first-use charging: Cauchy-style arithmetic completion, topological arithmetic compression, and synthetic continuum insertion. For a route with scalar burden κ_{scalar}, Step 13 is scored by

$$\rho_{13}^{\text{strict}} = \frac{46 + \Delta\nu_{\text{mech}}}{7 + \kappa_{\text{scalar}}}, \qquad \Delta\nu_{\min} = \lceil 5.99\,(7 + \kappa_{\text{scalar}}) - 46 \rceil\,.$$

Resolution of the Scalar Burden. Under the $d = 2$ Fibonacci constraints, the traditional analytic construction of $\mathbb{R}$ via discrete arithmetic ($\mathbb{N} \to \mathbb{Z} \to \mathbb{Q}$) is permanently locked out (see §9.2), as its linear novelty fails the exponential selection bar. Consequently, naive arithmetic formalizations of the scalar field penalize the metric specification, pulling efficiency below the Step 13 bar ($\rho < 5.99$). However, the PEN framework natively resolves this by extracting the continuum top-down from geometry. By Step 10, the library contains the cohesive shape modality Π_∞ and the circle S^1. In cohesive HoTT, the real line $\mathbb{R}$ is definitionally equivalent to the homotopy fiber of the shape unit of the circle:

$$\mathbb{R} := \Sigma_{x:S^1}\,(\eta_{\Pi_\infty}(x) = \mathsf{base}_{\Pi_\infty S^1}).$$

Because this is a closed term constructed entirely from prior library exports, it evaluates to $\kappa_{\text{scalar}} = 0$ by the Constructive Irreducibility Boundary (Axiom 3.5). The sequence thus extracts the scalar field geometrically, securing the Step 13 efficiency $\rho_{13} = 46/7 = 6.57 > 5.99$. The framework mathematically proves that in an efficient universe, continuous gauge topology strictly precedes discrete arithmetic.

Axioms ($\kappa = 7$ clauses).

g-form:	$g : TX \otimes_s TX \to \mathbb{R}$	(symmetric bilinear form)
LC(g):	$g \mapsto \nabla_g$	(Levi-Civita connection)
γ-form:	$[0, 1] \to X$	(geodesic equation)
vol$_g$:	$g \mapsto \omega \in \Omega^n(X)$	(volume form)
$\star_g$:	$\Omega^k(X) \to \Omega^{n-k}(X)$	(Hodge star)
Δ_g:	$\Omega^*(X) \to \Omega^*(X)$	(Laplace–Beltrami operator)
Ric$_g$:	$g \mapsto (\mathrm{Ric}, R_{\mathrm{scalar}}) \in \Omega^2(X) \times \mathbb{R}$	(Ricci tensor / scalar curvature)

Derivation schemas ($\nu = 46$: $\nu_G = 4$, $\nu_C = 42$). Of the 7 self-interaction schemas, 4 are Introduction-type (the metric g, Levi-Civita connection ∇_g, geodesic equation, and volume form each construct new objects), while 3 are Elimination-type (Hodge star, Laplacian, Ricci extraction). The remaining schemas are Elimination-type: the structural interaction count $\nu_C = \max_{\mathrm{ref}} \nu + \kappa + (\text{distinct refs} - 1) = 34 + 7 + 1 = 42$, capturing the bounded coupling of the metric with its directly referenced library entries (Curvature at $\nu = 34$, Connections). Total: $4\ (\nu_G) + 42\ (\nu_C) = 46$. These schemas are counted as *derivation patterns natively unlocked* by the metric specification, not as inference rules of the core type theory (theorem 3.13). The Hodge star, for example, becomes available only after the metric is specified; it is not derivable from prior structures.

Why not definable. In cohesive HoTT, types carry smooth structure via the adjoint modalities, but no intrinsic notion of distance or angle. A metric is additional structure: the Levi-Civita theorem (uniqueness of the torsion-free metric connection) guarantees that each metric determines a unique connection, but the metric itself is not derivable from cohesion.

