IB Subject(s): Mathematics

Extended Essay

3D Parametric Equations:

How successfully can the use of three dimensional parametric equations be applied to model and predict a satellite's orbit in Earth's outer atmosphere?

Word Count: 3965

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Introduction

Research Question: How well can three dimensional parametric equations be used to model and predict the movement of the International Space Station?

Parametric equations are considered some of the most applicable uses of Calculus in the modern world. By using two different variables in a function of time, they allow for us to better understand the movement of objects no matter their dependence. Primarily based on a coordinate grid such as x and y, parametric equations are used to express the coordinates of the points that make a curve or surface. By interpreting two dimensional parametric equations into three dimensional parametric equations the sole addition of the variable z is created relating to time. Three dimensional parametric equations are the same as two dimensional parametric equations with the exception of another plane of space. This plane allows for the equation to represent the path of an object in three dimensional space or to represent a three dimensional object in space.

Throughout my life I had been fascinated by space and how our world orbits around something that orbits around something else. However, I found through my studies in Physics and Calculus that both have equations to model such events. Thus I wanted to see if I could model an orbit only with the use of Calculus in the form of an extended essay as to see how well Calculus works independent of the concepts of Physics. For the purpose of this extended essay, the International Space Station (ISS) will be used as the satellite as it is one of the only satellites with a consistently updating GPS for referencing and testing purposes. All other satellites either require administrative access to their locations or are undisclosed; however, the satellite itself should not impact the results, as the purpose of this extended essay is to explain the modelling process and explain how successfully the final model worked.

Background

The construction of the International Space Station began in 1998 and was completed in 2011. Completely attached in space, multiple countries combined their efforts to launch parts to make the giant station. It has currently been accessed by 18 different countries to help discover the nature of space and how life varies outside of Earth's 'natural' state. The station flies approximately 400 kilometers above the surface of the Earth at approxiamtely 28,000 kilometers per hour.

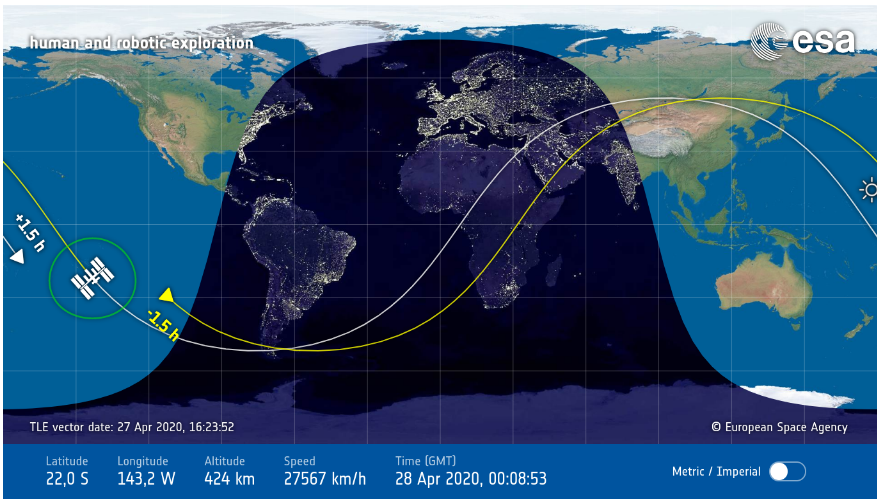

Figure 1: The ISS positioning system at 17:08 on 4/27/2020

As shown in Figure 1, the ISS follows a curvature similar to a negative sine function in basic Calculus. This will be referred to later as a component of a position equation which will be the basis for the creation of the three dimensional representation.

