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Source: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, May 8, 1956, Vol. 235, No. 1202 (May 8, 1956), pp. 408-418

Published by: Royal Society

Stable URL: https://www.jstor.org/stable/99979



# The Brueckner theory of nuclear structure

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(*Communicated by R. E. Peierls, F.R.S.—Received* 16 *January* 1956—*Read* 3 *May* 1956)

The object of the Brueckner theory is to calculate the properties of atomic nuclei from a knowledge of the forces which act between nucleons in free space. The method involves constructing a nuclear model having the same energy as the nucleus, and it also gives a relation between the nuclear model and the actual nucleus which can be used to investigate detailed aspects of the nucleus in circumstances where the model is insufficient.

Previous papers on the Brueckner theory have used a formalism applicable only to an infinite medium of nuclear matter and this has involved some inconsistencies. The present paper establishes the theory in a form which applies to a nucleus of finite size and therefore provides a basis for calculating from first principles the properties of an actual nucleus. The nuclear model on which the method is based is closely related to the nuclear shell model and is determined by a set of coupled self-consistent equations which take into account the strong short-range character of nuclear forces. These self-consistent equations have also been derived by H. A. Bethe who describes methods of solving them in a paper to be published in the *Physical Review*. The present paper concentrates on the derivation and meaning of the equations for the model, on their relation to a self-consistent variational procedure, on the relation between the model and the nucleus and on discussing correction terms neglected in the construction of the model.

The justification of the theory depends primarily on the operation of the exclusion principle between nucleons, but it also involves a physical assumption related to the absence of clustering in the nucleus.

## 1. Introduction

It is the purpose of this paper to establish the Brueckner theory in a form which applies to a nucleus of finite size. This new formulation is necessary in order to give a precise meaning to all quantities and equations which are used and also to clarify the basis of the variational procedure on which many of the calculations are based.

The starting point of the theory is the many-body Schrödinger equation for a system of nucleons in which it is assumed that interactions consist entirely of two-body forces, and these are taken so that they agree with the two-body scattering data for nucleons in free space. It is shown that a formal solution of this many-body equation can be constructed from the wave function for a nuclear model. This nuclear model is simple provided that certain correction terms can be neglected. The energy of the model is determined by a set of self-consistent equations which represent a considerable improvement on the Hartree method when it is applicable, and the present method will also apply in situations where the Hartree method is not valid—for example, when particles interact through strong short-range forces.

In the present paper it is not assumed that the reader is familiar with the succession of papers by Brueckner and his collaborators, but for reference purposes the scope of those papers is outlined at the end of this introduction. The form of the theory which is developed here leads to a nuclear model and correction terms which are the same as those used by H. A. Bethe in a forthcoming paper on the Brueckner theory

(Bethe 1956). The present paper is intended to provide a theoretical basis for the nuclear model and the correction terms and to establish its relation to the nuclear wave function. The properties of the model itself are investigated in the paper by Bethe who considers in particular the method of solving the self-consistent equations and the estimation of the size of the correction terms.

In § 2 a formal solution of the many-body Schrödinger equation is obtained in terms of a model wave function without making any approximations. In § 3 the formal definitions are made precise so that they apply to a nucleus of finite size. The approximate equations for the nuclear model are obtained and the correction terms are exhibited and discussed. In § 4 the self-consistent equations for the model are related to a variational procedure, and a method of going over to the limit of an infinite nuclear medium is obtained. This limit is particularly suitable for discussing the phenomenon of nuclear saturation where the effects of the nuclear surface are unimportant.

It is emphasized that the operation of the exclusion principle is of great importance in the nucleus and it contributes largely to the justification of the neglect of correction terms which are omitted in the approximate equations which define the model. It is essential that the exclusion principle should be allowed for when solving the equations for the model, and it is shown by Bethe (1956) that this leads to solutions which will differ considerably from those used by Brueckner and collaborators.

It is convenient to give now an outline of previous papers, and the author wishes to emphasize that although the body of the paper contains few references much of it depends directly or indirectly on the contents of these papers, and the author is correspondingly indebted to his former colleagues in Indiana and particularly to K. A. Brueckner.

