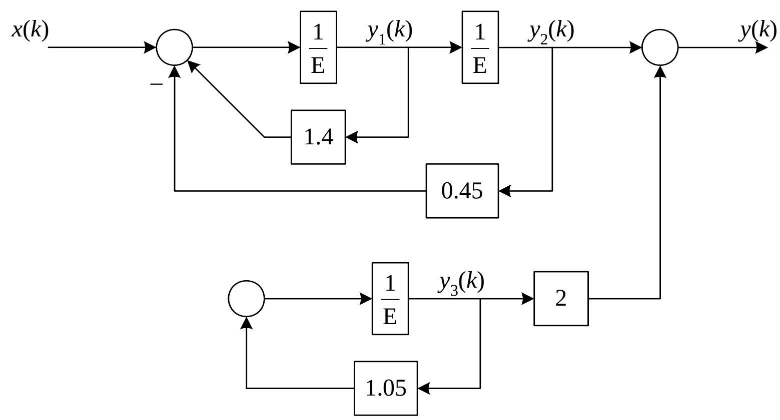
Signals and Systems
Fall 2012

Solution to Homework Assignment #4

1.



(a) From the diagram, we have:

- $y_2(k+1) = y_1(k)$

  $\Rightarrow E\{y_2(k)\} = y_1(k)$ (1)

- $y_1(k+1) = 1.4y_1(k) - 0.45y_2(k) + x(k)$

  $\Rightarrow (E-1.4)\{y_1(k)\} = -0.45y_2(k) + x(k)$ (2)

  Substituting (1) into (2) yields

  $\Rightarrow (E-1.4)(E)\{y_2(k)\} = -0.45y_2(k) + x(k)$

  $\Rightarrow (E^2 - 1.4E + 0.45)\{y_2(k)\} = x(k)$ (3)

- $y(k) = 2y_3(k) + y_2(k)$ (4)

  $y_3(k+1) = 1.05y_3(k)$

  $\Rightarrow (E-1.05)\{y_3(k)\} = 0$ (5)

  $\Rightarrow y_3(k) = m \times (1.05)^k$. Note that m depends on the initial condition.

Observe that $y_3(k)$ does not depend on the input $x(k)$. Furthermore, we have $y(k) = y_2(k) + 2y_3(k)$. As such, the operational transfer function can be obtained by using just the component $y_2(k)$ in the output. In particular, substituting (6) into (3) yields

$$(E^2 - 1.4E + 0.45)\{y(k)\} = x(k)$$

Finally, we obtain the operational transfer function of the system as:

$$H(E) = \frac{1}{E^2 - 1.4E + 0.45}$$

(b) MATLAB Program for the system in the given diagram:

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%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Assignment 4
% Problem 1. b.

%Setting up the variable and input
for i=1:21
    k(i)=i-1;
    x(i)=1;
end

%part (c) : zero input
%initial conditions
y1(1)=1; y2(1)=1;y3(1)=1;
y(1)=y2(1)+2*y3(1);

%calculating the values
for i=1:20
    y1(i+1)=1.4*y1(i)-0.45*y2(i);
    y2(i+1)=y1(i);
    y3(i+1)=1.05*y3(i);
    y(i+1)=y2(i+1)+2*y3(i+1);
end

figure(1)
subplot(221),stem(k,y1,'o');grid, title('y_1')
subplot(222),stem(k,y2,'o');grid, title('y_2')
subplot(223),stem(k,y3,'o');grid, title('y_3')
subplot(224),stem(k,y,'o');grid, title('y')

            %part (d): zero initial state
            %initial conditions
            y1(1)=0; y2(1)=0; y3(1)=0;
            y(1)=y2(1)+2*y3(1);

            %calculating the values
            for i=1:20
                y1(i+1)=1.4*y1(i)-0.45*y2(i)+x(i);
                y2(i+1)=y1(i);
                y3(i+1)=1.05*y3(i);
                y(i+1)=y2(i+1)+2*y3(i+1);
            end

