Signals and Systems
Fall 2012

Solution to Homework Assignment #5

1.

(a) $(E^2+3E+2)\{y(k)\} = x(k)$

The characteristic polynomial is

$$D(E) = E^2 + 3E + 2 = (E+1)(E+2)$$

It can be seen that we have two roots $-1, -2$, which are not inside the unit circle. The system is therefore unstable.

(b) $(10E^2+3E+2)\{y(k)\} = (E-2)\{x(k)\}$

We have

$$D(E) = 10E^2 + 3E + 2$$

The two conjugate roots are:

$$\frac{-3 \pm \sqrt{9-80}}{20} = \frac{-3}{20} \pm j\frac{\sqrt{71}}{20} = -0.15 \pm j0.4218$$

It is not hard to verify that these roots are inside the unit circle. Hence, the system is stable.

(c) $(E^2+1.2E+0.2)\{y(k)\} = x(k)$

We have:

$$D(E) = E^2 + 1.2E + 0.2 = (E+1)(E+0.2)$$

We therefore have two roots -1 and -0.2. Since the root -1 is on the unit circle, the system is marginally stable. It is not BIBO stable.

(d) $(E^3+3E^2+3E+1)\{y(k)\} = (E^2-0.1E)\{x(k)\}$

We have:

$$D(E) = E^3 + 3E^2 + 3E + 1 = (E+1)^3$$

Obviously, we have a repeated root on the unit circle. Therefore, the system is not BIBO stable

2. (a) Given the operation transfer function

$$H_1(\mathrm{E}) = \frac{E}{E - 0.5}.$$

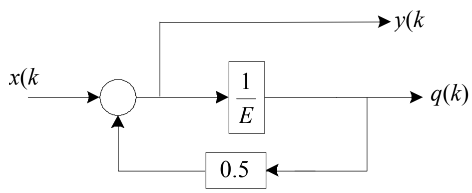
We have:

$$H_1(E) = \frac{1}{1 - 0.5(1/E)}$$

$$\Rightarrow \left(1 - 0.5 \times \frac{1}{E}\right)\{y(k)\} = x(k)$$

$$\Rightarrow y(k) - 0.5y(k-1) = x(k)$$

The corresponding simulation diagram is plotted below (Matlab codes to generate impulse and step responses are provided in the next page):



(b) Given the operation transfer function

$$H_2(\mathrm{E}) = \frac{E^4 + 0.5E^3 + 0.25E^2 + 0.125E + 0.0625}{E^4}.$$

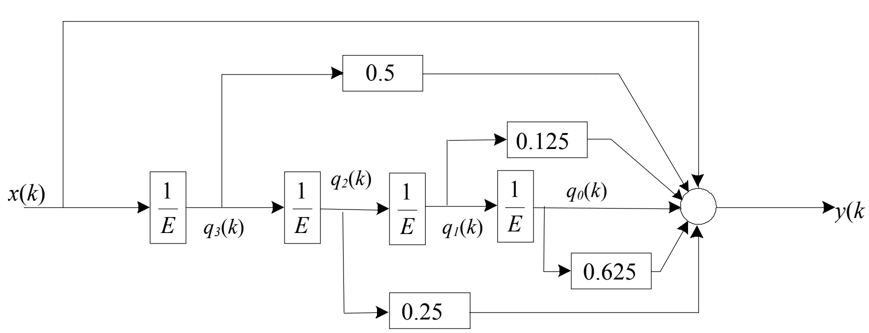
It is not hard to verify that we have:

$$H_2(E) = 1 + 0.5 \times \frac{1}{E} + 0.25 \times \frac{1}{E^2} + 0.125 \times \frac{1}{E^3} + 0.0625 \times \frac{1}{E^4}$$

$$\Rightarrow y(k) = \left(1 + 0.5 \times \frac{1}{E} + 0.25 \times \frac{1}{E^2} + 0.125 \times \frac{1}{E^3} + 0.0625 \times \frac{1}{E^4}\right)\{x(k)\}$$

$$\Rightarrow y(k) = x(k) + 1 + 0.5x(k-1) + 0.25x(k-2) + 0.125x(k-3) + 0.0625x(k-4)$$

The simulation diagram is then given as:



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%%%%%%%%%%%%SIMULATION FOR IMPULSE RESPONSE AND STEP
      FUNCTION%%%%%%%%%%%%%%%%%%
% Setup the variable and input
% xi = impulse input; xu = unit step input
for i = 1:21
    k(i)  = i-1;
    xu(i) = 1;
    xi(i) = 0;
end
xi(1) = 1;

