Fresnel diffraction from two-slit apertures

Theoretical analysis

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1. Theoretical description

1.1. Interference

We start by reviewing the basics of interference. When two waves (or fields) superpose, they form characteristic bright and dark patterns. More precisely, the superposition of two waves is called *interference*. Let's say that these waves are $\mathbf{E}_1$ and $\mathbf{E}_2$. If both waves interact in a particular point of space at the same time, the resulting wave would be their addition:

$$\mathbf{E}_{\text{total}} = \mathbf{E}_1 + \mathbf{E}_2. \tag{1.1}$$

The validity of this property comes from the linearity of the wave equation (Fowles, 1989; Saleh and Teich, 1991). However, this is not valid all the time. The orientation of the electric or magnetic fields is essential for the linearity of the superposition to hold. Also, this condition is approximately valid inside a material Fowles (1989); Pedrotti et al. (2006). For our purposes, we will consider only scenarios where this property is true. For instance, consider the following *monochromatic waves*:

$$\begin{aligned} \mathbf{E}_1(\mathbf{r}, t) &= \mathbf{E}_{01} \cos(\mathbf{k}_1 \cdot \mathbf{r} - \omega t + \varphi_1), \\ \mathbf{E}_2(\mathbf{r}, t) &= \mathbf{E}_{02} \cos(\mathbf{k}_2 \cdot \mathbf{r} - \omega t + \varphi_2). \end{aligned} \tag{1.2}$$

In equation (1.2),

- $\mathbf{E}_{01}, \mathbf{E}_{02}$ represent the amplitude of their respective waves.
- φ represents the phase;
- $\mathbf{r}$ represents the position vector;
- $\mathbf{k} = (k_x, k_y, k_z)$ represents the wave vector, where $|\mathbf{k}| = k = 2\pi/\lambda$, k being the *wave number*;
- ω represents the angular frequency.
- t represents the time.

For monochromatic waves, both the amplitude and phase do not depend on the position, but for other waves this may not be true. Going back to the superposition, the resultant wave would be

$$\begin{aligned} \mathbf{E}_{\text{total}}(\mathbf{r}, t) &= \mathbf{E}_1(\mathbf{r}, t) + \mathbf{E}_2(\mathbf{r}, t) \\ &= \mathbf{E}_{01} \cos(\mathbf{k}_1 \cdot \mathbf{r} - \omega t + \varphi_1) + \mathbf{E}_{02} \cos(\mathbf{k}_2 \cdot \mathbf{r} - \omega t + \varphi_2). \end{aligned} \tag{1.3}$$

Another relevant concept is the *irradiance* I. The irradiance is a *physical quantity that measures the average energy impinging on a surface*, and it is defined as the square of the magnitude of the field,

$$I(\mathbf{r}) = |\mathbf{E}_{\text{total}}(\mathbf{r})|^2 = \mathbf{E}_{\text{total}}(\mathbf{r}) \cdot \mathbf{E}^*_{\text{total}}(\mathbf{r}). \tag{1.4}$$

where * denotes the complex conjugate. The resultant intensity corresponds to

$$I = |\mathbf{E}_{01}|^2 + |\mathbf{E}_{02}|^2 + 2\mathbf{E}_{01} \cdot \mathbf{E}_{02} \cos(\delta). \tag{1.5}$$

From here, $2\mathbf{E}_1 \cdot \mathbf{E}_2 \cos(\delta)$ is referred to as the *interference term.* Notice that the interference term is $\cos(\delta)$-dependent, where δ is the relative phase between waves,

$$\delta = \varphi_1 - \varphi_2. \tag{1.6}$$

Accordingly, if $\cos(\delta) = 1$, that is, $\delta = 2k, k \in \mathbb{Z}$, the amplitude is maximum; moreover, when $\cos(\delta) = -1$, the amplitude is minimum, which happens when $\delta = (2k+1)\pi, k \in \mathbb{Z}$. In other words, the interference relies on the polarization of the fields. Once we are aware of the basic idea of interference and diffraction, we can start to develop a more formal mathematical treatment for the problem.

1.2. Diffraction theory

One intriguing consequence of interference occurs when light passes through an obstacle. For centuries, it has been known that if light passes through an aperture in an opaque screen, the shadow will not exhibit complete sharpness instantly, contrasting with the expected behavior from geometrical optics. Instead, the light bends and becomes a second source of propagation. This is called *diffraction.* In this fashion, diffraction is regarded as *the deviation of the rectilinear path of light rays, casting bright and dark fringes*, as Sommerfeld explained (Born and Wolf, 2019; Fowles, 1989; Goodman, 2005). Here, we discuss the foundations of diffraction theory.

