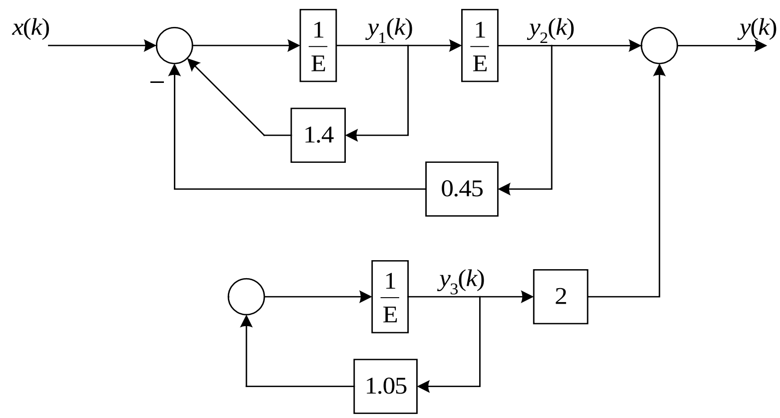
Signals and Systems
Fall 2013

Solution to Homework Assignment #4

1.



(a) From the diagram, we have:

- $y_2(k+1) = y_1(k)$

  $\Rightarrow E\{y_2(k)\} = y_1(k)$ (1)

- $y_1(k+1) = 1.4y_1(k) - 0.45y_2(k) + x(k)$

  $\Rightarrow (E-1.4)\{y_1(k)\} = -0.45y_2(k) + x(k)$ (2)

  Substituting (1) into (2) yields

  $\Rightarrow (E-1.4)(E)\{y_2(k)\} = -0.45y_2(k) + x(k)$

  $\Rightarrow (E^2 - 1.4E + 0.45)\{y_2(k)\} = x(k)$ (3)

- $y(k) = 2y_3(k) + y_2(k)$ (4)

  $y_3(k+1) = 1.05y_3(k)$

  $\Rightarrow (E-1.05)\{y_3(k)\} = 0$ (5)

  $\Rightarrow y_3(k) = m \times (1.05)^k$. Note that m depends on the initial condition.

Observe that $y_3(k)$ does not depend on the input $x(k)$. Furthermore, we have $y(k) = y_2(k) + 2y_3(k)$. As such, the operational transfer function can be obtained by using just the component $y_2(k)$ in the output. In particular, substituting (6) into (3) yields

$$(E^2 - 1.4E + 0.45)\{y(k)\} = x(k)$$

Finally, we obtain the operational transfer function of the system as:

$$H(E) = \frac{1}{E^2 - 1.4E + 0.45}$$

(b) MATLAB Program for the system in the given diagram:

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%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Assignment 4
% Problem 1. b.

%Setting up the variable and input
for i=1:21
    k(i)=i-1;
    x(i)=1;
end

%part (c) : zero input
%initial conditions
y1(1)=1; y2(1)=1;y3(1)=1;
y(1)=y2(1)+2*y3(1);

%calculating the values
for i=1:20
    y1(i+1)=1.4*y1(i)-0.45*y2(i);
    y2(i+1)=y1(i);
    y3(i+1)=1.05*y3(i);
    y(i+1)=y2(i+1)+2*y3(i+1);
end

figure(1)
subplot(221),stem(k,y1,'o');grid, title('y_1')
subplot(222),stem(k,y2,'o');grid, title('y_2')
subplot(223),stem(k,y3,'o');grid, title('y_3')
subplot(224),stem(k,y,'o');grid, title('y')

            %part (d): zero initial state
            %initial conditions
            y1(1)=0; y2(1)=0; y3(1)=0;
            y(1)=y2(1)+2*y3(1);

