Imperial College London

CIVE 40006 Structural Mechanics

Lecture 9 – Plastic theory & collapse mechanisms

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Lecture 9

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- Plastic collapse
- Plastic load factor
- Virtual work
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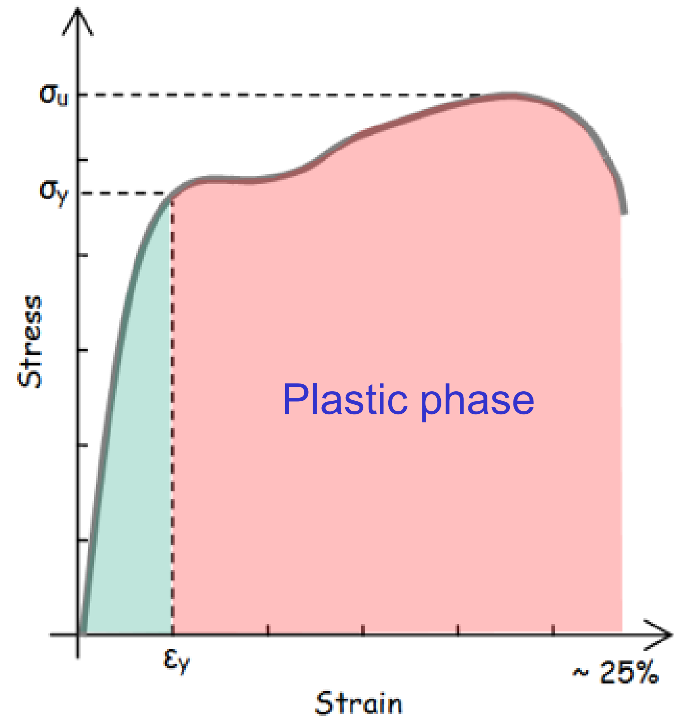
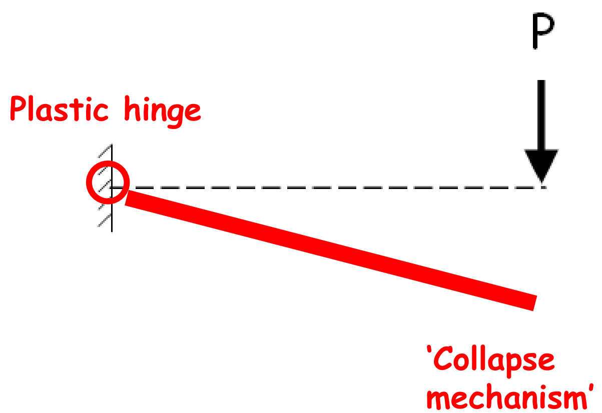
Chapter 5



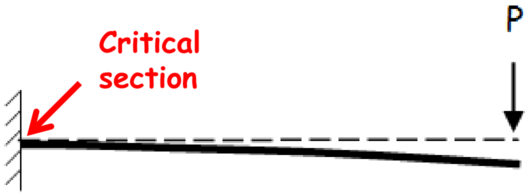


Plastic

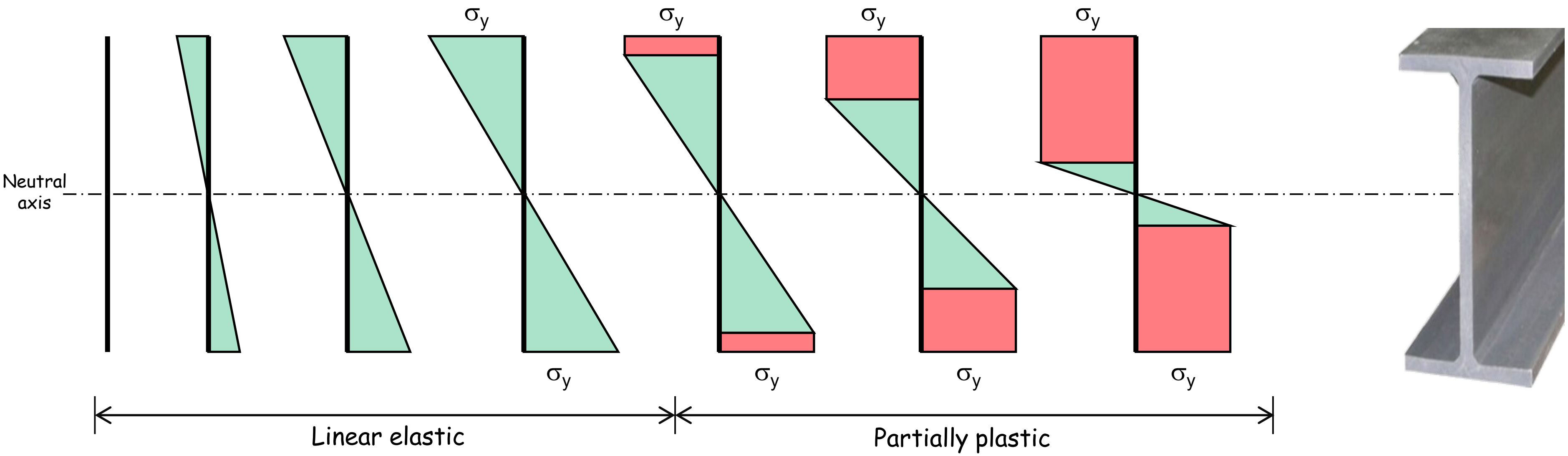
- Extreme loads
- Stress > yield stress (σ_y)
- Non-linear stress-strain behaviour
- Large and permanent deformations
- Ultimate failure

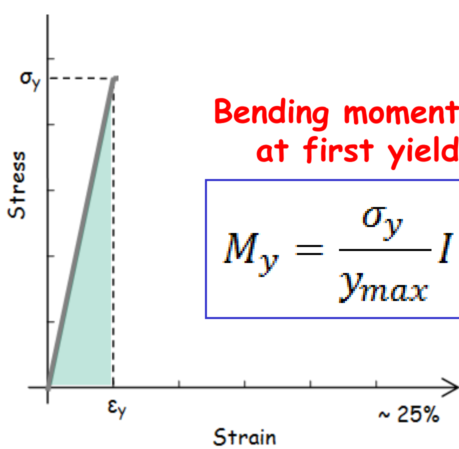
Formation of plastic hinge

(Fig. 5.2, Page 49)



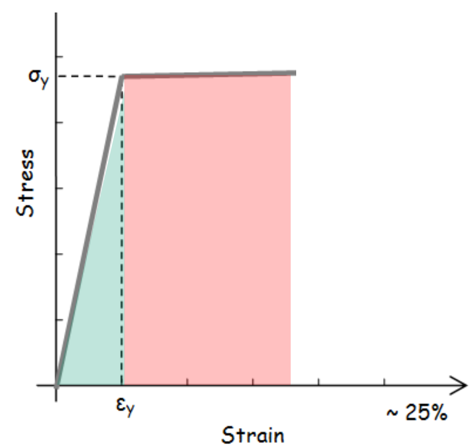
Change in stress distribution at critical section as load P is increased...



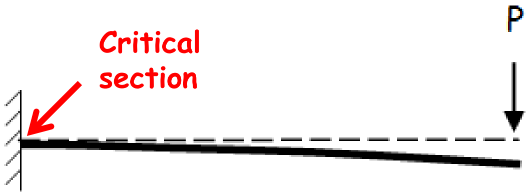


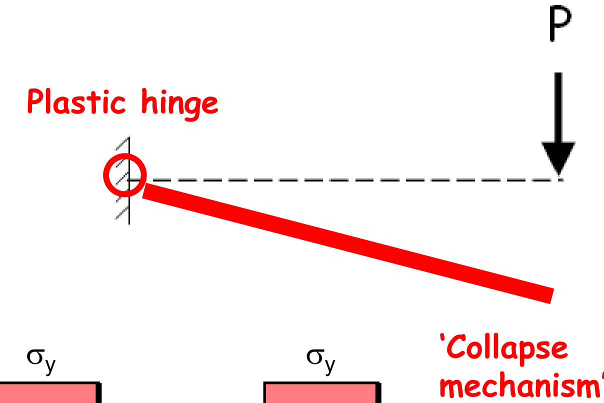
Bending moment at first yield

$$M_y = \frac{\sigma_y}{y_{max}} I$$

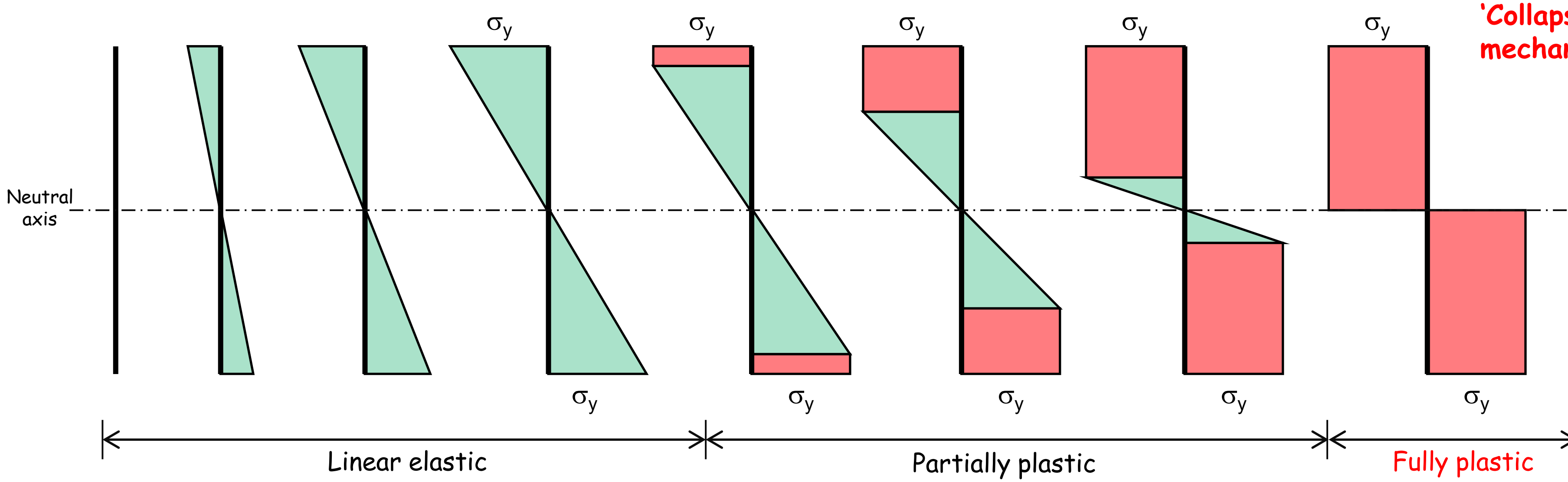


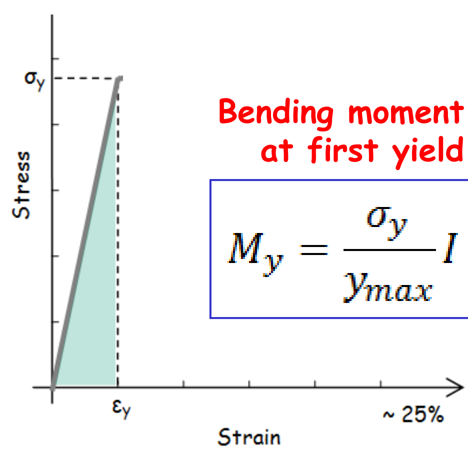
Formation of plastic hinge





Change in stress distribution at critical section as load P is increased...





