$$\frac{1}{x^a} = x^{-a}$$
$$\sqrt[a]{x} = x^{\frac{1}{a}}$$

$$\ln M^k = k \ln M$$
$$\ln MN = \ln M + \ln N$$
$$\ln \frac{M}{N} = \ln M - \ln N$$

$$\ln x = \log_e x$$

$$\ln e^x = x$$
$$e^{\ln x} = x$$
$$\ln 1 = 0$$
$$\ln e = 1$$
$$\ln b = \frac{\log_a b}{\log_a e}$$

$$\begin{aligned} e^x &= \lim_{n\to\infty}(1+\frac{x}{n})^n \\ &= \lim_{t\to 0}(1+tx)^{\frac{1}{t}} \\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= \sum_{r=0}^{\infty} \frac{x^r}{r!} \end{aligned}$$

(Let a be a number including ∞ and $-\infty$,
k be constant)

$$\lim_{x\to a} k = k$$
$$\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$$
$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$
$$\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$
$$\lim_{x\to a}[f(x)^2] = [\lim_{x\to a} f(x)]^2$$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ for } \lim_{x\to a} g(x) \neq 0$$
$$\lim_{x\to a} f[g(x)] = f[\lim_{x\to a} g(x)]$$

$$\lim_{x\to 0} \frac{\sin\theta}{\theta} = \lim_{x\to 0} \frac{\theta}{\sin\theta} = 1$$
$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to -\infty} \frac{1}{x} = 0$$
$$\lim_{x\to\infty} \frac{a}{x} = a \lim_{x\to\infty} \frac{1}{x} = 0$$
$$\lim_{x\to\infty} \frac{1}{x^2} = (\lim_{x\to\infty} \frac{1}{x})^2 = 0^2 = 0$$

$$\lim_{x\to\infty} a^x = \infty \qquad \text{when } a > 1$$
$$\cdots = 0 \qquad \text{when } 0 < a < 1$$

$$\lim_{x\to -\infty} a^x = \lim_{x\to\infty} a^{-x} = \lim_{x\to\infty} \frac{1}{a^x} = 0 \qquad \text{when } a > 1$$
$$\cdots = \infty \qquad \text{when } 0 < a < 1$$

L'Hopital's rule: (when $\frac{f(x)}{g(x)}$ is undefined, e.g. $\frac{0}{0}$)

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Case 1: $\lim_{x\to\infty} \frac{ax^{\text{bigger}} + \cdots}{bx^{\text{smaller}} + \cdots}$ does not exist.

Case 2: $\lim_{x\to\infty} \frac{ax^{\text{smaller}} + \cdots}{bx^{\text{bigger}} + \cdots} = 0$

Case 3: $\lim_{x\to\infty} \frac{ax^{\text{same}} + \cdots}{bx^{\text{same}} + \cdots} = \frac{a}{b}$

Examples

Target: denominator $\neq 0$

$$\lim_{x\to 5}\frac{x^2}{x+1} = \frac{25}{6}$$

$$\lim_{x\to 0}\frac{x^3+3x^2+2x}{x^2-3x} = \lim_{x\to 0}\frac{x(x^2+3x+2)}{x(x-3)} = \lim_{x\to 0}\frac{x^2+3x+2}{x-3} = -\frac{2}{3}$$

$$\lim_{x\to 1}\frac{2x-2}{\sqrt{x+3}-2} = \lim_{x\to 1}\frac{2x-2}{\sqrt{x+3}-2}\cdot\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} = \lim_{x\to 0}\frac{2(x-1)(\sqrt{x+3}+2)}{x-1} = 8$$

$$\lim_{x\to\infty}\frac{4^{x+1}}{17^{\frac{x}{2}}} = \lim_{x\to\infty}\frac{4^x\cdot 4}{(17^{\frac{1}{2}})^x} = \lim_{x\to\infty}(\frac{4}{17^{\frac{1}{2}}})^x\cdot 4 = 0\cdot 4 = 0$$

$$\lim_{x\to\infty}\frac{2x^3+x^2+7}{x^2+x-1} = \lim_{x\to\infty}\frac{\frac{2x^3+x^2+7}{x^2}}{\frac{x^2+x-1}{x^2}} = \cdots = \lim_{x\to\infty} 2x+1$$

When $x\to\infty$, $2x\to\infty$.

