UNIVERSITY OF ENERGY AND NATURAL RESOURCES

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

ELNG 305: Classical control systems

LECTURER

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Control system Modeling

After completing this lecture, the student will be able to:

- Find the Laplace transform of time functions and the inverse Laplace transform.
- Find the transfer function from a differential equation and solve the differential equation using the transfer function.
- Find the transfer function for linear, time-invariant electrical networks

Introduction to control system Modeling

- In lecture 1, we discussed the analysis and design sequence that included obtaining the system's schematic and demonstrated this step for a position control system.
- To obtain a schematic, the control systems engineer must often make many simplifying assumptions in order to keep the ensuing model manageable and still approximate physical reality.
- The next step is to develop mathematical models from schematics of physical systems.
- We will discuss two methods: (1) transfer functions in the frequency domain and (2) state equations in the time domain

Transfer functions in the frequency domain

- In every modeling case, the first step in developing a mathematical model is to apply the fundamental physical laws of science and engineering like
- For example, when we model electrical networks, Ohm’s law and Kirchhoff’s laws, which are basic laws of electric networks, will be applied initially. We will sum voltages in a loop or sum currents at a node.
- When we study mechanical systems, we will use Newton’s laws as the fundamental guiding principles. Here we will sum forces or torques.
- From these equations we will obtain the relationship between the system’s output and input.

Transfer functions in the frequency domain

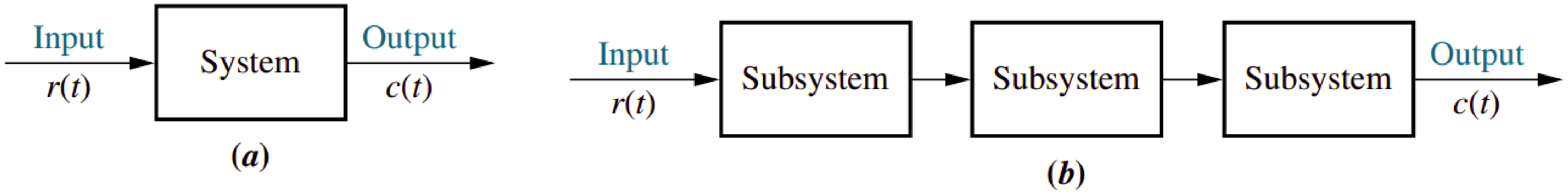
- Differential equation can describe the relationship between the input and output of a system.

$$\frac{d^m c(t)}{dt^n} + d_{n-1}\frac{d^{m-1}c(t)}{dt^{n-1}} + \cdots + d_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1}r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

- Many systems can be approximately described by this equation, which relates the output, c(t), to the input, r(t), by way of the system parameters, ai and bj. We assume the reader is familiar with differential equations.
- Although the differential equation relates the system to its input and output, it is not a satisfying representation from a system perspective.
- This is because we see that the system parameters, which are the coefficients, as well as the output, c(t), and the input,r(t), appear throughout the equation.

Transfer functions in the frequency domain

- We would prefer a mathematical representation such as that shown in Figure 2.1(a), where the input, output, and system are distinct and separate parts.
- Also, we would like to represent conveniently the interconnection of several sub-systems. For example, we would like to represent cascaded interconnection in Figure 2.1(b), where a mathematical function, called a transfer function, is inside each block, and block functions can easily be combined to yield Figure 2.1 (a) for ease of analysis and design.
- This convenience cannot be obtained with the differential equation



Laplace Transform

- A system represented by a differential equation is difficult to model as a block diagram.
- Thus, we now lay the groundwork for the Laplace transform, with which we

can represent the input, output, and system as separate entities.

- Further, their interrelationship will be simply algebraic.

Laplace Transform

- Let us first define the Laplace transform and then show how it simplifies the representation of physical systems.
- The Laplace transform is defined as

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$

where $s = \sigma + j\omega$, a complex variable. Thus, knowing $f(t)$ and that the integral in Eq. (2.1) exists, we can find a function, $F(s)$, that is called the *Laplace transform* of $f(t)$.[1]

PROBLEM: Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

SOLUTION: Since the time function does not contain an impulse function, we can replace the lower limit of Eq. (2.1) with 0. Hence,

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}\,dt = \int_{0}^{\infty} Ae^{-at}e^{-st}\,dt = A\int_{0}^{\infty} e^{-(s+a)t}\,dt$$

$$= -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{A}{s+a} \qquad (2.3)$$

Laplace Transform

- . If we use the tables, we do not have to use Eq. (2.2), which requires complex integration, to find f(t) given F(s).
- Table 2.1 shows the results for a representative sample of functions.

TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^n + 1}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$

Inverse Laplace Transform

In addition to the Laplace transform table, Table 2.1, we can use Laplace transform theorems, listed in Table 2.2, to assist in transforming between f(t) and F(s). In the next example, we demonstrate the use of the Laplace transform theorems shown in Table 2.2 to find f(t) given F(s).

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem

Inverse Laplace Transform

Item no.	Theorem		Name
7.	$\mathscr{L}\left[\dfrac{df}{dt}\right]$	$= sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\dfrac{d^2f}{dt^2}\right]$	$= s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\dfrac{d^nf}{dt^n}\right]$	$= s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right]$	$= \dfrac{F(s)}{s}$	Integration theorem
11.	$f(\infty)$	$= \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+)$	$= \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

Inverse Laplace Transform

PROBLEM: Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

SOLUTION: For this example we make use of the frequency shift theorem, Item 4 of Table 2.2, and the Laplace transform of $f(t) = tu(t)$, Item 3 of Table 2.1. If the inverse transform of $F(s) = 1/s^2$ is $tu(t)$, the inverse transform of $F(s+a) = 1/(s+a)^2$ is $e^{-at}tu(t)$. Hence, $f_1(t) = e^{-3t}tu(t)$.

Partial-Fraction Expansion

To find the inverse Laplace transform of a complicated function, we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term. The result is called a partial-fraction expansion.

- We will now consider three cases and show for each case how an $F(s)$ can be expanded into partial fractions

Partial-Fraction Expansion

Case 1. Roots of the Denominator of F(s) Are Real and Distinct

In general, then, given an $F(s)$ whose denominator has real and distinct roots, a partial-fraction expansion,

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)}$$
$$= \frac{K_1}{(s+p_1)} + \frac{K_2}{(s+p_2)} + \cdots + \frac{K_m}{(s+p_m)} + \cdots + \frac{K_n}{(s+p_n)} \tag{2.11}$$

can be made if the order of $N(s)$ is less than the order of $D(s)$. To evaluate each residue, K_i, we multiply Eq. (2.11) by the denominator of the corresponding partial fraction. Thus, if we want to find K_m, we multiply Eq. (2.11) by $(s+p_m)$ and get

Partial-Fraction Expansion

$$(s+p_m)F(s) = \frac{(s+p_m)N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)}$$
$$= (s+p_m)\frac{K_1}{(s+p_1)} + (s+p_m)\frac{K_2}{(s+p_2)} + \cdots + K_m + \cdots$$
$$+ (s+p_m)\frac{K_n}{(s+p_n)} \tag{2.12}$$

If we let s approach $-p_m$, all terms on the right-hand side of Eq. (2.12) go to zero except the term K_m, leaving

$$\left.\frac{\cancel{(s+p_m)}N(s)}{(s+p_1)(s+p_2)\cdots\cancel{(s+p_m)}\cdots(s+p_n)}\right|_{s\to -p_m} = K_m \tag{2.13}$$

The following example demonstrates the use of the partial-fraction expansion to solve a differential equation. We will see that the Laplace transform reduces the task of finding the solution to simple algebra.

Partial-Fraction Expansion

Case 1. Roots of the Denominator of *F(s)* Are Real and Distinct An example of an $F(s)$ with real and distinct roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)} \tag{2.7}$$

The roots of the denominator are distinct, since each factor is raised only to unity power. We can write the partial-fraction expansion as a sum of terms where each factor of the original denominator forms the denominator of each term, and constants, called *residues*, form the numerators. Hence,

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)} \tag{2.8}$$

To find K_1, we first multiply Eq. (2.8) by $(s+1)$, which isolates K_1. Thus,

$$\frac{2}{(s+2)} = K_1 + \frac{(s+1)K_2}{(s+2)} \tag{2.9}$$

Letting s approach -1 eliminates the last term and yields $K_1 = 2$. Similarly, K_2 can be found by multiplying Eq. (2.8) by $(s+2)$ and then letting s approach -2; hence, $K_2 = -2$.