Bending moment at first yield

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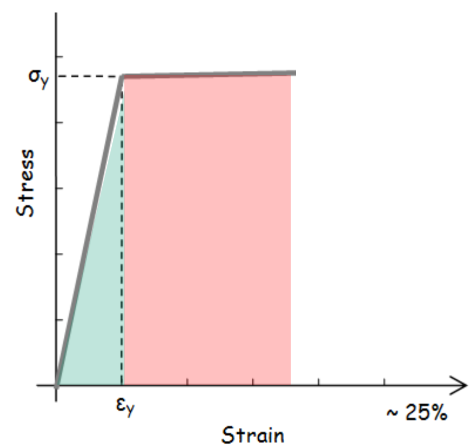
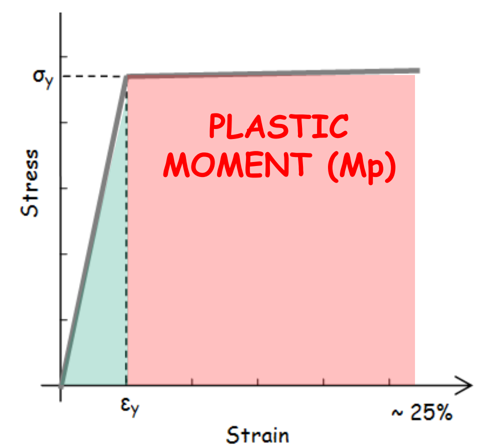
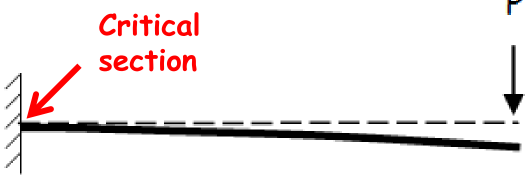
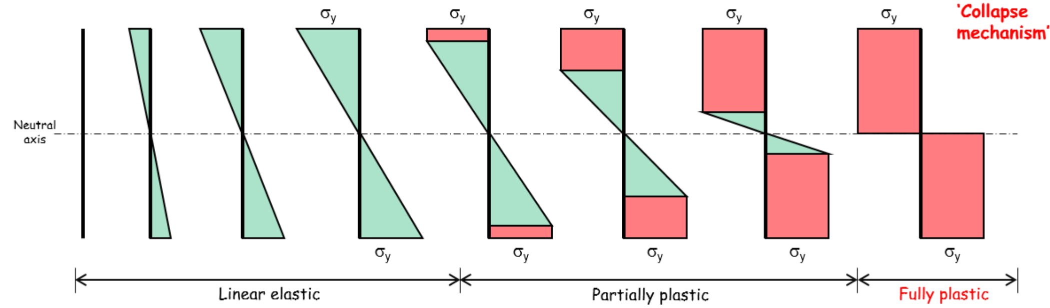
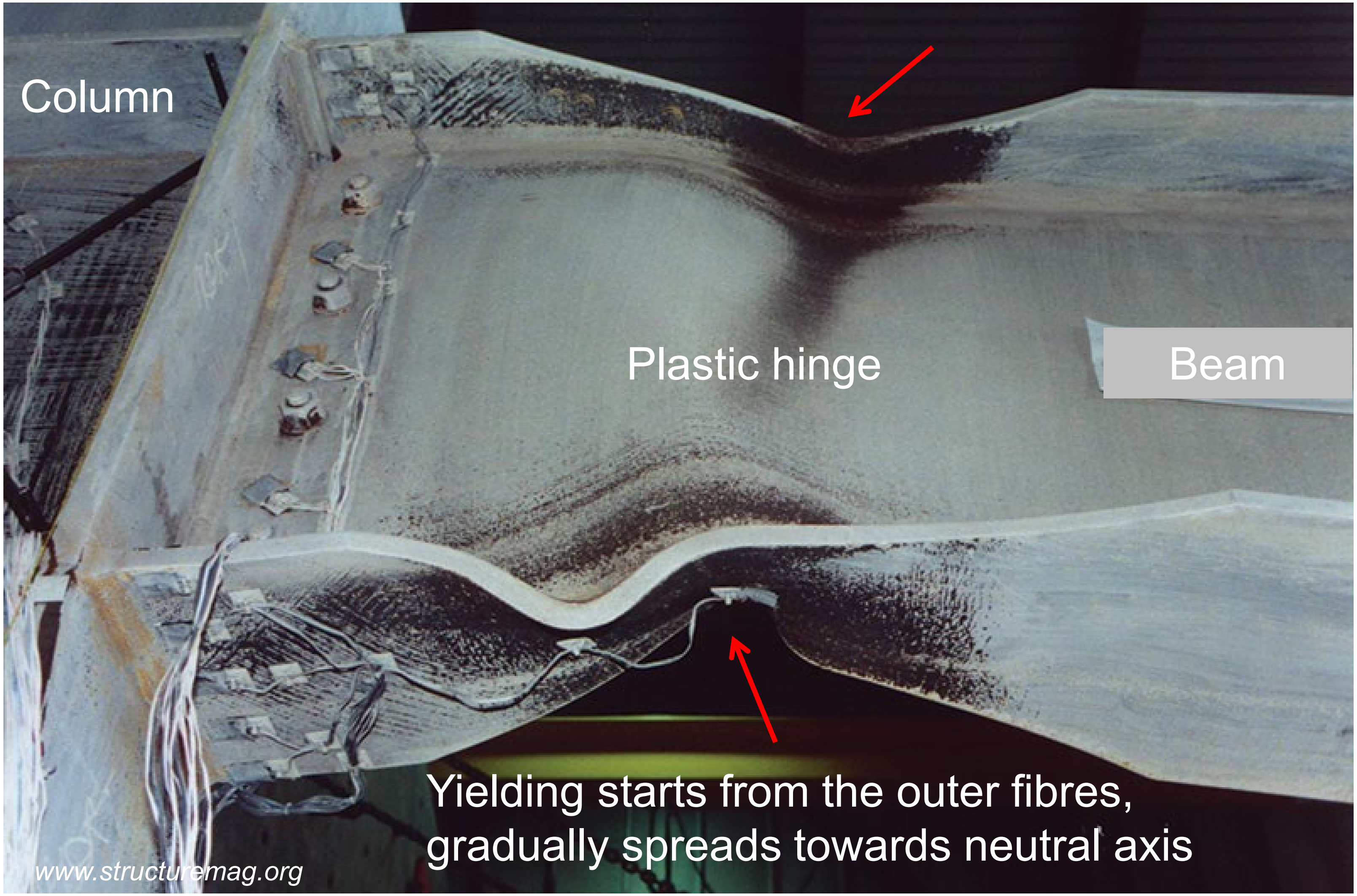





Critical section
P


Neutral axis
σy
'Collapse mechanism'
Linear elastic
Partially plastic
Fully plastic


Column
Plastic hinge
Beam
Yielding starts from the outer fibres,
gradually spreads towards neutral axis
www.structuremag.org

Plastic moment (Revision)

The plastic moment M_p can be found by taking moments of resultant forces about the plastic neutral axis. This is relatively straightforward because we have a uniform tensile stress on one side of the neutral axis and uniform compressive stress on the other.

Taking moments about the plastic neutral axis (Fig 5.3):

$$M_p = C\left(\frac{y_1}{2}\right) + T\left(\frac{y_2}{2}\right) = \sigma_y A_1\left(\frac{y_1}{2}\right) + \sigma_y A_2\left(\frac{y_2}{2}\right) \tag{5.4}$$

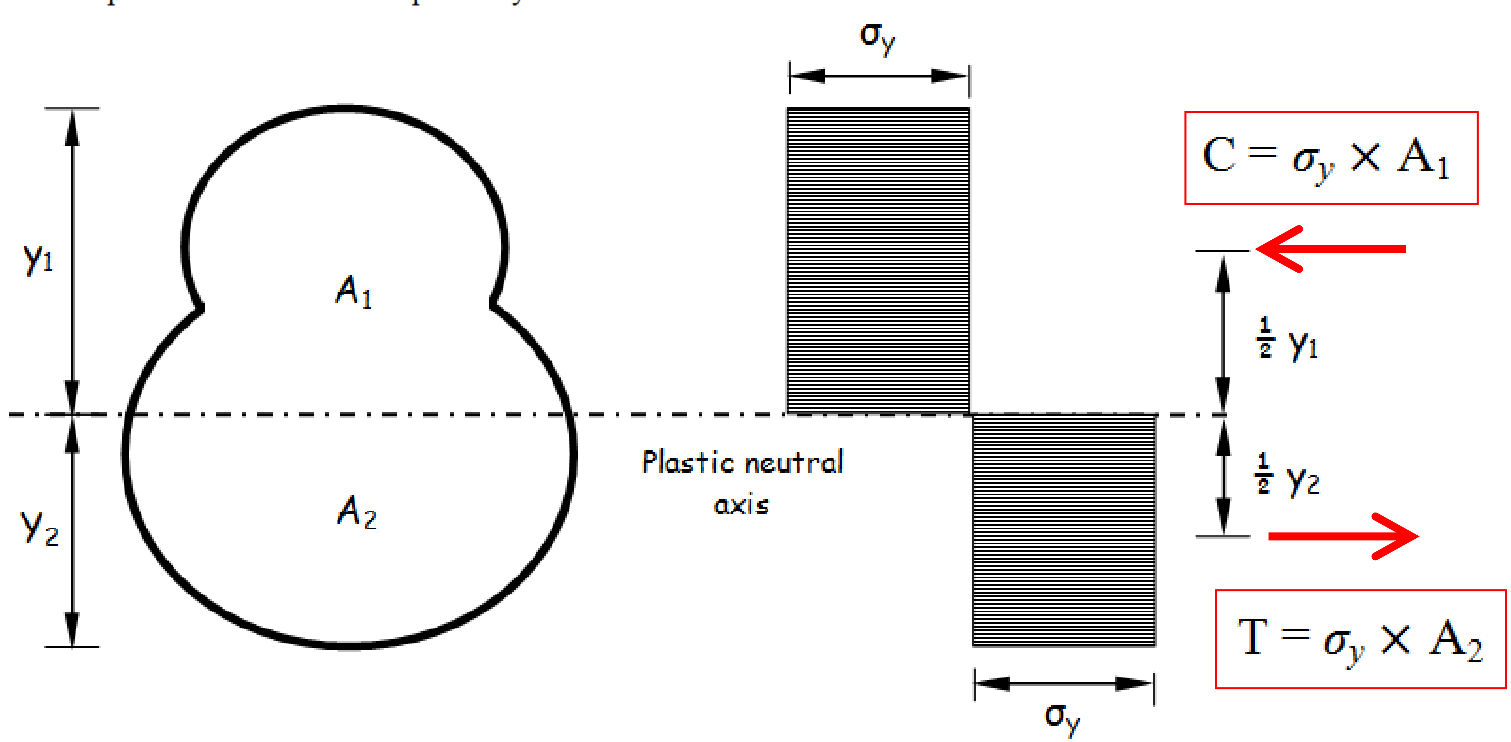
Which can also be expressed as:

$$\boxed{M_p = \sum C_i y_i + \sum T_i y_i} \tag{5.5}$$

Or

$$M_p = \sigma_y \sum (A_{Ci} y_i + A_{Ti} y_i) \tag{5.6}$$

Where y_i is the distance of the resultant forces from the plastic neutral axis, A_{Ci} and A_{Ti} are the regions of the beam in compression and tension respectively.



Plastic moment (Revision)

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Which can also be expressed as:

$$\boxed{M_p = \sum C_i y_i + \sum T_i y_i}$$

Or $$M_p = \sigma_y \sum (A_{Ci} y_i + A_{Ti} y_i)$$

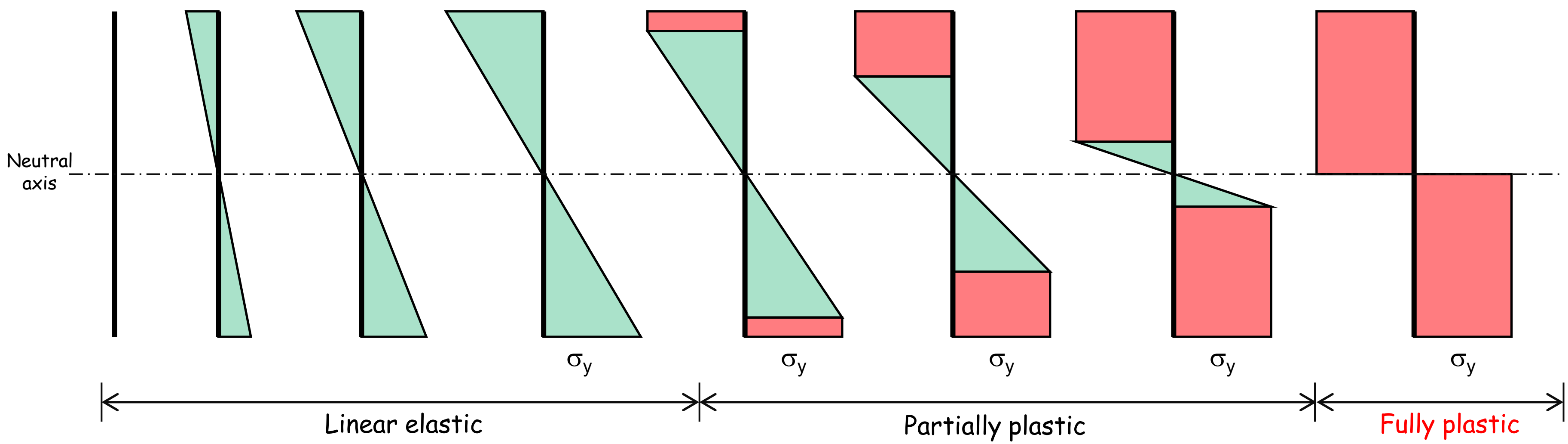
Shape factor

$$\boxed{\alpha = \frac{M_p}{M_y} = \frac{Z_p}{Z_e}}$$

$$\boxed{M_p = Z_p \sigma_y}$$

$$\boxed{M_y = Z_e \sigma_y}$$

Where y_i is the distance of the resultant forces from the plastic neutral axis, A_{Ci} and A_{Ti} are the regions of the beam in compression and tension respectively.



Bending moment at first yield

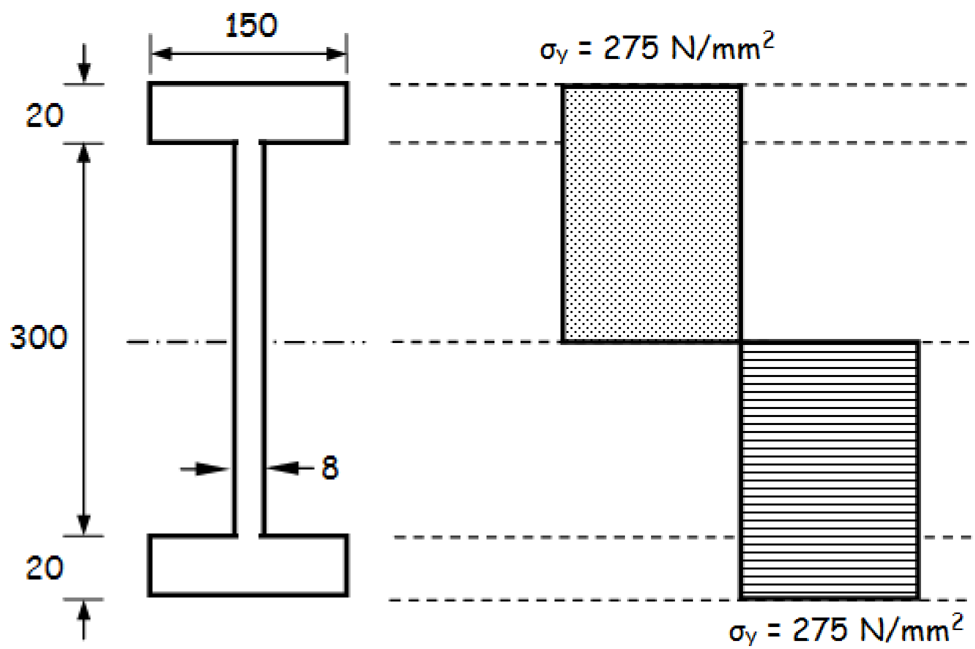
$$\boxed{M_y = \frac{\sigma_y}{y_{max}} I}$$