            %calculating the values
            for i=1:20
                y1(i+1)=1.4*y1(i)-0.45*y2(i)+x(i);
                y2(i+1)=y1(i);
                y3(i+1)=1.05*y3(i);
                y(i+1)=y2(i+1)+2*y3(i+1);
            end

            figure(2)
            subplot(221),stem(k,y1,'o');grid, title('y_1')
            subplot(222),stem(k,y2,'o');grid, title('y_2')
            subplot(223),stem(k,y3,'o');grid, title('y_3')
            subplot(224),stem(k,y,'o');grid, title('y')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
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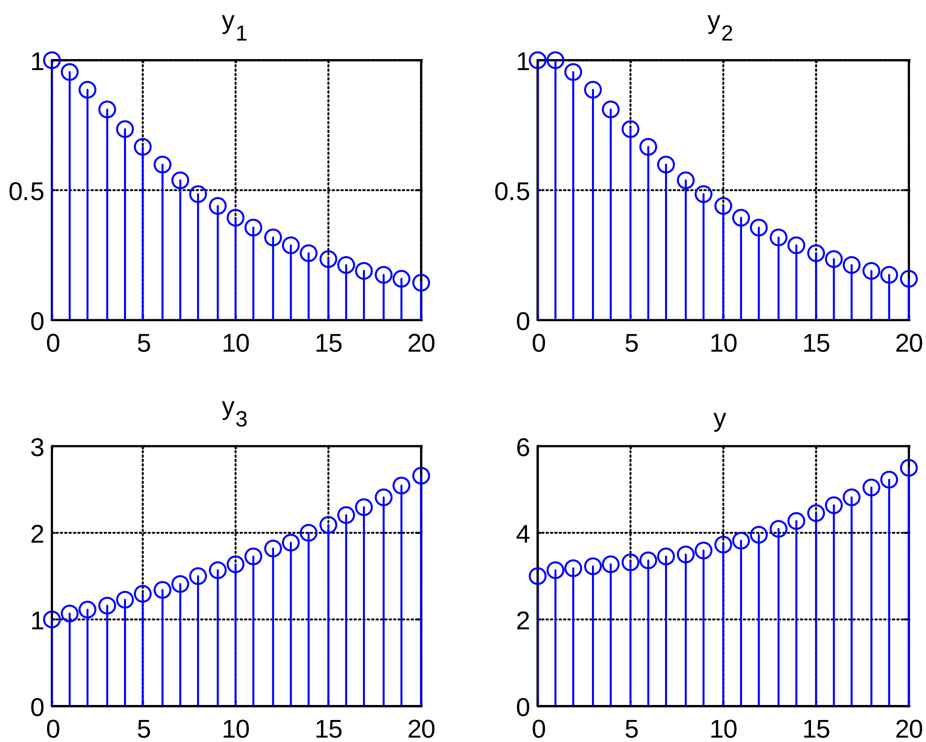
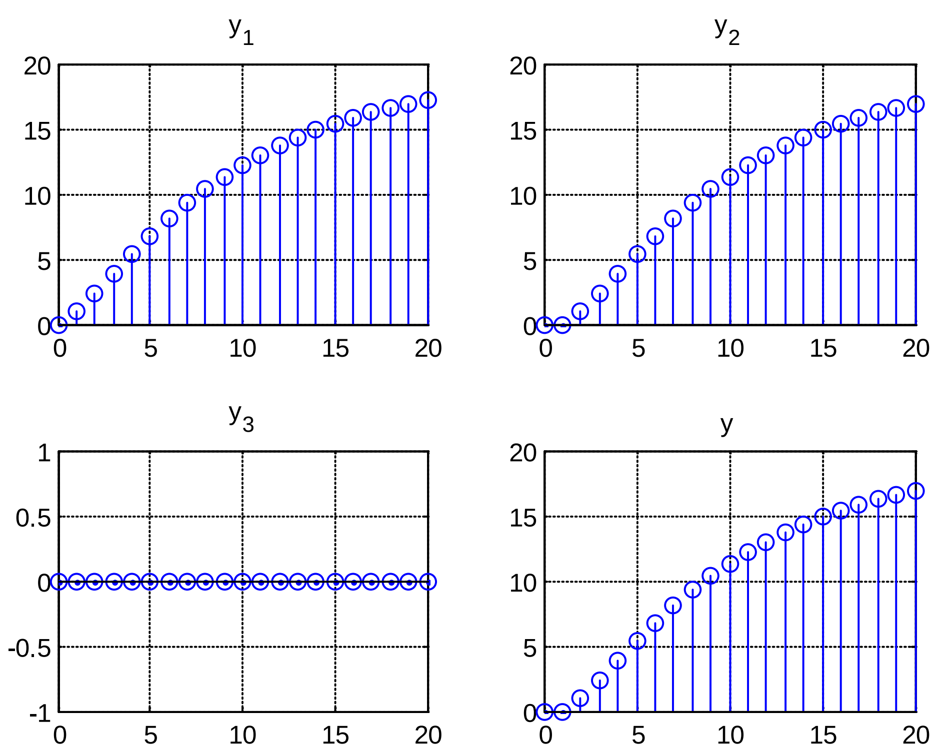
(c) With the initial conditions $y_1(0) = y_2(0) = 1$, $y_3(0) = 1$, the zero-input responses of $y_1(k)$, $y_2(k)$, $y_3(k)$, and $y(k)$, respectively, are plotted below.