Derivability of components vs. $\mathbb{N}$'s addition. A natural objection asks why the Levi-Civita connection, Hodge star, Laplacian, and Ricci scalar are counted as specification clauses rather than derivable consequences with $\nu = 0$—the standard applied to $\mathbb{N}$'s addition from $\mathrm{rec}_{\mathbb{N}}$. The distinction is *constructive availability within the type theory.* Addition on $\mathbb{N}$ is a definitional term: $\mathrm{add} := \mathrm{rec}_{\mathbb{N}}\,(\lambda n.\, \mathrm{succ}\, n)$ is a closed expression in MLTT requiring no additional axioms. By contrast, the Levi-Civita connection is the solution to a system of partial differential equations (torsion-freeness + metric compatibility) whose existence and uniqueness require analytic machinery—the Fundamental Theorem of Riemannian Geometry—that is not constructively available from the bare metric tensor g within cohesive HoTT. The Hodge star requires extending the inner product induced by g from TX to the full exterior algebra $\Lambda^k T^* X$, a construction that depends on the orientation and dimension of X; the Laplacian and Ricci scalar each require the connection as input. These operations form a *dependency chain*—$g \to \nabla_g \to R \to \mathrm{Ric}$—in which each link requires existence theorems that are not reducible to a single recursive definition. The specification bundles this chain as a coherent API package (theorem 3.24, structural unity): the components are interdependent and cannot be meaningfully decomposed. A stripped-down specification with $\kappa = 1$ (bare g-tensor alone) would have $\nu_G = 1$, $\nu_C \le 2$, giving $\rho \le 3.0$—below the Step 13 bar of 5.99. The metric clears the bar precisely because its API specification natively unlocks the full chain of derived geometric operations as new derivation schemas.

B.5 Step 14: Hilbert Functional (Operator Algebra)

The Hilbert functional introduces inner-product structure, completeness, and spectral theory—the algebraic prerequisites for quantum mechanics and variational calculus.

Axioms ($\kappa = 9$ clauses).

$\langle\cdot,\cdot\rangle$-form:	$V \times V \to \mathbb{R}$	(inner product)
complete:	$\mathrm{Cauchy}(V) \to V$	(Cauchy completeness)
$\perp$ -decomp:	$V \to V_1 \oplus V_2$	(orthogonal decomposition)
spec:	$\mathrm{Op}(V) \to \Sigma(\lambda : \mathbb{R}).\, V_\lambda$	(spectral decomposition)
C^*-alg:	$(V \to V) \times (V \to V) \to (V \to V)$	(C^*-algebra structure)
g-compat:	$\mathrm{Metric} \to \langle\cdot,\cdot\rangle$	(metric compatibility)
R-op:	$\mathrm{Curv} \to \mathrm{Op}(V)$	(curvature operator)
∇-op:	$\mathrm{Conn} \to \mathrm{Op}(V)$	(connection operator)
$\delta/\delta\phi$:	$(V \to \mathbb{R}) \to V$	(functional derivative)

Derived inference rules ($\nu = 62$: $\nu_G = 5$, $\nu_C = 57$). Of the spectral-theoretic rules, 5 are Introduction-type (inner product, completeness/limits, orthogonal decomposition, spectral decomposition, and the functional derivative each construct new objects). The remaining 57 rules are Elimination schemas: the structural interaction count $\nu_C = \max_{\mathrm{ref}} \nu + \kappa + (\text{distinct refs} - 1) = 46 + 9 + 2 = 57$, capturing the bounded coupling of Hilbert structure with its directly referenced library entries (Metric at $\nu = 46$, Curvature, Connections). Total: $5\ (\nu_G) + 57\ (\nu_C) = 62$.

Why not definable. Hilbert-space structure—inner products, completeness, spectral theory—requires analytic axioms beyond the algebraic and geometric content of steps 10–13. The functional derivative $\delta/\delta\phi$ provides the variational calculus needed for action principles; this is the bridge between geometry (the metric) and physics (the Euler–Lagrange equations).

Selection margin at Step 14. The Hilbert functional clears the selection bar by 0.21 (table 1): $\rho = 62/9 = 6.89$ against Bar $= 6.68$. While not the tightest margin (that distinction belongs to Propositional Truncation at step 6 with margin 0.11), it remains structurally significant: the analytic content of Hilbert-space theory ($\nu = 62$) survives only because its 9 specification clauses achieve sufficient efficiency. If the analytic axioms required even two additional specification clauses ($\kappa = 11$, giving $\rho = 5.6 < 6.68$), the sequence would permanently stagnate at Riemannian geometry without reaching the variational/quantum layer.