PLASTIC MOMENT (Mp)

Example 5.1

The figure below shows an I-beam taken from Example 3.3. The beam is made of steel with a yield stress of 275 N/mm^2.

a) Calculate the bending moment at first yield M_y and the fully plastic moment M_p
b) The plastic section modulus and shape factor



From Ex 3.3, second moment of area I = 171.8 x 10^6 mm^4

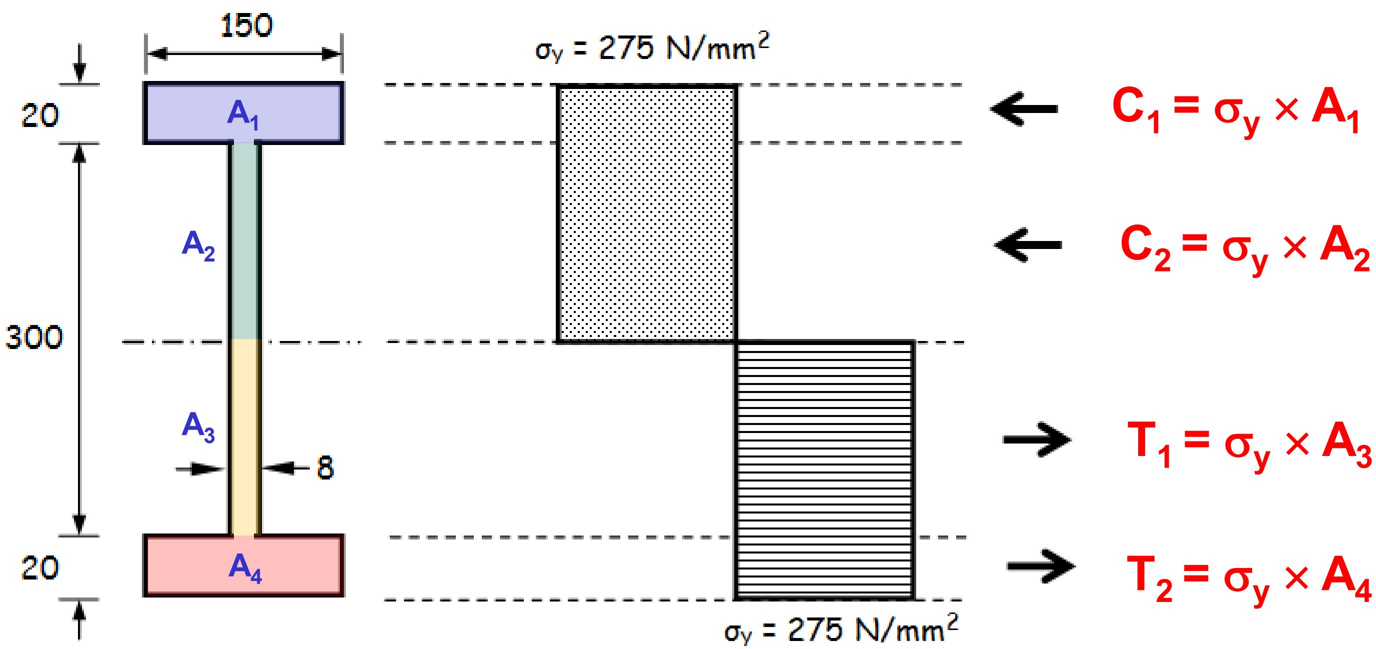
Bending moment at first yield: $$M_y = \frac{\sigma_y}{y_{max}} I$$

$$M_y = \frac{275}{170} \times 171.8 \times 10^6 = 277.9 \times 10^6 \; Nmm$$

Example 5.1

The figure below shows an I-beam taken from Example 3.3. The beam is made of steel with a yield stress of 275 N/mm^2.

a) Calculate the bending moment at first yield M_y and the fully plastic moment M_p
b) The plastic section modulus and shape factor



Fully plastic moment M_p is obtained by taking moments of resultant forces about the NA

$$M_p = \sum C_i y_i + \sum T_i y_i$$

$$M_p = 275[(20 \times 150 \times 160) + (8 \times 150 \times 75)] \times 2$$

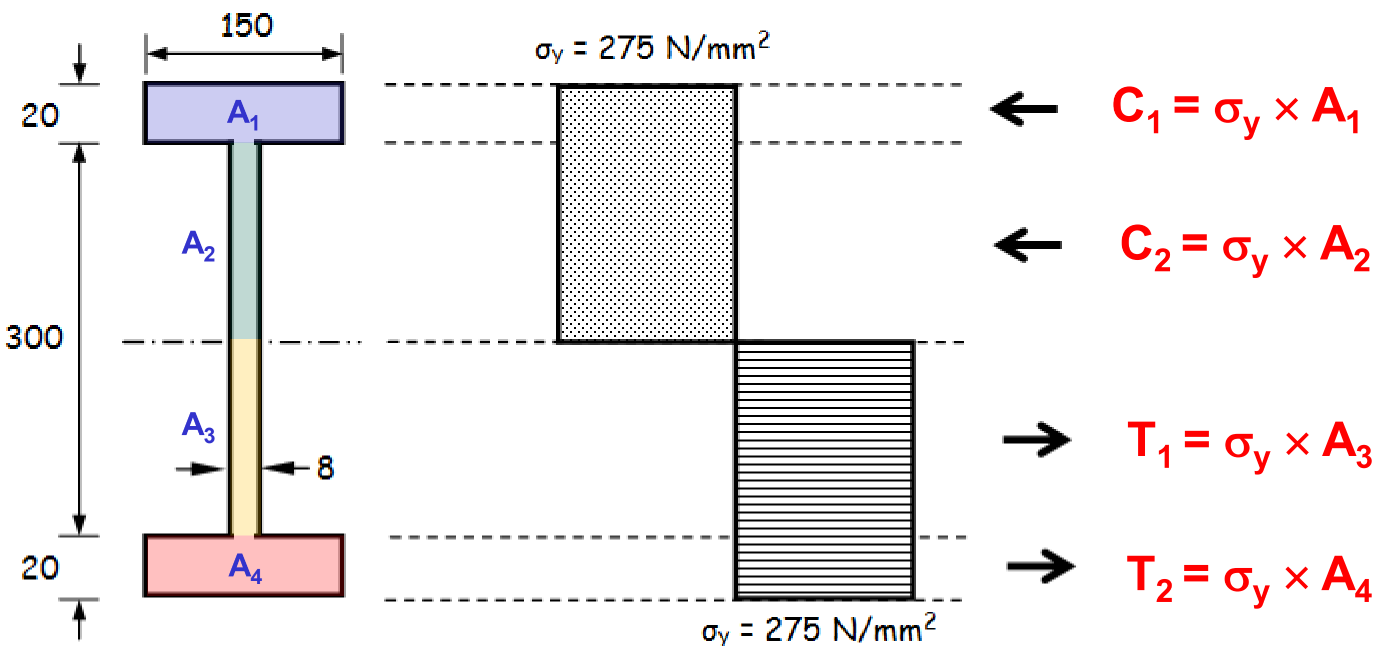
$$M_p = 313.5 \times 10^6 \; Nmm$$

$$Note\ that:\ M_p \gg M_y = 277.9 \times 10^6 \; Nmm$$

Example 5.1

The figure below shows an I-beam taken from Example 3.3. The beam is made of steel with a yield stress of 275 N/mm^2.

a) Calculate the bending moment at first yield M_y and the fully plastic moment M_p
b) The plastic section modulus and shape factor

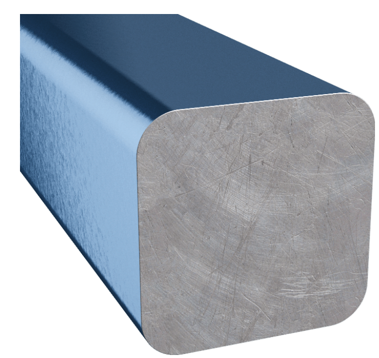


$$M_y = 277.9 \times 10^6 \ Nmm$$

$$M_p = 313.5 \times 10^6 \ Nmm$$

$$Shape\ factor, \propto = \frac{M_p}{M_y} = \frac{313.5}{277.9} = 1.128$$

$\propto \sim 1.15$

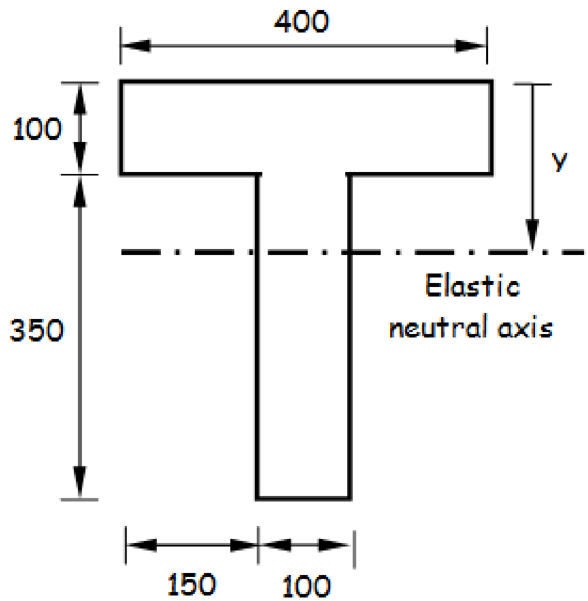


$\propto = 1.5$

Example 5.2

The figure below shows a T-beam from Example 2.4. The beam is made of steel with a yield stress of 275 N/mm^2.

a) Calculate the bending moment at first yield M_y and the fully plastic moment M_p
b) The plastic section modulus and shape factor



From Ex 2.4, $y = 155$ mm
$I = 1.335 \times 10^9$ mm^4

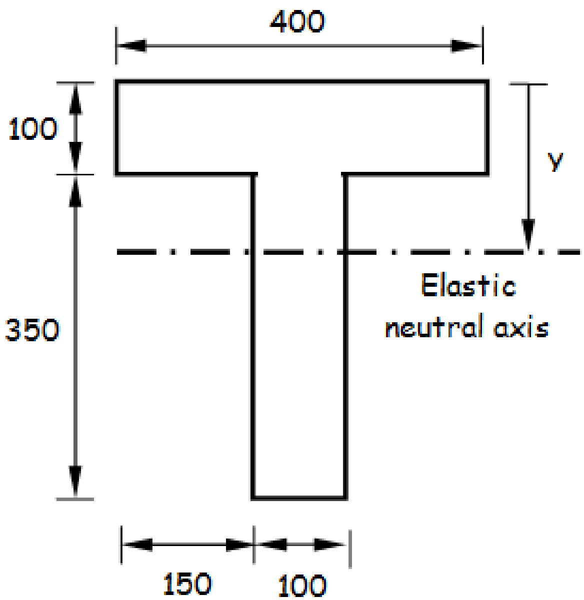
Bending moment at first yield: $$M_y = \frac{\sigma_y}{y_{max}} I$$

$$M_y = \frac{275}{(450 - 155)} \times 1.335 \times 10^9 = 1.24 \times 10^9 \; Nmm$$

Example 5.2

The figure below shows a T-beam from Example 2.4. The beam is made of steel with a yield stress of 275 N/mm^2.

a) Calculate the bending moment at first yield M_y and the fully plastic moment M_p
b) The plastic section modulus and shape factor



Fully plastic moment M_p is obtained by taking moments of resultant forces about the plastic NA (axis that divides the section into equal halves)

Total cross-sectional area $A = (100 \times 400) + (350 \times 100) = 75{,}000\ \text{mm}^2$

Area of flange $= (100 \times 400) = 40{,}000\ \text{mm}^2$

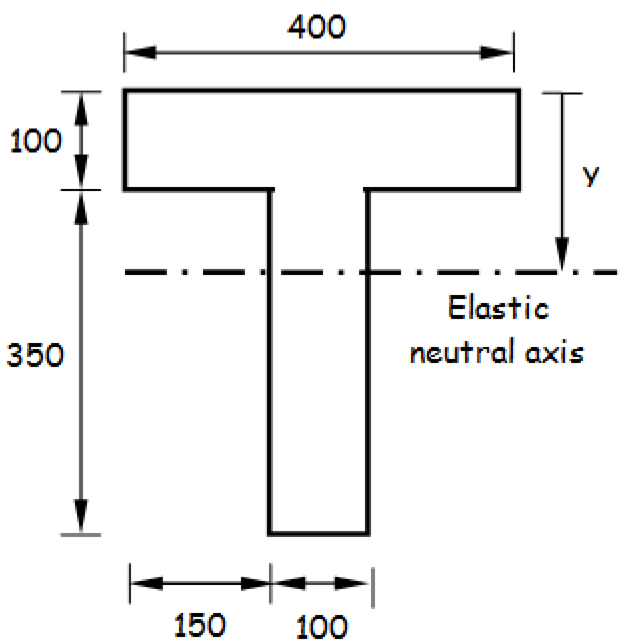
Let y_p = distance from the top of the beam to plastic NA

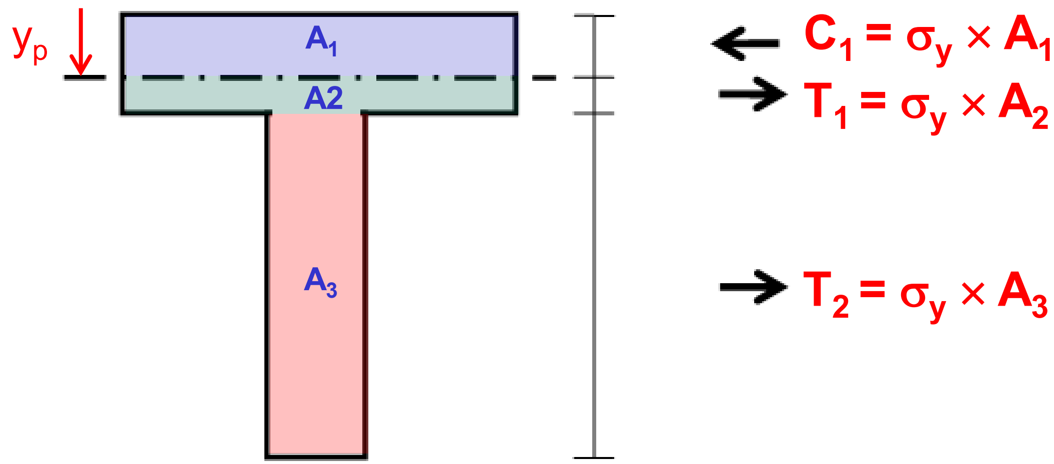
$(400 \times y_p) = \frac{1}{2} \times 75{,}000 \qquad \therefore y_p = 93.75\ \text{mm}$