(d) Given that the input $x(k)$ is the unit step sequence, the zero-state responses of $y_1(k)$, $y_2(k)$, $y_3(k)$, and $y(k)$, respectively, are plotted below:



2.

(a) $(E-0.5)\{y(k)\} = 0$, $y(0) = 7$.

$$\Rightarrow y(k+1) = 0.5y(k),\ y(0) = 7$$

This is the 1st-order homogeneous difference equation. We can implement some iterations to find the general form of the solution:

$$y(1) = 0.5y(0) = 0.5\times 7$$

$$y(2) = 0.5y(1) = (0.5)^2\times 7$$

$$y(3) = 0.5y(2) = (0.5)^3\times 7$$

$$\vdots$$

$$\boxed{y(k) = 7\times(0.5)^k}$$

Note that you can also just simply using a general form that we studied in class to end up with the same result.

(b) $(E-1)\{y(k)\} = 3\cdot(0.5)^k$, $y(0) = 0$.

This is the forced difference equation. Its solution has the form of:

$$y(k) = y_c(k) + y_p(k).$$

- Finding the particular solution:

  The annihilator operator is $D_A(E) = E - 0.5$, since $(E-0.5)\{0.5^k\} = 0$

  Using $D_A(E)$ on the forced equation yields

  $$(E-0.5)(E-1)\left\{y\left(k\right)\right\} = 0$$

  The particular solution is the term associated with the factor $(E-0.5)$

  $$y_p(k) = A\times 0.5^k$$

  Note that we do not have any common root between $D_A(E)$ and the characteristic polynomial (E-1). By substituting the above form into the original equation, we can determine the value of A as follows:

$$(E-1)\{A\times 0.5^k\} = 3\times 0.5^k$$

$$A\times 0.5^{k+1} - A\times 0.5^k = 3\times 0.5^k$$

$$A\times 0.5 - A = 3$$

$$A = -6$$

  Hence, $y_p(k) = -6\times(0.5)^k$

- Finding the complementary solution:

  The solution for the equation $(E-1)\left\{y\left(k\right)\right\} = 0$ has the form:

  $$y_c(k) = m\times(1)^k$$

The complete solution is : $y(k) = y_p(k) + y_c(k) = m\times(1)^k - 6\times(0.5)^k$

Applying initial condition $y(0) = 0$,we obtain $m = 6$.

Finally, the solution can be expressed as:

$$\boxed{y(k) = 6 - 6(0.5)^k}$$

(c) $(E^2+3E+2)\{y(k)\} = 0$, $y(0) = 1$, $y(1) = 0$

$$\Rightarrow (E+1)(E+2)\{y(k)\} = 0$$

This is the 2nd-ordered-homogeneous difference equation, and the solutions of the equation have the form of:

$$y(k) = m_1 \times (-1)^k + m_2 \times (-2)^k$$

By using initial conditions, we have:

$$\begin{matrix} y(0) = m_1 \times (-1)^0 + m_2 \times (-2)^0 = 1 \\ y(1) = m_1 \times (-1)^1 + m_2 \times (-2)^1 = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} m_1 = 2 \\ m_2 = -1 \end{matrix}$$

The final solution can then be expressed as:

$$\boxed{y(k) = 2(-1)^k - (-2)^k}$$

(d) $(E^2+1)\{y(k)\} = 3 \cdot 2^k$, $y(0) = y(1) = 0$.