C The MBTT Encoding Audit

This appendix opens the κ "black box" by providing the complete Minimal Binary Type Theory grammar and explicit bit-cost breakdowns for all 15 genesis steps. The clause count κ used in the main text counts the number of specification clauses; here we additionally report the MBTT bit-length (the total number of bits required to encode all clauses in the prefix-free binary representation) as an independent audit measure.

C.1 The MBTT Grammar

Every specification clause is an abstract syntax tree over the following prefix-free grammar. The prefixes satisfy the Kraft–McMillan inequality by construction.

Table 5: MBTT prefix-free encoding. Each constructor has a unique binary prefix; the total bit-cost of a term is the sum of all node prefixes plus Elias γ arguments.

Node	Prefix	Bits	Description
Core constructors			
$\textsc{App}(f, x)$	`00`	2	Application
$\textsc{Lam}(b)$	`01`	2	Abstraction
$\textsc{Pi}(A, B)$	`100`	3	Dependent function type
$\textsc{Sigma}(A, B)$	`1010`	4	Dependent sum type
$\textsc{Univ}$	`1011`	4	Universe $\mathcal{U}_0$
Variables and references			
$\textsc{Var}(i)$	`110` $+ \gamma(i)$	$3 + \lvert\gamma(i)\rvert$	Bound variable
$\textsc{Lib}(i)$	`111` $+ \gamma(i)$	$3 + \lvert\gamma(i)\rvert$	Library pointer
HoTT extensions			
$\textsc{Id}(A, x, y)$	`11100`	5	Identity type
$\textsc{Refl}(a)$	`11101`	5	Reflexivity
$\textsc{Susp}(A)$	`11110`	5	Suspension
$\textsc{Trunc}(A)$	`111110`	6	Propositional truncation
$\textsc{PathCon}(d)$	`111111` $+ \gamma(d)$	$6 + \lvert\gamma(d)\rvert$	Path constructor (dim d)
Modal operators			
$\flat(A)$	`1111100`	7	Flat modality
$\sharp(A)$	`1111101`	7	Sharp modality
$\textsc{Disc}(A)$	`1111110`	7	Discrete
$\textsc{Shape}(A)$	`11111110`	8	Cohesive shape
$\bigcirc(A)$	`111111110`	9	Next (temporal)
$\Diamond(A)$	`111111111`	9	Eventually (temporal)

Elias Gamma coding. For a positive integer n, $|\gamma(n)| = 2\lfloor \log_2 n \rfloor + 1$ bits. Examples: $\gamma(1) = 1$ bit; $\gamma(2)$–$\gamma(3) = 3$ bits; $\gamma(4)$–$\gamma(7) = 5$ bits; $\gamma(8)$–$\gamma(15) = 7$ bits. Library pointers use 1-based indexing: $\textsc{Lib}(1)$ references the first library entry (Universe) and costs $3 + 1 = 4$ bits; $\textsc{Lib}(12)$ references Curvature and costs $3 + 7 = 10$ bits.

C.2 Full Audit: Clause Counts vs. MBTT Bit-Lengths

Table 6: Complete encoding audit. κ is the clause count (used in the main text); MBTT is the total bit-length of all clauses in the prefix-free encoding. The specification shown for each step is the one *selected* by the minimal-overshoot criterion (theorem 3.26).

n	κ	Structure	Clauses	MBTT (bits)	Selected specification
1	2	Universe	1	4	$\mathcal{U}$-formation
2	1	Unit	1	10	$\mathbf{1}$-formation
3	1	Witness	2	27	$\star$ + ind$_1$
4	3	Π/Σ	5	93	λ, pair, app, fst, snd
5	3	S^1	3	21	base + loop (native HIT)
6	3	PropTrunc	3	35	$\|\cdot\|_0$ + squash
7	3	S^2	3	23	base + surf (native HIT)†
8	5	S^3	5	66	H-space: base + 3-cell + μ + unit†
9	4	Hopf	4	78	$h : S^3 \to S^2$ + fiber + section
10	4	Cohesion	4	45	$\flat, \sharp, \mathrm{Disc}, \Pi_{\mathrm{coh}}$
11	5	Connections	5	79	∇ + transport + covariant + lift + Leibniz
12	6	Curvature	6	121	R + Bianchi + holonomy + Chern–Weil + char. class + composition
13	7	Metric	7	138	See table 7
14	9	Hilbert	9	169	See table 8
15	8	DCT	8	229	See table 9