Example 5.2

The figure below shows a T-beam from Example 2.4. The beam is made of steel with a yield stress of 275 N/mm^2.

a) Calculate the bending moment at first yield M_y and the fully plastic moment M_p
b) The plastic section modulus and shape factor





Fully plastic moment M_p is obtained by taking moments of resultant forces about the NA

$$M_p = \sum C_i y_i + \sum T_i y_i$$

$$M_p = 275\left[(400 \times 93.75 \times 93.75/2) + \left(400 \times 6.25 \times \frac{6.25}{2}\right) + \left(100 \times 350 \times (\frac{350}{2} + 6.25\right)\right]$$

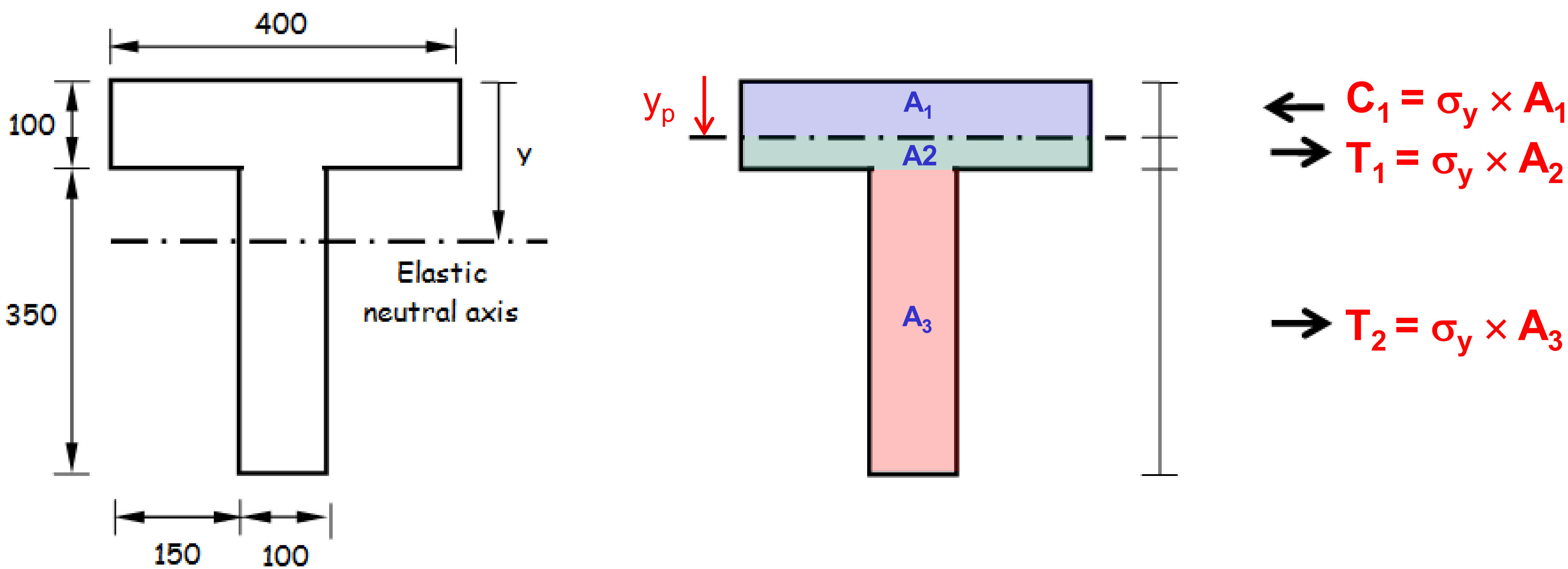
$$M_p = 2.23 \times 10^9\ Nmm$$

$Note\ that\colon\ M_p \gg M_y = 1.24 \times 10^9\ Nmm$

Example 5.2

The figure below shows a T-beam from Example 2.4. The beam is made of steel with a yield stress of 275 N/mm².

a) Calculate the bending moment at first yield M_y and the fully plastic moment M_p
b) The plastic section modulus and shape factor



Plastic section modulus, Z_p is the sum of area x lever arm

$$Z_p = \sum A_{Ci} y_i + \sum A_{Ti} y_i$$

$$Z_p = \left[(400 \times 93.75 \times 93.75/2) + \left(400 \times 6.25 \times \frac{6.25}{2} \right) + \left(100 \times 350 \times (\frac{350}{2} + 6.25 \right) \right]$$

$$Z_p = 8.11 \times 10^6 \ mm^3$$

Example 5.2

The figure below shows a T-beam from Example 2.4. The beam is made of steel with a yield stress of 275 N/mm^2.

a) Calculate the bending moment at first yield M_y and the fully plastic moment M_p
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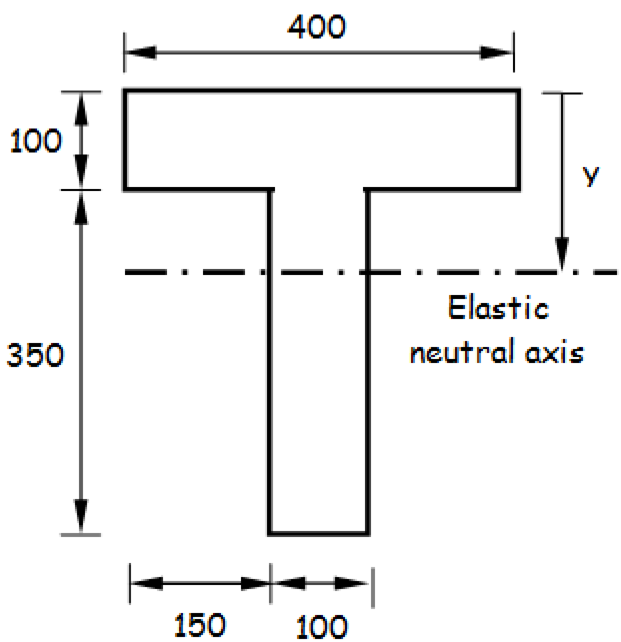
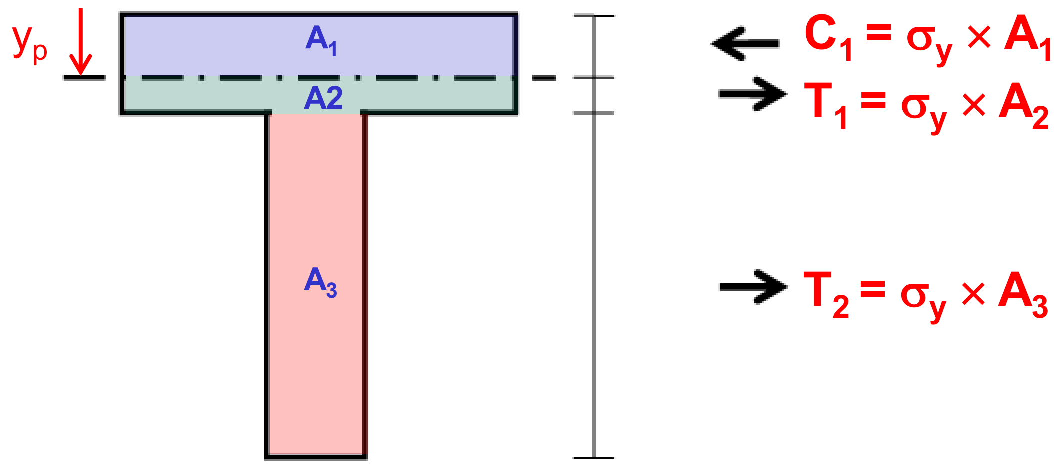


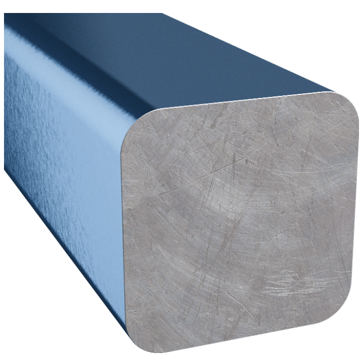


$$M_y = 1.24 \times 10^9 \ Nmm$$

$$M_p = 2.23 \times 10^9 \ Nmm$$

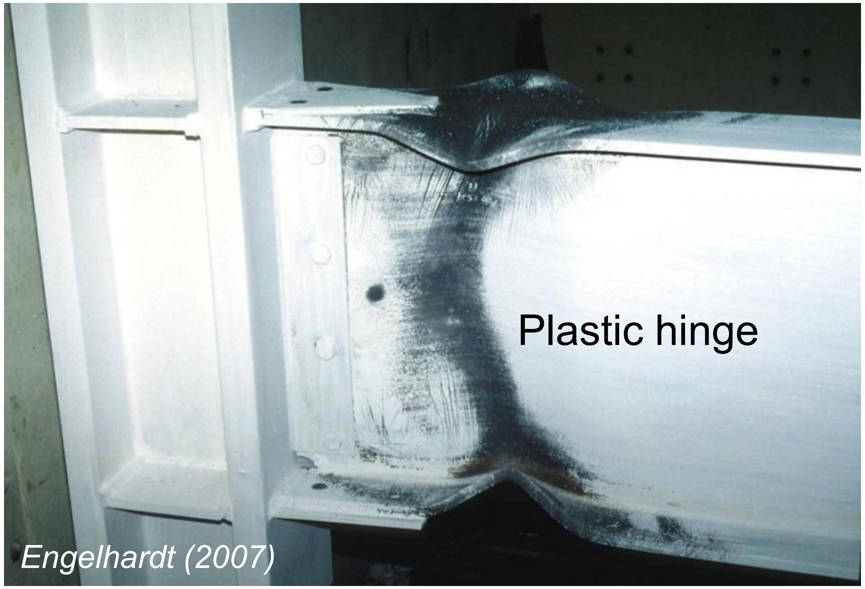
$$Shape\ factor, \propto = \frac{M_p}{M_y} = \frac{2.23}{1.24} = 1.798$$

$\propto \sim 1.15$



$\propto = 1.5$

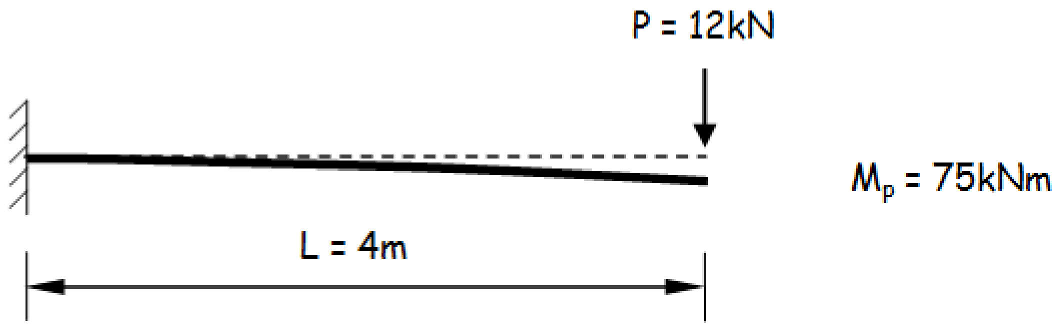
$\propto = 1.8$



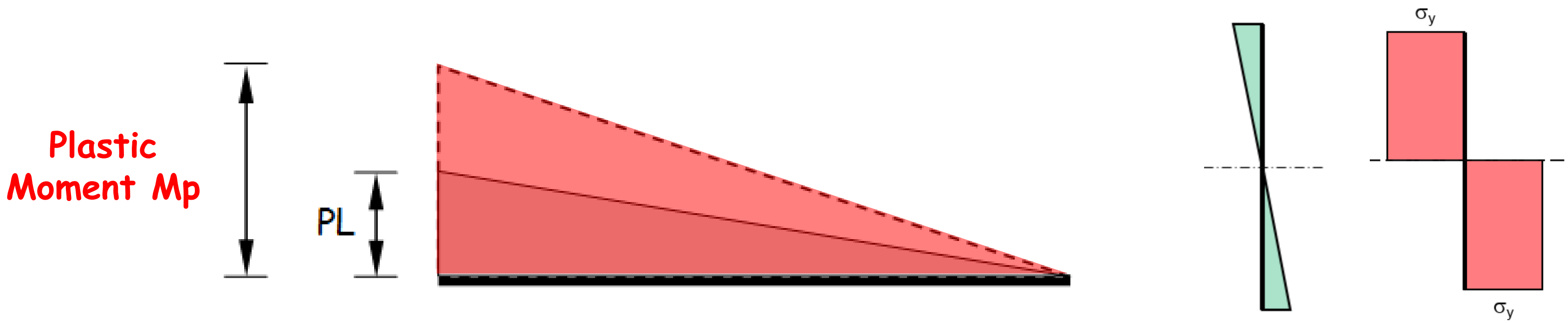
Structure collapses when sufficient number of hinges form to produce a mechanism.