The algebra of the present theory is a modification of that developed by Watson (1953) and by Watson & Francis (1953) for studying the multiple scattering of a fast particle through an atomic nucleus. The work of Watson was adapted for a particle in a bound state and applied to the problem of nuclear saturation by Brueckner, Levinson & Mahmoud (1954) and by Brueckner (1954 *a*, *b*). The mathematical basis for this method was discussed by Brueckner & Levinson (1955), and its relation to the general theory of nuclear models was considered by Eden & Francis (1954). The relation of the method to correlations in the nucleus and their effect on high-energy nuclear reactions, to the nuclear shell model, and to the optical model, was investigated by Brueckner, Eden & Francis (1955 *a*, *b*, *c*). The equations for a model for a finite-sized nucleus and the basis for the variational method were given by the present author (Eden 1955), and further mathematical difficulties in the method were considered by Brueckner (1955) in relation to a linked cluster expansion.

## 2. A formal solution of the nuclear many-body problem

It is assumed that the Schrödinger equation for a system of $A$ nucleons can be written

$$\left[E_0' - \sum_{i=1}^{A} T_i - \sum_{i<j}^{A} v_{ij}\right]\Psi_0(1, \ldots, A) = 0, \tag{2·1}$$

where $T_i$ denotes the kinetic energy of the $i$th nucleon and $v_{ij}$ is the potential between the $i$th and $j$th nucleons. It is to be assumed that $v_{ij}$ is the same as the potential

between two nucleons in free space, so it can be chosen phenomenologically to fit nucleon-nucleon scattering data. The exclusion principle must be satisfied so the nuclear wave function must be an antisymmetrized solution of equation (2·1). The suffix 0 on $\Psi_0(1, \ldots, A)$ denotes a particular eigenstate, and it will be assumed that there is no degeneracy.

We will now try to relate the wave function $\Psi_0(1, \ldots, A)$ to a simpler wave function $\Phi_0(1, \ldots, A)$ which closely resembles the shell model wave function. This will be achieved by writing

$$\Psi_0(1, \ldots, A) = F\Phi_0(1, \ldots, A), \tag{2·2}$$

where $F$ is a certain operator, and then determining the equation for $\Phi_0(1, \ldots, A)$ by substituting the right-hand side of (2·2) for $\Psi_0$ in (2·1). The operator $F$ is defined by

$$F = 1 + \frac{Q}{e} \sum_{i<j}^{A} I_{ij} F_{ij}, \tag{2·3}$$

$$F_{ij} = 1 + \frac{Q}{e} \sum_{l,m \neq i,j}^{A} I_{lm} F_{lm}, \tag{2·4}$$

$$e = E_0 - \sum_{i=1}^{A} T_i - \sum_{i<j}^{A} \bar{t}_{ij}, \tag{2·5}$$

$$I_{ij} = t_{ij} - \bar{t}_{ij}, \tag{2·6}$$

$$t_{ij} = v_{ij} + v_{ij} \frac{Q}{e_{ij}} t_{ij}. \tag{2·7}$$

The energy $E_0$ differs from $E_0'$ by a small amount which will be discussed later. The operator $Q$ in the above equations is a projection operator defined to exclude states for which $e$ or $e_{ij}$ may be singular and also to exclude states for which the exclusion principle is not satisfied. The operator $e_{ij}$ denotes a part of $e$ and will be defined later so that it depends only on the states of particles $i$ and $j$. The operator $\bar{t}_{ij}$ denotes a kind of average interaction. For the present, $Q$, $e_{ij}$ and $\bar{t}_{ij}$ will not be defined more precisely, as it will appear later that their precise definitions can be determined by the requirement that the equation for $\Phi_0(1, \ldots, A)$ which results from (2·1) and (2·2) shall approximate to a simple equation.

The equation for $\Phi_0(1, \ldots, A)$ is

$$\left[E_0' - \sum_{i=1}^{A} T_i - \Sigma v_{ij}\right] F\Phi_0(1, \ldots, A) = 0. \tag{2·8}$$

The term containing $v_{ij}$ can be transformed by using equations (2·3) and (2·4),

$$F = F_{ij} + \frac{Q}{e} I_{ij} F_{ij}, \tag{2·9}$$

$$v_{ij} F = v_{ij} F_{ij} + v_{ij} \frac{Q}{e} I_{ij} F_{ij} \tag{2·10}$$

$$= \left(t_{ij} - v_{ij} \frac{Q}{e_{ij}} t_{ij}\right) F_{ij} + v_{ij} \frac{Q}{e} (t_{ij} - \bar{t}_{ij}) F_{ij} \tag{2·11}$$

$$= \bar{t}_{ij} F - \bar{t}_{ij} \frac{Q}{e} I_{ij} F_{ij} + I_{ij} F_{ij} - v_{ij} \frac{Q}{e} \bar{t}_{ij} F_{ij} + v_{ij} \left(\frac{Q}{e} - \frac{Q}{e_{ij}}\right) t_{ij} F_{ij}. \tag{2·12}$$