            figure(2)
            subplot(221),stem(k,y1,'o');grid, title('y_1')
            subplot(222),stem(k,y2,'o');grid, title('y_2')
            subplot(223),stem(k,y3,'o');grid, title('y_3')
            subplot(224),stem(k,y,'o');grid, title('y')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
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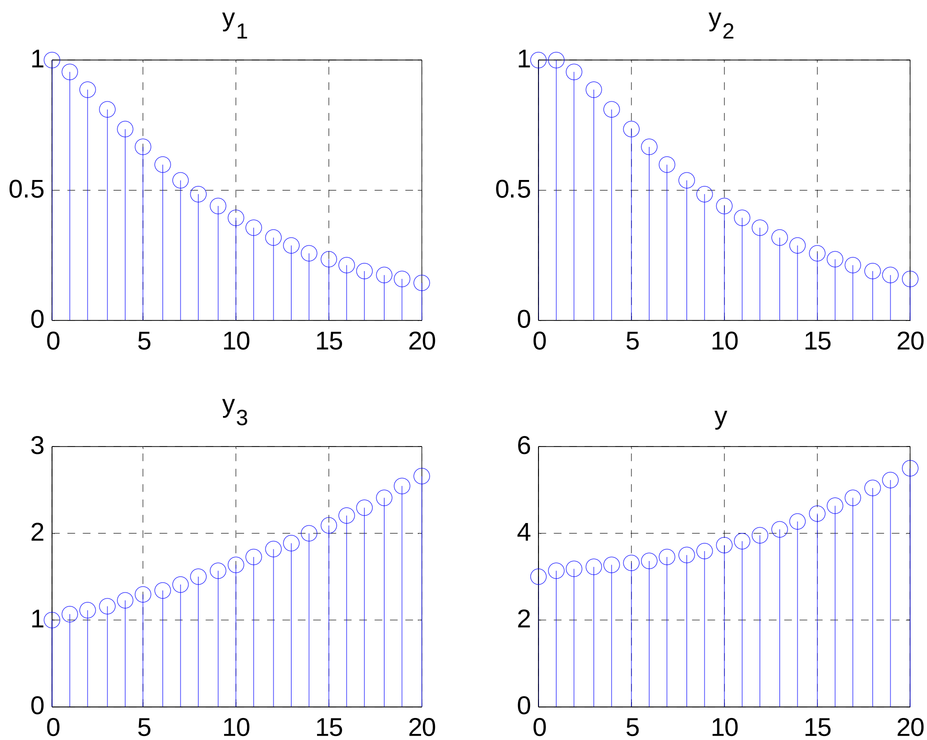
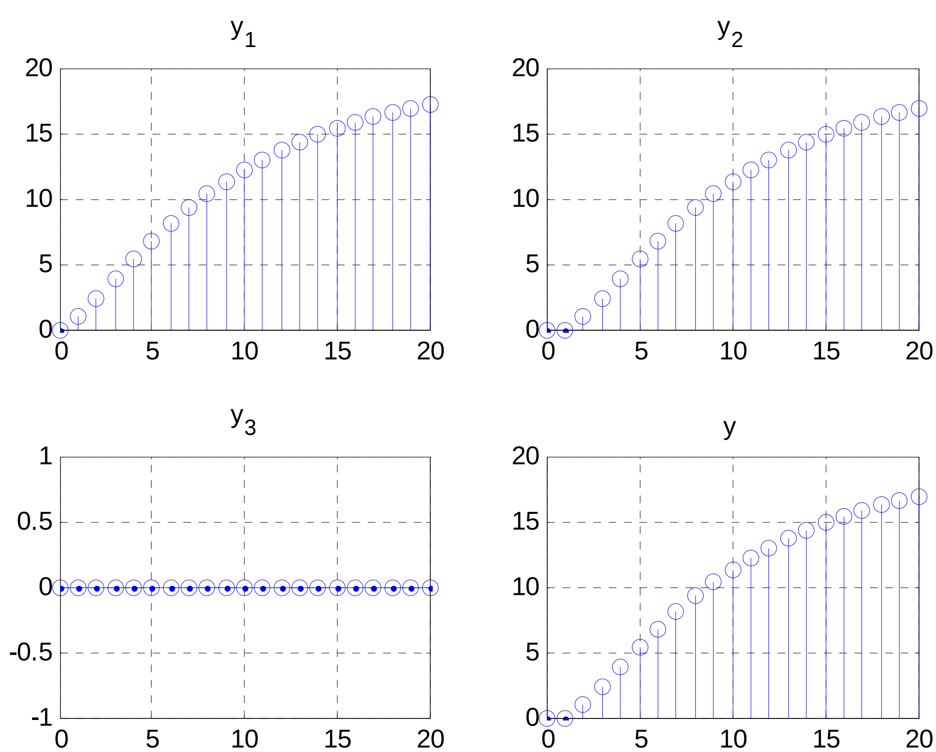
(c) With the initial conditions $y_1(0) = y_2(0) = 1$, $y_3(0) = 1$, the zero-input responses of $y_1(k)$, $y_2(k)$, $y_3(k)$, and $y(k)$, respectively, are plotted below.



(d) Given that the input $x(k)$ is the unit step sequence, the zero-state responses of $y_1(k)$, $y_2(k)$, $y_3(k)$, and $y(k)$, respectively, are plotted below:



2.