5.4 Plastic collapse and load factor

Working load

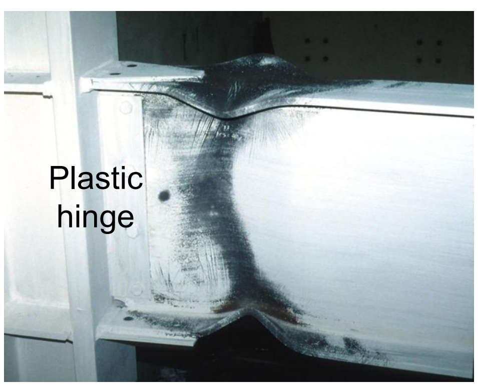


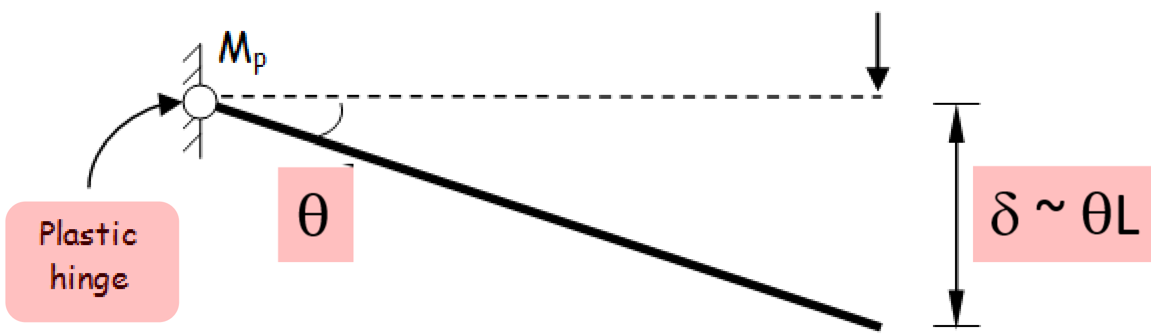
(Bending moment diagram)



(Collapse mechanism)

Critical Load: P_c





Load factor:

$$\boxed{\lambda = \frac{P_c}{P}}$$

Virtual work

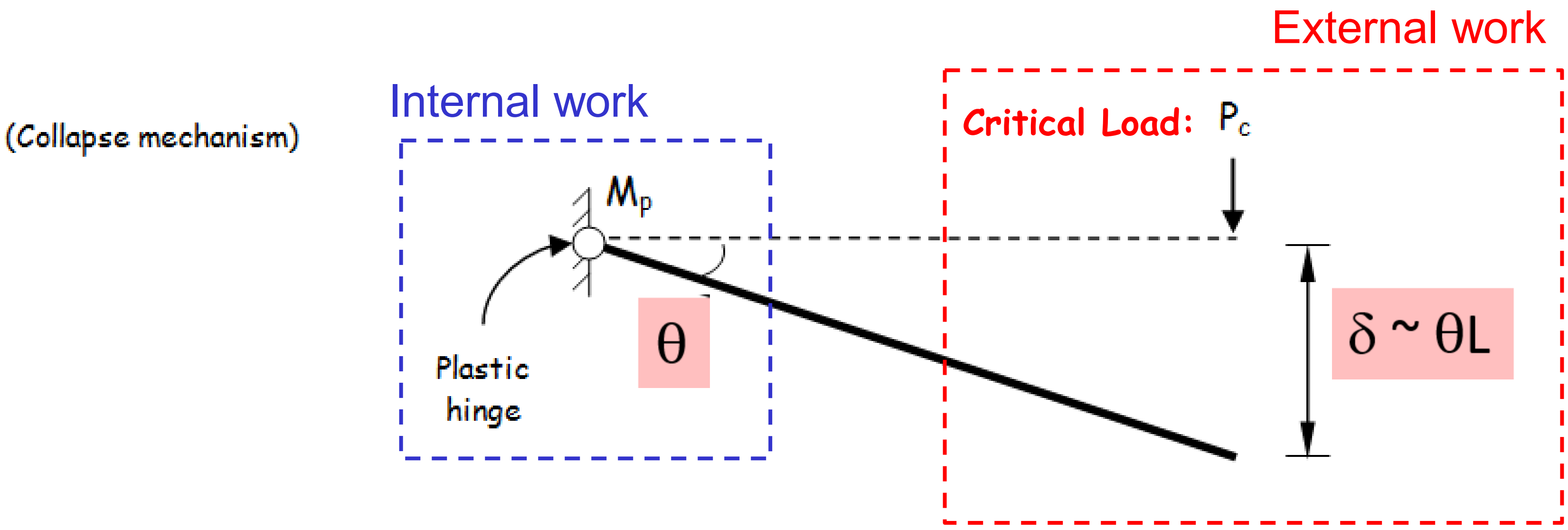
A powerful method of analysis to determine the critical load P_c uses the principle of virtual work, which states that for a structure which is in equilibrium and given a small virtual displacement, the sum of work done by the external forces is equal to the work done by internal forces.

When the beam collapses, all work done by the external load is used to rotate the plastic hinge. Applying energy conservation and ignoring energy losses, the work done by the external load is equal to the work done by the internal moment to rotate the plastic hinge.

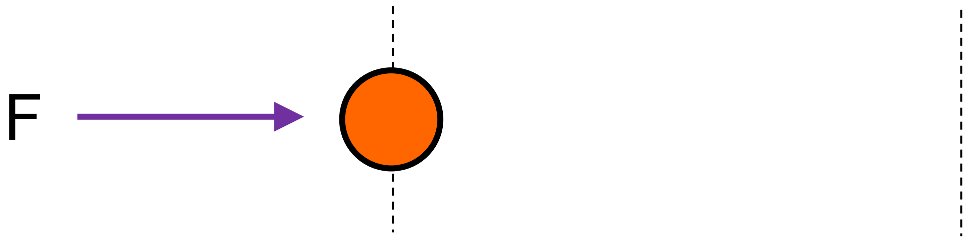
$$External\ work = \sum P \times \delta$$

$$Internal\ work = \sum M_p \times \theta$$

$$\therefore \quad \sum P \times \delta = \sum M_p \times \theta \qquad (5.11)$$



Work done (J) = Force (N) $\times$ Displacement (m)



Work done (J) = Moment (Nm) $\times$ rotation

Virtual work

A powerful method of analysis to determine the critical load P_c uses the principle of virtual work, which states that for a structure which is in equilibrium and given a small virtual displacement, the sum of work done by the external forces is equal to the work done by internal forces.

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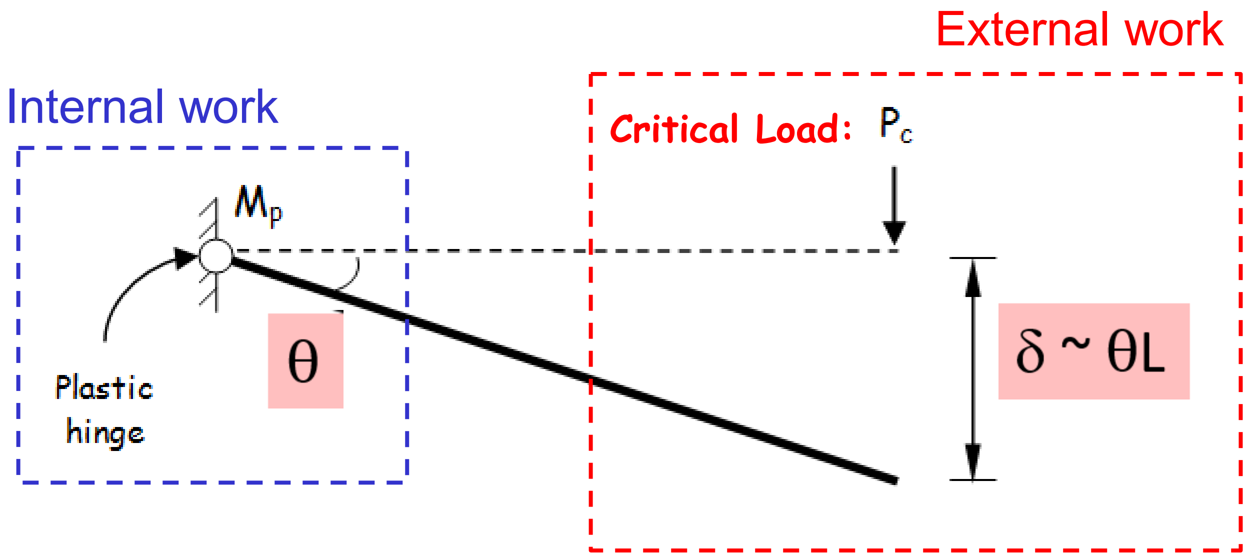
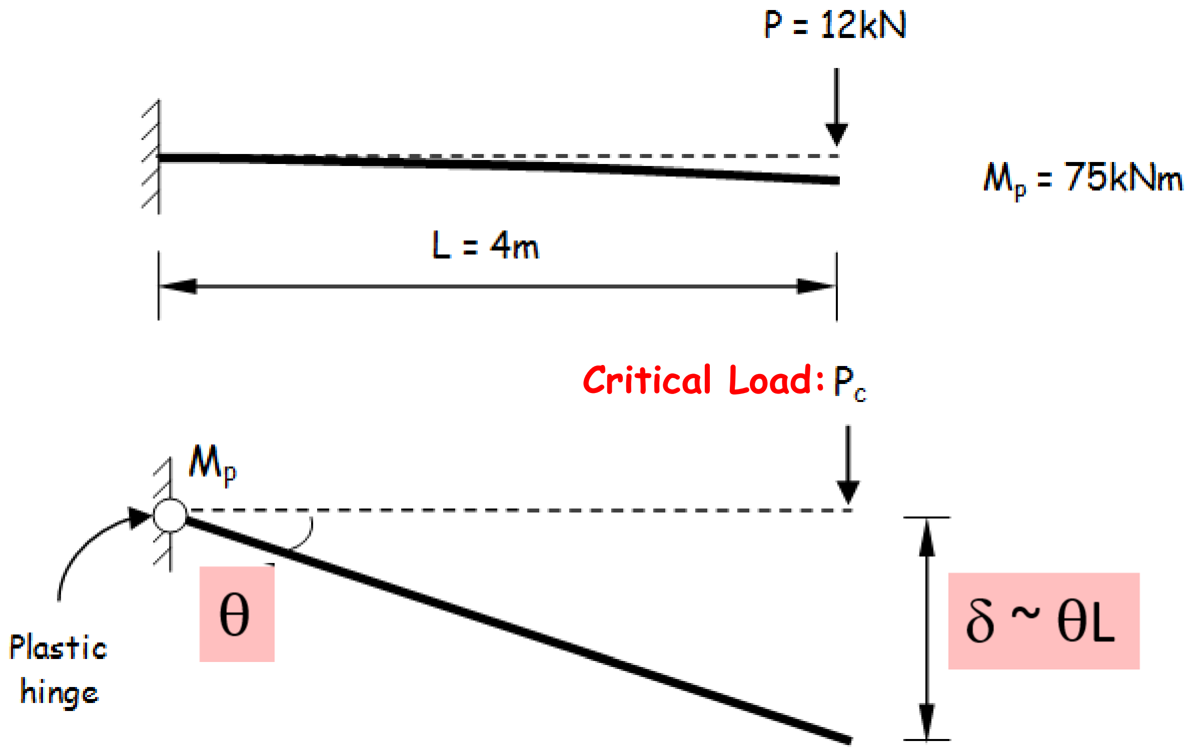
$$External\ work = \sum P \times \delta$$

$$Internal\ work = \sum M_p \times \theta$$

$$\therefore \quad \sum P \times \delta = \sum M_p \times \theta \tag{5.11}$$

(Collapse mechanism)





(Fig. 5.5)

Load factor:

$$\boxed{\lambda = \frac{P_c}{P}}$$

Therefore, (p. 56)

External work = load × displacement $= P_c \times \theta L$

Internal work = work done by hinge $= M_p \times \theta$

Equating internal work to external work

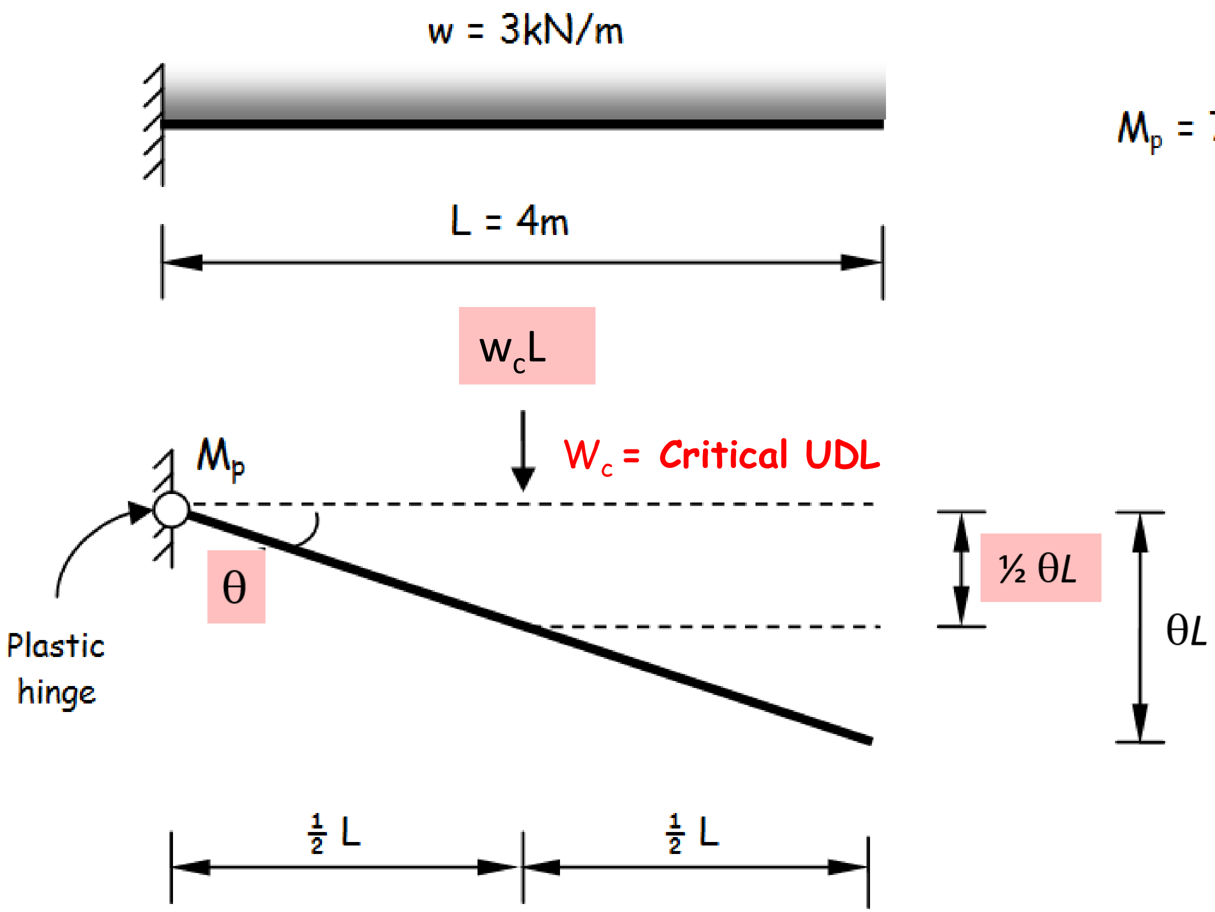
$$P_c \times \theta L = M_p \times \theta$$

Thus, $P_c = \dfrac{M_p}{L} = \dfrac{75}{4} = 18.75kN$

And, $\lambda = \dfrac{P_c}{P} = \dfrac{18.75}{12} = 1.56$

- Note that the angle θ does not appear in the final answer.