From (2·5) and (2·3), $$eF\Phi_0 = e\Phi_0 + Q\Sigma I_{ij}F_{ij}\Phi_0. \tag{2·13}$$

Using equations (2·8), (2·12), (2·5) and (2·13) it can be deduced that the nuclear wave function $\Psi_0(1, \ldots, A)$ given by (2·2) will exactly satisfy (2·1) provided that $\Phi_0(1, \ldots, A)$ satisfies the equation

$$\left[E_0 - \sum_{i=1}^{A} T_i - \sum_{i<j}^{A} \bar{t}_{ij}\right] \Phi_0(1, \ldots, A) = w\Phi_0(1, \ldots, A) - (E_0' - E_0)F\Phi_0, \tag{2·14}$$

where $w$ is given by

$$w = (1-Q)\Sigma I_{ij}F_{ij} - \Sigma\left(\bar{t}_{ij}\frac{Q}{e}I_{ij} + v_{ij}\frac{Q}{e}\bar{t}_{ij}\right)F_{ij} + \Sigma v_{ij}\left(\frac{Q}{e} - \frac{Q}{e_{ij}}\right)t_{ij}F_{ij}. \tag{2·15}$$

This completes the derivation of a formal solution to (2·1). Its usefulness will depend on a number of conditions being satisfied by the transformation and by the transformed wave function so that the equations to be solved are both consistent and soluble in an appropriate approximation in practice. These conditions will vary according to the properties of the nucleus which are being studied. For example, in studying nuclear saturation the energy and density of the nucleus defined by (2·1) must be determined. It will be shown that the energy can be determined from equation (2·14) provided $w$ can be treated as a small perturbation, and that the smallness of $w$ can be shown to follow from certain reasonable physical assumptions. The nuclear density can also be found from (2·14) provided that the transformation $F$ does not appreciably change the boundaries given by (2·1) and (2·14).

In order to discuss the solution of (2·14) it is necessary to make the definitions more precise. This will be done in the next section which is concerned primarily with setting up well-defined equations for $\Phi_0(1, \ldots, A)$ when $w$ can be neglected, and secondly discusses the corrections due to $w$.

## 3. Self-consistent equations for the model wave function

In this section it will be shown that $Q$, $e_{ij}$ and $\bar{t}_{ij}$ can be defined so that $w$ in (2·14) can be neglected (subject to the correctness of certain physical assumptions which have to be made about the nucleus). These definitions will be chosen so that when $w$ is neglected, $\Phi_0(1, \ldots, A)$ is given by a Slater determinant of single-particle wave functions. It is convenient to begin by neglecting $w$ and to discuss it later as a correction to the model.

When $w$ is neglected the model wave function $\Phi_0$ will satisfy the equation

$$\left\{E_0 - \sum_{i=1}^{A} T_i - \sum_{i<j}^{A} \bar{t}_{ij}\right\} \Phi_0(1, \ldots, A) = 0. \tag{3·1}$$

It is now desired to choose a definition for $\bar{t}_{ij}$ so that (i) this equation has a solution of the form

$$\Phi_0(1, \ldots, A) = (A!)^{-\frac{1}{2}} \det \phi_i^0, \tag{3·2}$$

where $\phi_i^0$ is a single-particle wave function. For a nucleus of finite size this also implies that (ii) the solution must consist of wave functions having a finite extent in space, this being determined by the finite range of $\bar{t}_{ij}$ in space and not by the imposition of extra boundary conditions. Finally, (iii) $\bar{t}_{ij}$ must be such that $w$ can be shown to be small.

It is not possible to take $\bar{t}_{ij} = t_{cij}$ as was done in earlier presentations of the Brueckner theory since this does not satisfy the requirement (ii) above. This can be seen by noting that $t_{cij}$ is defined to be completely diagonal with respect to the single-particle states, $t_{cij} = (\phi_i\phi_j, t_{ij}\phi_i\phi_j)$. Thus the potential $V_i$ (see (3·4)) will be diagonal with respect to $\phi_i$, i.e. like the energy eigenvalue $E_i$, $V_i$ will be just a number in this representation. It follows that the solution $\phi_i$ of (3·3) would have to be an eigenstate of the kinetic energy operator $T_i$; but since this cannot be true of a wave function $\phi_i$ which has a finite extension in space we deduce that $t_{cij}$ is not a suitable form for $\bar{t}_{ij}$ for a nucleus of finite size. Therefore $\bar{t}_{ij}$ must be defined so that it leads to a single-particle potential $V_i$ which has both diagonal and off-diagonal matrix elements with respect to the single-particle wave functions $\phi_i$.

