UNIVERSITY OF ENERGY AND NATURAL RESOURCES

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

ELNG 305: Classical control systems

LECTURER

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The Transfer Function

- In the previous section we defined the Laplace transform and its inverse.
- We presented the idea of the partial-fraction expansion and applied the concepts to the solution of differential equations.
- We are now ready to formulate the system representation by establishing a viable definition for a function that algebraically relates a system's output to its input.
- This function will allow separation of the input, system, and output into three separate and distinct parts, unlike the differential equation.
- The function will also allow us to algebraically combine mathematical

representations of subsystems to yield a total system representation.

The Transfer Function

- The general nth-order, linear, time-invariant differential equation can be written as;

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

where c(t) is the output, r(t) is the input, and the ai's, bi's, and the form of the differential equation represent the system.

$$\begin{aligned} & a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition} \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{terms involving } c(t) \\ & = b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition} \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{terms involving } r(t) \qquad (2.51) \end{aligned}$$

The Transfer Function

- EQ 2.51 can be rewritten as;

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)R(s) \qquad (2.52)$$

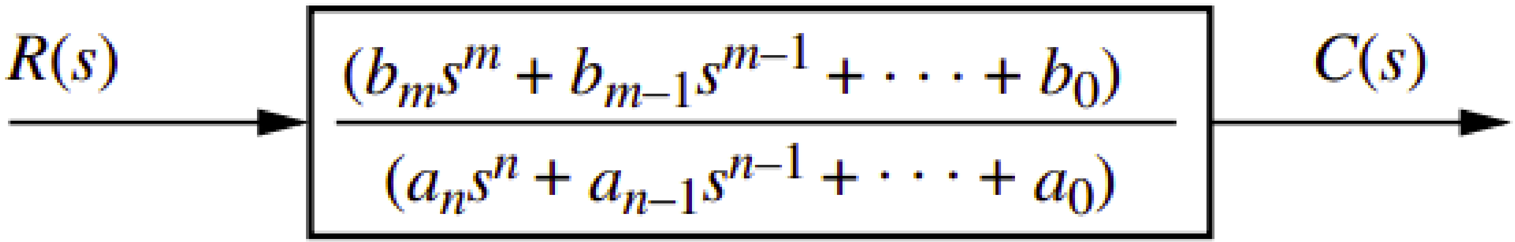
Now form the ratio of the output transform, $C(s)$, divided by the input transform, $R(s)$:

$$\boxed{\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}} \qquad (2.53)$$

Notice that Eq. (2.53) separates the output, C(s), the input, R(s), and the system, the ratio of polynomials in s on the right. We call this ratio, G(s), the transfer function and evaluate it with zero initial conditions

The Transfer Function

- The transfer function can be represented as a block diagram, as shown in below, with the input on the left, the output on the right, and the system transfer function inside the block.



we can find the output, C(s) by using

$$\boxed{C(s) = R(s)G(s)} \tag{2.54}$$

The Transfer Function

PROBLEM: Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t) \qquad (2.55)$$

b. find the response, $c(t)$ to an input,

$r(t) = u(t)$, a unit step, assuming zero initial conditions.

SOLUTION: Taking the Laplace transform of both sides, assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s) \qquad (2.56)$$

The transfer function, $G(s)$, is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2} \qquad (2.57)$$

The Transfer Function

$$C(s) = R(s)G(s) = \frac{1}{s(s+2)} \tag{2.58}$$

Expanding by partial fractions, we get

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2} \tag{2.59}$$

Finally, taking the inverse Laplace transform of each term yields

$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} \tag{2.60}$$

PROBLEM: Find the ramp response for a system whose transfer function is

$$G(s) = \frac{s}{(s+4)(s+8)}$$

Electrical Network Transfer Functions

In this section, we formally apply the transfer function to the mathematical modeling of electric circuits including passive networks and operational amplifier circuits.

- We will consist of three passive linear components: resistors, capacitors, and inductors.
- After that, then combine electrical components into circuits, decide on the input and output, and find the transfer function.
- Our guiding principles are Kirchhoff's laws.
- We sum voltages around loops or sum currents at nodes, depending on which technique involves the least effort in algebraic manipulation, and then equate the result to zero

Electrical Network Transfer Functions

- From these relationships we can write the differential equations for the circuit.
- Then we can take the Laplace transforms of the differential equation and finally solve for the transfer function

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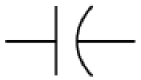
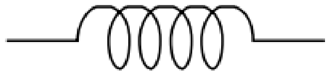
TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

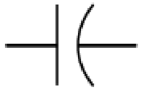
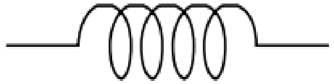
Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Electrical Network Transfer Functions

Table 2.3 summarizes the components and the relationships between voltage and current and between voltage and charge under zero initial conditions.

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Electrical Network Transfer Functions (Single Loop via the Differential Equation)

PROBLEM: Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$ in Figure 2.3.

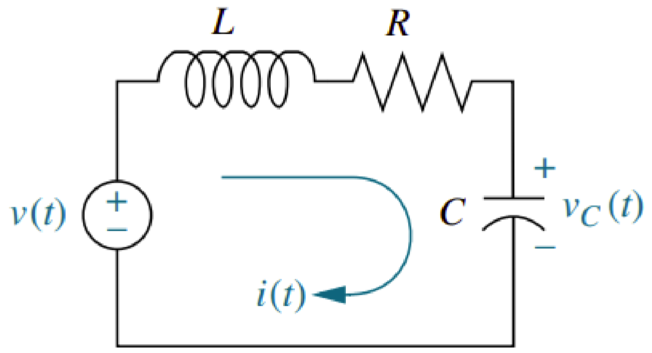


FIGURE 2.3 *RLC* network

- In any problem, the designer must first decide what the input and output should be
- The problem statement, however, is clear in this case: We are to treat the capacitor voltage as the output and the applied voltage as the input

Electrical Network Transfer Functions (Single Loop via the Differential Equation)

Summing the voltages around the loop, assuming zero initial conditions, yields the integro-differential equation for this network as

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \tag{2.61}$$

Changing variables from current to charge using $i(t) = dq(t)/dt$ yields

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \tag{2.62}$$

From the voltage-charge relationship for a capacitor in Table 2.3,

$$q(t) = Cv_C(t) \tag{2.63}$$

Substituting Eq. (2.63) into Eq. (2.62) yields

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \tag{2.64}$$

Electrical Network Transfer Functions (Single Loop via the Differential Equation)

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \tag{2.65}$$

Solving for the transfer function, $V_C(s)/V(s)$, we obtain

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{2.66}$$

as shown in Figure 2.4.