(Fig. 5.6, p.57)

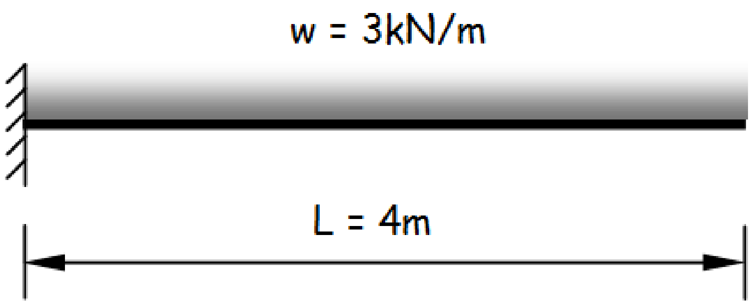


Now consider the same cantilever beam but with the point load replaced with a uniformly distributed load w as shown in Fig. 5.6. The uniformly distributed load w is gradually increased up to a critical value w_c when collapse occurs.

If the angle of rotation at the plastic hinge is taken as θ, the average deflection caused by the uniformly distributed load is $\theta \times \frac{1}{2} L$.

External work = load × displacement $\quad = w_c L \times \frac{1}{2} \theta L$

Internal work = work done by hinge $\quad = M_p \times \theta$



(Fig. 5.6, p.57)

External work = load × displacement $= w_c L \times ½\, \theta L$

Consider a small element of width δx located at a distance x from the fixed end

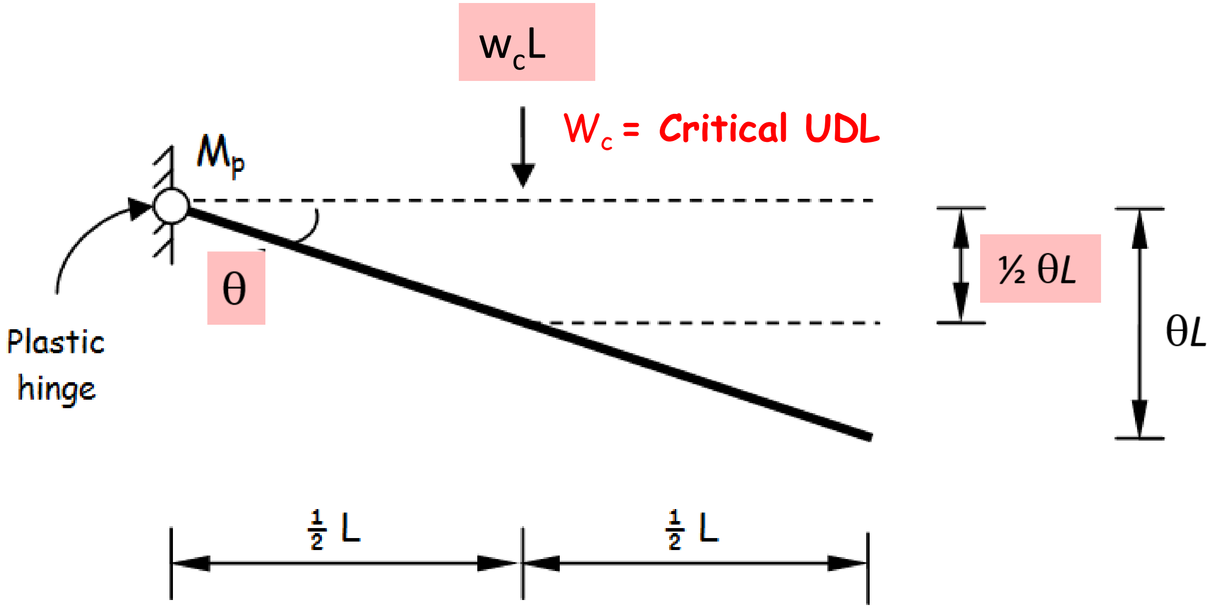
External work done to this small element $= w\ \delta x\ (\theta x)$

Total external work done,

$$\mathrm{W} = \int_0^L w\ \delta x\ (\theta x)$$

$$\mathrm{W} = \left[\frac{w x^2 \theta}{2}\right]_0^L$$

$$\mathrm{W} = \frac{w L^2 \theta}{2}$$



Equating internal work to external work

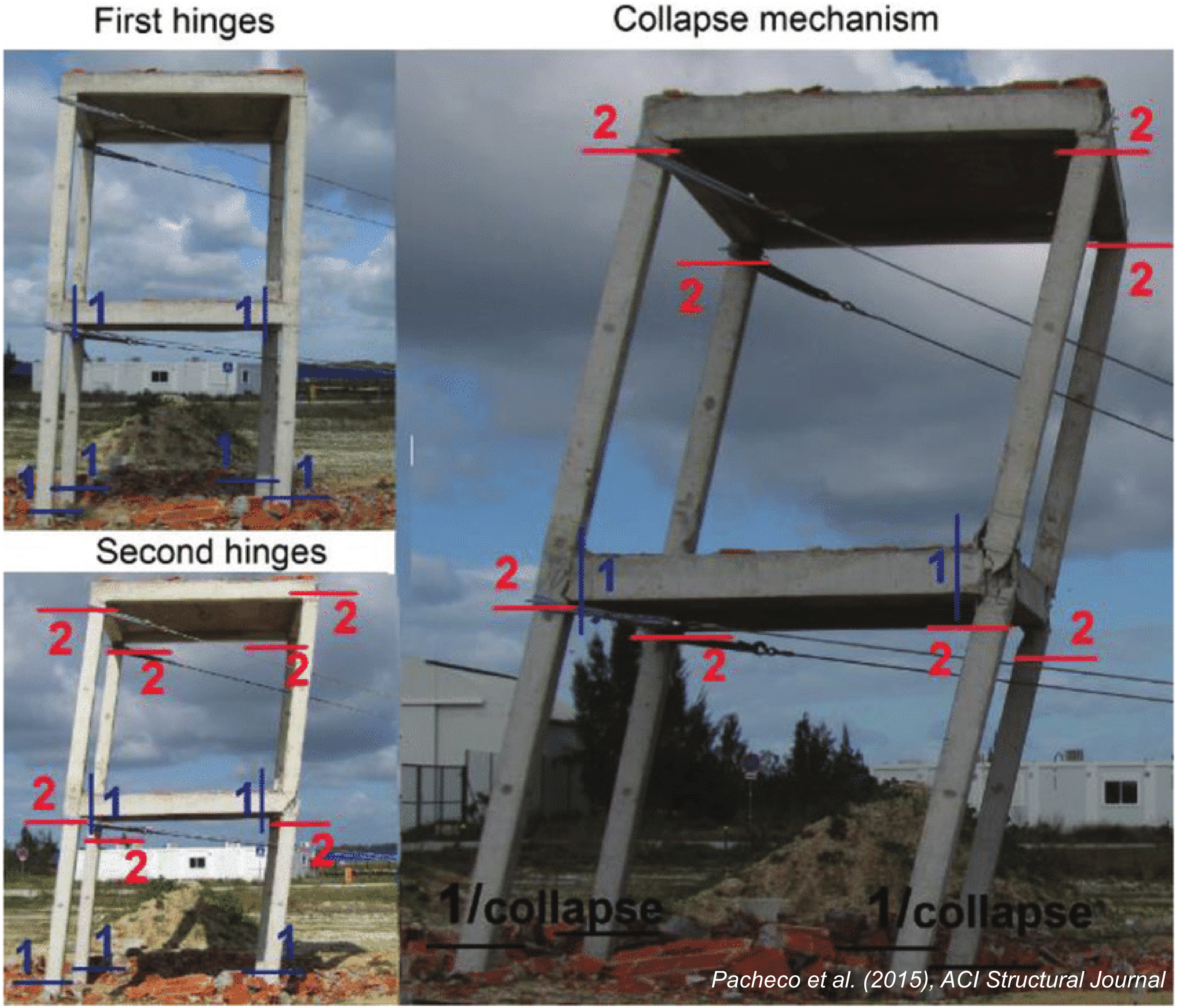
$$w_c L \times \frac{\theta L}{2} = M_p \times \theta$$

Thus, $$w_c = \frac{2M_p}{L^2} = \frac{2 \times 75}{4^2} = 9.375kNm$$

$$\lambda = \frac{w_c}{w} = \frac{9.375}{3} = 3.125$$

Note that collapse can only occur when sufficient number of hinges has form to reduce a structure to a mechanism.

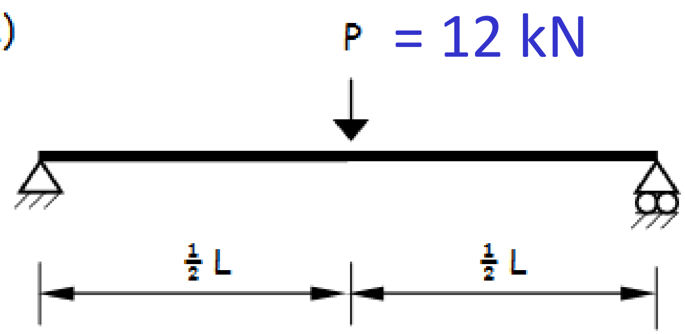
- For a statically determinate structure, only one plastic hinge is required to cause collapse.
- For a statically indeterminate structure, additional hinges will be required to cause collapse.

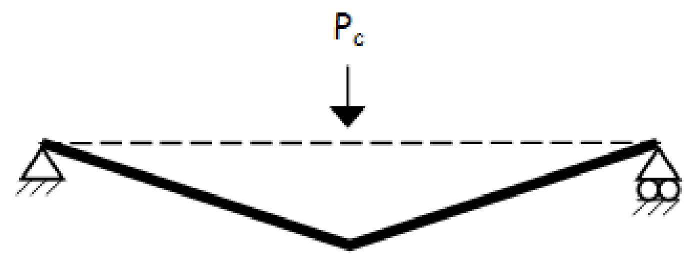

First hinges
Second hinges
Collapse mechanism
1
2
1/collapse
Pacheco et al. (2015), ACI Structural Journal

Example 5.3

The beam shown below has a uniform cross section and supports a working load P (= 12kN) at mid-span. How many hinges would it require to form a collapse mechanism? Determine the collapse load and load factor if L = 4m and the beam has a fully plastic moment of M_p (= 75kNm) throughout.

a)





(Collapse mechanism)

Number of plastic hinge =

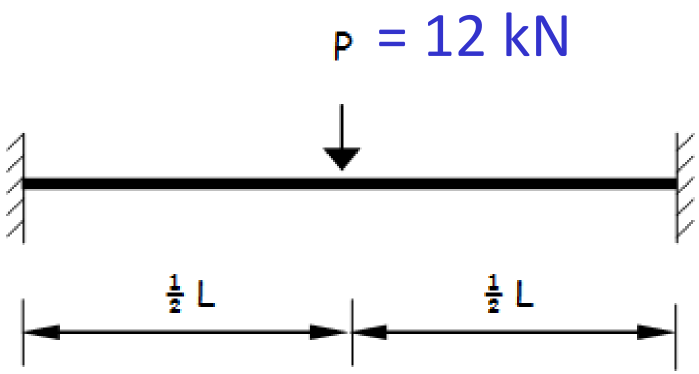
External work = load × displacement = $P_c \times ½\, \theta\, L$

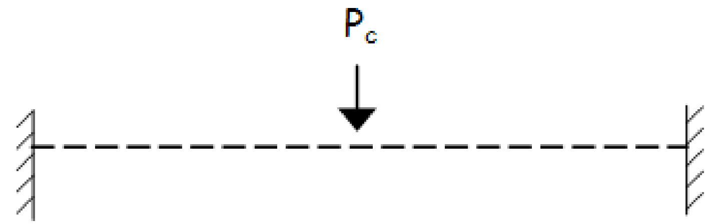
Internal work = work done by hinge = $M_p \times (\theta + \theta)$

Equating internal work to external work, $P_c = 4M_p / L$

$P_c = 4\ (75/4) = 75$ kN

b)





(Collapse mechanism)