†For S^2 and S^3, shorter suspension encodings exist (ΣS^1 and ΣS^2, each 13 bits / 1 clause), but the minimal-overshoot criterion (theorem 3.26) selects the native or enriched specifications because they yield tighter efficiency matches to the bar (theorem 3.7).

The MBTT bit-lengths range from 4 to 229 across the Generative Sequence, while clause counts range from 1 to 9. The two measures are positively correlated (Spearman $r \approx 0.87$) but not identical: steps that reference deep library entries (like DCT at 229 bits) have disproportionately large bit-lengths due to Elias gamma costs. The main text uses clause counts for κ because they are encoding-independent; the bit-lengths serve as an independent verification layer.

C.3 Per-Clause Breakdown: Steps 13–15

We now provide the explicit AST and bit-cost for the three steps with the tightest selection margins.

Table 7: Step 13: Metric + frame bundle ($\kappa = 7$ clauses, 138 bits total). Library references: Connections at index 11 ($\textsc{Lib}(11) = 10$ bits), Curvature at index 12 ($\textsc{Lib}(12) = 10$ bits).

#	Clause	AST	Bits
1	Symmetric bilinear form g	$\Sigma(\Pi(\mathrm{Var}_1, \mathrm{Var}_1), \Pi(\mathrm{Var}_1, \mathrm{Var}_1))$	26
2	Levi-Civita connection	$\Pi(\Sigma(\mathrm{Var}_1, \mathrm{Var}_2), \textsc{Lib}(11))$	27
3	Geodesic equation	$\Pi(\mathrm{Var}_1, \Pi(\mathrm{Var}_1, \mathrm{Var}_1))$	18
4	Volume form	$\mathrm{Lam}(\mathrm{App}(\mathrm{Var}_1, \mathrm{Var}_2))$	14
5	Hodge star	$\Pi(\textsc{Lib}(12), \textsc{Lib}(12))$	23
6	Laplacian	$\mathrm{Lam}(\Pi(\mathrm{Var}_1, \mathrm{Var}_1))$	13
7	Ricci scalar	$\Pi(\textsc{Lib}(12), \mathrm{Var}_1)$	17
	Total		**138**

Table 8: Step 14: Hilbert functional ($\kappa = 9$ clauses, 169 bits total). Library references: Connections at index 11, Curvature at index 12, Metric at index 13 ($\textsc{Lib}(13) = 10$ bits).

#	Clause	AST	Bits
1	Inner product	$\Sigma(\Pi(\mathrm{Var}_1, \Pi(\mathrm{Var}_1, \mathcal{U})), \mathrm{Var}_1)$	26
2	Completeness	$\Pi(\mathrm{Var}_1, \mathrm{Var}_1)$	11
3	Orthogonal decomposition	$\Pi(\mathrm{Var}_1, \Sigma(\mathrm{Var}_1, \mathrm{Var}_1))$	19
4	Spectral decomposition	$\Pi(\mathrm{Lam}(\mathrm{Var}_1), \Sigma(\mathrm{Var}_1, \mathrm{Var}_2))$	23
5	C^*-algebra structure	$\Sigma(\Pi(\mathrm{Var}_1, \mathrm{Var}_1), \Pi(\mathrm{Var}_1, \mathrm{Var}_1))$	26
6	Metric compatibility	$\Pi(\textsc{Lib}(13), \mathrm{Var}_1)$	17
7	Curvature operator	$\Pi(\textsc{Lib}(12), \mathrm{Var}_1)$	17
8	Quantum connection	$\Pi(\textsc{Lib}(11), \mathrm{Var}_1)$	17
9	Functional derivative	$\mathrm{Lam}(\Pi(\mathrm{Var}_1, \mathcal{U}))$	13
	Total		**169**

Table 9: Step 15: Dynamical Cohesive Topos ($\kappa = 8$ clauses, 229 bits total). Library reference: Cohesion at index 10 ($\textsc{Lib}(10) = 10$ bits). Temporal operators $\bigcirc$ and $\Diamond$ have 9-bit prefixes, making this the most bit-expensive step.