In order to make a precise definition of $\bar{t}_{ij}$ it is desirable to fix our attention on some definite representation so that its matrix elements can be prescribed. First, however, the relation between $\bar{t}_{ij}$ and the wave function $\phi_i$ must be noted; it is obtained by comparing (3·1) and (3·2). These give

$$(E_i - T_i - V_i)\,\phi_i = 0, \tag{3·3}$$

where

$$V_i = \sum_{j=1}^{A} (\phi_j^0, \bar{t}_{ij}\phi_j^0). \tag{3·4}$$

It is useful now to introduce a notation due to Bethe (1956); this is related to the above formalism by making the assumption that $V_i$ is some definite given potential (having the form $(r \mid V \mid r')$ in co-ordinate space, for example). The given potential substituted in (3·3) will determine a complete set of single-particle wave functions which will be denoted by $\mid n_i)$ for various values of $n_i$. Anticipating that the term $w$ can be neglected the $i$th nucleon in the model will occupy a definite state which will be denoted by $\mid n_i^0)$. The exclusion principle is to be satisfied and a definite configuration is chosen for $A$ particles in different states. In Bethe's notation these states are written

$$\mid n_1^0), \mid n_2^0), \ldots, \mid n_A^0). \tag{3·5}$$

This configuration will be called the 'chosen configuration', and the aim of the method is to identify it with the solution $\Phi_0(1, \ldots, A)$ of (3·1).

In order for $w$ (see (2·15)) to be small, $\bar{t}_{ij}$ must contain all the diagonal part of $t_{ij}$, and we have shown above that for a nucleus of finite size it is necessary also that $\bar{t}_{ij}$ should contain matrix elements which are off-diagonal with respect to one particle at a time. It must not contain matrix elements which are off-diagonal with respect to both particles $i$ and $j$, since it can then be shown that $w$ is not small or alternatively that the model is not simple. Finally, it is necessary that $\bar{t}_{ij}$ should be genuinely a two-body operator as the suffixes $i$ and $j$ suggest. These conditions are sufficient to show that the matrix elements of $\bar{t}_{ij}$ with respect to particles $i$ and $j$ in different states of the complete set $\mid n_i)$ must have the form

$$(n_i' n_j \mid \bar{t} \mid n_i n_j) = (n_i' n_j^0 \mid t \mid n_i n_j^0) \quad \text{for} \quad n_i' \neq n_i, \tag{3·6a}$$

$$(n_i n_j' \mid \bar{t} \mid n_i n_j) = (n_i^0 n_j' \mid t \mid n_i^0 n_j) \quad \text{for} \quad n_j' \neq n_j, \tag{3·6b}$$

$$(n_i n_j \mid \bar{t} \mid n_i n_j) = (n_i n_j \mid t \mid n_i n_j), \tag{3·6c}$$

$$(n_i' n_j' \mid \bar{t} \mid n_i n_j) = 0 \quad \text{for} \quad n_i' \neq n_i \quad \text{and} \quad n_j' \neq n_j. \tag{3·6d}$$

In these definitions it is specified that $\bar{t}_{ij}$ shall be given by these matrix elements of $t_{ij}$ evaluated with all other particles in the chosen configuration.

It is necessary to regard each particle $i$ as associated with a definite state of the chosen configuration. This can be done by selecting one product wave function from the Slater determinant $\Phi_0$ which describes the model; in this product the particle $i$ is in a definite state. This procedure fixes the definitions with respect to each product wave function and therefore to the Slater determinant as a whole.

It follows from these definitions that if $\Phi_0$ is identified with the chosen configuration then the only matrix elements of $\bar{t}_{ij}$ which occur in (3·1) are those of the type $(n_i n_j^0 \mid \bar{t} \mid n_i^0 n_j^0)$, which are just those which add up to give $V_i$ acting on $\mid n_i^0)$.