Each component part of Eq. (2.8) is an $F(s)$ in Table 2.1. Hence, $f(t)$ is the sum of the inverse Laplace transform of each term, or

$$f(t) = (2e^{-t} - 2e^{-2t})u(t) \tag{2.10}$$

Partial-Fraction Expansion

PROBLEM: Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t) \tag{2.14}$$

SOLUTION: Substitute the corresponding $F(s)$ for each term in Eq. (2.14), using Item 2 in Table 2.1, Items 7 and 8 in Table 2.2, and the initial conditions of $y(t)$ and $dy(t)/dt$ given by $y(0-) = 0$ and $\dot{y}(0-) = 0$, respectively. Hence, the Laplace transform of Eq. (2.14) is

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s} \tag{2.15}$$

Solving for the response, $Y(s)$, yields

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)} \tag{2.16}$$

Partial-Fraction Expansion

To solve for $y(t)$, we notice that Eq. (2.16) does not match any of the terms in Table 2.1. Thus, we form the partial-fraction expansion of the right-hand term and match each of the resulting terms with $F(s)$ in Table 2.1. Therefore,

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)} \tag{2.17}$$

where, from Eq. (2.13),

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0} = 1 \tag{2.18a}$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s\to -4} = -2 \tag{2.18b}$$

Partial-Fraction Expansion

$$K_3 = \left.\frac{32}{s(s+4)}\right|_{s\to -8} = 1 \qquad (2.18c)$$

Hence,

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)} \qquad (2.19)$$

Since each of the three component parts of Eq. (2.19) is represented as an $F(s)$ in Table 2.1, $y(t)$ is the sum of the inverse Laplace transforms of each term. Hence,

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t) \qquad (2.20)$$

The $u(t)$ in Eq. (2.20) shows that the response is zero until $t = 0$. Unless otherwise specified, all inputs to systems in the text will not start until $t = 0$. Thus, output responses will also be zero until $t = 0$. For convenience, we will leave off the $u(t)$ notation from now on. Accordingly, we write the output response as

$$y(t) = 1 - 2e^{-4t} + e^{-8t} \qquad (2.21)$$

Partial-Fraction Expansion

Case 2. Roots of the Denominator of F(s) Are Real and Repeated

In general, then, given an $F(s)$ whose denominator has real and repeated roots, a partial-fraction expansion,

$$\begin{aligned} F(s) &= \frac{N(s)}{D(s)} \\ &= \frac{N(s)}{(s+p_1)^r(s+p_2)\cdots(s+p_n)} \\ &= \frac{K_1}{(s+p_1)^r} + \frac{K_2}{(s+p_1)^{r-1}} + \cdots + \frac{K_r}{(s+p_1)} \\ &\quad + \frac{K_{r+1}}{(s+p_2)} + \cdots + \frac{K_n}{(s+p_n)} \end{aligned} \tag{2.27}$$

can be made if the order of $N(s)$ is less than the order of $D(s)$ and the repeated roots are of multiplicity r at $-p_1$. To find K_1 through K_r for the roots of multiplicity greater than unity, first multiply Eq. (2.27) by $(s+p_1)^r$ getting $F_1(s)$, which is

Partial-Fraction Expansion

$$F_1(s) = (s+p_1)^r F(s)$$
$$= \frac{(s+p_1)^r N(s)}{(s+p_1)^r (s+p_2)\cdots(s+p_n)}$$
$$= K_1 + (s+p_1)K_2 + (s+p_1)^2 K_3 + \cdots + (s+p_1)^{r-1} K_r$$
$$+ \frac{K_{r+1}(s+p_1)^r}{(s+p_2)} + \cdots + \frac{K_n(s+p_1)^r}{(s+p_n)} \qquad (2.28)$$

Immediately, we can solve for K_1 if we let s approach $-p_1$. We can solve for K_2 if we differentiate Eq. (2.28) with respect to s and then let s approach $-p_1$. Subsequent differentiation will allow us to find K_3 through K_r. The general expression for K_1 through K_r for the multiple roots is

$$K_i = \frac{1}{(i-1)!} \left. \frac{d^{i-1} F_1(s)}{ds^{i-1}} \right|_{s \to -p_1} \qquad i = 1, 2, \ldots, r; \quad 0! = 1 \qquad (2.29)$$

Partial-Fraction Expansion

Case 2. Roots of the Denominator of *F(s)* Are Real and Repeated An example of an $F(s)$ with real and repeated roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)^2} \tag{2.22}$$

The roots of $(s+2)^2$ in the denominator are repeated, since the factor is raised to an integer power higher than 1. In this case, the denominator root at -2 is a *multiple root* of *multiplicity* 2.

