Signals and Systems
Fall 2012

**Solution to Homework Assignment #8**

1.

(a) (E+0.5){$y(k)$} = 0, $y(0) = 2$.

$$\Rightarrow y(k+1)+0.5y(k)=0$$

Taking the z-transform, we have:

$$\begin{aligned} z\left[Y(z)-y(0)\right]+0.5Y(z)&=0 \\ zY(z)-2z+0.5Y(z)&=0 \end{aligned} \quad \Rightarrow \ Y_{ZI}(z)=\frac{2z}{z+0.5}$$

It then follows that:

$$\boxed{y(k)=y_{ZI}(k)=2(-0.5)^k u_s(k)}$$

(b) $(E^2-3E+2)\{y(k)\} = u_s(k)$, $y(0) = 2$, $y(1) = 1$.

$$z^2\left[Y(z)-y(0)-z^{-1}y(1)\right]-3z\left[Y(z)-y(0)\right]+2Y(z)=U_S(z)$$

$$\left(z^2-3z+2\right)Y(z)=U_S(z)+z\cdot y(1)+\left(z^2-3z\right)y(0)$$

$$\begin{aligned} Y(z)&=\underbrace{\frac{1}{z^2-3z+2}\left(\frac{z}{z-1}\right)}_{ZSR}+\underbrace{\frac{z}{z^2-3z+2}+\frac{z^2-3z}{z^2-3z+2}\cdot 2}_{ZIR} \\ &=\underbrace{\frac{z}{(z-2)(z-1)^2}}_{ZSR}+\underbrace{\frac{z}{(z-2)(z-1)}+\frac{2z^2-6z}{(z-2)(z-1)}}_{ZIR} \\ &=\underbrace{\frac{z}{(z-2)(z-1)^2}}_{ZSR}+\underbrace{\frac{2z^2-5z}{(z-2)(z-1)}}_{ZIR} \end{aligned}$$

Using partial-fraction expansion, we have:

- $Y_{ZSR}(z)=\dfrac{z}{(z-2)(z-1)^2}=A\dfrac{z}{z-2}+B\dfrac{z}{z-1}+C\dfrac{z}{(z-1)^2}$

Multiply $Y_{ZSR}(z)$ by $(z-2)$ and set $z=2$, we have: $\dfrac{2}{(2-1)^2}=A\times 2\Rightarrow A=1$

Multiply $Y_{ZSR}(z)$ by $(z-1)^2$ and set $z=1$, we have: $\dfrac{1}{(1-2)}=C\times 1\Rightarrow C=-1$

Then let $z=\infty$, we have: $0=A+B\Rightarrow B=-1$

$$\Rightarrow Y_{ZSR}(z) = \frac{z}{z-2} - \frac{z}{z-1} - \frac{z}{(z-1)^2}$$

Taking the inverse z-transform yields:

$$\boxed{y_{ZS}(k) = 2^k u_s(k) - u_s(k) - k u_s(k)}$$

- $Y_{ZIR}(z) = \dfrac{2z^2 - 5z}{(z-2)(z-1)} = A\dfrac{z}{z-2} + B\dfrac{z}{z-1}$

Multiply $Y_{ZIR}(z)$ by $(z-2)$ and set $z = 2$: $\dfrac{2\times 2^2 - 5\times 2}{(2-1)} = A\times 2 \Rightarrow A = -1$

Multiply $Y_{ZIR}(z)$ by $(z-1)$ and set $z = 1$: $\dfrac{2\times 1^2 - 5\times 1}{(1-2)} = B\times 1 \Rightarrow B = 3$

$$\Rightarrow Y_{ZIR}(z) = \frac{-z}{z-2} + \frac{3z}{z-1}$$

Taking the inverse z-transform yields:

$$\boxed{y_{ZI}(k) = -(2)^k u_s(k) + 3u_s(k)}$$

Finally, the complete response is given as:

$$\boxed{\begin{aligned} y(k) &= y_{ZS}(k) + y_{ZI}(k) = 2^k u_s(k) - u_s(k) - k u_s(k) - (2)^k u_s(k) + 3u_s(k) \\ &= -k u_s(k) + 2u_s(k) = (2-k) u_s(k) \end{aligned}}$$

