Signals and Systems
Fall 2013

Solution to Homework Assignment #3

1.

(a) $y(k+1) + y(k) = x(k),\ y(0) = 0,\ x(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}.$

We can rewrite the equation as follows:

$$\Rightarrow y(k+1) = -y(k) + x(k)$$

By using the initial condition, we have:

$$y(0) = 0$$
$$y(1) = -y(0) + x(0) = -0 + 1 = 1$$
$$y(2) = -y(1) + x(1) = -1 + 1 = 0$$
$$y(3) = -y(2) + x(2) = -0 + 1 = 1$$

After taking a few iterations, it can be easily seen that the sequence continues to oscillate between 0 and 1.

(b) $y(k+1) + y(k) = 0,\ y(0) = 1.$

The system can be described as:

$$\Rightarrow y(k+1) = -y(k)$$

Taking a few iterations:

$$y(1) = -y(0) = -1$$
$$y(2) = -y(1) = 1$$
$$y(3) = -y(2) = -1$$
$$y(4) = -y(3) = 1$$

Similar to (a), after a few iterations, the sequence will oscillate alternatively between -1 and 1.

(c) $y(k+2) - y(k+1) - 2y(k) = x(k+1) + x(k),\ y(0) = y(1) = 0,\ x(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}.$

The equation can be written as below:

$$\Rightarrow y(k+2) = y(k+1) + 2y(k) + x(k+1) + x(k)$$

Taking a few iterations:

$x(0) = 1 \quad y(0) = 0$

$x(1) = 0 \quad y(1) = 0$

$$x(2)=0 \quad y(2)=y(1)+2y(0)+x(1)+x(0)=1$$
$$x(3)=0 \quad y(3)=y(2)+2y(1)+x(2)+x(1)=1$$
$$y(4)=y(3)+2y(2)=3$$
$$y(5)=y(4)+2y(3)=5$$
$$y(6)=y(5)+2y(4)=11$$
$$y(7)=y(6)+2y(5)=21$$
$$y(8)=y(7)+2y(6)=43$$

The sequence grows significantly without bound.

(d) $y(k+2) - y(k+1) - 2y(k) = 0$, $y(0) = 1$, $y(1) = 0$.

We can rewrite the equation as follow:

$$\Rightarrow y(k+2)=y(k+1)+2y\left(k\right)$$

Inserting the initial conditions and taking a few calculations:

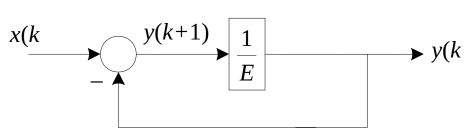
$$y(0)=1$$
$$y(1)=0$$
$$y(2)=y(1)+2y(0)=2$$
$$y(3)=y(2)+2y(1)=2$$
$$y(4)=y(3)+2y(2)=6$$
$$y(5)=y(4)+2y(3)=10$$
$$y(6)=y(5)+2y(4)=22$$
$$y(7)=y(6)+2y(5)=42$$
$$y(8)=y(7)+2y(6)=86$$

The sequence also grows unboundedly. When $k>1$, its values are as twice as the results obtained in (c).

2.

(a) $y(k+1)+y\left(k\right)=x\left(k\right) \quad \Rightarrow y(k+1)=-y\left(k\right)+x\left(k\right)$

The simulation diagram is plotted below.

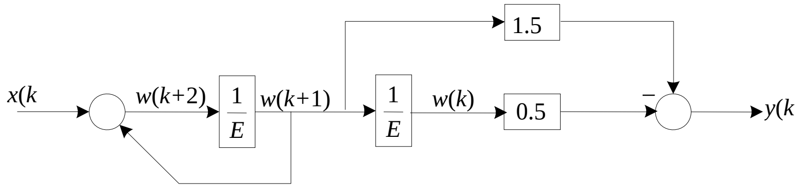


(b) $y(k+2) - y(k+1) = 1.5x(k+1) - 0.5x(k)$

$$\Rightarrow (E^2-E)\{y(k)\}=(1.5E-0.5)\{x\left(k\right)\}$$

$$\Rightarrow (E^2-E)\{w(k)\}=\{x\left(k\right)\} \text{ and } y(k)=(1.5E-0.5)\{w\left(k\right)\}$$

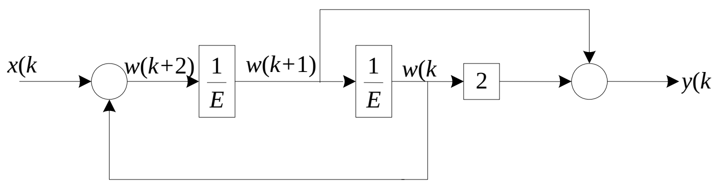
The simulation diagram can then be easily sketched as:



(c) $(E^2 - 1)\{y(k)\} = (E+2)\{x(k)\}$.

$\Rightarrow (E^2 - 1)\{w(k)\} = x(k)$ and $y(k) = (E+2)\{w(k)\}$.

The corresponding simulation diagram is plotted below.