Next, $e_{ij}$ is defined so that the equation for $t_{ij}$ can be made precise. Let $E_i^0$ be the energy of $\mid n_i^0)$ corresponding to the given potential $V_i$. Then $e_{ij}$ is defined by

$$e_{ij} = E_i^0 + E_j^0 - T_i - T_j - V_i - V_j. \tag{3·7}$$

The state $\mid n_i) \mid n_j)$ will be an eigenstate of $e_{ij}$ belonging to the eigenvalue $(E_i^0 + E_j^0 - E_i - E_j)$. Thus the equation for $t_{ij}$, (2·7), can be written explicitly in this representation as

$$(n_i' n_j' \mid t \mid n_i n_j) = (n_i' n_j' \mid v \mid n_i n_j) + \sum_{n_i'' n_j''}^{Q} \frac{(n_i' n_j' \mid v \mid n_i'' n_j'')\,(n_i'' n_j'' \mid t \mid n_i n_j)}{E_i^0 + E_j^0 - E_i'' - E_j''}. \tag{3·8}$$

It is also specified that all particles except $i$ and $j$ are in the chosen configuration and further that both the intermediate states $\mid n_i'')$, $\mid n_j'')$ shall lie outside the chosen configuration (this requirement is indicated by the superscript $Q$ on the summation sign). It is assumed that this will be sufficient to remove the singular term from the sum and questions of degeneracy will not be considered here.

The self-consistent equations for the model are now completely defined by the requirement that $\mid n_i)$ and $\phi_i$ shall be the same wave function for all $i$. The single-particle potential $V_i$ must be chosen so that it determines a complete set of wave functions $\mid n_i)$ (or $\phi_i$) from (3·3); these then permit one to define the chosen configuration and calculate $t_{ij}$ from (3·8); then $V_i$ can be calculated from the equation

$$V_i = \sum_{j=1}^{A} (\phi_j^0, t_{ij} \phi_j^0), \tag{3·9}$$

which, by (3·6), is identical with (3·4). The original form of $V_i$ must be chosen so that the calculated form (3·9) gives the same potential. The problem of solving these self-consistent equations will be discussed further in § 4, where it is related to a variational method which simplifies the problem for a large nucleus.

The form of the correction term $w$ and its contribution to the energy of (2·14) as compared with (3·1) will now be considered. The first-order energy contribution is

$$(\Phi_0(1, \ldots, A), w\Phi_0(1, \ldots, A)). \tag{3·10}$$

It will be noted that $e$ defined by (2·5) and (3·6) is a diagonal operator in the representation given by the single-particle wave functions $\mid n_i)$. For the self-consistent solution, $\Phi_0$ will be a determinant of these wave functions in the chosen configuration. The only factors in $w$ which can excite or de-excite particles are $I_{ij}$ (either explicitly or in the $F$ operators), $v_{ij}$ and $\bar{t}_{ij}$, but $\bar{t}_{ij}$ can change the state of one

particle only. The operator $Q$ projects out of the chosen configuration. Hence the first term of $w$ when substituted in (3·10) gives

$$\sum_{i,j,k}\left(\Phi_0, I_{ij}\frac{Q}{e}I_{jk}\frac{Q}{e}I_{ki}\Phi_0\right)+\text{higher-order terms.} \tag{3·11}$$

This term corresponds to an interaction between three (or more) nucleons in the nucleus and has been shown to be small by Brueckner & Levinson (1955) and by Bethe (1956). It will be important only if position correlations between three or more nucleons in the nucleus have an important effect on the energy. If such correlations were very important $w$ could not be neglected, and it might be a poor approximation to estimate its relative size by an expansion in terms of the solutions of (3·1). However, the calculations of the magnitude of the first term in (3·11) do show that it is not inconsistent to neglect these correlation terms.

The lowest-order contribution to (3·10) from the second term of $w$ (see (2·15)) will be

$$\sum_{i<j}\left(\Phi_0, \left\{\bar{t}_{ij}\frac{Q}{e}I_{ij}+v_{ij}\frac{Q}{e}\bar{t}_{ij}\right\}\Phi_0\right). \tag{3·12}$$

Since $\bar{t}_{ij}$ can only excite one particle out of the chosen configuration and $I_{ij}$ cannot de-excite just one particle (if all others are in the chosen configuration) the first term of (3·12) will be zero. Since $Q$ has been defined to exclude all but doubly excited states, $Q\bar{t}_{ij}$ cannot have any matrix elements to $\Phi_0$. Hence the lowest-order contribution (3·12) is zero. In higher-order terms like

$$\sum_{i,j,k}\left(\Phi_0, v_{ij}\frac{Q}{e}\bar{t}_{ij}\frac{Q}{e}I_{jk}\frac{Q}{e}I_{ki}\Phi_0\right) \tag{3·13}$$

will be non-zero.