1.2.1 Kirchhoff's diffraction theory

Suppose there is a function $V(\mathbf{r}, t)$, not necessarily monochromatic (Born and Wolf, 2019), that is a solution of the time-dependent wave equation, and could be express as a Fourier transform,

$$V(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(\mathbf{r}) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}\omega \, . \tag{1.7}$$

Likewise, $U(\mathbf{r})$ is a solution of the time-independent wave equation, or *Helmholzt equation*, and could be expressed in terms of $V(\mathbf{r}, t)$ through the inverse Fourier transform,

$$U(\mathbf{r}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(\mathbf{r}, t) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}t \, . \tag{1.8}$$

As mentioned previously, both equations satisfy the respective wave equations:

$$\begin{aligned} \nabla^2 V(\mathbf{r}, t) &= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} V(\mathbf{r}, t), \\ (\nabla^2 + k^2) U(\mathbf{r}) &= 0. \end{aligned} \tag{1.9}$$

Also, let $\mathcal{S}$ be a closed surface that bounds a volume $\mathcal{V}$ where we set a point of observation P; r is the distance from P to an arbitrary point (x, y, z). We propose the function V (dependent only of the position) to be of the form

$$V(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{r}. \tag{1.10}$$

The wave vector is such that $\mathbf{k} \cdot \mathbf{r} = kr$. Nevertheless, if we set P at $r = 0$, then V will be discontinuous in contradiction to our function smoothness assumptions; to avoid the discontinuity, we exclude the point P from the integration by surrounding it with a small sphere of radius ε and then approach the singularity by taking the limit as $\varepsilon \to 0$. Said that, recall Green's theorem:

$$\iiint\limits_{\mathcal{V}} (U\nabla^2 V - V\nabla^2 U)\, \mathrm{d}^3x = \oiint\limits_{\mathcal{S}} \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n} \right) \mathrm{d}S\,, \tag{1.11}$$

where

$$\frac{\partial f}{\partial n} \equiv \nabla f \cdot \hat{\mathbf{n}} \tag{1.12}$$

represents the normal derivative at the surface $\mathcal{S}$ directed outward from the inside of the volume $\mathcal{S}$. If we want the inward direction (our case) we add a negative sign.

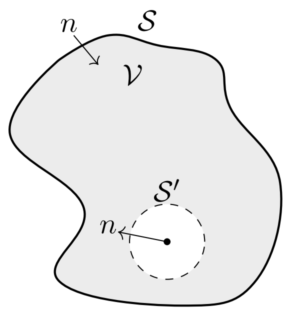

Figure 1: Region of integration. This region is a closed region of surface $\mathcal{S}$ and volume $\mathcal{V}$ that bounds the point of observation. Also, the normal vector to the surface is taken to be inwards.

Since U and V are solutions to the Helmholtz equation, the left-hand side of equation (1.11) goes to zero, yielding

$$\oiint\limits_{\mathcal{S}} \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n} \right) \mathrm{d}S = 0. \tag{1.13}$$

Substituting the equation (1.10) in equation (1.13), we get

$$\oiint\limits_{S} \left[U\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right) - \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\frac{\partial U}{\partial n} \right] \mathrm{d}S = 0.$$

As expected, we must integrate in both $\mathcal{S}$ and $\mathcal{S}'$,

$$\oiint\limits_{\mathcal{S}} \left[U\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right) - \frac{\mathrm{e}^{ikr}}{r}\frac{\partial U}{\partial n} \right] \mathrm{d}S + \oiint\limits_{\mathcal{S}'} \left[U\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right) - \frac{\mathrm{e}^{ikr}}{r}\frac{\partial U}{\partial n} \right] \mathrm{d}S' = 0.$$

We isolate for the integral over $\mathcal{S}$:

$$\begin{aligned}
\oiint_{\mathcal{S}} \left[U \frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{ikr}}{r}\right) - \frac{\mathrm{e}^{ikr}}{r}\frac{\partial U}{\partial n}\right] \mathrm{d}S &= -\oiint_{\mathcal{S}'} \left[U \frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right) - \frac{\mathrm{e}^{ikr}}{r}\frac{\partial U}{\partial n}\right] \mathrm{d}S' \\
&= -\oiint_{\mathcal{S}'} \left[U \left(\frac{\mathrm{i}kr - 1}{r^2}\right) \mathrm{e}^{\mathrm{i}kr}(\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}) - \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\frac{\partial U}{\partial n}\right] \mathrm{d}S' \\
&= -\oiint_{\mathcal{S}'} \left[U(\mathrm{i}kr - 1) - r\frac{\partial U}{\partial r}\right] \mathrm{e}^{\mathrm{i}kr} \frac{\mathrm{d}S'}{r^2}(\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}).
\end{aligned}$$