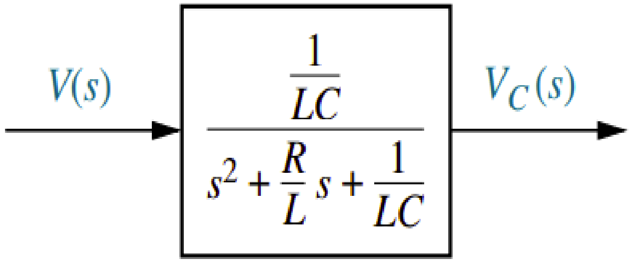


FIGURE 2.4 Block diagram of series *RLC* electrical network

Let us now develop a technique for simplifying the solution for future problems. First, take the Laplace transform of the equations in the voltage-current column of Table 2.3 assuming zero initial conditions.

Electrical Network Transfer Functions (Transfer Function—Single Loop via Transform Methods)

For the capacitor,

$$\boxed{V(s) = \frac{1}{Cs} I(s)}$$

For the resistor,

$$\boxed{V(s) = RI(s)}$$

For the inductor,

$$\boxed{V(s) = LsI(s)}$$

Now define the following transfer function:

$$\boxed{\frac{V(s)}{I(s)} = Z(s)}$$

Electrical Network Transfer Functions (Transfer Function—Single Loop via Transform Methods)

Let us now demonstrate how the concept of impedance simplifies the solution for the transfer function. The Laplace transform of Eq. (2.61), assuming zero initial conditions, is

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s) \qquad (2.71)$$

Notice that Eq. (2.71), which is in the form

$$[\text{Sum of impedances}]I(s) = [\text{Sum of applied voltages}]$$

Notice that the circuit of Figure 2.5 could have been obtained immediately from the circuit of Figure 2.3 simply by replacing each element with its impedance. We call this altered circuit the transformed circuit

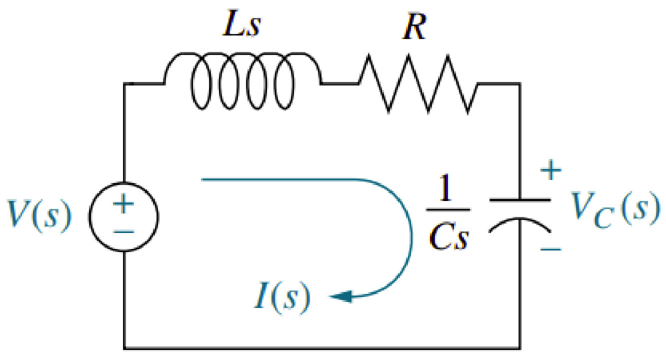


FIGURE 2.5 Laplace-transformed network

Electrical Network Transfer Functions (Transfer Function—Single Loop via Transform Methods)

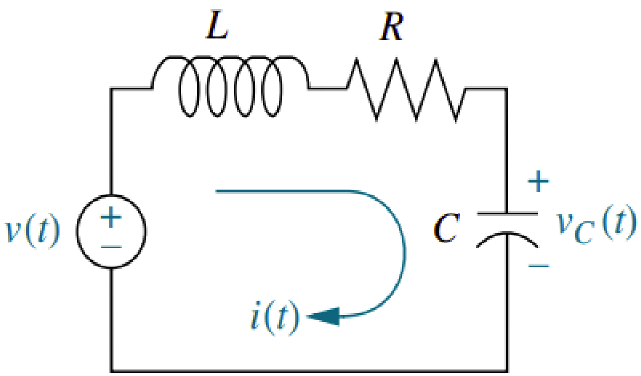
We notice that the transformed circuit leads immediately to Eq. (2.71) if we add impedances in series as we add resistors in series.

- Thus, rather than writing the differential equation first and then taking the Laplace transform, we can draw the transformed circuit and obtain the Laplace transform of the differential equation simply by applying Kirchhoff's voltage law to the transformed circuit.
- We summarize the steps as follows:

1. Redraw the original network showing all time variables, such as v(t), i(t), and vC(t), as Laplace transforms V(s), I(s), and VC(s), respectively.
2. Replace the component values with their impedance values. This replacement is similar to the case of dc circuits, where we represent resistors with their resistance values

Electrical Network Transfer Functions (Transfer Function—Single Loop via Transform Methods)

Repeat Example 2.6 using mesh analysis and transform methods without writing a differential equation.



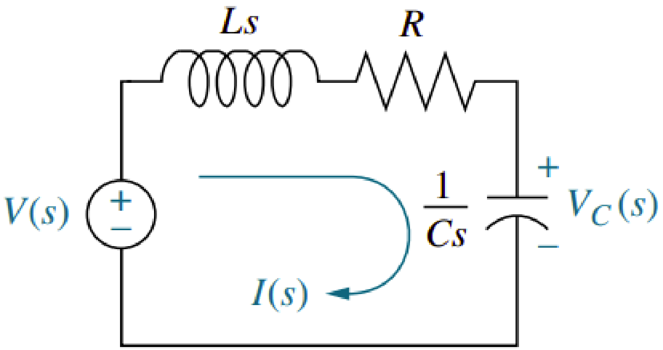


FIGURE 2.5 Laplace-transformed network

Electrical Network Transfer Functions (Transfer Function—Single Node via Transform Methods)

- Transfer functions also can be obtained using Kirchhoff's current law and summing currents flowing from nodes.
- We call this method nodal analysis.
- We now demonstrate this principle by applying Fig 2.5 using Kirchhoff's current law and the transform methods just described to bypass writing the differential equation

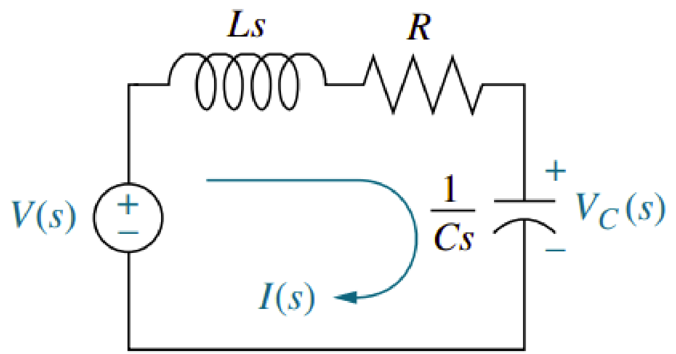


FIGURE 2.5 Laplace-transformed network

Electrical Network Transfer Functions (Transfer Function—Single Node via Transform Methods)