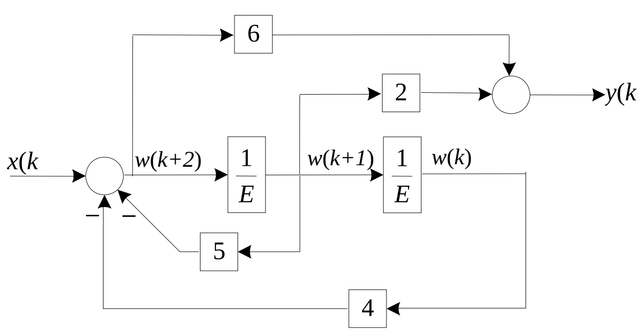


(d) $y(k) = \dfrac{6E^2 + 2E}{E^2 + 5E + 4}\{x(k)\}$

$\Rightarrow (E^2 + 5E + 4)\{y(k)\} = (6E^2 + 2E)\{x(k)\}$

$\Rightarrow (E^2 + 5E + 4)\{w(k)\} = x(k)$ and $y(k) = (6E^2 + 2E)\{w(k)\}$.

The simulation diagram is then given as:

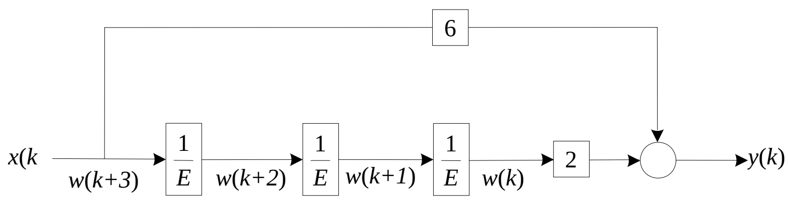


(e) $y(k) = \dfrac{6E^3 + 2}{E^3}\{x(k)\}$

$\Rightarrow E^3\{y(k)\} = (6E^3 + 2)\{x(k)\}$

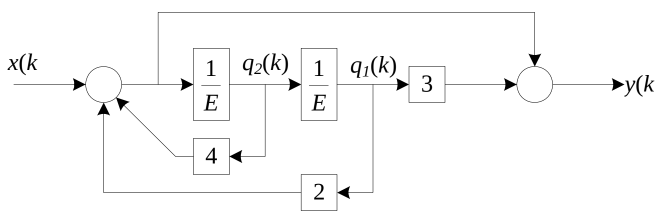
$\Rightarrow E^3\{w(k)\} = x(k)$ and $y(k) = (6E^3 + 2)\{w(k)\}$

The simulation diagram of the system is:



3.

(a)



By setting $q_1(k)$ and $q_2(k)$ after delay operators as above, we have:

- $q_1(k+1) = q_2(k) \;\Rightarrow E\{q_1(\omega)\} = q_2(k) \;\Rightarrow q_1(k) = \dfrac{1}{E}\{q_2(k)\}$
- $q_2(k+1) = 2q_1(k) + 4q_2(k) + x(k)$

$$E\{q_2(k)\} = \frac{2}{E}\{q_2(k)\} + 4q_2(k) + x(k)$$

$$\Rightarrow \left(E - 4 - \frac{2}{E}\right)\{q_2(k)\} = x(k)$$

$$\Rightarrow q_2(k) = \frac{1}{E - 4 - \dfrac{2}{E}}\{x(k)\} = \frac{E}{E^2 - 4E - 2}\{x(k)\}$$

and $q_1(k) = \dfrac{1}{E}\{q_2(k)\} = \dfrac{1}{E^2 - 4E - 2}\{x(k)\}$

Now,

$$y(k) = 3q_1(k) + q_2(k+1) = 3q_1(k) + Eq_2(k)$$

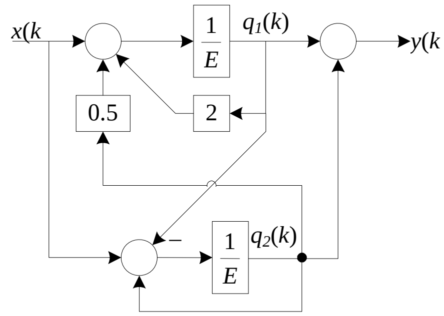
$$= \frac{3}{E^2 - 4E - 2}\{x(k)\} + \frac{E^2}{E^2 - 4E - 2}\{x(k)\}$$

Hence, $y(k) = \dfrac{E^2 + 3}{E^2 - 4E - 2}\{x(k)\}$

Therefore, the operational transfer function of the system is:

$$H(E) = \frac{E^2 + 3}{E^2 - 4E - 2}.$$

(b)



By setting $q_1(k)$ and $q_2(k)$ as above, we have:

- $q_1(k+1) = 2q_1(k) + 0.5q_2(k) + x(k)$

  $\Rightarrow (E-2)\{q_1(k)\} = 0.5q_2(k) + x(k)$

  $\Rightarrow q_1(k) = \dfrac{0.5}{E-2}\{q_2(k)\} + \dfrac{1}{E-2}\{x(k)\}$

- $q_2(k+1) = -q_1(k) + q_2(k) + x(k)$

$$\Rightarrow E\{q_2(k)\} = -\left[\frac{0.5}{E-2}\{q_2(k)\} + \frac{1}{E-2}\{x(k)\}\right] + q_2(k) + x(k)$$

$$\Rightarrow E\{q_2(k)\} = \left[1 - \frac{0.5}{E-2}\right]\{q_2(k)\} + \left[1 - \frac{1}{E-2}\right]\{x(k)\}$$

Hence,

$$q_2(k) = \frac{1 - \dfrac{1}{E-2}}{E - 1 + \dfrac{0.5}{E-2}}\{x(k)\}$$

$$= \frac{(E-2)-1}{E(E-2)-(E-2)+0.5}\{x(k)\}$$

$$= \frac{E-3}{E^2-3E+2.5}\{x(k)\}$$

On the other hand, we have:

$$y(k) = q_1(k) + q_2(k) = \frac{1}{E-2}\left[0.5\,q_2(k) + x(k)\right] + q_2(k)$$

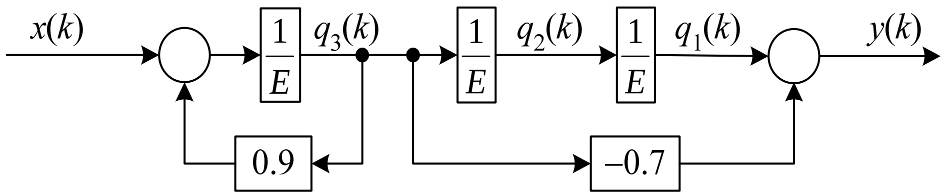
$$= \frac{1}{E-2}\left[(E-1.5)\{q_2(k)\} + x(k)\right]$$

$$= \frac{1}{E-2}\left[\frac{(E-1.5)(E-3) + E^2 + 3E - 2.5}{E^2 + 3E - 2.5}\right]\{x(k)\}$$

Finally, the operational transfer function of the system can be obtained as follows:

$$H(E) = \frac{1}{E-2}\left[\frac{2E^2 - 7.5E + 7}{E^2 - 3E + 2.5}\right] = \frac{2E - 3.5}{E^2 - 3E + 2.5}$$

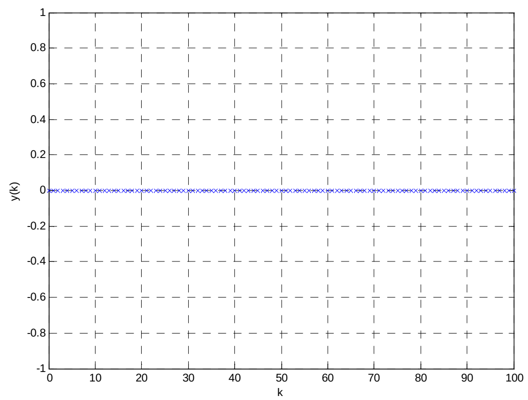
4.



From the diagram, we obtain the following expressions:

- $q_3(k+1) = x(k) + 0.9q_3(k)$
- $q_3(k) = q_2(k+1)$
- $q_2(k) = q_1(k+1)$
- $y(k) = q_1(k) - 0.7q_3(k)$

(a)



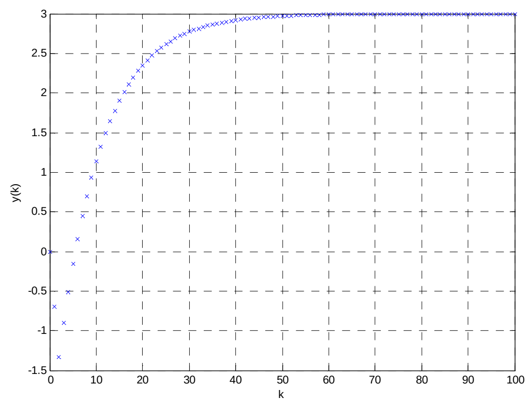

```
%Assignment 3
%Problem 4. a.

%Define Input
N=101;
time=0:N-1;
x=0*ones(1,N); % zero input.

% Initial Conditions
q1=0;
q2=0;
q3=0;
y=q1-0.7*q3;

%Iteration
for k=1:N-1
    q1(k+1)=q2(k);
    q2(k+1)=q3(k);
    q3(k+1)=0.9*q3(k)+x(k);
    y(k+1)=q1(k+1)-0.7*q3(k+1);
end
plot(time,y,'x')
hold on
grid on
```

(b)

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%Assignment 3
%Problem 4. b.

%Define Input
N=101;
time=0:N-1;
x=1*ones(1,N); % unit-step input

% Initial Conditions
q1=0;
q2=0;
q3=0;
y=q1-0.7*q3;

%Iteration
for k=1:N-1
    q1(k+1)=q2(k);
    q2(k+1)=q3(k);
    q3(k+1)=0.9*q3(k)+x(k);
    y(k+1)=q1(k+1)-0.7*q3(k+1);
end
plot(time,y,'x')
hold on
grid on
```