We notice the presence of the *solid angle*

$$\mathrm{d}\Omega = \frac{\mathrm{d}S'}{r^2}(\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}), \tag{1.14}$$

therefore

$$\oiint_{\mathcal{S}} \left[U \frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right) - \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\frac{\partial U}{\partial n}\right] \mathrm{d}S = -\oiint_{\Omega} \left[U(\mathrm{i}k\varepsilon - 1) - \varepsilon\frac{\partial U}{\partial n}\right] \mathrm{e}^{\mathrm{i}k\varepsilon}\, \mathrm{d}\Omega\,.$$

At this point, it is possible to take the limit as $\varepsilon \to 0$:

$$\oiint_{\mathcal{S}} \left[U \frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right) - \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\frac{\partial U}{\partial n}\right] \mathrm{d}S = -\lim_{\varepsilon \to 0}\oiint_{\Omega} \left[U(\mathrm{i}k\varepsilon - 1) - \varepsilon\frac{\partial U}{\partial n}\right] \mathrm{e}^{\mathrm{i}k\varepsilon}\, \mathrm{d}\Omega = \oiint_{\Omega} U\, \mathrm{d}\Omega\,.$$

The right-hand size evaluates to $U(4\pi)$, so

$$U(P) = \frac{1}{4\pi}\oiint_{\mathcal{S}} \left[U \frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right) - \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\frac{\partial U}{\partial n}\right] \mathrm{d}S \tag{1.15}$$

represents the *Kirchhoff-Helmholtz theorem* and the function $U(P)$ is called *disturbance*.

1.2.2 Kirchhoff's diffraction theory on a planar screen

With the previous result, we have the mathematical tools to study the diffraction on an arbitrary aperture (Fig. 2). Making use of equation (1.15), we determine the diffraction contribution on the paths $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$:

$$\begin{aligned}
U(P) = \frac{1}{4\pi}&\iint_{\mathcal{A}} \left[U \frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right) - \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\frac{\partial U}{\partial n}\right] \mathrm{d}S \\
+ \frac{1}{4\pi}&\iint_{\mathcal{B}} \left[U \frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right) - \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\frac{\partial U}{\partial n}\right] \mathrm{d}S \\
+ \frac{1}{4\pi}&\iint_{\mathcal{C}} \left[U \frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right) - \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\frac{\partial U}{\partial n}\right] \mathrm{d}S\,.
\end{aligned} \tag{1.16}$$

One thing worth mentioning is that the Kirchhoff-Helmholtz formulation works if the aperture dimensions are larger than the wavelength itself (Goodman, 2005).

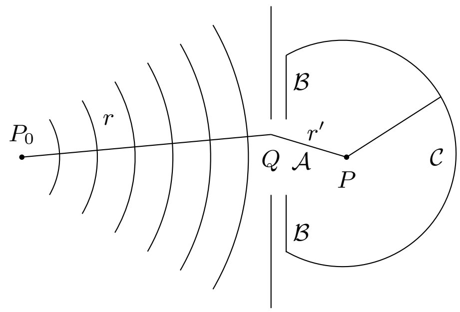

Figure 2: Fresnel-Kirchhoff diffraction (Born and Wolf, 2019). For the integration, it is necessary to consider the propagation of the waves through the aperture $\mathcal{A}$, the walls of the aperture (denoted by $\mathcal{B}$), and the wavefront $\mathcal{C}$.

Now, we proceed to establish what is known as *Kirchhoff's boundary conditions*. We establish that diffraction at $\mathcal{A}$ will be unaltered, and $\mathcal{B}$ will not have disturbance at all.

$$\begin{aligned} \mathcal{A}: \quad U = U(\mathbf{r}) \quad & \frac{\partial U}{\partial n} = \frac{\partial U(\mathbf{r})}{\partial n}, \\ \mathcal{B}: \quad U = 0 \qquad & \frac{\partial U}{\partial n} = 0. \end{aligned} \tag{1.17}$$

Thus, the perturbance for $\mathcal{A}$ refers to

$$U(\mathbf{r}) = \frac{U_0 \mathrm{e}^{\mathrm{i}kr}}{r}, \tag{1.18}$$

$$\frac{\partial U(\mathbf{r})}{\partial n} = \frac{U_0 \mathrm{e}^{\mathrm{i}kr}}{r} \left(\mathrm{i}k - \frac{1}{r} \right) \cos(\hat{\mathbf{n}}, \hat{\mathbf{r}}), \tag{1.19}$$

where U_0 is a constant. The same happens for $\mathbf{r}'$,

$$U(\mathbf{r}') = \frac{U_0 \mathrm{e}^{\mathrm{i}kr'}}{r}, \tag{1.20}$$

$$\frac{\partial U(\mathbf{r}')}{\partial n} = \frac{U_0 \mathrm{e}^{\mathrm{i}kr'}}{r'} \left(\mathrm{i}k - \frac{1}{r'} \right) \cos\left(\hat{\mathbf{n}}, \hat{\mathbf{r}}'\right), \tag{1.21}$$

where $(\hat{\mathbf{n}}, \hat{\mathbf{r}})$ denotes the angle between the normal surface of integration and the position. In summary, these conditions imply the following (Goodman, 2005):

1. The disturbance U and its normal derivative $\partial U/\partial n$ in $\mathcal{A}$ are the same as if there was no aperture.