Number of plastic hinge =

External work = load × displacement $= P_c \times \frac{1}{2}\, \theta L$

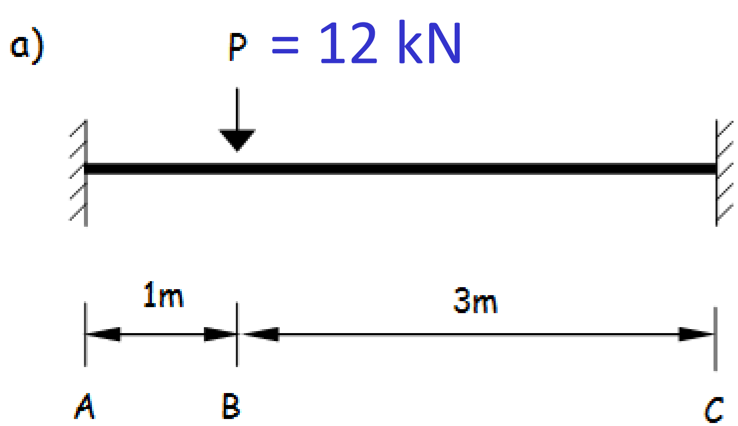
Internal work = work done by hinge $= M_p \times (4\theta)$

Equating internal work to external work, $P_c = 8M_p / L$

$$P_c = 8\ (75/4) = 150 \text{ kN}$$

Example 5.4

The beam shown below has a uniform cross section and supports a point load P (= 12kN) at B. Determine the collapse load and load factor if the beam has a fully plastic moment of M_p (= 75kNm) throughout.



$$\boxed{\begin{aligned}\theta\,(1) &= \alpha\,(3)\\ \therefore\ \alpha &= \theta/3\end{aligned}}$$

(Collapse mechanism)

External work = load × displacement = $P_c \times \theta$

Internal work = work done by hinge = $M_p\,(2\theta + 2\alpha) = M_p\,(2\theta + 2\theta/3)$

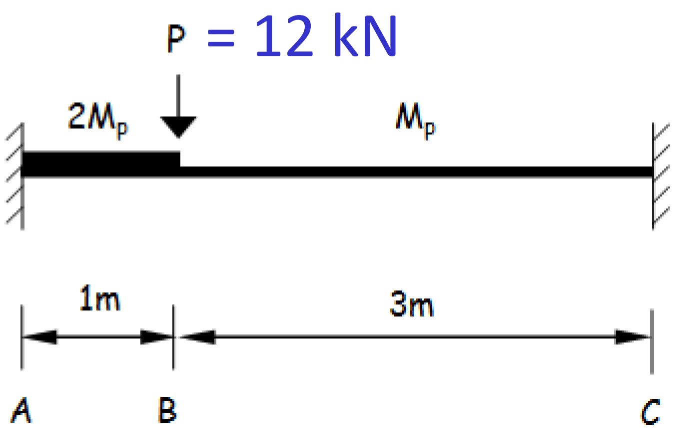
Equating internal work to external work,

$$P_c = 8M_p/3$$

$$P_c = 8(75/3) = 200 \text{ kN}$$

If the beam is strengthened such that it has a plastic moment capacity of $2M_p$ in span AB, calculate the new collapse load and load factor.

b)



(Collapse mechanism)

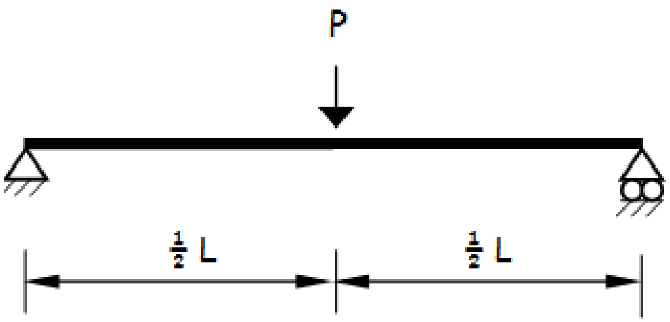
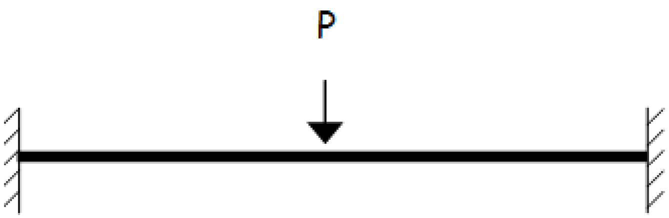
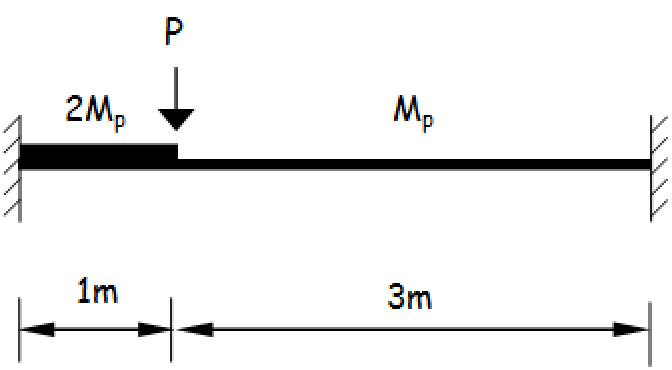
External work = load × displacement = $P_c \times \theta$

Internal work = work done by hinge = $2M_p\theta + M_p(\theta + \alpha) + M_p\alpha = 11M_p\theta/3$

Equating internal work to external work, $P_c = 11M_p/3$

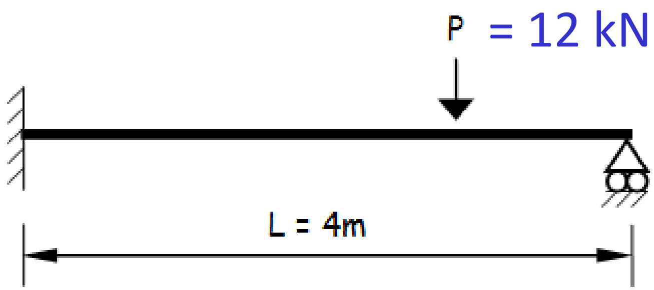
$P_c = 11(75/3) = 275$ kN

Example 5.3 & 5.4: Summary

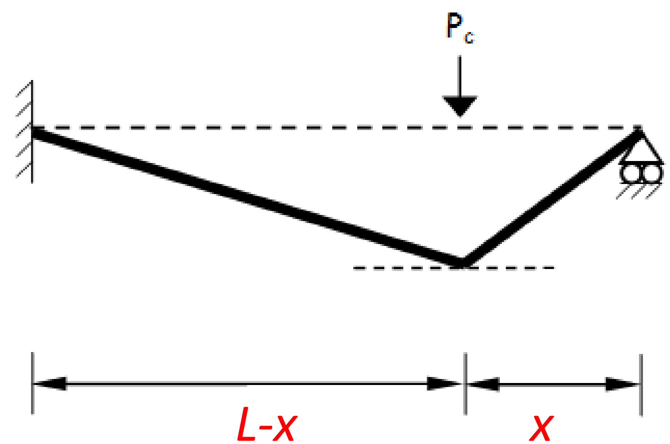
	# of hinges	Collapse load	Load factor
		75 kN	6.25
		150 kN	12.5
		200 kN	16.7
		275 kN	22.9

Example 5.5

The figure below shows a propped cantilever supporting a working load P (= 12kN) that can be applied anywhere along the length of the beam. Determine the position of the load that gives the lowest collapse load. Calculate the collapse load and load factor. The beam has a fully plastic moment of M_p (= 75kNm) throughout.



$$\theta x = \alpha (L - x)$$
$$\therefore \alpha = \theta x / (L - x)$$



(Collapse mechanism)

External work $= P_c \times \theta x$

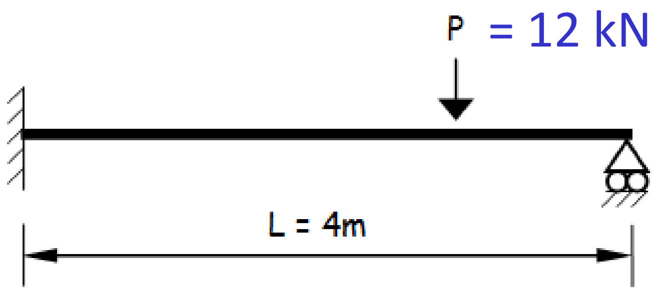
Internal work $= M_p[\alpha + \alpha + \theta] = M_p\left[\dfrac{x\theta}{(L-x)} + \dfrac{x\theta}{(L-x)} + \theta\right]$

Equating internal work to external work and rearranging,

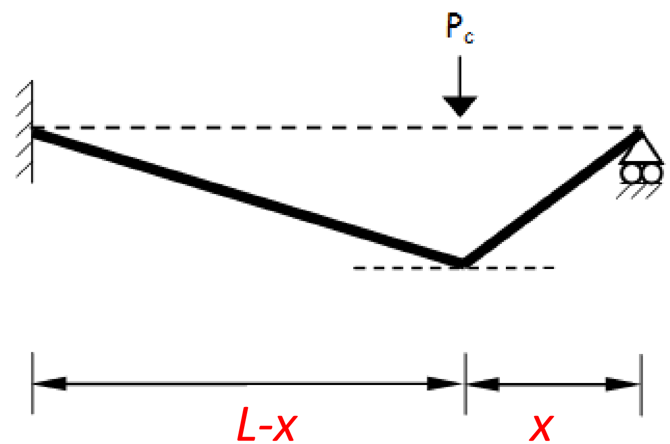
$$P_c = M_p\left[\frac{L+x}{x(L-x)}\right]$$

Example 5.5

The figure below shows a propped cantilever supporting a working load P (= 12kN) that can be applied anywhere along the length of the beam. Determine the position of the load that gives the lowest collapse load. Calculate the collapse load and load factor. The beam has a fully plastic moment of M_p (= 75kNm) throughout.



$$\theta\, x = \alpha\, (L - x)$$
$$\therefore\ \alpha = \theta\, x \,/\, (L - x)$$



(Collapse mechanism)

Equating internal work to external work and rearranging,

$$P_c = M_p \left[\frac{L + x}{x\,(L - x)} \right]$$

The lowest collapse load can now be obtained by differentiating the above with respect to the variable x,

$$\frac{dP_c}{dx} = M_p \left[\frac{x(L - x) - (L - 2x)(L + x)}{x^2 (L - x)^2} \right] = 0$$

$$\frac{dP_c}{dx} = M_p\left[\frac{x(L-x)-(L-2x)(L+x)}{x^2(L-x)^2}\right] = 0$$

This reduces to a quadratic equation that can be solved to determine x

$$x(L-x)-(L-2x)(L+x) = 0$$

$$x^2 + 2xL - L^2 = 0$$

$$x = 0.414L = 0.414 \times 4 = 1.656m$$

The lowest collapse load

$$P_c = M_p\left[\frac{L+x}{x(L-x)}\right] = 5.83\frac{M_p}{L} = 109.3kN$$

$$\lambda = \frac{P_c}{P} = \frac{109.3}{12} = 9.1$$

5.5 Multiple collapse mechanisms

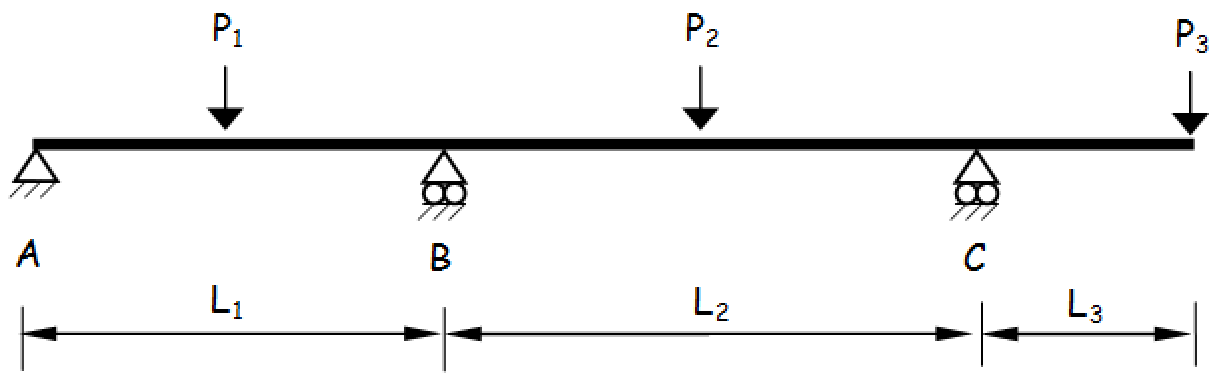
The principle of virtual work can also be applied to multi-span beams supporting several loads. However, these beams have more than one possible collapse mechanism and each must be investigated to determine the most critical one. The most critical mechanism is the one that has the lowest load factor. This depends on the relative magnitudes of the loads, span size and the number of hinges required.

5.5 Multiple collapse mechanisms

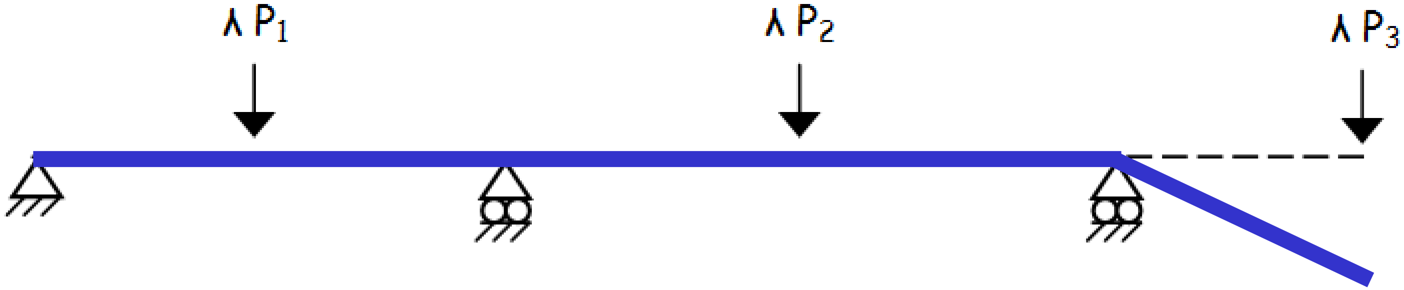
The principle of virtual work can also be applied to multi-span beams supporting several loads. However, these beams have more than one possible collapse mechanism and each must be investigated to determine the most critical one. The most critical mechanism is the one that has the lowest load factor. This depends on the relative magnitudes of the loads, span size and the number of hinges required.

