Signals and Systems
Fall 2012

Solution to Homework Assignment #7

1.

(a) $F(z) = 2 + 3z^{-1} + 6z^{-3} + 4z^{-7}$

Taking the inverse z-transform:

$$\boxed{f(k) = 2\delta(k) + 3\delta(k-1) + 6\delta(k-3) + 4\delta(k-7)}$$

(b) $F(z) = \dfrac{z+1}{(z-1)(z-e^{-2T})}$, $T = \dfrac{1}{2}$

$$\Rightarrow F(z) = \frac{z+1}{(z-1)(z-e^{-1})}$$

Using partial-fraction expansion, we have

$$F(z) = \frac{z+1}{(z-1)(z-e^{-1})} = A\frac{z}{z-1} + B\frac{z}{z-e^{-1}} + C$$

$$= \frac{(A+B+C)z^2 + \left[-Ae^{-1} - B - C(1+e^{-1})\right] + Ce^{-1}}{(z-1)(z-e^{-1})}$$

Equating the numerator of the fraction $F(z)$:

$$\begin{cases} A+B+C=0 \\ -Ae^{-1} - B - C(1+e^{-1}) = 1 \\ Ce^{-1} = 1 \end{cases} \Rightarrow \begin{cases} A = 2/(1-e^{-1}) \\ B = -(1+e)/(1-e^{-1}) \\ C = e \end{cases}$$

This yields:

$$F(z) = \left(\frac{2}{1-e^{-1}}\right)\left(\frac{z}{z-1}\right) - \left(\frac{e+1}{1-e^{-1}}\right)\left(\frac{z}{z-e^{-1}}\right) + e$$

Taking the inverse z-transform by looking up from table, we obtain:

$$\boxed{f(k) = \frac{2}{(1-e^{-1})}u_s(k) - \left(\frac{e+1}{1-e^{-1}}\right)e^{-k}u_s(k) + e\delta(k)}$$

(c) $F(z) = \dfrac{z+1}{z^6(z-1)} = \dfrac{1}{z^6}\left(\dfrac{z}{z-1} + \dfrac{1}{z-1}\right) = \dfrac{1}{z^6}\left(\dfrac{z}{z-1}\right) + \dfrac{1}{z^7}\left(\dfrac{z}{z-1}\right)$

Taking the inverse z-transform yields:

$$\boxed{f(k) = E^{-6}\left\{u_s(k) + u_s(k-1)\right\} = u_s(k-6) + u_s(k-7)}$$

(d) $F(z)=\dfrac{z+1}{z^2-3z+2}=\dfrac{z+1}{(z-1)(z-2)}$

Analyzing $F(z)$ as following:

$$F(z)=\frac{z+1}{(z-1)(z-2)}=A\frac{z}{z-1}+B\frac{z}{z-2}+C$$
$$=\frac{(A+B+C)z^2+(-2A-B-3C)z+2C}{(z-1)(z-2)}$$

Equating the numerator of the fraction $F(z)$:

$$\begin{cases}A+B+C=0\\-2A-B-3C=1\\2C=1\end{cases}\Rightarrow\begin{cases}A=-2\\B=3/2\\C=1/2\end{cases}$$

This yields:

$$F(z)=-2\left(\frac{z}{z-1}\right)+\frac{3}{2}\left(\frac{z}{z-2}\right)+\frac{1}{2}$$

Taking the inverse z-transform by looking up from table, we obtain:

$$\boxed{f(k)=-2u_s(k)+\frac{3}{2}\left(2^k\right)u_s(k)+\frac{1}{2}\delta(k)}$$

(e) $F(z)=\dfrac{z^2+2z+1}{(z+0.5)^3(z-1)}=A\dfrac{z}{z-1}+B\dfrac{z}{z+0.5}+C\dfrac{(-0.5)z}{(z+0.5)^2}+D\dfrac{2(-0.5)z}{(z+0.5)^3}+E$

in which we used the forms from the z-transform table.