2. The disturbance U and its normal derivative $\partial U/\partial n$ are negligible in $\mathcal{B}$.

However, Kirchhoff boundary conditions are problematic because they restrict simultaneously the perturbation and its normal derivative. In parallel, a first idea is that, if the radius of $\mathcal{C}$ is considerably large, then it will not contribute to the diffraction because the disturbance decays as $1/R$.

This is not necessarily true because the area grows as R^2, and we require another condition to neglect the integral (Born and Wolf, 2019). Another argument is that the source starts to irradiate at $t = t_0$. Therefore, for a time $t > t_0$, the field would have only covered a distance of $c(t - t_0)$ from P since the wave has a finite velocity; this suggests that $\mathcal{C}$ could not contribute to the diffraction. Nevertheless, this goes against the monochromatic case, and then it is not sufficient either. A deeper analysis of the boundary conditions is thus needed.

Return to equation (1.10). If $r = R$, that is, on $\mathcal{C}$, if R is sufficiently large, the normal derivative is approximately equal to

$$\frac{\partial V}{\partial n} = \left(\frac{\mathrm{i}kR - 1}{R^2}\right) \mathrm{e}^{\mathrm{i}kR} \approx \frac{\mathrm{i}k\mathrm{e}^{\mathrm{i}kR}}{R} = \mathrm{i}kV.$$

Using this approximation, we realize that

$$\oiint_{\mathcal{S}} \left(V\frac{\partial U}{\partial n} - U\frac{\partial V}{\partial n}\right) \approx \oiint_{\Omega} \left[V\frac{\partial U}{\partial n} - U(\mathrm{i}kV)\right] \mathrm{d}S = \oiint_{\Omega} V \left(\frac{\partial U}{\partial n} - \mathrm{i}kU\right) R^2 \, \mathrm{d}\Omega \,.$$

From the last expression, we get the *Sommerfeld radiation condition*,

$$\lim_{R\to\infty} R \left(\frac{\partial U}{\partial n} - \mathrm{i}kU\right) = 0. \tag{1.22}$$

The use of this condition is that it ensures the presence of waves emanating from the source instead of waves coming from the infinite.

1.2.3 Limit of the Kirchhoff-Helmholtz formulation

Once again, if we consider the Kirchhoff's boundary conditions, then the disturbance at P would be

$$U(P) = \frac{1}{4\pi} \oiint_{\mathcal{A}} \left[\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\frac{\partial U}{\partial n} - U\frac{\partial}{\partial n}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\right)\right] \mathrm{d}S \,. \tag{1.23}$$

As it has already been discussed, the Kirchhoff-Helmholtz formulation requires boundary conditions in the function and in its normal derivative, or mathematical speaking, it requires *Cauchy boundary conditions* to ensure a unique solution. Nevertheless, neither of the two conditions are always satisfied:

1. The aperture does perturb the field in $\mathcal{A}$.
2. The shadow in $\mathcal{B}$ extends to a distance of several wavelengths, which will affect the distance.

Both *Dirichlet* and *Neumann boundary conditions* are conditions imposed on differential equations to guarantee a unique solution. The Dirichlet boundary condition establishes which values belong to the solution of a differential equation over a closed surface; conversely, the Neumann boundary condition establishes the behavior of the normal derivative of the solution over a closed surface. To determine a physical solution, one of the two conditions must be met, but not both of them at the same time. The reason is that Cauchy boundary conditions are satisfied for an open surface, which is not our case of study (Jackson, 1998). Regardless of the contradictions that arise from the Kirchhoff-Helmholtz formulation, it produces outstanding experimental results if the aperture dimensions are larger in comparison to a wavelength.

1.3. Fresnel-Kirchhoff formulation

Previously, we established the Kirchhoff boundary conditions. If $k \gg 1/r$, we notice that

$$U(P) = \frac{U\mathrm{e}^{\mathrm{i}kr}}{r}\left(\mathrm{i}k - \frac{1}{r}\right)\cos(\hat{\mathbf{n}}, \hat{\mathbf{r}}) \approx \frac{\mathrm{i}kU_0\mathrm{e}^{\mathrm{i}kr}}{r}\cos(\hat{\mathbf{n}}, \hat{\mathbf{r}}).$$