SOLUTION: The transfer function can be obtained by summing currents flowing out of the node whose voltage is $V_C(s)$ in Figure 2.5. We assume that currents leaving the node are positive and currents entering the node are negative. The currents consist of the current through the capacitor and the current flowing through the series resistor and inductor. From Eq. (2.70), each $I(s) = V(s)/Z(s)$. Hence,

$$\frac{V_C(s)}{I/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0 \tag{2.76}$$

where $V_C(s)/(1/Cs)$ is the current flowing out of the node through the capacitor, and $[V_C(s) - V(s)]/(R + Ls)$ is the current flowing out of the node through the series resistor and inductor. Solving Eq. (2.76) for the transfer function, $V_C(s)/V(s)$, we arrive at the same result as Eq. (2.66).

Electrical Network Transfer Functions (Transfer Function—Single Loop via Voltage Division)

- Transfer functions also can be obtained using voltage division.
- The voltage across any component is some proportion of the input voltage.
- The impedance of the component can be divided by the sum of the

Impedances in the circuit

We can demonstrate this using figure 2.5

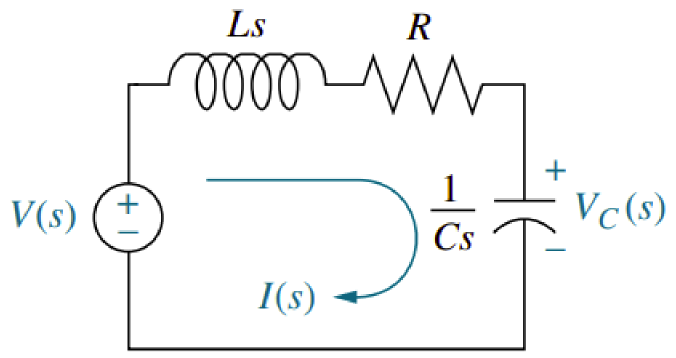


FIGURE 2.5 Laplace-transformed network

Electrical Network Transfer Functions (Transfer Function—Single Loop via Voltage Division)

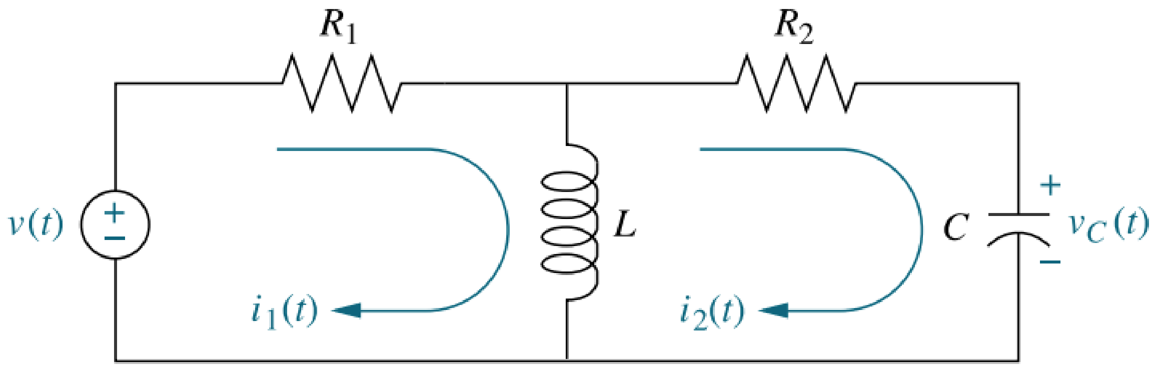
SOLUTION: The voltage across the capacitor is some proportion of the input voltage, namely the impedance of the capacitor divided by the sum of the impedances. Thus,

$$V_C(s) = \frac{1/Cs}{\left(Ls + R + \frac{1}{Cs}\right)} V(s) \tag{2.77}$$

Solving for the transfer function, $V_C(s)/V(s)$, yields the same result as Eq. (2.66).

Review Examples 2.6 through 2.9. Which method do you think is easiest for this circuit?

Electrical Network Transfer Functions (Complex Circuits via Mesh Analysis)



(a)

$$\begin{bmatrix}\text{Sum of}\\\text{impedances}\\\text{around Mesh 1}\end{bmatrix} I_1(s) - \begin{bmatrix}\text{Sum of}\\\text{impedances}\\\text{common to the}\\\text{two meshes}\end{bmatrix} I_2(s) = \begin{bmatrix}\text{Sum of applied}\\\text{voltages around}\\\text{Mesh 1}\end{bmatrix} \quad (2.83a)$$

$$-\begin{bmatrix}\text{Sum of}\\\text{impedances}\\\text{common to the}\\\text{two meshes}\end{bmatrix} I_1(s) + \begin{bmatrix}\text{Sum of}\\\text{impedances}\\\text{around Mesh 2}\end{bmatrix} I_2(s) = \begin{bmatrix}\text{Sum of applied}\\\text{voltages around}\\\text{Mesh 2}\end{bmatrix} \quad (2.83b)$$

Electrical Network Transfer Functions (Complex Circuits via Mesh Analysis)

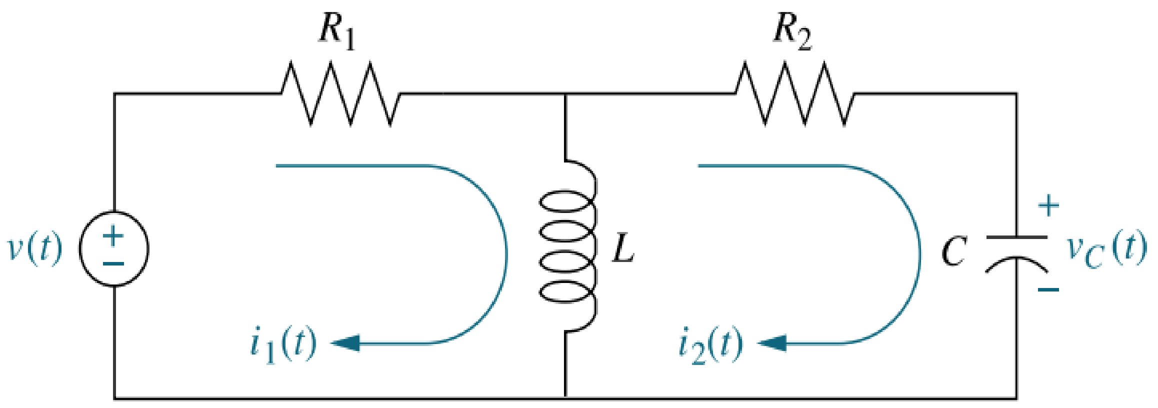
To solve complex electrical networks—those with multiple loops and nodes—using mesh analysis, we can perform the following steps:

1. Replace passive element values with their impedances.
2. Replace all sources and time variables with their Laplace transform.
3. Assume a transform current and a current direction in each mesh.
4. Write Kirchhoff's voltage law around each mesh.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.