If the single-particle wave functions $\phi_i$ are all normalized in a volume proportional to the number $A$ of nucleons then it is proved by Bethe (1956) that terms of this type which contain a factor $\bar{t}_{ij}$ give an energy contribution which is independent of the number of nucleons; thus it is of order $1/A$ compared with the main energy term in (3·1) (this anticipates the result that the model saturates with an energy proportional to $A$). It should, however, be noted that a proof that such terms as (3·13) are of relative order $1/A$ is not in itself a complete justification of the neglect of the corresponding terms in $w$, since it is not clear what happens to higher-order terms in the perturbation series, particularly since the exclusion principle becomes less effective for these terms due to there being many 'holes' in the chosen configuration for complicated intermediate states. It is basic to the whole method that $w$ does not cause a major perturbation in the equation for the model. If $w$ was not negligible the following exact form of the energy shift would have to be considered:

$$\Delta E = (\Phi_1, w\Phi_0)/(\Phi_1, \Phi_0), \tag{3·14}$$

where $\Phi_1$ denotes an exact solution of (2·14) and $\Phi_0$ is a solution of (3·1). This term represents the complete sum of the perturbation series for the energy shift and may give an indication of the circumstances in which the total effect of higher-order terms in the series may be important. If, for example, $w$ causes clustering in the model, then $\Phi_1$ could not be written as a Slater determinant so the estimates of the

$1/A$ character of certain terms in $w$ cannot be made as before. There may be a difference between clustering in the model and clustering in the nucleus, since the operator $F$ certainly introduces strong correlations into the nuclear wave function as compared with the model wave function. However, it seems very reasonable to assume that if there is no clustering in the nucleus then there will be none in the model. The assumption that there is no clustering in the nucleus would appear to be a sufficient (but perhaps not a necessary) condition for the model wave function $\Phi_1$ to approximate to the Slater determinant $\Phi_0$. The discussion of the perturbation series shows that this assumption is consistent with the Brueckner theory, but the discussion is not adequate to conclude that the absence of clustering is proved by the Brueckner theory.

There is one further correction term in $w$ which has not yet been considered. In lowest order the last term in (2·15) leads to an energy contribution

$$\sum_{i<j}\left(\Phi_0, v_{ij}\left(\frac{Q}{e}-\frac{Q}{e_{ij}}\right)t_{ij}\Phi_0\right). \tag{3·15}$$

$Q$ is defined so that in the intermediate state both $i$ and $j$ are excited out of the chosen configuration, and since $v_{ij}$ and $t_{ij}$ are two-body operators no other particles can be excited. It is therefore necessary to consider matrix elements of $e$ between states with two particles excited. From the definitions, (2·5), (3·6) and (3·3), for large $A$,

$$e\,|\,n_in_j) \doteq \left(E_i^0+E_j^0-T_i-T_j-\sum_{k=1}^{A}\bar{t}_{ik}-\sum_{k=1}^{A}\bar{t}_{jk}\right)|\,n_in_j). \tag{3·16}$$

In deriving (3·16) the following relation has been used:

$$\text{for } l,m\neq i,j,\quad \bar{t}_{lm}\,|\,n_in_j) = \bar{t}_{lm}\,|\,n_i^0n_j^0), \tag{3·17}$$

since the matrix elements of $\bar{t}_{lm}$ must be evaluated from those of $t_{lm}$ having all particles (with the possible exception of $l$, $m$) in the chosen configuration. Thus the $\bar{t}_{lm}$ act like parts of $V_l$ and the relation

$$(E_l^0-T_l-V_l)\,|\,n_l^0) = 0 \tag{3·18}$$

can be used in establishing (3·16). Since particle $k$ (for $k\neq i,j$) in (3·16) is in the state $|\,n_k^0)$, (3·16) reduces to

$$e_{ij}\,|\,n_in_j) \tag{3·19}$$

with neglect of a term of order $1/A$. This term is

$$(n_in_j\,|\,t\,|\,n_in_j)-(n_i^0n_j^0\,|\,t\,|\,n_i^0n_j^0) \tag{3·20}$$

and arises from the purely diagonal part of $\bar{t}_{ij}$. Its neglect here is justified by its $1/A$ dependence. However, in higher-order contributions this 'reaction off the energy shell' will cause significant departures of $e$ from $e_{ij}$, but it is likely that their contribution to the energy correction $\Delta E$ will be small. In higher-order contributions from $w$, matrix elements will occur with three or more particles excited giving