If this happens, the disturbance could be expressed as

$$U(P) = -\frac{\mathrm{i}U_0\mathrm{e}^{-\mathrm{i}\omega t}}{\lambda}\iint_{\mathcal{A}} \frac{\mathrm{e}^{\mathrm{i}k(r+r')}}{rr'}\frac{[\cos(\hat{\mathbf{r}}, \hat{\mathbf{n}}) - \cos(\hat{\mathbf{r}}', \hat{\mathbf{n}})]}{2}\,\mathrm{d}S\,. \tag{1.24}$$

Here,

$$K(\theta) = \frac{\cos(\hat{\mathbf{r}}, \hat{\mathbf{n}}) - \cos(\hat{\mathbf{r}}', \hat{\mathbf{n}})}{2} \tag{1.25}$$

is known as the *oblique factor*, and equation (1.24) is the *Fresnel-Kirchhoff diffraction formula*. This expression has an intriguing feature. From equation (1.25), a source P_0 produces in the observation point P the same effect that P produces in P_0. This is called *Helmholtz reciprocity theorem* (Born and Wolf, 2019). Going back to Fresnel-Kirchhoff formulation, we could drop out the time factor for the sake of simplicity. We now propose an arbitrary aperture as the following figure:

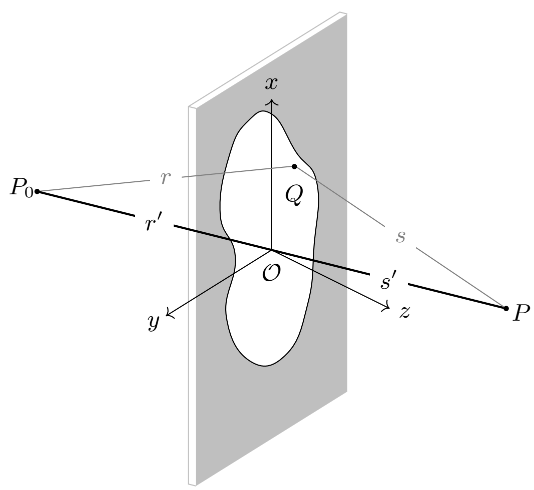

Figure 3: Arbitrary aperture. We consider an arbitrary point that lies in the xy plane with coordinates (ξ, η), where the diffraction will be studied. This `TikZ` figure is a modified version of the one shown here.

The points P_0 and P have as coordinates (x_0, y_0, z_0) and (x, y, z), respectively, and a point located in the aperture has as coordinate (ξ, η). We have the following distances:

$$\begin{aligned} r^2 &= (x_0 - \xi)^2 + (y_0 - \eta)^2 + z_0^2, \\ s^2 &= (x - \xi)^2 + (y - \eta)^2 + z^2, \\ (r')^2 &= x_0^2 + y_0^2 + z_0^2. \\ (s')^2 &= x^2 + y^2 + z^2. \end{aligned} \tag{1.26}$$

These expressions lead to

$$\begin{aligned} r^2 &= (r')^2 - 2(x_0\xi + y_0\eta) + \xi^2 + \eta^2, \\ s^2 &= (s')^2 - 2(x\xi + y\eta) + \xi^2 + \eta^2. \end{aligned} \tag{1.27}$$

If we say that the aperture dimensions are considerably smaller that r' and s', we are in position to expand the last two equations in power series (with the binomial expansion).

$$\begin{aligned} r &\sim r' - \frac{x_0\xi + y_0\eta}{r'} + \frac{\xi^2+\eta^2}{2r'} - \frac{(x_0\xi + y_0\eta)^2}{2(r')^3} - \cdots \\ s &\sim s' - \frac{x\xi + y\eta}{s'} + \frac{\xi^2+\eta^2}{2s'} - \frac{(x\xi + y\eta)^2}{2(s')^3} - \cdots \end{aligned} \tag{1.28}$$

With these expressions[1], we could write the Fresnel-Kirchhoff formula as a function of (ξ, η) by combining r and s such that

$$f(\xi,\eta) = -\frac{x_0\xi + y_0\eta}{r'} - \frac{x\xi + y\eta}{s'} + \frac{\xi^2+\eta^2}{s'} + \frac{\xi^2+\eta^2}{2r'} + \frac{\xi^2+\eta^2}{2s'} - \frac{(x_0\xi + y_0\eta)^2}{2(r')^3} - \frac{(x\xi + y\eta)^2}{2(s')^3} \cdots \tag{1.29}$$

With this, the Fresnel-Kirchhoff formula turns in

$$U(P) = -\frac{\mathrm{i}\cos(\delta)}{\lambda} \frac{U_0 \mathrm{e}^{\mathrm{i}k(r'+s')}}{r's'} \iint\limits_{\mathcal{A}} \mathrm{e}^{\mathrm{i}kf(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \tag{1.30}$$