Electrical Network Transfer Functions (Complex Circuits via Mesh Analysis)

Transfer Function—Multiple Loops

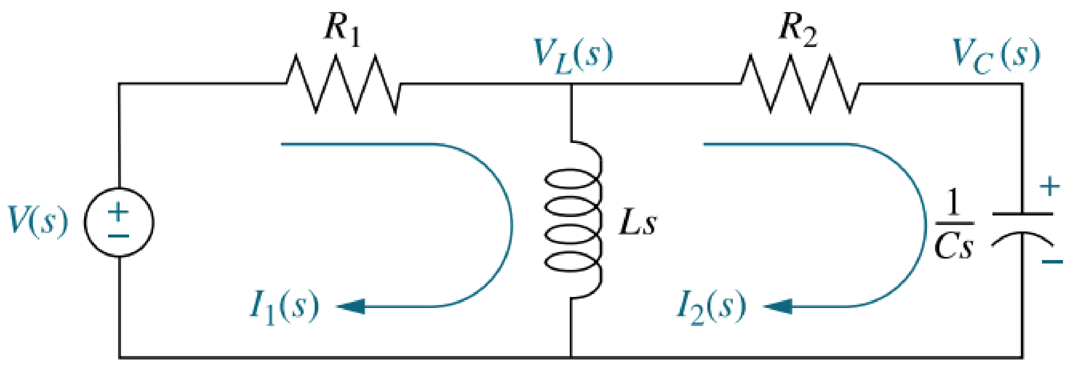
PROBLEM: Given the network of Figure 2.6(a), find the transfer function, $I_2(s)/V(s)$.



(a)

Electrical Network Transfer Functions (Complex Circuits via Mesh Analysis)

SOLUTION: The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions. The result is shown in Figure 2.6(b). The circuit with which we are dealing requires two simultaneous equations to solve for the transfer function. These equations can be found by summing voltages around each mesh through which the assumed currents, $I_1(s)$ and $I_2(s)$, flow. Around Mesh 1, where $I_1(s)$ flows,



(*b*)

Electrical Network Transfer Functions (Complex Circuits via Mesh Analysis)

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s) \tag{2.78}$$

Around Mesh 2, where $I_2(s)$ flows,

$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0 \tag{2.79}$$

Combining terms, Eqs. (2.78) and (2.79) become simultaneous equations in $I_1(s)$ and $I_2(s)$:

$$(R_1 + Ls)I_1(s) \qquad - LsI_2(s) = V(s) \tag{2.80a}$$

$$-LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right) I_2(s) = 0 \tag{2.80b}$$

We can use Cramer's rule (or any other method for solving simultaneous equations) to solve Eq. (2.80) for $I_2(s)$.[4] Hence,

Electrical Network Transfer Functions (Complex Circuits via Mesh Analysis)

We can use Cramer's rule (or any other method for solving simultaneous equations) to solve Eq. (2.80) for $I_2(s)$.[4] Hence,

$$I_2(s) = \frac{\begin{vmatrix} (R_1 + Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta} \tag{2.81}$$

where

$$\Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \frac{1}{Cs}\right) \end{vmatrix}$$

Forming the transfer function, $G(s)$, yields

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1} \tag{2.82}$$

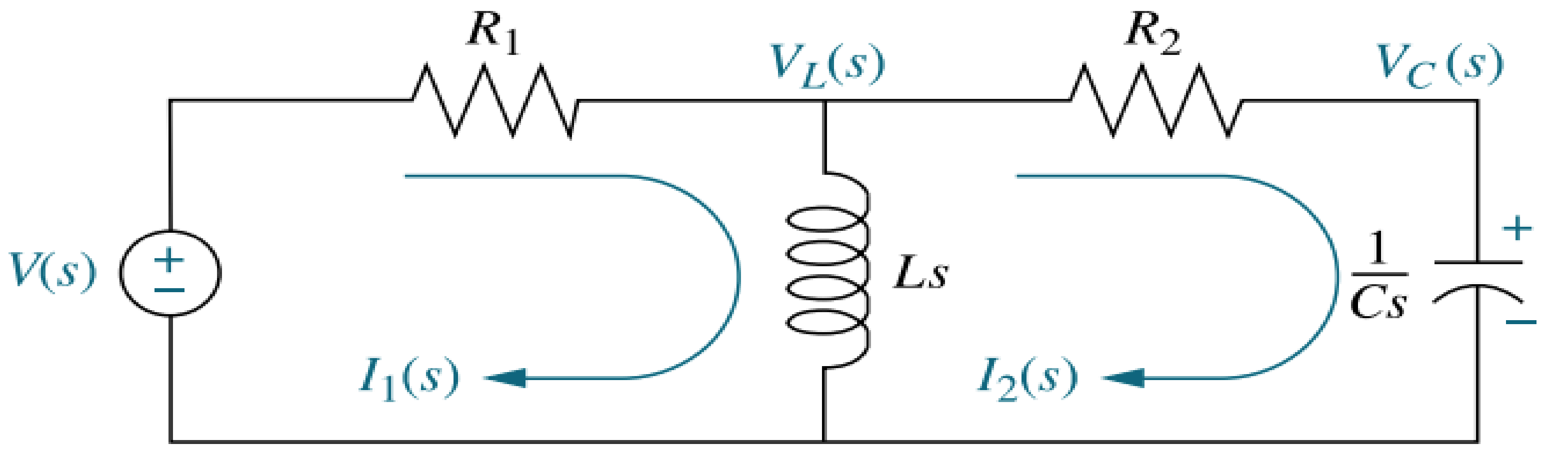
Electrical Network Transfer Functions (Complex Circuits via Nodal Analysis)

- Often, the easiest way to find the transfer function is to use nodal analysis rather than mesh analysis. The number of simultaneous differential equations that must be written is equal to the number of nodes whose voltage is unknown.
- For multiple nodes we use Kirchhoff's current law and sum currents flowing from each node. Again, as a convention, currents flowing from the node are assumed to be positive, and currents flowing into the node are assumed to be negative
- When writing nodal equations, it can be more convenient to represent circuit elements by their admittance

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)}$$

Electrical Network Transfer Functions (Complex Circuits via Nodal Analysis)

PROBLEM: Find the transfer function, $V_C(s)/V(s)$, for the circuit in Figure 2.6(*b*). Use nodal analysis.