(c) $(E^2 - 2E + 2)\{y(k)\} = (2)^{-k} u_s(k)$, $y(0) = y(1) = 0$.

$$z^2\left[Y(z) - y(0) - z^{-1}y(1)\right] - 2z\left[Y(z) - y(0)\right] + 2Y(z) = \frac{z}{z-1/2}$$

It can be seen that we only have the zero-state response.

$$\Rightarrow y(k+2) - 2y(k+1) + 2y(k) = 2^{-k} u_s(k),\ y(0) = y(1) = 0$$

By taking the inverse z-transform, we have:

$$\left(z^2 - 2z + 2\right) Y_{ZSR}(z) = \frac{z}{z-1/2} \quad \Rightarrow Y_{ZSR}(z) = \frac{z}{(z-1/2)\left(z^2 - 2z + 2\right)}$$

It is easy to see that the roots of $z^2 - 2z + 2$ are

$$\frac{2 \pm \sqrt{4-8}}{2} = 1 \pm j = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right)$$

Furthermore, by using partial-fraction expansion, we obtain:

$$Y_{ZSR}(z) = \frac{4}{5}\frac{z}{z-1/2} + \frac{-\frac{4}{5}z^2 + \frac{6}{5}z}{z^2 - 2z + 2} = \frac{4}{5}\left(\frac{z}{z-1/2}\right) - \frac{4}{5}\left(\frac{z^2 - z}{z^2 - 2z + 2}\right) + \frac{2}{5}\left(\frac{z}{z^2 - 2z + 2}\right)$$

Using the following z-transforms,

$$g(k)=\left|\sqrt{2}\right|^k \cos\left(k\frac{\pi}{4}\right)u_s(k)\Leftrightarrow G(z)=\frac{z\left(z-\left|\sqrt{2}\right|\cos(\pi/4)\right)}{z^2-\left(2\left|\sqrt{2}\right|\cos(\pi/4)\right)z+\left|\sqrt{2}\right|^2}=\frac{z(z-1)}{z^2-2z+2}$$

$$g(k)=\left|\sqrt{2}\right|^k \sin\left(k\frac{\pi}{4}\right)u_s(k)\Leftrightarrow G(z)=\frac{z\left(\left|\sqrt{2}\right|\sin(\pi/4)\right)}{z^2-\left(2\left|\sqrt{2}\right|\cos(\pi/4)\right)z+\left|\sqrt{2}\right|^2}=\frac{z}{z^2-2z+2}$$

we end up with the following time-domain sequence:

$$\boxed{y(k)=y_{ZS}(k)=\frac{4}{5}\left(\frac{1}{2}\right)^k u_s(k)-\frac{4}{5}\left(\sqrt{2}\right)^k\cos\left[k\frac{\pi}{4}\right]u_s(k)+\frac{2}{5}\left(\sqrt{2}\right)^k\sin\left(k\frac{\pi}{4}\right)u_s(k)}$$

Alternative method:

We can express $Y_{ZSR}(z)$ as follows:

$$Y_{ZSR}(z)=\frac{4}{5}\frac{z}{z-1/2}+\frac{-\frac{4}{5}z^2+\frac{6}{5}z}{z^2-2z+2}$$

It then follows that

$$\boxed{y(k)=y_{ZS}(k)=\frac{4}{5}\left(\frac{1}{2}\right)^k u_s(k)-\frac{4}{5}\left(\sqrt{2}\right)^{k+1}\sin\left[(k+1)\frac{\pi}{4}\right]u_s(k)+\frac{6}{5}\left(\sqrt{2}\right)^k\sin\left(k\frac{\pi}{4}\right)u_s(k)}$$