We use δ to express the angle between the normal to the screen and the line $\overline{PP_0}$. We define the *direction cosines* as follows:

$$\begin{aligned} l_0 &= -\frac{x_0}{r'}, \qquad & l &= \frac{x}{s'}, \\ m_0 &= -\frac{y_0}{r'}, \qquad & m &= \frac{y}{s'}, \end{aligned} \tag{1.31}$$

so, $f(\xi, \eta)$ can be written as

$$f(\xi,\eta) = (l_0 - l)\xi + (m_0 - m)\eta + \frac{1}{2}\left[\left(\frac{1}{r'} + \frac{1}{s'}\right)(\xi^2+\eta^2)\frac{(l_0\xi + m_0\eta)^2}{r'} - \frac{(l\xi + m\eta)^2}{s'}\right] \cdots \tag{1.32}$$

In equation (1.29), we will have *Fresnel diffraction* if the quadratic terms can be neglected, and *Fraunhofer diffraction* if these terms cannot be neglected, which is of our interest for this situation.

1.3.1 Fresnel diffraction

Hitherto, we have explained the theory of diffraction following Kirchhoff's approach. Also, we introduced the origin of Fraunhofer and Fresnel diffraction. With this in mind, we proceed to study Fresnel diffraction in deeper detail. We return to equation (1.30) and use Euler's identity,

$$U(P) = \kappa \underbrace{\iint\limits_{\mathcal{A}} \cos[kf(\xi,\eta)]\,\mathrm{d}\xi\,\mathrm{d}\eta}_{\text{C}} + \mathrm{i}\kappa \underbrace{\iint\limits_{\mathcal{A}} \sin[kf(\xi,\eta)]\,\mathrm{d}\xi\,\mathrm{d}\eta}_{\text{S}}\,. \tag{1.33}$$

[1] We are going to use s instead of r', as it was denoted in figure 2.

From equation (1.33), we define

$$\kappa = -\frac{\mathrm{i}U_0\cos(\delta)}{\lambda}\frac{\mathrm{e}^{\mathrm{i}kr(r'+s')}}{r's'}.$$

Because we know what the disturbance is, we can compute the *irradiance* $I = |U(P)|^2$, which is

$$I(P) = I_0(C^2+S^2). \tag{1.34}$$

With $I_0 = |\kappa|^2$. If we put the aperture $\mathcal{A}$ on the xy plane, we could consider the figure 4. This implies that $l = l_0 = \sin(\delta)$, $m = m_0 = 0$, and $n = n_0 = \cos(\delta)$ (where $n = n_0$ refers to the third direction cosine and δ is, again, the angle between $\overline{PP_0}$ and z). Thus, equation (1.32) is reduced to

$$f(\xi,\eta) = \frac{1}{2}\left(\frac{1}{r'}+\frac{1}{s'}\right)\left[\xi^2\cos^2(\delta)+\eta^2\right]\cdots \tag{1.35}$$

If we keep only the first terms (i.e. the terms shown in equation (1.35)), the integrals will be expressed in function of these terms. To simplify the computations, we use a change of variables by dividing equation (1.35) in two parts. Recall that $k = 2\pi/\lambda$. To simplify the integrals, the following substitutions are proposed:

$$\frac{\pi}{\lambda}\left(\frac{1}{r'}+\frac{1}{s'}\right)\xi^2\cos^2(\delta) = \frac{\pi}{2}u^2 \tag{1.36}$$

$$\frac{\pi}{\lambda}\left(\frac{1}{r'}+\frac{1}{s'}\right)\eta^2 = \frac{\pi}{2}v^2 \tag{1.37}$$

Finally, we need an equivalent to $\mathrm{d}\xi\,\mathrm{d}\eta$, which happens to be

$$\mathrm{d}\xi\,\mathrm{d}\eta = \frac{\lambda}{2\left(\frac{1}{r'}+\frac{1}{s'}\right)\cos(\delta)}\,\mathrm{d}u\,\mathrm{d}v\,.$$

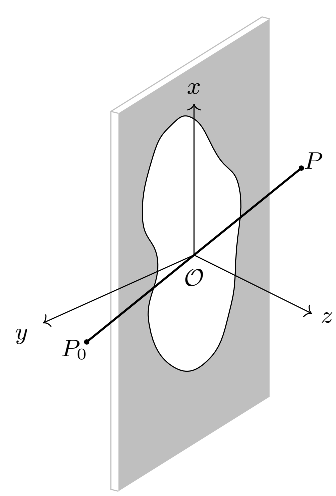

Figure 4: Diffraction at an arbitrary aperture. Now, for simplicity, the line $\overline{PP_0}$ is taken such that its projection lies on the x direction.