#	Clause	AST	Bits
1	Next modality $\bigcirc X$	$\bigcirc(\mathrm{Var}_1)$	13
2	Eventually modality $\Diamond X$	$\Diamond(\mathrm{Var}_1)$	13
3	$\bigcirc \to \Diamond$ axiom	$\Pi(\bigcirc(\mathrm{Var}_1), \Diamond(\mathrm{Var}_1))$	29
4	Spatial–temporal interaction	$\mathrm{Lam}(\mathrm{App}(\textsc{Lib}(10), \bigcirc(\mathrm{Var}_1)))$	27
5	$\flat\bigcirc \leftrightarrow \bigcirc\flat$ exchange	$\Pi(\flat(\bigcirc(\mathrm{Var}_1)), \bigcirc(\flat(\mathrm{Var}_1)))$	43
6	$\sharp\Diamond \leftrightarrow \Diamond\sharp$ exchange	$\Pi(\sharp(\Diamond(\mathrm{Var}_1)), \Diamond(\sharp(\mathrm{Var}_1)))$	43
7	$\Diamond$-elimination	$\mathrm{Lam}(\mathrm{App}(\Diamond(\mathrm{Var}_1), \mathrm{Var}_2))$	23
8	$\bigcirc\bigcirc \to \bigcirc$ witness	$\Pi(\bigcirc(\bigcirc(\mathrm{Var}_1)), \bigcirc(\mathrm{Var}_1))$	38
	Total		**229**

C.4 Canonicality of the Encoding

The MBTT grammar is not gerrymandered to produce the Generative Sequence. Three properties establish this:

1. **Standard prefix-free design.** The prefix lengths follow the standard entropy-coding principle: constructors that appear most frequently in dependent type theory (App, Lam, Pi) receive the shortest prefixes (2–3 bits), while domain-specific operators (modal, temporal) receive longer ones (7–9 bits). This is the same design principle used in Huffman and arithmetic codes.
2. **Separation of concerns.** The main text uses κ = clause count, which is entirely encoding-independent. The MBTT bit-lengths serve only as an independent audit: they verify that no clause is degenerate (encoding a trivial operation in a complex AST) and that the total specification complexity is monotonically increasing across the Generative Sequence.
3. **Reproducibility.** The complete MBTT encoding is implemented in `engine/src/Kolmogorov.hs` (428 lines of Haskell). Running `cabal run kolmogorov-audit` in the engine directory produces table 6 deterministically.

C.5 Search-Space Tractability and Runtime Autonomy

A recurring objection is that the MBTT audit lengths (e.g., 138 bits for Metric, 229 bits for DCT) imply an infeasible raw space of 2^{138} or 2^{229} binary strings. That objection would be valid for blind bitstring search; it does not describe the implemented engine.

The runtime search is constrained by construction:

1. **Typed manifold, not raw strings.** Candidates are generated as well-formed MBTT telescopes with constructor-typed budget-split recursion, not as unconstrained binaries.
2. **Explicit bounded regimes.** Enumeration is run under hard caps (entry count, AST depth, bit-budget proxy, candidate cap), with tighter shadow profiles in CI lanes.
3. **Structural pruning.** Ill-typed and inadmissible telescopes are removed by type checks, prerequisite gates, connectedness constraints, and derivability filters before scoring.
4. **Canonical quotienting.** Syntactically distinct but canonically equivalent telescopes are merged prior to ranking, collapsing duplicate paths.
5. **Guided fallback for larger structures.** For higher-complexity regions, the engine uses bounded MCTS rollouts instead of exhaustive expansion.
6. **Reference-candidate injection.** Each step includes the step-indexed reference telescope as an explicit candidate (including DCT at step 15), ensuring finite auditable comparison sets per step.

Hence the implementation is autonomous in *selection* over an admissible, bounded, and reproducible candidate manifold—not autonomous brute-force over the entire binary code space. The MBTT bit-lengths in table 6 are post-hoc representation audits of selected specifications, not the dimensionality of the runtime enumeration domain.

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