+ Multiply by $(z-1)$, set $z=1$:

$$(z-1)F(z)=\frac{z^2+2z+1}{(z+0.5)^3}=Az+B\frac{z(z-1)}{z+0.5}+C\frac{(-0.5)z(z-1)}{(z+0.5)^2}$$
$$+D\frac{2(-0.5)z(z-1)}{(z+0.5)^3}+E(z-1)$$

$$\frac{1+2+1}{(1+0.5)^3}=A.1+B.0+C.0+D.0+E.0\Rightarrow A=\frac{32}{27}$$

+ Multiply by $(z+0.5)^3$, set $z=-0.5$:

$$(z+0.5)^3F(z)=\frac{z^2+2z+1}{(z-1)}=A\frac{z(z+0.5)^3}{z-1}+Bz(z+0.5)^2+C(-0.5)z(z+0.5)$$
$$+D.2(-0.5)z+E(z+0.5)^3$$

$$\frac{(-0.5)^2+2(-0.5)+1}{(-0.5-1)}=A.0+B.0+C.0+D.2(-0.5)(-0.5)+E.0\Rightarrow D=-\frac{1}{3}$$

+ Set $z=0$:

$$\frac{1}{(-1)(0.5)^3} = A.0 + B.0 + C.0 + D.0 + E(0.5)^3 \Rightarrow E = -8$$

+ Set $z=\infty$:

$$0 = A + B + E \Rightarrow B = -E - A = 8 - \frac{32}{27} = \frac{184}{27}$$

+ Set $z=-1$:

$$0 = A.\frac{1}{2} + B.\frac{-1}{(-1/2)} + C.\frac{(-1).(-1/2)}{(-1/2)^2} + D.\frac{2.(-1/2)(-1)}{(-1/2)^3} + E$$

$$\Rightarrow 0 = \frac{1}{2}A + 2B + 2C - 8D + E$$

$$\Rightarrow C = \frac{-120}{27}$$

Substituting A,B,C,D and E and taking inverse z-transform yields:

$$\boxed{f(k) = \left[\frac{32}{27} + \frac{184}{27}\left(-\frac{1}{2}\right)^k - \frac{120}{27}k\left(-\frac{1}{2}\right)^k - \frac{1}{3}k(k-1)\left(-\frac{1}{2}\right)^{k-1}\right]u_s(k) - 8\delta(k)}$$

(f) $F(z) = \dfrac{z+1}{z^2-2z+2}$

Poles: $1 \pm j = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right)$

From the table we have

$$g(k) = \left|\sqrt{2}\right|^k \cos\left(k\frac{\pi}{4}\right)u_s(k) \Leftrightarrow G(z) = \frac{z\left(z - \left|\sqrt{2}\right|\cos(\pi/4)\right)}{z^2 - \left(2\left|\sqrt{2}\right|\cos(\pi/4)\right)z + \left|\sqrt{2}\right|^2} = \frac{z(z-1)}{z^2-2z+2}$$

$$g(k) = \left|\sqrt{2}\right|^k \sin\left(k\frac{\pi}{4}\right)u_s(k) \Leftrightarrow G(z) = \frac{z\left(\left|\sqrt{2}\right|\sin(\pi/4)\right)}{z^2 - \left(2\left|\sqrt{2}\right|\cos(\pi/4)\right)z + \left|\sqrt{2}\right|^2} = \frac{z}{z^2-2z+2}$$

One form of the inverse z-transform:

$$F(z) = \frac{z}{z^2-2z+2} + \frac{1}{z^2-2z+2} = \frac{z}{z^2-2z+2} + \frac{1}{z}\left(\frac{z}{z^2-2z+2}\right)$$

$$\boxed{\Rightarrow f(k) = \left(\sqrt{2}\right)^k \sin\left(k\frac{\pi}{4}\right)u_s(k) + \left(\sqrt{2}\right)^{k-1}\sin\left((k-1)\frac{\pi}{4}\right)u_s(k-1)}$$

Another form of the inverse z-transform:

$$F(z)=\frac{z}{z^2-2z+2}+\frac{1}{z^2-2z+2}=\frac{1}{z}\left(\frac{z(z-1)}{z^2-2z+2}\right)+\frac{2}{z}\left(\frac{z}{z^2-2z+2}\right)$$

$$\boxed{\Rightarrow f(k)=\left(\sqrt{2}\right)^{k-1}\cos\left((k-1)\frac{\pi}{4}\right)u_s(k-1)+2\left(\sqrt{2}\right)^{k-1}\sin\left((k-1)\frac{\pi}{4}\right)u_s(k-1).}$$

2.