(*b*)

Electrical Network Transfer Functions (Complex Circuits via Nodal Analysis)

SOLUTION: For this problem, we sum currents at the nodes rather than sum voltages around the meshes. From Figure 2.6(*b*) the sum of currents flowing from the nodes marked $V_L(s)$ and $V_C(s)$ are, respectively,

$$\frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0 \quad (2.85a)$$

$$CsV_C(s) + \frac{V_C(s) - V_L(s)}{R_2} = 0 \quad (2.85b)$$

Rearranging and expressing the resistances as conductances,[5] $G_1 = 1/R_1$ and $G_2 = 1/R_2$, we obtain,

$$\left(G_1 + G_2 + \frac{1}{Ls}\right)V_L(s) \qquad - G_2V_C(s) = V(s)G_1 \quad (2.86a)$$

$$-G_2V_L(s) + (G_2 + Cs)V_C(s) = 0 \quad (2.86b)$$

Electrical Network Transfer Functions (Complex Circuits via Nodal Analysis)

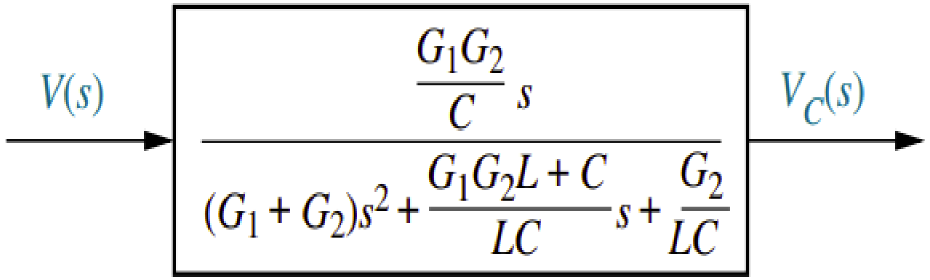


FIGURE 2.7 Block diagram of the network of Figure 2.6

Solving for the transfer function, $V_C(s)/V(s)$, yields

$$\frac{V_C(s)}{V(s)}=\frac{\dfrac{G_1G_2}{C}s}{(G_1+G_2)s^2+\dfrac{G_1G_2L+C}{LC}s+\dfrac{G_2}{LC}} \tag{2.87}$$

as shown in Figure 2.7.

Electrical Network Transfer Functions (Complex Circuits via Nodal Analysis) (voltage to current transform method)

- Another way to write node equations is to replace voltage sources by current sources.
- A voltage source presents a constant voltage to any load; conversely, a current source delivers a constant current to any load.
- Practically, a current source can be constructed from a voltage source by placing a large resistance in series with the voltage source.
- Theoretically, we rely

on *Norton's theorem*, which states that a voltage source, $V(s)$, in series with an impedance, $Z_s(s)$, can be replaced by a current source, $I(s) = V(s)/Z_s(s)$, in parallel with $Z_s(s)$.

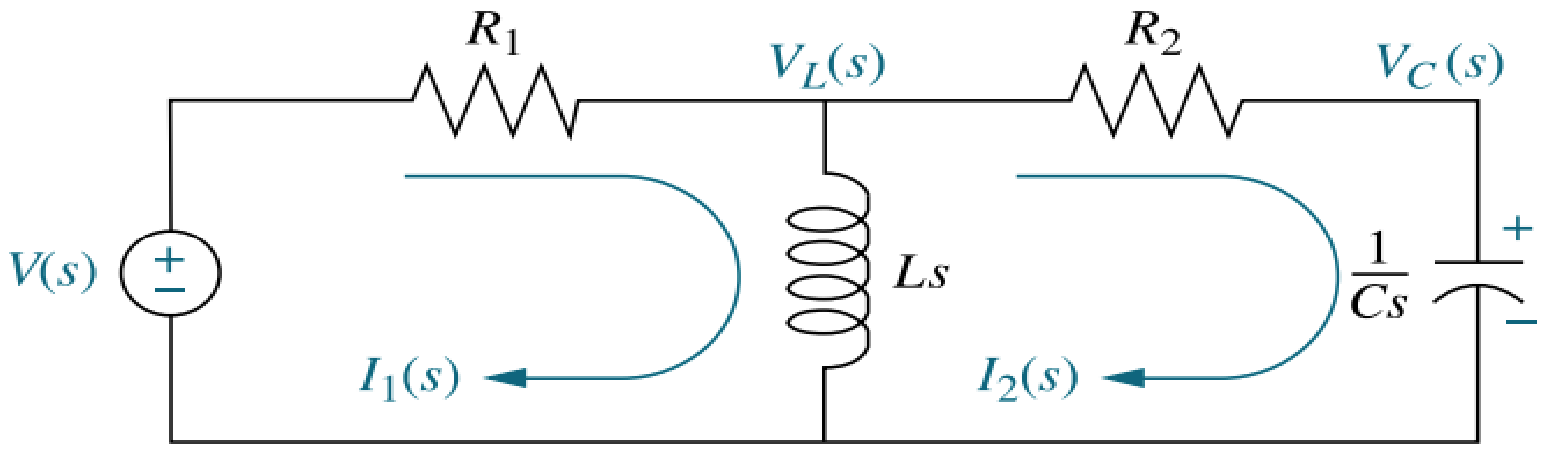
Electrical Network Transfer Functions (Complex Circuits via Nodal Analysis) (voltage to current transform method)

In order to handle multiple-node electrical networks, we can perform the following steps:

1. Replace passive element values with their admittances.
2. Replace all sources and time variables with their Laplace transform.
3. Replace transformed voltage sources with transformed current sources.
4. Write Kirchhoff's current law at each node.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.