2. Use a digital computer to plot the frequency response of the system

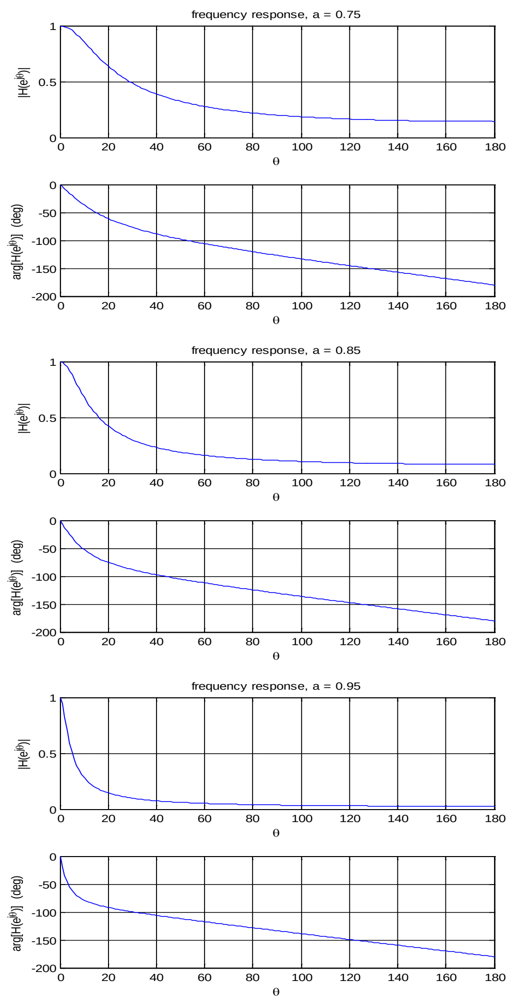
$$H(z)=\frac{1-a}{z-a}$$

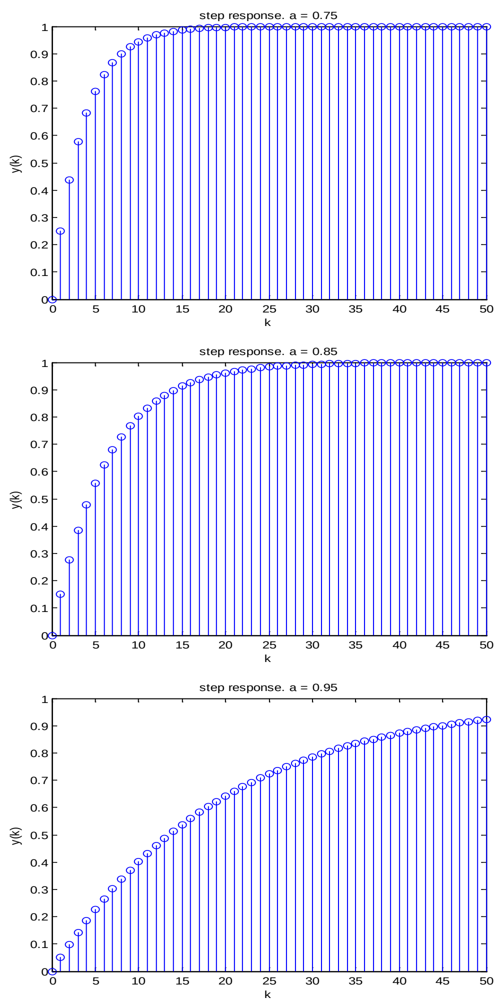
for the cases $a = 0.95$, $a = 0.85$, and $a = 0.75$. Comment on the effect of the pole location on the magnitude and phase of the frequency response.

```
%Assignment 8%
%Problem 2 %
% Matlab code for step response

a=0.75;
N=50;
k=[0:N];
y=0*k;
for i=1:N
    y(i+1)=a*y(i)+(1-a);
end
stem(k,y)
xlabel('k')
ylabel('y(k)')
title(['step response. a =
',num2str(a)])
```

```
% Matlab code for frequency response

a=0.95;
theta=0:180;
thrad=theta*pi/180;
j=sqrt(-1);
z=exp(j*thrad);
for i=1:181
    H(i)=(1-a)/(z(i)-a);
end
subplot(2,1,1)
plot(theta,abs(H)),grid
xlabel('\theta')
ylabel('|H(e^{j\theta})|')
title(['frequency response, a =
',num2str(a)])
subplot(2,1,2)
```

```
plot(theta,(180/pi)*angle(H)),grid
xlabel('\theta')
ylabel('arg[H(e^{j\theta})]  (deg)')
```