This leads to the diffraction integrals:

$$\mathrm{C} = b \iint_{\mathcal{A}} \cos\left[\frac{\pi(u^2+v^2)}{2}\right] \mathrm{d}u\, \mathrm{d}v\,, \tag{1.38}$$

$$\mathrm{S} = b \iint_{\mathcal{A}} \sin\left[\frac{\pi(u^2+v^2)}{2}\right] \mathrm{d}u\, \mathrm{d}v\,. \tag{1.39}$$

These integrals are special functions, as it will be discuss in more detail in the next subsection.

1.3.2 Fresnel integrals for rectangular slits

Let $\mathcal{A}$ be a rectangular type of aperture. Let's say that the limits of integration are u_1 and u_2 for u and v_1 and v_2 for v. Instead of writing the previous integrals as a double integral, we could write the disturbance as the product of two integrals (applying Fubini's theorem because of the constant limits). We now do the same strategy explained for the general aperture.

$$U(P) = -\frac{\mathrm{i}U_0\cos(\delta)}{\lambda}\frac{\mathrm{e}^{\mathrm{i}k(r'+s')}}{r's'} \iint_{\mathcal{A}} \mathrm{e}^{\mathrm{i}kf(\xi,\eta)}\, \mathrm{d}\xi\, \mathrm{d}\eta\,. \tag{1.40}$$

Next, we use the function $f(\xi)$,

$$f(\xi) = \frac{1}{2}\left(\frac{r'+s'}{r's'}\right)\xi^2\cos^2(\delta) + \frac{1}{2}\left(\frac{r'+s'}{r's'}\right)\eta^2 \tag{1.41}$$

in equation (1.40):

$$U(P) = -\frac{\mathrm{i}U_0\cos(\delta)}{\lambda}\frac{\mathrm{e}^{\mathrm{i}k(r'+s')}}{r's'} \iint_{\mathcal{A}} \mathrm{e}^{\mathrm{i}k\left\{\frac{1}{2}\left(\frac{1}{r'}+\frac{1}{s'}\right)\left[\xi^2\cos^2(\delta)+\eta^2\right]\right\}}\, \mathrm{d}\xi\, \mathrm{d}\eta\,. \tag{1.42}$$

Once again, we use the substitutions (1.36), (1.37), and (1.38),

$$\begin{aligned} U(P) &= -\frac{\mathrm{i}U_0\cos(\delta)\mathrm{e}^{\mathrm{i}k(r'+s')}}{\lambda r's'\left[2\left(\frac{1}{r'}+\frac{1}{s'}\right)\cos(\delta)\right]} \int_{v_1}^{v_2}\int_{u_1}^{u_2} \mathrm{e}^{\frac{\mathrm{i}\pi(u^2+v^2)}{2}}\, \mathrm{d}u\, \mathrm{d}v \\ &= -\frac{\mathrm{i}U_0\mathrm{e}^{\mathrm{i}k(r'+s')}}{2(r'+s')} \int_{v_1}^{v_2}\int_{u_1}^{u_2} \mathrm{e}^{\frac{\mathrm{i}\pi(u^2+v^2)}{2}}\, \mathrm{d}u\, \mathrm{d}v\,. \end{aligned}$$

For the sake of simplicity, the term outside the integral is expressed as ε_0 (keeping the two factor). With Fubini's theorem, we rewrite the disturbance as

$$\begin{aligned} U(P) &= \varepsilon_0 \left[\int_{v_1}^{v_2} \mathrm{e}^{\frac{\mathrm{i}\pi v^2}{2}}\, \mathrm{d}v\right]\left[\int_{u_1}^{u_2} \mathrm{e}^{\frac{\mathrm{i}\pi u^2}{2}}\, \mathrm{d}u\right] \\ &= \frac{\varepsilon_0}{2}\left[\mathrm{C}(v)+\mathrm{i}S(v)\right]_{v_1}^{v_2}\left[\mathrm{C}(u)+\mathrm{i}S(u)\right]_{u_1}^{u_2}\,. \end{aligned} \tag{1.43}$$

This expression is of particular interest for the model of a slit, which is about to be discussed.

1.3.3 Fresnel diffraction from two-slit apertures

With the theoretical background covered, we have the mathematical and physical tools to study the behavior of Fresnel diffraction, thence the Fresnel diffraction in two-slit apertures. As there are two slits, we must consider two disturbances because the light would pass through both slits and superpose on an observation point.

$$\begin{aligned} U(P) &= U_1 + U_2 \\ &= \frac{\varepsilon_0}{2}\left[\mathrm{C}(v) + \mathrm{i}S(v)\right]_{v_1}^{v_2}\left[\mathrm{C}(u) + \mathrm{iS}(u)\right]_{u_1}^{u_2} + \frac{\varepsilon_0}{2}\left[\mathrm{C}(v) + \mathrm{iS}(v)\right]_{v_1}^{v_2}\left[\mathrm{C}(u) + \mathrm{iS}(u)\right]_{u_3}^{u_4} \\ &= \frac{\varepsilon_0}{2}\left\{\left[\mathrm{C}(u) + \mathrm{iS}(u)\right]_{u_1}^{u_2} + \left[\mathrm{C}(u) + \mathrm{iS}(u)\right]_{u_3}^{u_4}\right\}\left[\mathrm{C}(v) + \mathrm{iS}(v)\right]_{v_1}^{v_2}. \end{aligned} \tag{1.44}$$