% Part (a)
% Initial conditions
qi(1) = 0; qu(1) = 0;

%calculating the values
for i = 1:21
    %Impulse Response
    qi(i+1) = 0.5*qi(i) + xi(i);
    yi(i) = qi(i+1);
    %Step Response
    qu(i+1) = 0.5*qu(i) + xu(i);
    yu(i) = qu(i+1);
end

figure(1)
subplot(211),stem(k,yi,'o');grid on
title('Part (a): Impulse Response')
subplot(212),stem(k,yu,'o');grid on
title('Part (a): Step Response')

% Part (b)
% Initial conditions
q0(1) = 0; q1(1) = 0; q2(1) = 0; q3(1) = 0;
% Impulse Response
yi(1) = 0.0625*q0(1) + 0.125*q1(1) + 0.25*q2(1) + 0.5*q3(1) + xu(1);

% Calculating the values for Step Response
for i = 1:20
    q3(i+1) = xu(i);
    q2(i+1) = q3(i);
    q1(i+1) = q2(i);
    q0(i+1) = q1(i);

    %Step Response
    yu(i+1) = 0.0625*q0(i+1) + 0.125*q1(i+1) + 0.25*q2(i+1) +...
        + 0.5*q3(i+1) + xu(i+1);
end

figure(2)
subplot(211),stem(k,yi,'o');grid on
title('Part (b): Impulse Response')
subplot(212),stem(k,yu,'o');grid on
title('Part (b): Step Response')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
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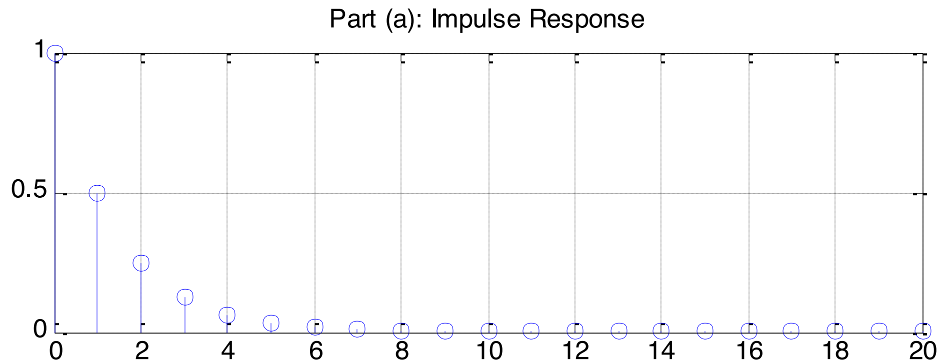
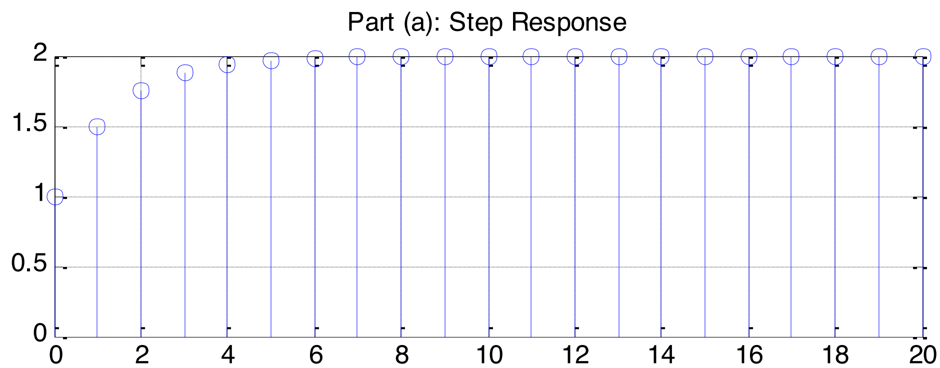
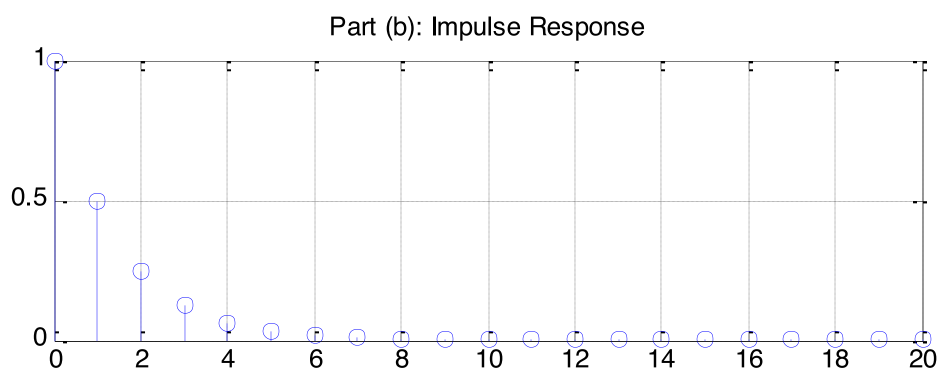
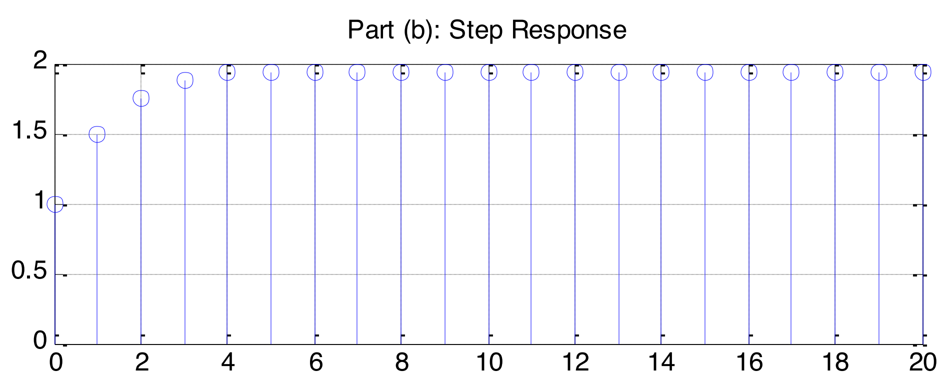