If a pole is closer to $z = 1$, the transient response decays more slowly, and the frequency response rolls off earlier.

3. The transfer function

$$G(z) = \frac{z^3 + 0.5z^2 + 0.25z + 0.125}{z^4}$$

is an FIR approximation to the transfer function

$$H(z) = \frac{1}{z - 0.5}.$$

(a) Plot the impulse response of each system.

With $G(z)$, we can simulate the system in time domain as follows:

$$E^3\{g(k)\}=\left(E^3+0.5E^2+0.25E+0.125\right)\{x(k)\}$$

$$\Rightarrow g(k+3)=x(k+3)+0.5x(k+2)+0.25x(k+1)+0.125x(k)$$

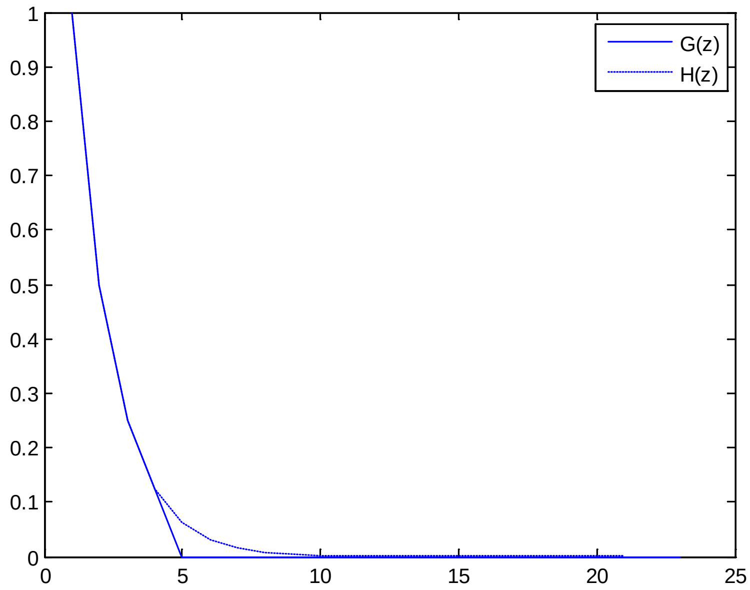
Then we can simulate $H(z)$ as follows:

$$(E-0.5)\{y(k)\}=E\{x(k)\}$$

$$\Rightarrow y(k+1)=0.5y(k)+x(k+1)$$

The Matlab code is given below:

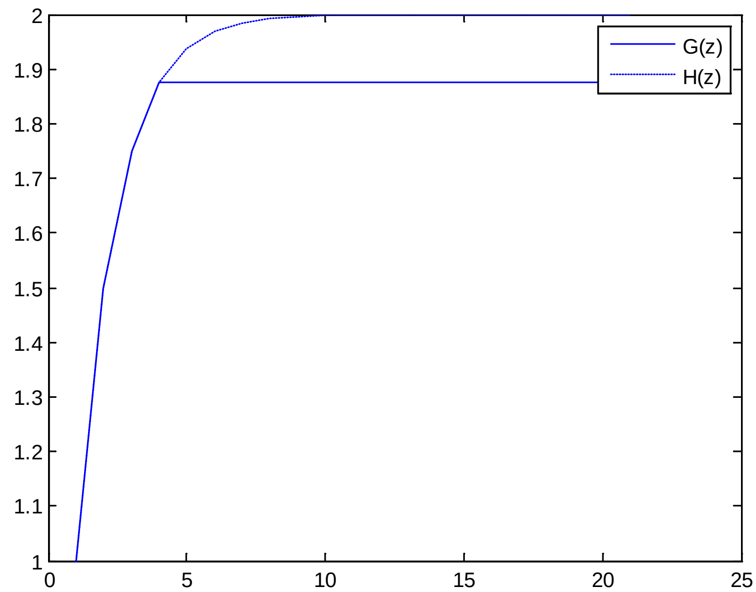
```
%Assignment 8%
%Problem 3 a%
x=zeros(25,1);  % setup all inputs to be zero
x(4)=1;         % let the first input to be 1 to simulate an impulse
                % note that all inputs up to 4 are just initial conditions
for k=1:20
    g(k+3)=x(k+3)+0.5*x(k+2)+0.25*x(k+1)+0.125*x(k);
end

x=zeros(21,1);  % setup all inputs to be zero
x(2)=1;         % let first input to be 1 to simulate an impulse.
                % Note that all inputs prior to 1 are just initial
                % conditions
y(1)=0;               % set up one initial condition

for k=1:20
    y(k+1)=0.5*y(k)+x(k+1);
end

for k=1:20
    g(k)=g(k+3);    % g(k) does not get its first input until k=4,
                    % so everything befor g(4) is just initial conditions
    y(k)=y(k+1);    % y(1) is just an initial condition, so our real
                    % impulse response begin at y(2). Here we sifht the
                    % indices so we can plot both graphs on the same plot.
end
hold off
plot(g)
hold on
plot(y,':')
axis([0 25 0 1])
legend('G(z)','H(z)')
```




(b) Plot the step response of each system. Determine the D.C. gain of each system from its step response.

We can use the same code above, but with all-one input (after the initial conditions are set to zero):

```
%Assignment 8%
%Problem 3 b%
x=ones(30,1);          % setup all inputs to be one
x(1:3)=0;              % setup initial conditions
for k=1:20
    g(k+3)=x(k+3)+0.5*x(k+2)+0.25*x(k+1)+0.125*x(k);
end

x=ones(30,1);           % setup all inputs to be one
x(1)=0;                 % setup initial conditions
y(1)=0;                 % setup one initial condition

for k=1:20
    y(k+1)=0.5*y(k)+x(k+1);
end

for k=1:20;
    g(k)=g(k+3);         % g(k) does not get its first input until k=4,
                         % so everything before g(4) is just initial condition
    y(k)=y(k+1);         % y(1) is just an initial condition, so our real
                         % impulse response begin at y(2).
end
                         % Here we shift the indices so we can plot both
                         % graphs on the same plot.
hold off
plot(g)
hold on
plot(y,':')
axis([0 25 1 2])
legend('G(z)','H(z)')
```

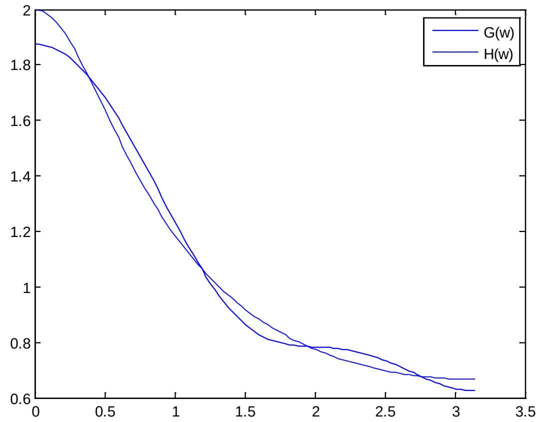



In this case, $H(z)$ has a DC gain of 2, while $G(z)$ has a DC gain of 1.875.