The characteristic equation $E^2 + 1 = 0$ has the $roots = \pm j = 1 \times e^{\pm j\frac{\pi}{2}}$

- Finding the particular solution:

  The annihilator operator is $D_A(E) = E - 2$, since $(E-2)\{2^k\} = 0$

  Using $D_A(E)$ on the forced equation yields

  $$(E-2)(E^2+1)\left\{y\left(k\right)\right\} = 0$$

  The particular solution is the term associated with the factor $(E-2)$

  $$y_p(k) = A \times 2^k$$

  Substitute into the original equation to determine the value of A:

  $$(E^2+1)\{A \times 2^k\} = 3 \times 2^k$$
  $$A \times 2^{k+2} + A \times 2^k = 3 \times 2^k$$
  $$A \times 2^2 + A = 3$$
  $$A = \frac{3}{5}$$

  Hence, $y_p(k) = \frac{3}{5} \times 2^k$

- Finding the complementary solution:

  From the root of the characteristic equation, $y_c(k)$ has the form:

$$y_c(k) = m_1 \times \cos\left(k\frac{\pi}{2}\right) + m_2 \times \sin\left(k\frac{\pi}{2}\right)$$

The complete solution is : $y(k) = y_p(k) + y_c(k) = \frac{3}{5} \times 2^k + m_1 \times \cos\left(k\frac{\pi}{2}\right) + m_2 \times \sin\left(k\frac{\pi}{2}\right)$

By using initial condition $y(0) = y(1) = 0$, we obtain $m_1 = -3/5$ and $m_2 = -6/5$

The final solution can then be expressed as:

$$\boxed{y(k) = \frac{3}{5} \times 2^k - \frac{3}{5} \times \cos\left(k\frac{\pi}{2}\right) - \frac{6}{5} \times \sin\left(k\frac{\pi}{2}\right)}$$

(e) $(E^2-1)\{y(k)\} = 0.5$, $y(0) = 1$, $y(1) = 2$.

$$\Rightarrow (E+1)(E-1)\{y(k)\} = 0.5$$

- Finding the complementary solution:

  The solution for the equation $(E^2-1)\{y(k)\} = 0$ has the form:

$$y_c(k) = m_1 \times (1)^k + m_2(-1)^k$$

- Finding the particular solution: Given that $f(k)$=0.5, we have $D_A(E) = E - 1$. Therefore, there is a common root between $D_A(E)$ and $D(E) = E^2 - 1$. As such, we need to use the following form for the particular solution:

$$y_p(k) = A \times k$$

  It then follows that

$$(E^2-1)\{A \times k\} = 0.5$$
$$A(k+2) - Ak = 0.5$$
$$A = 0.25$$

  Hence, the particular solution is: $y_p(k) = 0.25k$

The complete solution of the equation can then be express as:

$$y(k) = y_c(k) + y_p(k) = m_1 \times (1)^k + m_2(-1)^k + 0.25k$$

Now, using the initial conditions $y(0) = 1$, $y(1) = 2$, we obtain $m_1 = 11/8$ and $m_2 = -3/8$

The final solution can then be express as:

$$\boxed{y(k) = \frac{11}{8} - \frac{3}{8} \times (-1)^k + 0.25k}.$$

3.

(a) $(E^2+3E+2)\{y(k)\} = x(k)$

The characteristic polynomial is

$$D(E) = E^2 + 3E + 2 = (E+1)(E+2)$$

It can be seen that we have two roots $-1, -2$, which are not inside the unit circle. The system is unstable. Therefore, the system is not BIBO stable.

(b) $(10E^2+3E+2)\{y(k)\} = (E-2)\{x(k)\}$

We have

$$D(E) = 10E^2 + 3E + 2$$

The two conjugate roots are:

$$\frac{-3 \pm \sqrt{9-80}}{20} = \frac{-3}{20} \pm j\frac{\sqrt{71}}{20} = -0.15 \pm j0.4218$$

It is not hard to verify that these roots are inside the unit circle. Hence, the system is <u>stable</u>.

(c) $(E^2+1.2E+0.2)\{y(k)\} = x(k)$

We have:

$$D(E) = E^2 + 1.2E + 0.2 = (E+1)(E+0.2)$$

We therefore have two roots -1 and -0.2. Since the root -1 is on the unit circle, the system is marginally stable. Therefore, the system is not BIBO stable.

(d) $(E^3+3E^2+3E+1)\{y(k)\} = (E^2-0.1E)\{x(k)\}$

We have:

$$D(E) = E^3 + 3E^2 + 3E + 1 = (E+1)^3$$

Obviously, we have a repeated root on the unit circle. Therefore, the system is not BIBO stable.