(a) (E–0.5){$y(k)$} = 0, $y(0) = 7$.

$$\Rightarrow y(k+1) = 0.5y(k),\ y(0) = 7$$

This is the 1st-order homogeneous difference equation. We can implement some iterations to find the general form of the solution:

$$y(1) = 0.5y(0) = 0.5\times 7$$
$$y(2) = 0.5y(1) = (0.5)^2 \times 7$$
$$y(3) = 0.5y(2) = (0.5)^3 \times 7$$
$$\vdots$$
$$\boxed{y(k) = 7\times (0.5)^k}$$

Note that you can also just simply using a general form that we studied in class to end up with the same result.

(b) (E–1){$y(k)$} = $3\cdot(0.5)^k$, $y(0) = 0$.

This is the forced difference equation. Its solution has the form of:

$$y(k) = y_c(k) + y_p(k).$$

- Finding the particular solution:

  The annihilator operator is $D_A(E) = E - 0.5$, since $(E-0.5)\{0.5^k\} = 0$

  Using $D_A(E)$ on the forced equation yields

$$(E-0.5)(E-1)\left\{y\left(k\right)\right\} = 0$$

  The particular solution is the term associated with the factor $(E-0.5)$

$$y_p(k) = A\times 0.5^k$$

Note that we do not have any common root between $D_A(E)$ and the characteristic polynomial (E-1). By substituting the above form into the original equation, we can determine the value of A as follows:

$$(E-1)\{A\times 0.5^k\} = 3\times 0.5^k$$
$$A\times 0.5^{k+1} - A\times 0.5^k = 3\times 0.5^k$$
$$A\times 0.5 - A = 3$$
$$A = -6$$

  Hence, $y_p(k) = -6\times (0.5)^k$

- Finding the complementary solution:

  The solution for the equation $(E-1)\left\{y\left(k\right)\right\} = 0$ has the form:

$$y_c(k) = m\times (1)^k$$

The complete solution is : $y(k) = y_p(k) + y_c(k) = m\times (1)^k - 6\times (0.5)^k$

Applying initial condition $y(0) = 0$,we obtain $m = 6$.

Finally, the solution can be expressed as:

$$\boxed{y(k) = 6 - 6(0.5)^k}$$

(c) $(E^2+3E+2)\{y(k)\} = 0$, $y(0) = 1$, $y(1) = 0$

$$\Rightarrow (E+1)(E+2)\{y(k)\} = 0$$

This is the 2nd-ordered-homogeneous difference equation, and the solutions of the equation have the form of:

$$y(k) = m_1 \times (-1)^k + m_2 \times (-2)^k$$

By using initial conditions, we have:

$$\begin{matrix} y(0) = m_1 \times (-1)^0 + m_2 \times (-2)^0 = 1 \\ y(1) = m_1 \times (-1)^1 + m_2 \times (-2)^1 = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} m_1 = 2 \\ m_2 = -1 \end{matrix}$$

The final solution can then be expressed as:

$$\boxed{y(k) = 2(-1)^k - (-2)^k}$$

(d) $(E^2+1)\{y(k)\} = 3 \cdot 2^k$, $y(0) = y(1) = 0$.

The characteristic equation $E^2 + 1 = 0$ has the $roots = \pm j = 1 \times e^{\pm j\frac{\pi}{2}}$

- Finding the particular solution:

  The annihilator operator is $D_A(E) = E - 2$, since $(E-2)\{2^k\} = 0$

  Using $D_A(E)$ on the forced equation yields

$$(E-2)(E^2+1)\left\{y\left(k\right)\right\} = 0$$

  The particular solution is the term associated with the factor $(E-2)$

$$y_p(k) = A \times 2^k$$

  Substitute into the original equation to determine the value of A:

$$(E^2+1)\{A \times 2^k\} = 3 \times 2^k$$
$$A \times 2^{k+2} + A \times 2^k = 3 \times 2^k$$
$$A \times 2^2 + A = 3$$
$$A = \frac{3}{5}$$

  Hence, $y_p(k) = \frac{3}{5} \times 2^k$

- Finding the complementary solution:

  From the root of the characteristic equation, $y_c(k)$ has the form:

$$y_c(k) = m_1 \times \cos\left(k\frac{\pi}{2}\right) + m_2 \times \sin\left(k\frac{\pi}{2}\right)$$

The complete solution is : $y(k) = y_p(k) + y_c(k) = \frac{3}{5} \times 2^k + m_1 \times \cos\left(k\frac{\pi}{2}\right) + m_2 \times \sin\left(k\frac{\pi}{2}\right)$

By using initial condition $y(0) = y(1) = 0$, we obtain $m_1 = -3/5$ and $m_2 = -6/5$