(a) $F(z) = 2 + 3z^{-1} + 6z^{-3} + 4z^{-7}$

$$\Rightarrow F(z)=\frac{2z^7+3z^6+6z^4+4}{z^7}$$

All poles are at $z=0$. Hence, FVT applies.

$$\lim_{k\to\infty} f(k)=\lim_{z\to 1}(z-1)F(z)=0$$

(b) $F(z) = \dfrac{z+1}{(z-1)(z-e^{-1})}$

$z=1$ and $z=e^{-1}\approx 0.367$

FVT applies.

$$\lim_{k\to\infty} f(k)=\lim_{z\to 1}\frac{z+1}{z-e^{-1}}=\lim_{z\to 1}\frac{2}{1-e^{-1}}=3.16395$$

(c) $F(z) = \dfrac{z+1}{z^6(z-1)}$

Poles are at $z=0$ and $z=1$.

FVT applies.

$$\lim_{k\to\infty} f(k)=\lim_{z\to 1}\frac{z+1}{z^6}=2$$

(d) $F(z) = \dfrac{z+1}{z^2-3z+2}$

$z=2$ and $z=1$, FVT does not apply.

(e) $F(z) = \dfrac{z^2+2z+1}{(z+0.5)^3(z-1)}$

$z=-0.5$ and $z=1$, FVT applies.

$$\lim_{k\to\infty} f(k)=\lim_{z\to 1}\frac{z^2+2z+1}{(z+0.5)^3}=\lim_{z\to 1}\frac{4}{(1+0.5)^3}=\frac{32}{27}$$

(f) $F(z) = \dfrac{z+1}{z^2 - 2z + 2}$

Poles: $1 \pm j = \sqrt{2}\left(\cos\left(\dfrac{\pi}{4}\right) + j\sin\left(\dfrac{\pi}{4}\right)\right)$

FVT does not apply.

3.

(a) $h(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$, $\quad x(k) = u_s(k)$

$\Rightarrow h(k) = u_s(k)$

The z-transform of the impulse response and input unit step sequence is:

$H(z) = \dfrac{z}{z-1} \qquad X(z) = \dfrac{z}{z-1}$

Zero-state response:

$$Y(z) = H(z)X(z) = \frac{z^2}{(z-1)^2}$$

Taking the inverse z-transform yields:

$$\boxed{y(k) = (k+1)u_s(k+1) = (k+1)u_s(k)}$$

(b) $h(k) = \begin{cases} 1, & k = 0 \\ 2, & k = 1 \\ 3, & k = 2 \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow h(k) = \delta(k) + 2\delta(k-1) + 3\delta(k-2)$

The z-transform of impulse response:

$$H(z) = 1 + \frac{2}{z} + \frac{3}{z^2} = \frac{z^2 + 2z + 3}{z^2}$$

Then, the zero-state response is given by:

$$Y(z) = H(z)X(z) = \left(\frac{z^2 + 2z + 3}{z^2}\right)\left(\frac{z}{z-1}\right)$$

$$= \frac{z^2 + 2z + 3}{z(z-1)} = \frac{6z}{z-1} - \frac{3}{z} - 5$$

Taking the inverse z-transform, we obtain:

$$\boxed{y(k) = 6u_s(k) - 5\delta(k) - 3\delta(k-1)}$$

(c) $h(k) = \begin{cases} (0.9)^k, & k \geq 0 \\ 0, & k < 0 \end{cases} \qquad \Rightarrow h(k) = (0.9)^k u_s(k)$

The z-transform is follow then:

$$H(z) = \frac{z}{z-0.9}$$

The zero-state response:

$$Y(z) = H(z)X(z) = \frac{z^2}{(z-0.9)(z-1)}$$

$$= \left[\frac{Az}{z-0.9} + \frac{Bz}{z-1} + C\right]$$

+ Multiply $Y(z)$ by $(z-0.9)$ and set $z = 0.9$ we get $A = -9$

+ Multiply $Y(z)$ by $(z-1)$ and set $z = 1$ we get $B = 10$

+ Set $z = \infty$ we obtain $C = 0$

Then, $Y(z) = 10\frac{z}{z-1} - 9\frac{z}{z-0.9}$

Taking the inverse z-transform:

$$\boxed{y(k) = 10u_s(k) - 9(0.9)^k u_s(k).}$$

4.