$$e\,|\,n_in_jn_k) = (E_i^0+E_j^0+E_k^0-E_i-E_j-E_k)\,|\,n_in_jn_k), \tag{3·21}$$

which provides another difference between $e$ and $e_{ij}$. It has been shown by Brueckner (1955) that all higher-order corrections consist of linked interactions having an

energy contribution proportional to $A$ or smaller, and individually these terms are small. For their collective effect one is forced again to assume it to be small on the physical grounds that it cannot be large unless clusters occur in the actual nucleus. Although the expansion of the perturbation series for $\Delta E$ leads to unlinked interaction terms depending on $A^2$ or larger, Brueckner's paper (1955) shows that these occur in combinations which add up to zero; they can therefore be omitted.*

It can be concluded that there is a strong basis for assuming that the nuclear model will have energy eigenvalues near to those of the nucleus provided that it is evaluated self-consistently from equations (3·8), (3·9) and (3·3) (with $\phi_i = |\, n_i)$), for a nucleon-nucleon potential $v_{ij}$ chosen to fit observed scattering.

## 4. The variational method

If for the present a particular form for $t_{ij}$ is assumed, the self-consistent problem reduces to one of solving the equations

$$(E_i^0 - T_i - V_i) \,|\, n_i^0) = 0, \tag{4·1}$$

$$V_i = \sum_{j=1}^{A} (n_j^0 \,|\, t_{ij} \,|\, n_j^0). \tag{4·2}$$

These equations are precisely those given by the Hartree approximation to the many-body problem which has an interaction $t_{ij}$ between particles,

$$\left(E - \sum_{i=1}^{A} T_i - \sum_{i<j}^{A} t_{ij}\right) \chi(1, \ldots, A) = 0. \tag{4·3}$$

It must be emphasized that the energy $E$ in (4·3) will not give such a good approximation to the nuclear energy as the energy $E_0$ in (3·1). However, the fact that (4·1) and (4·2) give the Hartree solution to (4·3) suggests another method of obtaining the self-consistent solution to (4·1) and (4·2). Given the quantities $t_{ij}$ the wave functions $|\, n_i^0)$ can be obtained by minimizing the energy corresponding to (4·3) with respect to a set of trial wave functions $\chi_i$. The energy

$$\frac{(\det \chi_i, \{\Sigma T_i + \Sigma t_{ij}\} \det \chi_i)}{(\det \chi_i, \det \chi_i)} \tag{4·4}$$

will be a minimum when $\chi_i = |\, n_i^0)$, the self-consistent solution of (4·1) and (4·2).

In the present problem the situation is more complicated since $t_{ij}$ is not given but is determined itself by the self-consistent wave functions according to (3·8). The question arises whether, if the operators $t_{ij}$ are determined using the trial wave functions $\chi_i$ instead of $|\, n_i)$ in (3·8), the variational method will still give a minimum for (4·4) when (4·1) and (4·2) are satisfied. It is clear that this procedure will still be valid provided that the expectation value of $t_{ij}$ is itself stationary when (4·4) is stationary, that is, when $\chi_i = |\, n_i^0)$. If a variation of $\chi_i$ about this value is considered it will affect the diagonal part $(n_i^0 n_j^0 \,|\, t \,|\, n_i^0 n_j^0)$ of $t_{ij}$ only through its direct effect on $E_i^0$ in the denominator of (3·8) and its indirect effect on the other energy

* See also a forthcoming paper by H. A. Bethe and R. J. Eden entitled, 'Linked interactions in the nuclear many-body problem'.

eigenvalues $E_j^0$, $E_k'$, etc. Now $E_i^0$ is itself stationary when (4·4) is minimized and other energy eigenvalues in (3·8) have only a $1/A$ dependence on the state $\chi_i$. Hence, neglecting terms of order $1/A$, the expectation value of $t_{ij}$ is stationary for $\chi_i = |n_i^0)$.