We can write the partial-fraction expansion as a sum of terms, where each factor of the denominator forms the denominator of each term. In addition, each multiple root generates additional terms consisting of denominator factors of reduced multiplicity. For example, if

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)} \tag{2.23}$$

then $K_1 = 2$, which can be found as previously described. K_2 can be isolated by multiplying Eq. (2.23) by $(s+2)^2$, yielding

Partial-Fraction Expansion

$$\frac{2}{s+1} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3 \tag{2.24}$$

Letting s approach -2, $K_2 = -2$. To find K_3 we see that if we differentiate Eq. (2.24) with respect to s,

$$\frac{-2}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2} K_1 + K_3 \tag{2.25}$$

K_3 is isolated and can be found if we let s approach -2. Hence, $K_3 = -2$.

Each component part of Eq. (2.23) is an $F(s)$ in Table 2.1; hence, $f(t)$ is the sum of the inverse Laplace transform of each term, or

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t} \tag{2.26}$$

If the denominator root is of higher multiplicity than 2, successive differentiation would isolate each residue in the expansion of the multiple root.

Partial-Fraction Expansion

Case 3. Roots of the Denominator of F(s) Are Complex or Imaginary

In general, then, given an $F(s)$ whose denominator has complex or purely imaginary roots, a partial-fraction expansion,

$$\begin{aligned} F(s) = \frac{N(s)}{D(s)} &= \frac{N(s)}{(s+p_1)(s^2+as+b)\cdots} \\ &= \frac{K_1}{(s+p_1)} + \frac{(K_2 s + K_3)}{(s^2+as+b)} + \cdots \end{aligned} \tag{2.42}$$

can be made if the order of $N(s)$ is less than the order of $D(s)$ p_1 is real, and $(s^2 + as + b)$ has complex or purely imaginary roots. The complex or imaginary roots are expanded with $(K_2 s + K_3)$ terms in the numerator rather than just simply K_i, as in the case of real roots. The K_i's in Eq. (2.42) are found through balancing the coefficients of the equation after clearing fractions. After completing the squares on $(s^2 + as + b)$ and adjusting the numerator, $(K_2 s + K_3)/(s^2 + as + b)$ can be put into the form shown on the right-hand side of Eq. (2.36).

Partial-Fraction Expansion

Finally, the case of purely imaginary roots arises if $a = 0$ in Eq. (2.42). The calculations are the same.

Another method that follows the technique used for the partial-fraction expansion of $F(s)$ with real roots in the denominator can be used for complex and imaginary roots. However, the residues of the complex and imaginary roots are themselves complex conjugates. Then, after taking the inverse Laplace transform, the resulting terms can be identified as

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = cos\,\theta \tag{2.43}$$

and

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = sin\,\theta \tag{2.44}$$

Partial-Fraction Expansion

Case 3. Roots of the Denominator of $F(s)$ Are Complex or Imaginary An example of $F(s)$ with complex roots in the denominator is

$$F(s) = \frac{3}{s(s^2+2s+5)} \tag{2.30}$$

This function can be expanded in the following form:

$$\frac{3}{s(s^2+2s+5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2s+5} \tag{2.31}$$

K_1 is found in the usual way to be $\frac{3}{5}$. K_2 and K_3 can be found by first multiplying Eq. (2.31) by the lowest common denominator, $s(s^2+2s+5)$, and clearing the fractions. After simplification with $K_1 = \frac{3}{5}$, we obtain

$$3 = \left(K_2 + \frac{3}{5}\right)s^2 + \left(K_3 + \frac{6}{5}\right)s + 3 \tag{2.32}$$

Partial-Fraction Expansion

Balancing coefficients, $\left(K_2 + \frac{3}{5}\right) = 0$ and $\left(K_3 + \frac{6}{5}\right) = 0$. Hence $K_2 = -\frac{3}{5}$ and $K_3 = -\frac{6}{5}$. Thus,