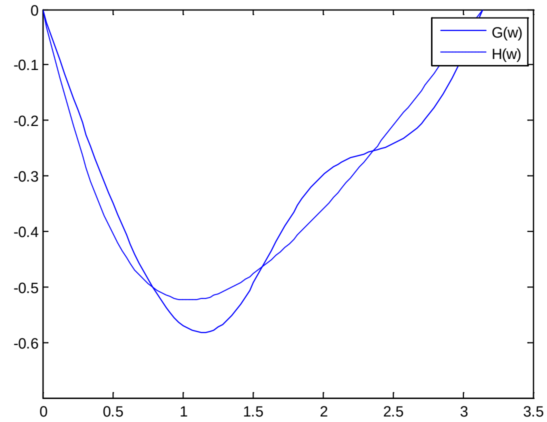
(c) Plot the frequency response of each system. Determine the D.C. gain of each system from its frequency response.

Let $z = e^{jw}$

```
%Assignment 8%
%Problem 3 c%
w =0:pi/100:pi;
g =((exp(j*w)).^3 + 0.5*(exp(j*w)).^2 +
0.25*(exp(j*w))+0.125)./((exp(j*w)).^3);
h = (exp(j*w))./(exp(j*w)-0.5);

hold off

figure(1)
plot(w,abs(g))
hold on
plot(w,abs(h),':')
axis([0 3.5 0.6 2])
legend('G(w)','H(w)')

figure(2)
plot(w,angle(g))
hold on
plot(w,angle(h),':')
axis([0 3.5 -0.7 0])
legend('G(w)','H(w)')
```




Magnitude of Frequency Response




Phase of Frequency Response

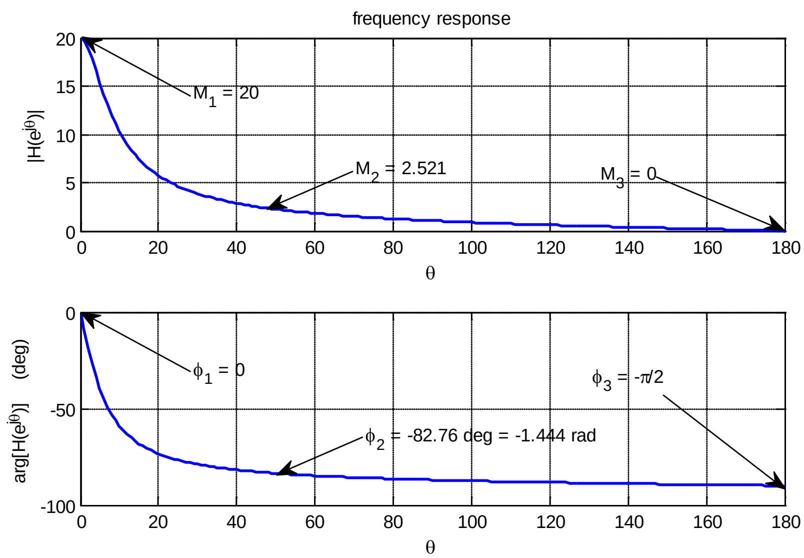
It can be observed that the DC gain of $H(z)$ is 2. For $G(z)$ , the gain is about 1.9. Obviously, the two DC gains fit well to those obtained from the frequency response.