Electrical Network Transfer Functions (Complex Circuits via Nodal Analysis) (voltage to current transform method)

PROBLEM: For the network of Figure 2.6, find the transfer function, $V_C(s)/V(s)$, using nodal analysis and a transformed circuit with current sources.



(b)

Electrical Network Transfer Functions (Complex Circuits via Nodal Analysis) (voltage to current transform method)

SOLUTION: Convert all impedances to admittances and all voltage sources in series with an impedance to current sources in parallel with an admittance using Norton's theorem.

Redrawing Figure 2.6(*b*) to reflect the changes, we obtain Figure 2.8, where $G_1 = 1/R_1$, $G_2 = 1/R_2$, and the node voltages—the voltages across the inductor and the capacitor—have been identified as $V_L(s)$ and $V_C(s)$, respectively. Using the general relationship, $I(s) = Y(s)V(s)$, and summing currents at the node $V_L(s)$,

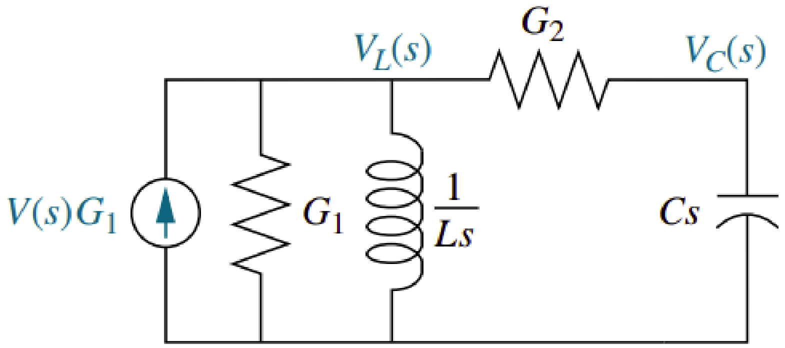


FIGURE 2.8 Transformed network ready for nodal analysis

Electrical Network Transfer Functions (Complex Circuits via Nodal Analysis) (voltage to current transform method)

$$G_1 V_L(s) + \frac{1}{Ls} V_L(s) + G_2[V_L(s) - V_C(s)] = V(s)G_1 \tag{2.88}$$

Summing the currents at the node $V_C(s)$ yields

$$CsV_C(s) + G_2[V_C(s) - V_L(s)] = 0 \tag{2.89}$$

Combining terms, Eqs. (2.88) and (2.89) become simultaneous equations in $V_C(s)$ and $V_L(s)$, which are identical to Eq. (2.86) and lead to the same solution as Eq. (2.87).

Electrical Network Transfer Functions (Complex Circuits via Nodal Analysis) (voltage to current transform method)

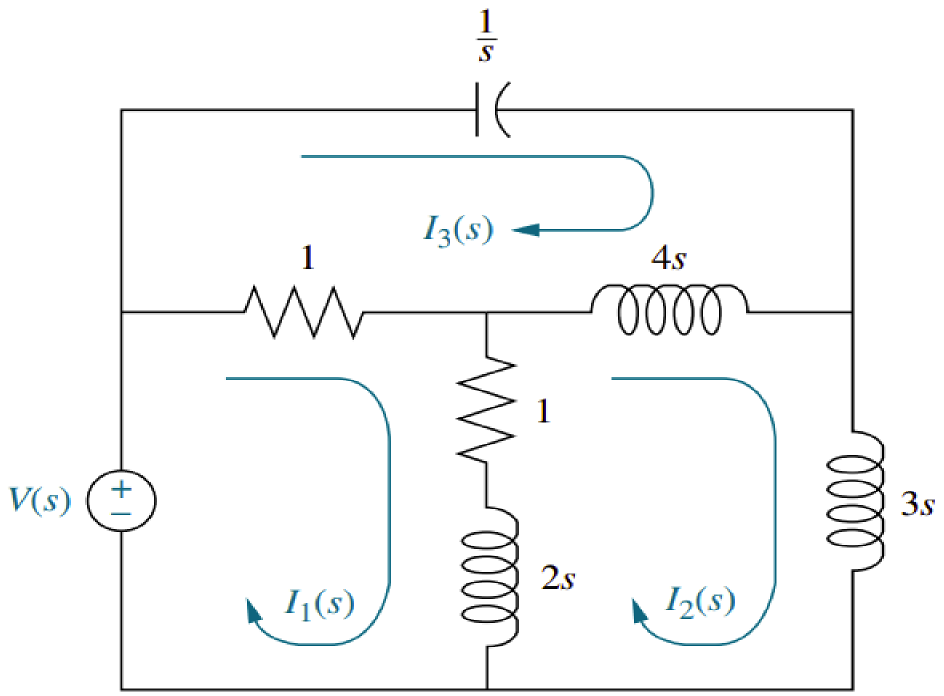
- In general the equations for the complex circuit can be written as.

$$\begin{bmatrix}\text{Sum of admittances}\\ \text{connected to Node 1}\end{bmatrix} V_L(s) - \begin{bmatrix}\text{Sum of admittances}\\ \text{common to the two}\\ \text{nodes}\end{bmatrix} V_C(s) = \begin{bmatrix}\text{Sum of applied}\\ \text{currents at Node 1}\end{bmatrix} \tag{2.90a}$$

$$-\begin{bmatrix}\text{Sum of admittances}\\ \text{common to the two}\\ \text{nodes}\end{bmatrix} V_L(s) + \begin{bmatrix}\text{Sum of admittances}\\ \text{connected to Node 2}\end{bmatrix} V_C(s) = \begin{bmatrix}\text{Sum of applied}\\ \text{currents at Node 2}\end{bmatrix} \tag{2.90b}$$

Electrical Network Transfer Functions (Complex Circuits via Nodal Analysis) (voltage to current transform method)

PROBLEM: Write, but do not solve, the mesh equations for the network shown in Figure 2.9.