Part (a): Impulse Response
1
0.5
0
0 2 4 6 8 10 12 14 16 18 20


Part (a): Step Response
2
1.5
1
0.5
0
0 2 4 6 8 10 12 14 16 18 20


Part (b): Impulse Response
1
0.5
0
0 2 4 6 8 10 12 14 16 18 20


Part (b): Step Response
2
1.5
1
0.5
0
0 2 4 6 8 10 12 14 16 18 20

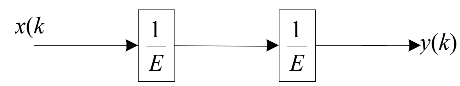
3. Find the simulation diagrams and transfer functions of the systems having the following impulse responses:

(a) $h(k) = \begin{cases} 1, & k = 2 \\ 0, & k \neq 2 \end{cases}$

The impulse response can be written as:

$h(k) = \delta(k-2)$

It then follows that



$\Rightarrow y(k) = x(k-2)$

$\Rightarrow y(k) = \left(\dfrac{1}{E^2}\right)\{x(k)\}$

Hence, the operational transfer function can be expressed as:

$$\boxed{H(E) = \frac{1}{E^2}}$$

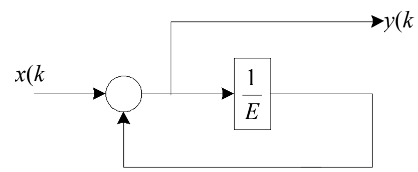
(The simulation diagram is depicted on the right)

(b) $h(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

The above impulse response can be interpreted as follows:

$h(k) = u_s(k) = \sum_{n=-\infty}^{k} \delta(n)$

Equivalently, $h(k) - h(k-1) = \delta(k)$



It then follows that:

$y(k) - y(k-1) = x(k)$

$\Rightarrow \left(1 - \dfrac{1}{E}\right)\{y(k)\} = x(k)$

The transfer function can then be expressed as:

$$\boxed{H(E) = \frac{1}{1 - 1/E} = \frac{E}{E-1}}$$

(The simulation diagram is depicted on the right).

(c) $h(k) = \begin{cases} 1, & k \geq 1 \\ 0, & k < 1 \end{cases}$

The impulse response can be interpreted as:

$h(k) = u_s(k-1)$

Note that $u_s(k-1) - u_s(k-2) = \delta(k-1)$

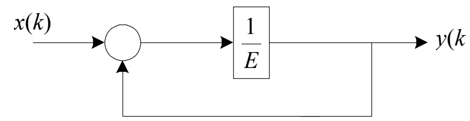
$$\Rightarrow h(k) - h(k-1) = \delta(k-1)$$

It then follows that:

$$y(k) - y(k-1) = x(k-1)$$

$$\Rightarrow \left(1 - \frac{1}{E}\right)\{y(k)\} = \frac{1}{E}x(k)$$

$$\Rightarrow (E-1)\{y(k)\} = x(k)$$



The transfer function of the system is:

$$\boxed{H(E) = \frac{1}{E-1}}$$

(The simulation diagram is depicted on the right).