4. Consider the discrete-time system having the transfer function

$$H(z) = \frac{z+1}{z-0.9}.$$

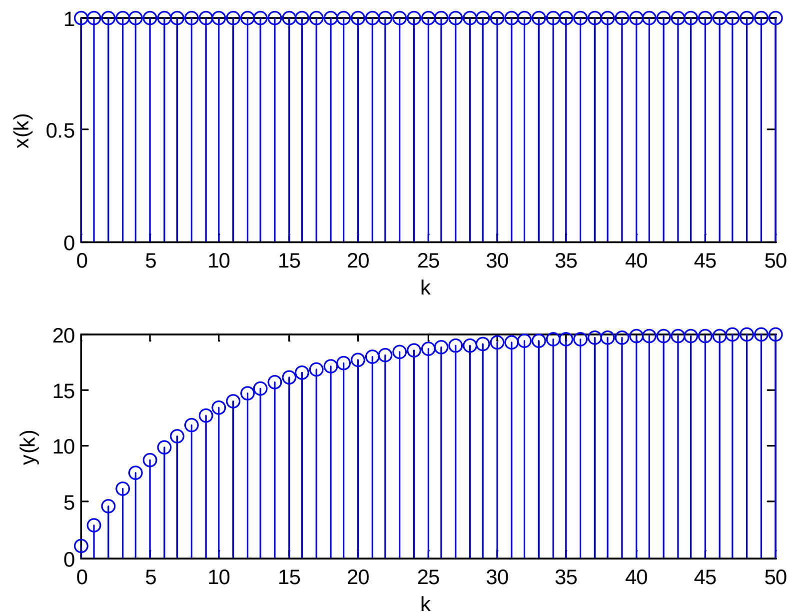
(a) Plot the magnitude and phase of the frequency response of the system.

```
%Assignment 8%
%Problem 4 a%
theta = 0:180;
thrad = theta*pi/180;
j = sqrt(-1);
z = exp(j*thrad);
for i =1:181
    H(i)=(z(i)+1)/(z(i)-0.9);
end
subplot(2,1,1)
plot(theta, abs(H)),grid
axis([0 180 0 20])
xlabel('\theta')
ylabel('|H(e^{j\theta})|')
title('frequency response')
subplot(2,1,2)
plot(theta,(180/pi)*angle(H))
,grid
axis([0 180 -100 0])
xlabel('\theta')
ylabel('arg[H(e^{j\theta})]
(deg)')
```