SOLUTION

$$+(2s+2)I_1(s) - (2s+1)I_2(s) - I_3(s) = V(s)$$

$$-(2s+1)I_1(s) + (9s+1)I_2(s) - 4sI_3(s) = 0$$

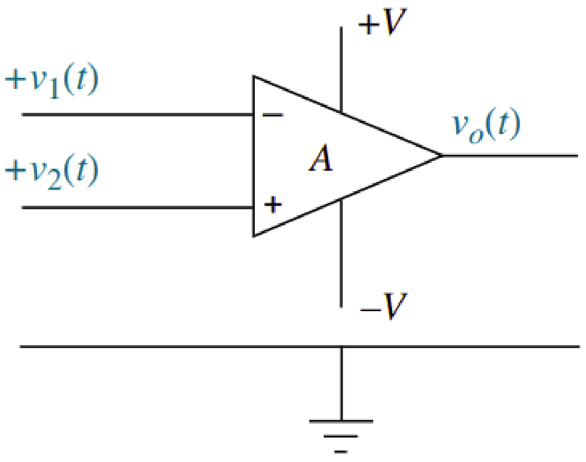
$$-I_1(s) - 4sI_2(s) + \left(4s+1+\frac{1}{s}\right)I_3(s) = 0$$

Electrical Network Transfer Functions (Modeling in the Frequency Domain)

Passive electrical circuits were the topic of discussion up to this point. We now discuss a class of active circuits that can be used to implement transfer functions. These are circuits built around an operational amplifier

An *operational amplifier*, pictured in Figure 2.10(*a*), is an electronic amplifier used as a basic building block to implement transfer functions. It has the following characteristics:

1. Differential input, $V_2(t) - v_1(t)$
2. High input impedance, $Z_i = \infty$ (ideal)
3. Low output impedance, $Z_o = 0$ (ideal)
4. High constant gain amplification, $A = \infty$ (ideal)



(a)

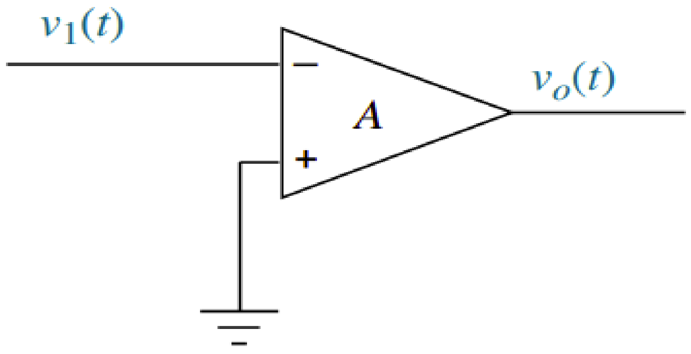
The output, $v_o(t)$, is given by

$$v_o(t) = A(v_2(t) - v_1(t))$$

Electrical Network Transfer Functions (Modeling in the Frequency Domain)

Inverting Operational Amplifier

If $v_2(t)$ is grounded, the amplifier is called an *inverting operational amplifier*, as shown in Figure 2.10(b). For the inverting operational amplifier, we have

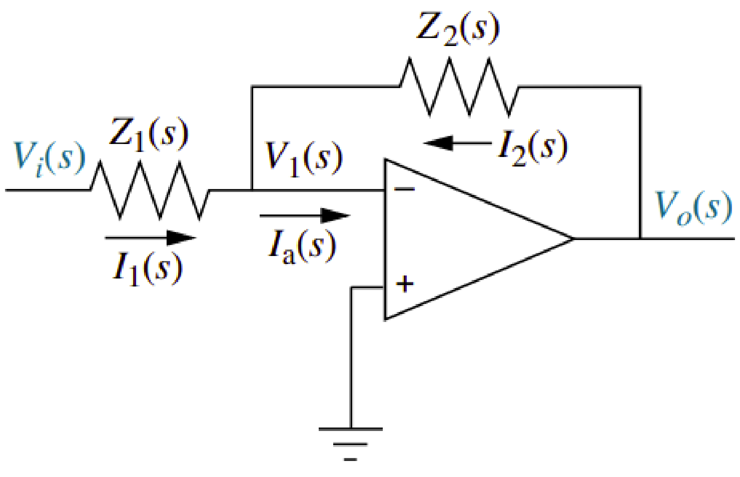


(b)

$$v_o(t) = -Av_1(t) \tag{2.96}$$

Electrical Network Transfer Functions (Modeling in the Frequency Domain)

If two impedances are connected to the inverting operational amplifier as shown in Figure 2.10(c), we can derive an interesting result if the amplifier has the characteristics mentioned in the beginning of this subsection. If the input impedance to the amplifier is high, then by Kirchhoff's current law, $I_a(s) = 0$ and $I_1(s) = -I_2(s)$.



(c)

Also, since the gain A is large, $v_1(t) \approx 0$. Thus, $I_1(s) = V_i(s)/Z_1(s)$, and $-I_2(s) = -V_o(s)/Z_2(s)$. Equating the two currents, $V_o(s)/Z_2(s) = -V_i(s)/Z_1(s)$, or the transfer function of the inverting operational amplifier configured as shown in Figure 2.10(c) is

$$\boxed{\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}} \tag{2.97}$$

Electrical Network Transfer Functions (Modeling in the Frequency Domain)

PROBLEM: Find the transfer function, $V_o(s)/V_i(s)$, for the circuit given in Figure 2.11.

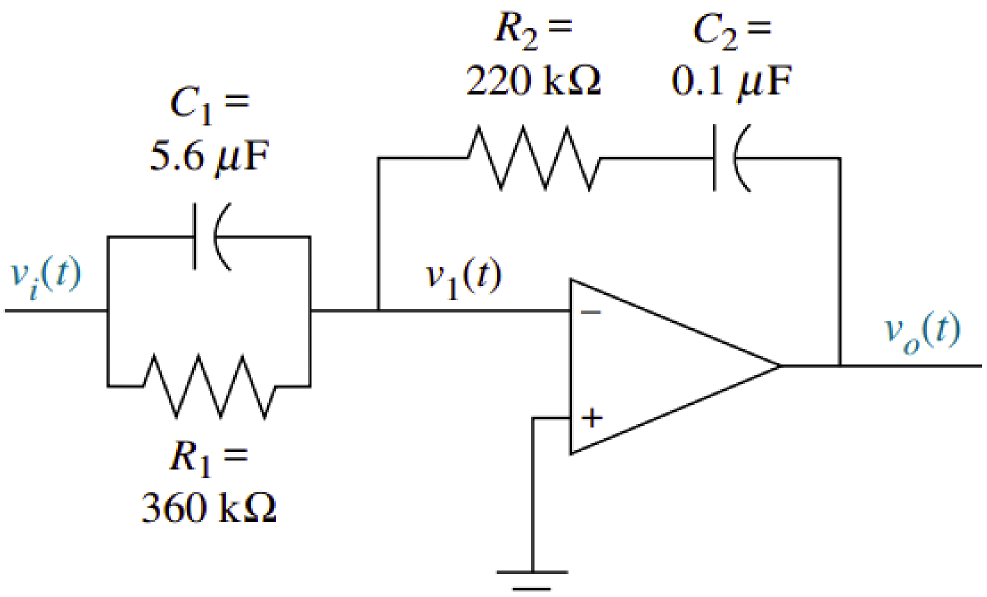


FIGURE 2.11 Inverting operational amplifier circuit for Example 2.14

Electrical Network Transfer Functions (Modeling in the Frequency Domain)