It is now possible to state a new procedure for solving the self-consistent equations for the nuclear model. First, define a complete set of one-particle trial wave functions $\chi_i$ and take $A$ of these to form the chosen configuration $\chi_i^0$ $(i = 1, 2, \ldots, A)$. Secondly, solve the equations for the diagonal part of $t_{ij}$ using $\chi_i$ instead of $|n_i)$ in (3·8), thus

$$(\chi_i^0\chi_j^0 \mid t \mid \chi_i^0\chi_j^0) = (\chi_i^0\chi_j^0 \mid v \mid \chi_i^0\chi_j^0) + \sum_{\chi_i\chi_j}^{Q} \frac{(\chi_i^0\chi_j^0 \mid v \mid \chi_i\chi_j)\,(\chi_i\chi_j \mid t \mid \chi_i^0\chi_j^0)}{E_i^0 + E_j^0 - E_i - E_j}. \tag{4·5}$$

The intermediate states $\chi_i\chi_j$ must lie outside the chosen configuration. The energy eigenvalues in this equation must be determined so that they are consistent with the solution; they are given by

$$E_i^0 = (\chi_i^0 \mid T_i + V_i \mid \chi_i^0) \tag{4·6}$$

(and similarly for the excited states $\chi_i$), where

$$V_i = \sum_{j=1}^{A} (\chi_j^0 \mid t_{ij} \mid \chi_j^0), \tag{4·7}$$

the sum being over the chosen configuration and excluding $j = i$.

For a given set of trial wave functions $\chi_i$, equations (4·5), (4·6) and (4·7) can in principle be solved self-consistently. In principle also, the complete matrix $t_{ij}$ corresponding to these trial wave functions could be obtained, but in practice only the expectation value is required. The final step in the new procedure for solving the self-consistent equations is to substitute into (4·4) the $(\chi_i^0\chi_j^0 \mid t \mid \chi_i^0\chi_j^0)$ obtained from (4·5) to (4·7) and minimize the expression (4·4) by varying the trial wave functions.

The above arguments show that the minimum value will correspond to $\chi_i^0 = |n_i^0)$, where $|n_i^0)$ satisfies (4·1) and (4·2). The consistency of (4·5) to (4·7) must be maintained whilst varying (4·4). It will be seen that (4·1) and (4·2) agree with (4·6) and (4·7) when $\chi_i^0 = |n_i^0)$, and this establishes the equivalence of the variational method to the coupled self-consistent equations. It is important to note that the self-consistent equations are satisfied only when (4·4) is minimized. Thus the nuclear model corresponds to the nucleus only at the minimum value of the energy (4·4). Neglecting degeneracy, there is a unique correspondence between the chosen configuration of the model and a state of the actual nucleus.

The easiest situation for guessing trial wave functions occurs with a very large nucleus when the $\chi_i$ can be taken to be plane waves satisfying periodic boundary conditions. The chosen configuration will then correspond to a particular set of these plane waves chosen to satisfy the exclusion principle. The energies are determined by (4·6) and (4·7), and in this representation $V_i$ must be a function of momentum only. In this form the problem becomes identical with that considered by Brueckner and his collaborators. However, the method of solution used by Brueckner involved neglecting the requirement that the exclusion principle must be satisfied in intermediate states in equation (4·5). The method of solution of these equations allowing for the exclusion principle is discussed by Bethe (1956) and will not be considered here.

## 5. Conclusion

If the motion of a nucleon in a nucleus involves approximately equal interaction with many other nucleons there will be no clustering of nucleons in the nucleus. The success of the nuclear shell model and of the optical model seems to confirm that this situation occurs in practice. In these circumstances the Brueckner theory of nuclear structure in the form proposed in this paper and in that of Bethe (1956) provides not only a method of constructing the nuclear shell model, but also gives the relation between the wave function for the model and the actual nuclear wave function. The Brueckner theory takes account of the strong two-body interactions which must take place between nucleons in the nuclear medium but treats the rest of the nucleus as exerting only an average effect on the interaction, thus it will be valid only if clustering in the nucleus is unimportant. The theory can be set up in terms of a variational method which provides considerable simplification in considering the properties of an infinite nuclear medium. The exclusion principle is of great importance in justifying the method and strongly affects the manner in which the interaction of each nucleon pair must be calculated.

This paper has been concerned with setting up the formalism of the Brueckner theory in a self-consistent manner so that it is applicable to a nucleus of finite size and so that all quantities are precisely defined.

The author is greatly indebted to Professor H. A. Bethe for helpful discussions and also for his suggestion that the first draft of the present work be modified to incorporate the precision of the definitions introduced by Professor Bethe in his own paper on the Brueckner theory. The author also wishes to thank Professor R. E. Peierls, F.R.S., for some useful comments and Professor L. Rosenfeld for some helpful discussions.

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