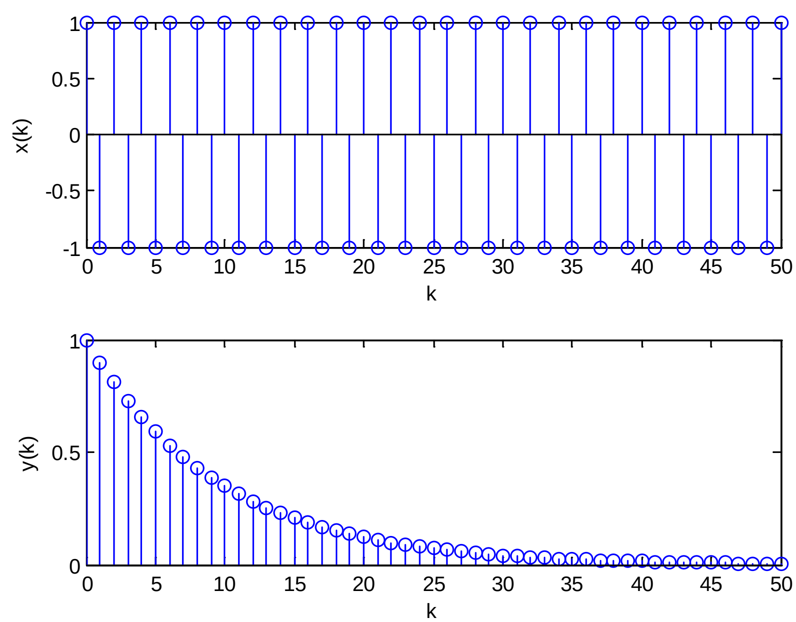

(b) Program the corresponding difference equation, and iterate to determine the responses to the inputs $x_1(k) = u_s(k)$, $x_2(k) = \cos(k\pi/4)\cdot u_s(k)$, and $x_3(k) = (-1)^k \cdot u_s(k)$.

```
%Assignment 8%
%Problem 4 b%

N = 50;
k =[0:N];
u = ones(1,N+1);
a = 1; % set a = 1 or -1
x = a.^k;
%x=cos(k*(pi/4)).*u;
y = 0*k;
y(1)=1;
for i=1:N
    y(i+1)=0.9*y(i) + x(i+1) + x(i);
end
subplot(2,1,1)
stem(k,x)
xlabel('k')
ylabel('x(k)')
subplot(2,1,2)
stem(k,y)
xlabel('k')
ylabel('y(k)')
```



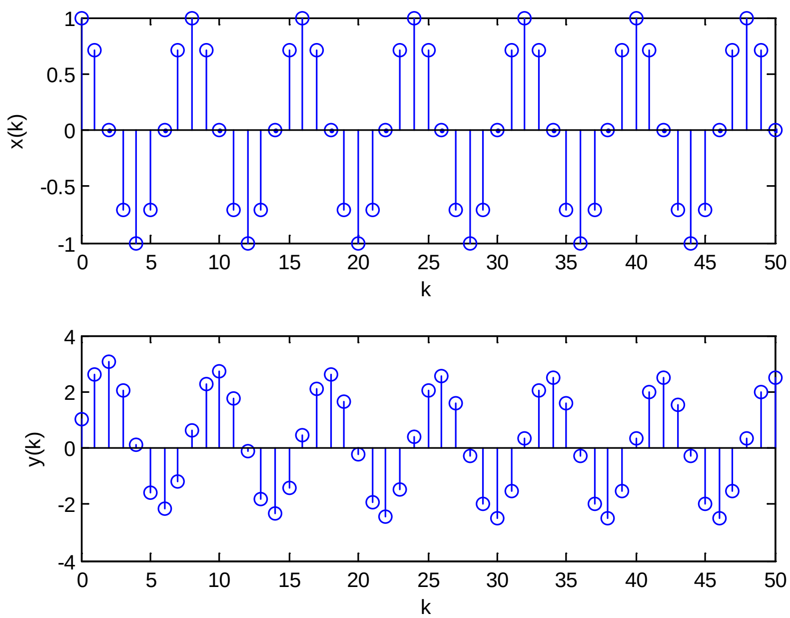

Input $x_1(k) = u_s(k)$ is constant $(\omega T = 0)$.

The steady-state output is $M_1 \cos(k\theta_1 + \phi_1) = 20\cos(k \cdot 0 + 0) = 20$ (please also see the plot of frequency response).




Input is $x_2(k) = (-1)^k \cdot u_s(k)$.

The steady-state output amplitude is $M_2 \cos(k\theta_2 + \phi_2) = 0\cos\left(k\pi - \frac{\pi}{2}\right)$, which is 0 (please also see the plot of frequency response).




Input is $x_3(k) = \cos(k\pi / 4) \cdot u_s(k)$, frequency-response at $\omega T = \theta = \dfrac{\pi}{4}$.

For long term, the steady-state output magnitude is

$M_3 \cos\left(k\theta_3 + \phi_3\right) = 2.521\cos\left(k\dfrac{\pi}{4} - 1.444rad\right)$ (please also seethe plot of frequency response. Note that $-82.76 \ \ \text{deg} = -1.444 \ \text{rad}$).

In summary, it can be seen that we have a peak, or a "high spot" in the frequency-response magnitude at $\omega T = 0$. Therefore, the amplitude response of the constant input is significantly enhanced. As $\omega T$ increases, the gain decreases, which can be observed from the amplitude response of $\cos\left(k\dfrac{\pi}{4}\right)u_s\left(k\right)$. At the high frequency $\omega T = \pi$, the amplitude response of $\left(-1\right)^k = \cos\left(k\pi\right)$ is suppressed in a significant manner.