$\therefore \lim_{x\to\infty}\frac{2x^3+x^2+7}{x^2+x-1}$ does not exist.

Target: form pattern

$$\lim_{x\to 0}\frac{1-\cos 2\theta}{(2\theta)^2} = \lim_{x\to 0}\frac{1-(1-2\sin^2\theta)}{4\theta^2} = \lim_{x\to 0}\frac{2\sin^2\theta}{4\theta^2} = \frac{1}{2}\lim_{x\to 0}\frac{4\sin^2\theta}{4\theta^2} = \frac{1}{2}(\lim_{x\to 0}\frac{\sin\theta}{\theta})^2 = \frac{1}{2}$$

$$\lim_{\theta\to\pi}\frac{2\cos\frac{\theta}{2}}{\pi-\theta} = \lim_{\theta\to\pi}\frac{2\sin(\frac{\pi}{2}-\frac{\theta}{2})}{\pi-\theta} = \lim_{\theta\to\pi}\frac{2\sin(\frac{\pi-\theta}{2})}{2(\frac{\pi-\theta}{2})} = \lim_{\theta\to\pi}\frac{\sin(\frac{\pi-\theta}{2})}{\frac{\pi-\theta}{2}} = 1$$

$$\lim_{x\to 0}\frac{e^{3x}-1}{2x} = \lim_{x\to 0}\frac{e^{3x}-1}{3x}\cdot\frac{3}{2} = \frac{3}{2}$$

$$\lim_{x\to 0}\frac{\sin x}{e^{2x}+3e^x-4} = \lim_{x\to 0}\frac{\frac{\sin x}{x}}{\frac{e^{2x}-1}{x}+\frac{3(e^x-1)}{x}} = \lim_{x\to 0}\frac{1}{\frac{2(e^{2x}-1)}{2x}+\frac{3(e^x-1)}{x}} = \frac{1}{2+3} = \frac{1}{5}$$

$$\lim_{n\to\infty}(1+\frac{1}{4n})^n$$

Let $k=4n$.

When $n\to\infty$, $k\to\infty$.

$$\lim_{n\to\infty}(1+\frac{1}{4n})^n = \lim_{k\to\infty}(1+\frac{1}{k})^{\frac{k}{4}} = \lim_{k\to\infty}[(1+\frac{1}{k})^k]^{\frac{1}{4}} = [\lim_{k\to\infty}(1+\frac{1}{k})^k]^{\frac{1}{4}} = e^{\frac{1}{4}}$$

$$\lim_{x\to-\infty}\frac{x^2-3x+7}{2x^2-6x+3} = \lim_{x\to-\infty}\frac{\frac{x^2-3x+7}{x^2}}{\frac{2x^2-6x+3}{x^2}} = \lim_{x\to-\infty}\frac{1-\frac{3}{x}+\frac{7}{x^2}}{2-\frac{6}{x}+\frac{3}{x^2}} = \frac{1}{2}$$

$$\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x-1}) = \lim_{x\to\infty}\frac{(\sqrt{x+1}-\sqrt{x-1})(\sqrt{x+1}+\sqrt{x-1})}{\sqrt{x+1}+\sqrt{x-1}} = \lim_{x\to\infty}\frac{(x+1)-(x-1)}{\sqrt{x+1}+\sqrt{x-1}} = \lim_{x\to\infty}\frac{2}{\sqrt{x+1}+\sqrt{x-1}}$$

When $x\to\infty$,

we have $(\sqrt{x+1}+\sqrt{x-1})\to\infty$.

$\therefore \lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x-1}) = 0$

$$\lim_{x\to-\infty}\frac{\sqrt{4x^2-x}}{x} = \lim_{x\to-\infty}\frac{\sqrt{4x^2-x}}{-\sqrt{x^2}} = -\lim_{x\to-\infty}\sqrt{\frac{4x^2-x}{x^2}} = -\lim_{x\to-\infty}\sqrt{4-\frac{1}{x}} = -\sqrt{4-0} = -2$$