SOLUTION: The transfer function of the operational amplifier circuit is given by Eq. (2.97). Since the admittances of parallel components add, $Z_1(s)$ is the reciprocal of the sum of the admittances, or

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 \times 10^{-6} s + \frac{1}{360 \times 10^3}} = \frac{360 \times 10^3}{2.016s + 1} \qquad (2.98)$$

For $Z_2(s)$ the impedances add, or

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s} \qquad (2.99)$$

Substituting Eqs. (2.98) and (2.99) into Eq. (2.97) and simplifying, we get

$$\frac{V_o(s)}{V_i(s)} = -1.232 \frac{s^2 + 45.95s + 22.55}{s} \qquad (2.100)$$

Electrical Network Transfer Functions (Modeling in the Frequency Domain)

Another circuit that can be analyzed for its transfer function is the noninverting operational amplifier circuit shown in Figure 2.12. To find the transfer function of the circuit

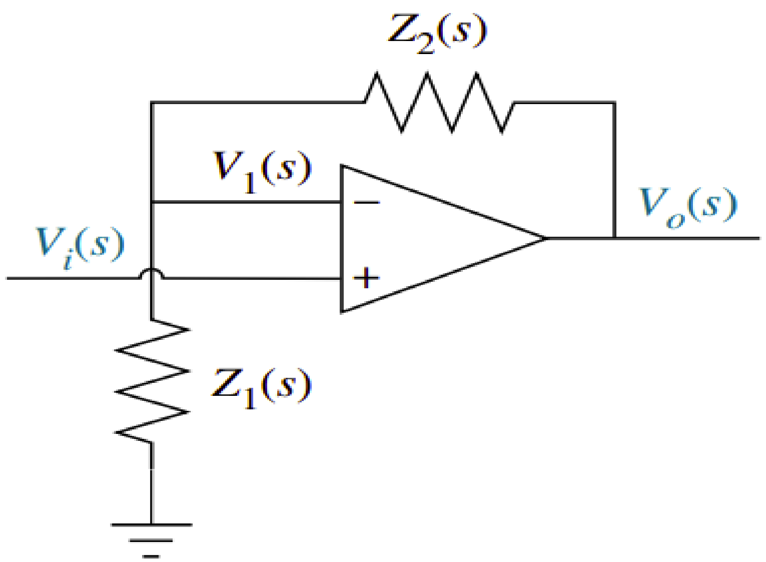


FIGURE 2.12 General noninverting operational amplifier circuit

$$V_o(s) = A(V_i(s) - V_1(s)) \tag{2.101}$$

But, using voltage division,

$$V_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_o(s) \tag{2.102}$$

Substituting Eq. (2.102) into Eq. (2.101), rearranging, and simplifying, we obtain

$$\frac{V_o(s)}{V_i(s)} = \frac{A}{1 + AZ_1(s)/(Z_1(s) + Z_2(s))} \tag{2.103}$$

For large A, we disregard unity in the denominator and Eq. (2.103) becomes

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}} \tag{2.104}$$

Electrical Network Transfer Functions (Modeling in the Frequency Domain)

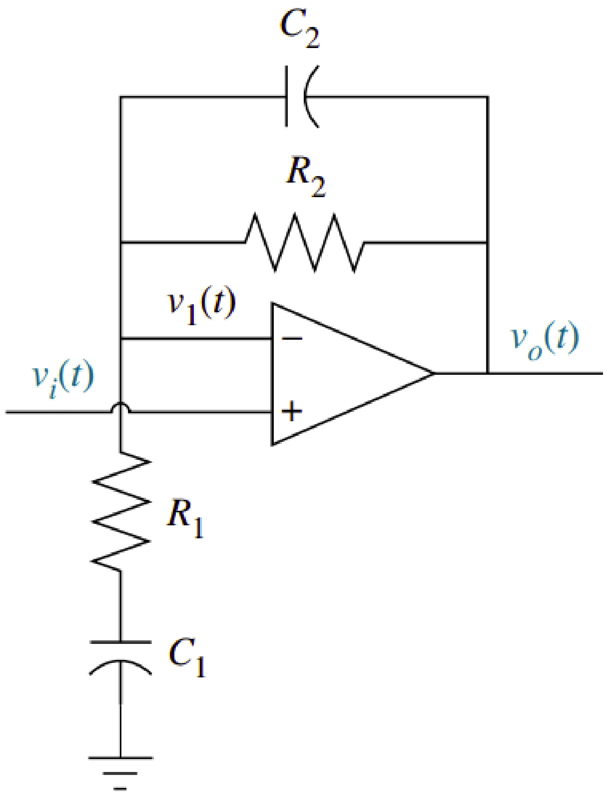


FIGURE 2.13 Noninverting operational amplifier circuit for Example 2.15

PROBLEM: Find the transfer function, $V_o(s)/V_i(s)$, for the circuit given in Figure 2.13.

SOLUTION: We find each of the impedance functions, $Z_1(s)$ and $Z_2(s)$, and then substitute them into Eq. (2.104). Thus,

$$Z_1(s) = R_1 + \frac{1}{C_1 s} \tag{2.105}$$

and

$$Z_2(s) = \frac{R_2(1/C_2 s)}{R_2 + (1/C_2 s)} \tag{2.106}$$

Substituting Eqs. (2.105) and (2.106) into Eq. (2.104) yields

$$\frac{V_o(s)}{V_i(s)} = \frac{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_2 + C_1 R_1)s + 1}{C_2 C_1 R_2 R_1 s^2 + (C_2 R_2 + C_1 R_1)s + 1} \tag{2.107}$$