The final solution can then be expressed as:

$$\boxed{y(k) = \frac{3}{5} \times 2^k - \frac{3}{5} \times \cos\left(k\frac{\pi}{2}\right) - \frac{6}{5} \times \sin\left(k\frac{\pi}{2}\right)}$$

(e) $(E^2-1)\{y(k)\} = 0.5$, $y(0) = 1$, $y(1) = 2$.

$$\Rightarrow (E+1)(E-1)\{y(k)\} = 0.5$$

- Finding the complementary solution:

  The solution for the equation $(E^2-1)\{y(k)\} = 0$ has the form:

$$y_c(k) = m_1 \times (1)^k + m_2(-1)^k$$

- Finding the particular solution: Given that $f(k)=0.5$, we have $D_A(E) = E-1$. Therefore, there is a common root between $D_A(E)$ and $D(E) = E^2 - 1$. As such, we need to use the following form for the particular solution:

$$y_p(k) = A \times k$$

  It then follows that

$$(E^2-1)\{A \times k\} = 0.5$$
$$A(k+2) - Ak = 0.5$$
$$A = 0.25$$

  Hence, the particular solution is: $y_p(k) = 0.25k$

The complete solution of the equation can then be express as:

$$y(k) = y_c(k) + y_p(k) = m_1 \times (1)^k + m_2(-1)^k + 0.25k$$

Now, using the initial conditions $y(0) = 1$, $y(1) = 2$, we obtain $m_1 = 11/8$ and $m_2 = -3/8$

The final solution can then be express as:

$$\boxed{y(k) = \frac{11}{8} - \frac{3}{8} \times (-1)^k + 0.25k}.$$

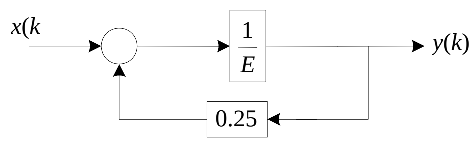
3.

(a) $y(k) = \dfrac{1}{E - 0.25}\{x(k)\}$ [Ans: $h(0) = 0$; $h(k) = (0.25)^{k-1}$, $k = 1,2,3,\cdots$.]

From the input-output equation, the transfer function can be expressed as:

$$H(E) = \frac{1}{E - 0.25} \Rightarrow y(k+1) = 0.25y(k) + x(k)$$

The diagram for the system is plotted below:



Impulse response:

$$y(0) = 0$$
$$y(1) = 0.25(0) + 1 = 1$$
$$y(2) = (0.25)(1) + 0 = 0.25$$
$$y(3) = (0.25)(0.25) = (0.25)^2$$
$$\vdots$$

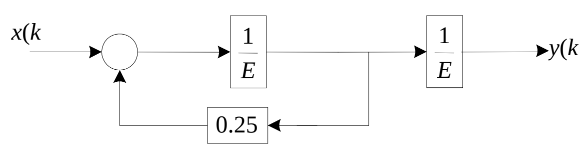
Therefore, the impulse response is: $h(k) = (0.25)^{k-1}u_s(k-1)$

(b) $y(k) = \dfrac{1}{E(E - 0.25)}\{x(k)\}$

The transfer function is given as:

$$H(E) = \frac{1}{E\left(E - 0.25\right)} \Rightarrow y(k+2) = 0.25y(k+1) + x(k)$$

The diagram for the system is plotted below:



Impulse response:

$$y(0) = 0$$
$$y(1) = 0$$
$$y(2) = 1$$
$$y(3) = 0.25$$
$$\vdots$$

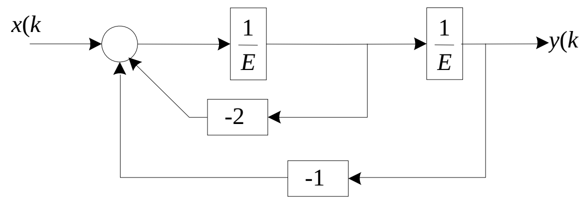
The impulse response is: $h(k) = (0.25)^{k-2}u_s(k-2)$

(c) $$y(k) = \frac{1}{E^2 + 2E + 1}\{x(k)\}$$

From the input-output equation, the transfer function is:

$$H(E) = \frac{1}{E^2 + 2E + 1} \Rightarrow y(k+2) = -2y(k+1) - y(k) + x(k)$$

The diagram for the system is shown below:



Impulse response:

$y(0) = 0$

$y(1) = 0$

$y(2) = x(0) = 1$

$y(3) = -2y(2) = -2$

$y(4) = -2y(3) - y(2) = 3$

$y(5) = -2y(4) - y(3) = -4$

$\vdots$

The impulse response is: $h(k) = (k-1)(-1)^{k-2}u_s(k-1)$