Thus, the total disturbance is

$$\begin{aligned} U(P) =& \frac{\varepsilon_0}{2}\left\{\left[\mathrm{C}(u_2) + \mathrm{C}(u_4) - \mathrm{C}(u_1) - \mathrm{C}(u_3)\right] + \mathrm{i}\left[\mathrm{S}(u_2) + \mathrm{S}(u_4) - \mathrm{S}(u_1) - \mathrm{S}(u_3)\right]\right\} \\ &\times \left\{\left[\mathrm{C}(v_2) - \mathrm{C}(v_1)\right] + \mathrm{i}\left[\mathrm{S}(v_2) - \mathrm{S}(v_1)\right]\right\}. \end{aligned} \tag{1.45}$$

The intensity would be the magnitude squared of the total disturbance. For the two-slit aperture, we need to consider $v_1 \to -\infty$ and $v_2 \to \infty$, which leads to

$$\begin{aligned} \mathrm{C}(\infty) &= \int_0^\infty \cos\left(\frac{\pi\phi^2}{2}\right)\mathrm{d}\phi\,, \\ \mathrm{S}(\infty) &= \int_0^\infty \sin\left(\frac{\pi\phi^2}{2}\right)\mathrm{d}\phi\,. \end{aligned} \tag{1.46}$$

Therefore, the disturbance reduces to

$$U(P) = \frac{\varepsilon_0\mathrm{i}(1+\mathrm{i})}{2}\left[\mathcal{F}(U_2) - \mathcal{F}(u_1) + \mathcal{F}(u_3) - \mathcal{F}(u_1)\right], \tag{1.47}$$

where $\mathcal{F}(x) = \mathrm{C}(x) + \mathrm{iS}(x)$. Since $(1+\mathrm{i}) \equiv \sqrt{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}$, the disturbance is

$$U(P) = \frac{\varepsilon_0\sqrt{2}\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{2}\left[\mathcal{F}(U_2) - \mathcal{F}(u_1) + \mathcal{F}(u_3) - \mathcal{F}(u_1)\right]. \tag{1.48}$$

Therefore, the intensity is

$$I(P) = \frac{I_0}{4}\left\{\left[\mathrm{C}(u_2) + \mathrm{C}(u_4) - \mathrm{C}(u_1) - \mathrm{C}(u_3)\right]^2 + \left[\mathrm{S}(u_2) + \mathrm{S}(u_4) - \mathrm{S}(u_1) - \mathrm{S}(u_3)\right]^2\right\}. \tag{1.49}$$

With this function, we will be able to model the behavior of the Fresnel diffraction in a two-slit aperture.

2. Numerical approach

Given the equation (1.49), we can start thinking about the numerical simulation. This work is based on the paper of Lock (1987). In such paper, some dimensions are proposed, and these dimensions consist of both slits of $2a$ width and their interior edges separated by a distance of $2d$. Also, the apertures is a distance D away from the viewing screen. The parameters u_1, u_2, u_3 and u_4 represent the dimensions of the slit, which corresponds to

$$
\begin{aligned}
u_2 = \alpha &= \left(\frac{2}{\lambda D}\right)^{\frac{1}{2}}(x + d + 2a)\,, \\
u_1 = \beta &= \left(\frac{2}{\lambda D}\right)^{\frac{1}{2}}(x + d)\,,
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
u_4 = \gamma &= \left(\frac{2}{\lambda D}\right)^{\frac{1}{2}}(x - d)\,, \\
u_3 = \delta &= \left(\frac{2}{\lambda D}\right)^{\frac{1}{2}}(x - d - 2a)\,.
\end{aligned}
\tag{2.2}
$$

We consider only the distance that is covered in each case. Also, at this point, the (normalized) irrandiance is

$$
\frac{I(P)}{I_0} = \frac{1}{4}\left[\mathrm{C}(\alpha) + \mathrm{C}(\gamma) - \mathrm{C}(\beta) - \mathrm{C}(\delta)\right]^2 + \left[\mathrm{S}(\alpha) + \mathrm{S}(\gamma) - \mathrm{S}(\beta) - \mathrm{S}(\delta)\right]^2 . \tag{2.3}
$$

For the visualization of this phenomenon, go to the `Jupyter notebook` uploaded in this folder.

Conflicts of interest

Some of the figures used in this document were based on the figures available in the following website (in French): https://femto-physique.fr/omp/transformee-de-fourier.php.

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