(a) $h(k) = (1/2)^k u_s(k) + \delta(k)$, $x(k) = u_s(k)$. $\qquad \Rightarrow H(z) = \frac{z}{z-0.5} + 1 = \frac{2z-0.5}{z-0.5}$

$x(k) = u_s(k) \qquad \Rightarrow X(z) = \frac{z}{z-1}$

The zero-state response is:

$$Y(z) = H(z)X(z) = \frac{z(2z-0.5)}{(z-0.5)(z-1)} = \frac{-z}{z-0.5} + \frac{3z}{z-1}$$

Taking the inverse z-transform:

$$\boxed{y(k) = -(0.5)^k u_s(k) + 3u_s(k)}$$

(b) Same answer with question 3.(b)

(c) $h(k) = u_s(k) \qquad \Rightarrow H(z) = \frac{z}{z-1}$

$x(k) = (1/2)^k u_s(k) + \delta(k)$. $\qquad \Rightarrow X(z) = \frac{z}{z-0.5} + 1 = \frac{2z-0.5}{z-0.5}$

The zero-state response:

$$Y(z) = H(z)X(z) = \frac{z(2z-0.5)}{(z-0.5)(z-1)} = \frac{-z}{z-0.5} + \frac{3z}{z-1}$$

Taking the inverse z-transform yields:

$$\boxed{y(k) = -(0.5)^k u_s(k) + 3u_s(k)}$$

(d) $h(k) = (1/2)^k \cos(k\pi/4) u_s(k)$, $\quad \Rightarrow H(z) = \dfrac{z\left(z - \frac{\sqrt{2}}{4}\right)}{z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}}$

$x(k) = u_s(k)$. $\quad \Rightarrow X(z) = \dfrac{z}{z-1}$

The zero-state response:

$$Y(z) = H(z)X(z) = \frac{z^2\left(z - \frac{\sqrt{2}}{4}\right)}{\left(z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}\right)(z-1)}$$

$$= 1.1907\left(\frac{z}{z-1}\right) - 0.1907\left(\frac{z(z-3.4142)}{z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}}\right)$$

$$= 1.1907\left(\frac{z}{z-1}\right) - 0.1907\left(\frac{z\left(z - \frac{\sqrt{2}}{4}\right)}{z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}} - \frac{3.0607z}{z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}}\right)$$

Taking the inverse z-transform:

$$\boxed{y(k) = 1.1907u_s(k) - 0.1907(0.5)^k \cos\left(k\frac{\pi}{4}\right)u_s(k) + 1.6509(0.5)^k \sin\left(k\frac{\pi}{4}\right)u_s(k).}$$

(e) $H(E) = \dfrac{2}{E-2}$ $\quad \Rightarrow H(z) = \dfrac{2}{z-2}$

$x(k) = k \cdot u_s(k)$. $\quad \Rightarrow X(z) = \dfrac{z}{(z-1)^2}$

The zero-state response:

$$Y(z) = H(z)X(z) = \frac{2z}{(z-2)(z-1)^2} = \frac{2z}{z-2} - \frac{2z}{z-1} - \frac{2z}{(z-1)^2}$$

Taking the inverse z-transform:

$$\boxed{y(k) = 2(2)^k u_s(k) - 2u_s(k) - 2ku_s(k)}$$

(f) $H(z) = \dfrac{z}{(z-0.25)^2}$, $X(z) = \dfrac{z}{z-0.25}$.

The zero-state response:

$$Y(z) = H(z)X(z) = \frac{z^2}{(z-0.25)^3}$$

Let $F(z) = \dfrac{z}{(z-0.25)^3}$. Taking the inverse z-transform of $F(z)$ using entry 10 from z-transform pair (table 1), we obtain:

$$f(k) = 0.5 \cdot k \cdot (k-1) \cdot 0.25^{k-2} \cdot u_s(k)$$

Since $f(k)=0$ when $k=0$, we have:

$$Y(z) = z \cdot \left[F(z) - f(0)\right]$$

Therefore,

$$\boxed{y(k) = f(k+1) = 0.5(k+1)k(0.25)^{k-1} u_s(k+1) = 0.5(k+1)k(0.25)^{k-1} u_s(k)}$$