(d) $h(k) = \begin{cases} 1, & k \geq 2 \\ 0, & k < 2 \end{cases}$

Similar to previous problem, we have: $h(k) = u_s(k-2)$

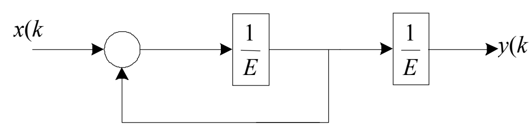
$$\Rightarrow h(k) - h(k-1) = \delta(k-2)$$

Therefore, we have:

$$y(k) - y(k-1) = x(k-2)$$

$$\Rightarrow \left(1 - \frac{1}{E}\right)\{y(k)\} = \frac{1}{E^2}x(k)$$

$$\Rightarrow (E-1)\{y(k)\} = \frac{1}{E}x(k)$$



The transfer function of the system is then expressed as:

$$\boxed{H(E) = \frac{1}{E(E-1)}}$$

(The simulation diagram is depicted on the right.)

(e) $h(k) = \begin{cases} 1, & k = 1 \\ -2, & k = 2 \\ 3, & k = 3 \\ 0 & \text{otherwise} \end{cases}$

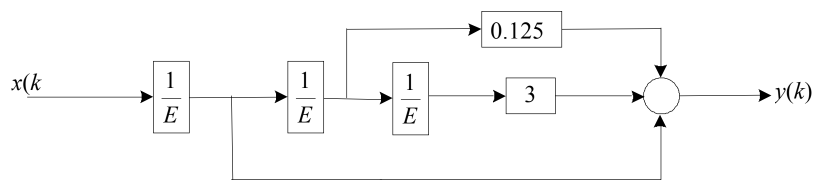
$$h(k) = \delta(k-1) - 2\delta(k-2) + 3\delta(k-3)$$

Hence,

$$y(k) = x(k-1) - 2x(k-2) + 3x(k-3)$$
$$= \left(\frac{1}{E} - \frac{2}{E^2} + \frac{3}{E^3}\right)\{x(k)\}$$

The transfer function and simulation diagram are given below:

$$\boxed{H(E) = \frac{E^2 - 2E + 3}{E^3}}$$



4.

(a) $h(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$,

$h(k) = u_s(k),\ x(k) = u_s(k)$

Applying the convolution operator yields

$$y(k) = \sum_{m=0}^{k} h(k-m)x(m) = \sum_{m=0}^{k} (1).(1) = k+1 \text{ for } k \geq 0$$

Finally, we obtain:

$\underline{y(k) = (k+1)u_s(k)}$

(b) $h(k) = \begin{cases} 1, & k = 1 \\ -2, & k = 2 \\ 3, & k = 3 \\ 0 & \text{otherwise} \end{cases}$,

$h(k) = \delta(k-1) - 2\delta(k-2) + 3\delta(k-3)$

$x(k) = u_s(k)$

Applying the convolution operator yields

$$y(k) = \sum_{m=0}^{k} h(k-m)x(m)$$
$$= \sum_{m=0}^{k} h(k-m)$$

$k = 1$: $h(0) + h(1) = 1$

$k = 2$: $h(0) + h(1) + h(2) = 1 - 2 = -1$

$k = 3$: $h(0) + h(1) + h(2) + h(3) = 1 - 2 + 3 = 2$

Finally, we have:

$$\underline{y(k) = u_s(k-1) - 2u_s(k-2) + 3u_s(k-3)}$$

(c) $h(k) = \begin{cases} (0.9)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$. [Ans: $y(k) = 10 - 9(0.9)^k, k = 0, 1, 2, \cdots$.]

Applying the convolution operator yields

$$y(k) = \sum_{m=0}^{k} h(k-m)x(m)$$

$$= \sum_{m=0}^{k} \underbrace{x(k-m)}_{=1} h(m)$$

$$= \sum_{m=0}^{k} (0.9)^m = \frac{1-(0.9)^{k+1}}{1-0.9}$$

$$= \frac{1-(0.9)(0.9)^k}{0.1} = 10 - 9(0.9)^k, k \geq 0$$

It then follows that:

$$y(k) = \left[10 - 9(0.9)^k\right] u_s(k)$$