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3/5}{s} - \frac{3}{5} \frac{s+2}{s^2 + 2s + 5} \tag{2.33}$$

The last term can be shown to be the sum of the Laplace transforms of an exponentially damped sine and cosine. Using Item 7 in Table 2.1 and Items 2 and 4 in Table 2.2, we get

$$\mathscr{L}[Ae^{-at}\cos\omega t] = \frac{A(s+a)}{(s+a)^2 + \omega^2} \tag{2.34}$$

Similarly,

$$\mathscr{L}[Be^{-at}\sin\omega t] = \frac{B\omega}{(s+a)^2 + \omega^2} \tag{2.35}$$

Partial-Fraction Expansion

Adding Eqs. (2.34) and (2.35), we get

$$\mathscr{L}[Ae^{-at}\cos\omega t + Be^{-at}\sin\omega t] = \frac{A(s+a)+B\omega}{(s+a)^2+\omega^2} \tag{2.36}$$

We now convert the last term of Eq. (2.33) to the form suggested by Eq. (2.36) by completing the squares in the denominator and adjusting terms in the numerator without changing its value. Hence,

$$F(s) = \frac{3/5}{s} - \frac{3}{5}\frac{(s+1)+(1/2)(2)}{(s+1)^2+2^2} \tag{2.37}$$

Comparing Eq. (2.37) to Table 2.1 and Eq. (2.36), we find

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) \tag{2.38}$$

Partial-Fraction Expansion

In order to visualize the solution, an alternate form of $f(t)$, obtained by trigonometric identities, is preferable. Using the amplitudes of the cos and sin terms, we factor out $\sqrt{1^2 + (1/2)^2}$ from the term in parentheses and obtain

$$f(t) = \frac{3}{5} - \frac{3}{5}\sqrt{1^2 + (1/2)^2}\,e^{-t}\left(\frac{1}{\sqrt{1^2 + (1/2)^2}}\cos 2t + \frac{1/2}{\sqrt{1^2 + (1/2)^2}}\sin 2t\right) \quad (2.39)$$

Letting $1/\sqrt{1^2 + (1/2)^2} = \cos\phi$ and $(1/2)/\sqrt{1^2 + (1/2)^2} = \sin\phi$,

$$f(t) = \frac{3}{5} - \frac{3}{5}\sqrt{1^2 + (1/2)^2}\,e^{-t}(\cos\phi\cos 2t + \sin\phi\sin 2t) \quad (2.40)$$

or

$$f(t) = 0.6 - 0.671e^{-t}\cos(2t - \phi) \quad (2.41)$$

where $\phi = \arctan 0.5 = 26.57^{\circ}$. Thus, $f(t)$ is a constant plus an exponentially damped sinusoid.

Partial-Fraction Expansion

F(s) can also be expanded in partial fractions as

$$F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3}{s(s+1+j2)(s+1-j2)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s+1+j2} + \frac{K_3}{s+1-j2} \quad (2.45)$$

Finding K_2,

$$K_2 = \left.\frac{3}{s(s+1-j2)}\right|_{s\to -1-j2} = -\frac{3}{20}(2+j1) \quad (2.46)$$

Similarly, K_3 is found to be the complex conjugate of K_2, and K_1 is found as previously described. Hence,

$$F(s) = \frac{3/5}{s} - \frac{3}{20}\left(\frac{2+j1}{s+1+j2} + \frac{2-j1}{s+1-j2}\right) \quad (2.47)$$

Partial-Fraction Expansion

from which

$$f(t) = \frac{3}{5} - \frac{3}{20}\left[(2 + j1)e^{-(1+j2)t} + (2 - j1)e^{-(1-j2)t}\right]$$

$$= \frac{3}{5} - \frac{3}{20}e^{-t}\left[4\left(\frac{e^{j2t} + e^{-j2t}}{2}\right) + 2\left(\frac{e^{j2t} + e^{-j2t}}{2j}\right)\right] \qquad (2.48)$$

Using Eqs. (2.43) and (2.44), we get

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) = 0.6 - 0.671e^{-t}\cos(2t - \phi) \qquad (2.49)$$

where $\phi = \arctan 0.5 = 26.57^{\circ}$.