Example 5.6

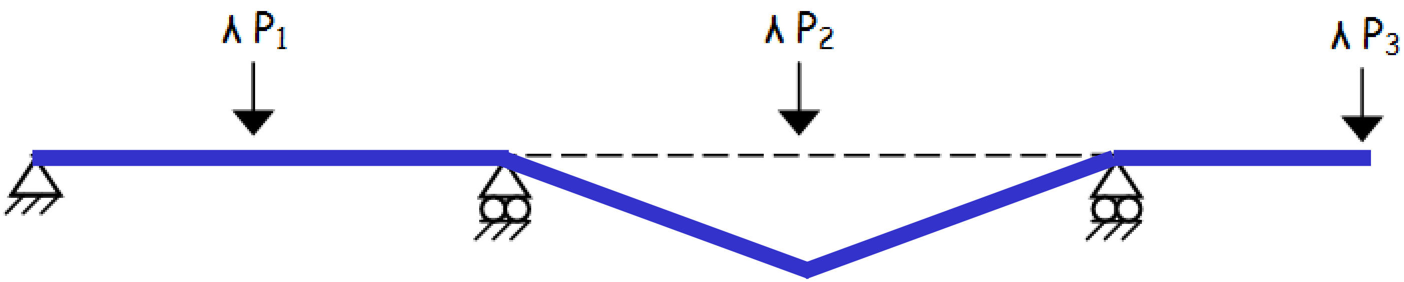
Sketch the possible collapse mechanisms and indicate the location of plastic hinges for the following beam.



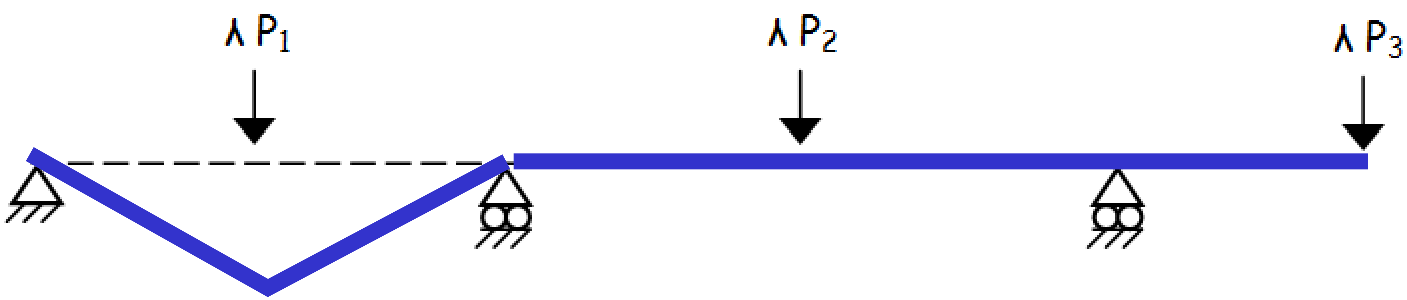
a)


λP_1
λP_2
λP_3

b)


λP_1
λP_2
λP_3

c)


λP_1
λP_2
λP_3

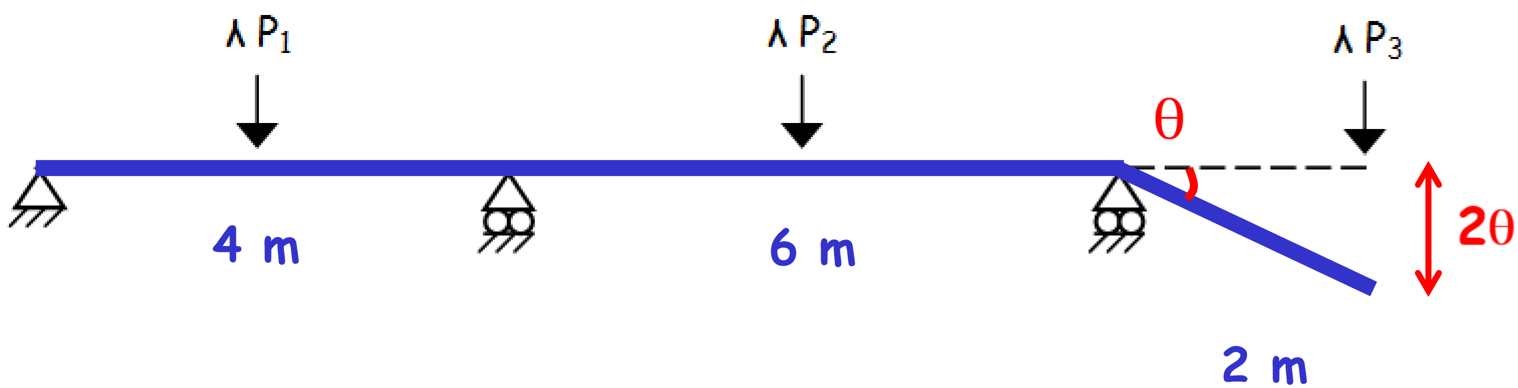
Example 5.7

Referring to the beam shown in Example 5.6, determine the most critical collapse mechanism and the collapse load factor ʎ for the following load cases

i) P_1 = 40kN, P_2 = 35kN and $\boxed{P_3 = 8\text{kN}}$
ii) P_1 = 25kN, P_2 = 38kN and $\boxed{P_3 = 15\text{kN}}$
iii) P_1 = 29kN, P_2 = 32 kN and $\boxed{P_3 = 12 \text{ kN}}$

Take L_1, L_2 and L_3 as 4m, 6m and 2m respectively. The beam has a plastic moment capacity of 75kNm throughout.

a)



Mechanism (a)

External work = $\lambda P_3\ (2\theta)$

Internal work = $M_p\ \theta$

Load factor ʎ = $M_p/2P_3$

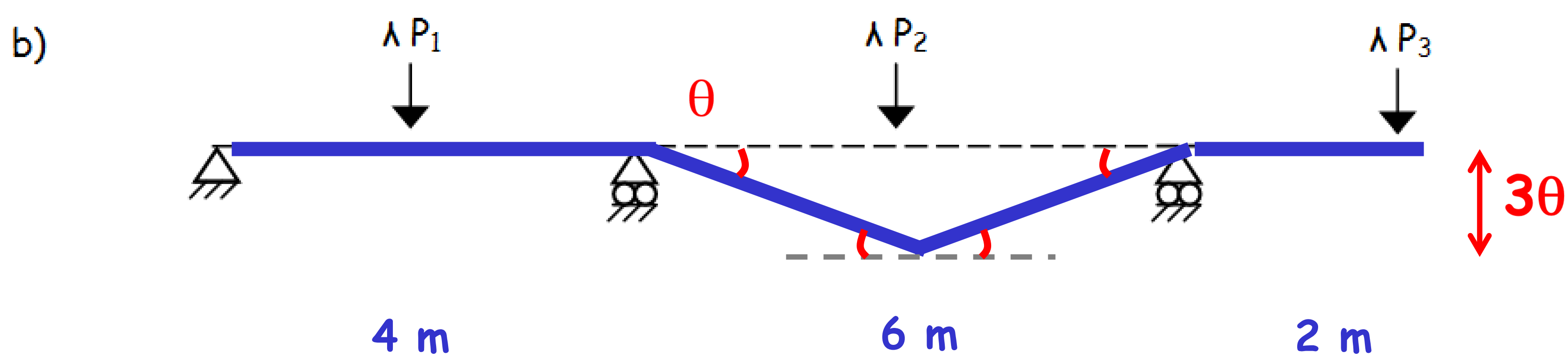
(Case i)	(Case ii)	(Case iii)
= 75/(2x8)	= 75/(2x15)	= 75/(2x12)
= 4.69	= 2.5	= 3.125

Example 5.7

Referring to the beam shown in Example 5.6, determine the most critical collapse mechanism and the collapse load factor ʎ for the following load cases

i) P_1 = 40kN, P_2 = 35kN and P_3 = 8kN
ii) P_1 = 25kN, P_2 = 38kN and P_3 =15kN
iii) P_1 = 29kN, P_2 = 32 kN and P_3 =12 kN

Take L_1, L_2 and L_3 as 4m, 6m and 2m respectively. The beam has a plastic moment capacity of 75kNm throughout.



Mechanism (b)

External work = λP_2 (3θ)

Internal work = M_p (4θ)

Load factor ʎ = $4M_p/3P_2$

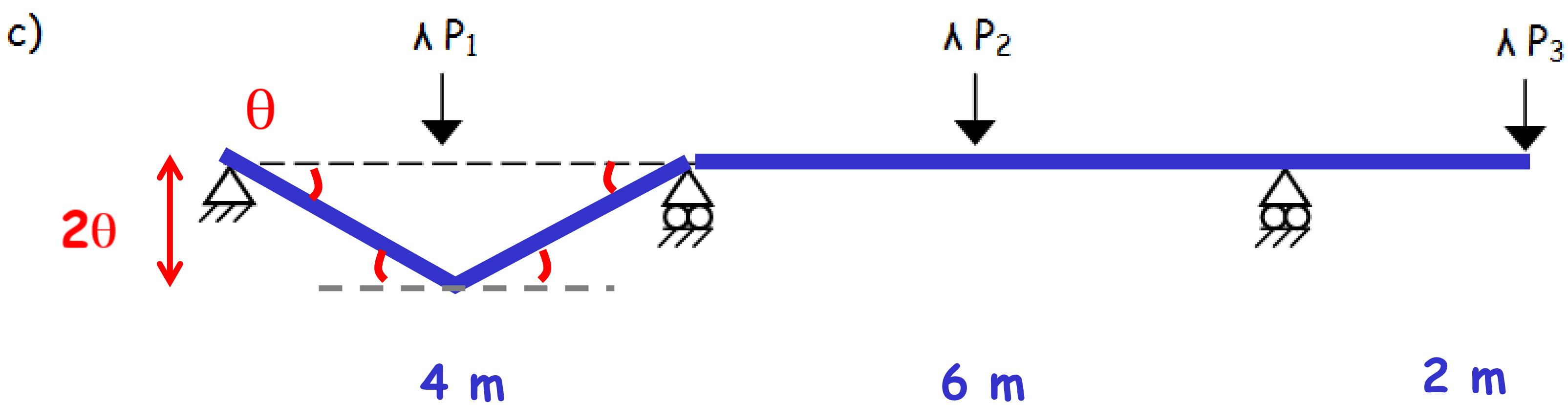
(Case i)	(Case ii)	(Case iii)
= 4(75)/(3x35)	= 4(75)/(3x38)	= 4(75)/(3x32)
= 2.86	= 2.63	= 3.125

Example 5.7

Referring to the beam shown in Example 5.6, determine the most critical collapse mechanism and the collapse load factor λ for the following load cases

i) $P_1 = 40kN$, $P_2 = 35kN$ and $P_3 = 8kN$
ii) $P_1 = 25kN$, $P_2 = 38kN$ and $P_3 = 15kN$
iii) $P_1 = 29kN$, $P_2 = 32$ kN and $P_3 = 12$ kN

Take L_1, L_2 and L_3 as 4m, 6m and 2m respectively. The beam has a plastic moment capacity of 75kNm throughout.



<u>Mechanism (c)</u>

External work = $\lambda P_1\ (2\theta)$

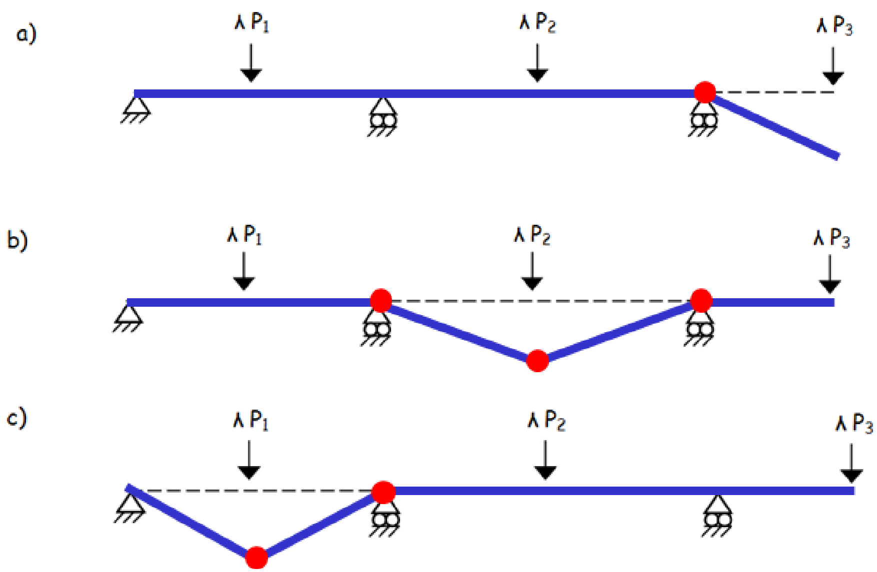
Internal work = $M_p\ (3\theta)$

Load factor λ = $3M_p/2P_1$

(Case i)	(Case ii)	(Case iii)
= 3(75)/(2x40)	= 3(75)/(2x25)	= 3(75)/(2x29)
= 2.81	= 4.5	= 3.88

Results:

Collapse mechanism	Load case (i)	Load case (ii)	Load case (iii)
(a)	4.69	2.5	3.125
(b)	2.86	2.63	3.125
(c)	2.81	4.5	3.879



i) P_1 = 40kN, P_2 = 35kN and P_3 = 8kN

ii) P_1 = 25kN, P_2 = 38kN and P_3 =15kN

iii) P_1 = 29kN, P_2 = 32 kN and P_3 =12 kN

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CIVE 40006 Structural Mechanics

Lecture 9 – Plastic theory & collapse mechanisms

14 Feb 2023

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