There exist two PDFs:

$$h(Z, A)$$

and

$$\ell(Z, A)$$

The bidirectional translation between $h(Z, A)$ and $\ell(Z, A)$ is given by:

$$P(A \to A')$$

such that:

$$\ell(Z', A') = \int P(A \to A')\, h(Z, A)\; dA \tag{1}$$

and similarly:

$$h(Z', A') = \int P(A \to A')\, \ell(Z, A)\; dA \tag{2}$$

For each PDF there the expectation value in A is:

$$\bar{A}_h = \int A\, h(Z, A)\, dA \tag{3}$$

and:

$$\bar{A}_\ell = \int A\, \ell(Z, A)\, dA \tag{4}$$

And for these particular PDFs it is known empirically that the sum of these expectation values equals a known constant, A_{tot}:

$$\bar{A}_h + \bar{A}_\ell = A_{tot} \tag{5}$$

Now say that one PDF is modified:

$$h'(Z, A) = h(Z, A) + \sigma(Z, A) \tag{6}$$

where $\sigma(Z, A)$ is a modifier that is selected such that the normalization of $h'(Z, A)$ is maintained. This modifier might for example be some statistical re-sampling process that increases values in $h(Z, A)$ in some regions and decreases it in some regions. This additive modifier is similar to that of the quantile function of a Gaussian distribution.

And then the other PDF is regenerated from the modified PDF:

$$\ell'(Z, A) = \int P(A' \to A)\, h'(Z', A')\; dA' \tag{7}$$

Now show that the sum of the expectation values of these modified PDFs is equal to that of the original PDFs:

$$\bar{A}'_h + \bar{A}'_\ell = A_{tot} \tag{8}$$

Calculate $\bar{A}'_h$:

$$\bar{A}'_h = \int A\, h'(Z,A)\, dA$$

$$\bar{A}'_h = \int A\, h(Z,A) + \sigma(Z,A)\, dA$$

$$\bar{A}'_h = \int A\, h(Z,A)\, dA + \int A\, \sigma(Z,A)\, dA$$

$$\bar{A}'_h = \bar{A}_h + \int A\, \sigma(Z,A)\, dA$$

$$\bar{A}'_h = \bar{A}_h + \Delta_h \tag{9}$$

where:

$$\Delta_h = \int A\, \sigma(Z,A)\, dA \tag{10}$$

Calculate $\bar{A}'_\ell$:

$$\bar{A}'_\ell = \int A\, \ell'(Z,A)\, dA$$

$$\bar{A}'_\ell = \int A \int P(A' \to A)\, h'(Z',A')\; dA'\, dA$$

$$\bar{A}'_\ell = \int A \int P(A' \to A) \left[h(Z',A') + \sigma(Z',A')\right]\; dA'\, dA$$

$$\bar{A}'_\ell = \int A \left[\int P(A' \to A)\, h(Z',A')\, dA' + \int P(A' \to A)\, \sigma(Z',A'))\; dA'\right] dA$$

$$\bar{A}'_\ell = \int A \left[\ell(Z,A) + \int P(A' \to A)\, \sigma(Z',A'))\; dA'\right] dA$$

$$\bar{A}'_\ell = \left[\int A\,\ell(Z,A)\,dA + \int A \int P(A' \to A)\,\sigma(Z',A'))\;dA'\,dA \right]$$

$$\bar{A}'_\ell = \left[\int A\,\ell(Z,A)\,dA + \int A \int P(A' \to A)\,\sigma(Z',A'))\;dA'\,dA \right]$$

$$\bar{A}'_\ell = \left[\bar{A}_\ell + \int A \int P(A' \to A)\,\sigma(Z',A')\;dA'\,dA \right]$$

$$\bar{A}'_\ell = \bar{A}_\ell + \Delta_\ell \tag{11}$$

where:

$$\Delta_\ell = \int A \int P(A' \to A)\,\sigma(Z',A')\;dA'\,dA \tag{12}$$

So then:

$$\bar{A}'_h + \bar{A}'_\ell = \bar{A}_h + \Delta_h + \bar{A}_\ell + \Delta_\ell \tag{13}$$

In order to have a non-trivial answer, it must be shown that Δ_h and Δ_ℓ are of equal and opposite magnitude:

$$\Delta_h - \Delta_\ell = 0 \tag{14}$$

Based on the definitions of Δ_h and Δ_ℓ, for to be true then it must also be true that:

$$\int P(A' \to A)\,\sigma(Z',A')\;dA' = -\sigma(Z,A) \tag{15}$$

The effect of acting $P(A' \to A)$ on $\sigma(Z',A')$ must be derived:

$$h(Z',A') = h'(Z',A') - \sigma(Z',A')$$

$$\ell(Z,A) = \int P(A' \to A)\,h(Z',A')\;dA'$$

$$\ell(Z,A) = \int P(A' \to A)\,[h'(Z',A') - \sigma(Z',A')]\;dA'$$

$$\ell(Z,A) = \int P(A' \to A)\,h'(Z',A')\,dA' - \int P(A' \to A)\,\sigma(Z',A')\,dA'$$

$$\int P(A' \to A)\, \sigma(Z', A')\, dA' = \int P(A' \to A)\, h'(Z', A')\, dA' \;\; - \ell(Z, A)$$

And so the value of Δ_ℓ is:

$$\Delta_\ell = \int A \left[\int P(A' \to A)\, h'(Z', A')\, dA' \;\; - \ell(Z, A) \right] dA$$

$$\Delta_\ell = \int A \int P(A' \to A)\, h'(Z', A')\, dA'\, dA \;\; - \int A\, \ell(Z, A)\, dA$$

$$\Delta_\ell = \int A\, \ell(Z, A)\, dA \;\; - \bar{A}_\ell$$

$$\Delta_\ell = \bar{A}_\ell - \bar{A}_\ell = 0 \tag{16}$$

Now the value of Δ_h must be derived:

$$\sigma(Z, A) = h'(Z, A) - h(Z, A)$$

$$\Delta_h = \int A\, \sigma(Z, A)\, dA$$

$$\Delta_h = \int A\, [h'(Z, A) - h(Z, A)]\, dA$$

$$\Delta_h = \int A\, h'(Z, A)\, dA - \int A\, h(Z, A)\, dA$$

$$\Delta_h = \bar{A}'_h - \bar{A}_h \tag{17}$$

So then:

$$\bar{A}'_h + \bar{A}'_\ell = \bar{A}_h + \Delta_h + \bar{A}_\ell + \Delta_\ell$$

$$\bar{A}'_h + \bar{A}'_\ell = \bar{A}_h + \bar{A}'_h - \bar{A}_h + \bar{A}_\ell + 0$$

$$\bar{A}'_h + \bar{A}'_\ell = \bar{A}'_h + \bar{A}_\ell$$