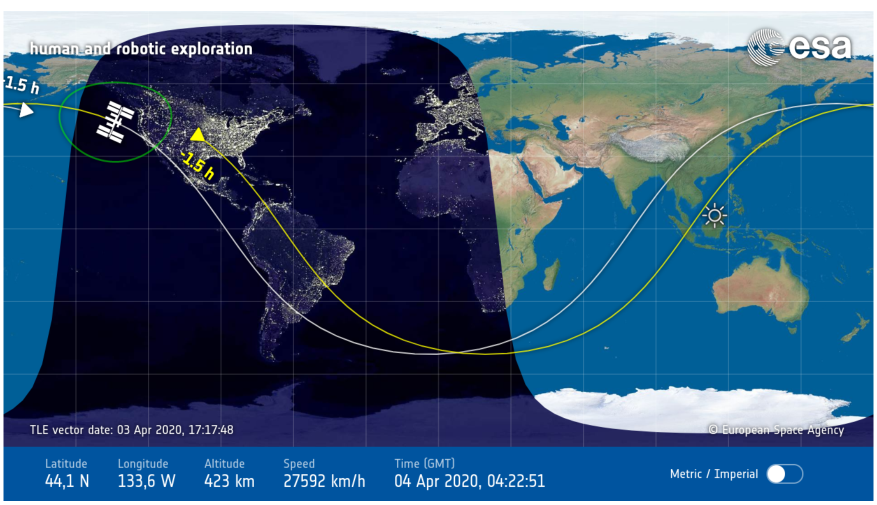

Figure 2: The ISS positioning system at 17:17 on 4/3/2020

Figure 2 shows that the same curvature pattern shown in Figure 1 does not stay constant throughout the station's travel. Similarly, this figure shows a cosine function; however, the sole representation of these cannot be used alone to express the positioning of the satellite, not if the model will want to work for times longer than three weeks into the future.

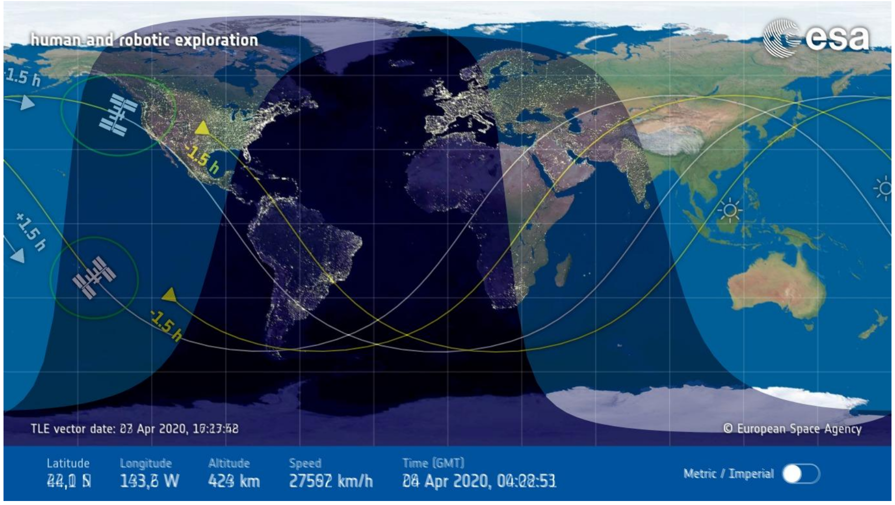

Figure 3: Figures 1 & 2 overlayed

Figure 3 shows the ISS at both times in one image. This figure furthers the reasoning for why a model following the traditional sine/cosine laws would provide inaccurate results. For if the model were based off of Figure 2, then 24 days later the model would be completely different from the true model (Figure 1). Therefore, if a natural sine or cosine function is going to be used for modeling, a factor changing the x positioning will be necessary. For example:

$cos(x)$ → Natural expression

$cos(x + 1)$ → Translated expression 1 unit to the left

Thus, the two dimensional initial expression used for modelling will consist of:

$$cos(x + h)$$

where h is a constant, real number that changes the orbital's initial condition.

The ISS circles the Earth about every 1.5 hours and completes a total of 16 circles per day. Staying at a constant 400 km above the surface of Earth, the ISS does vary in height through its sine/cosine pathway.

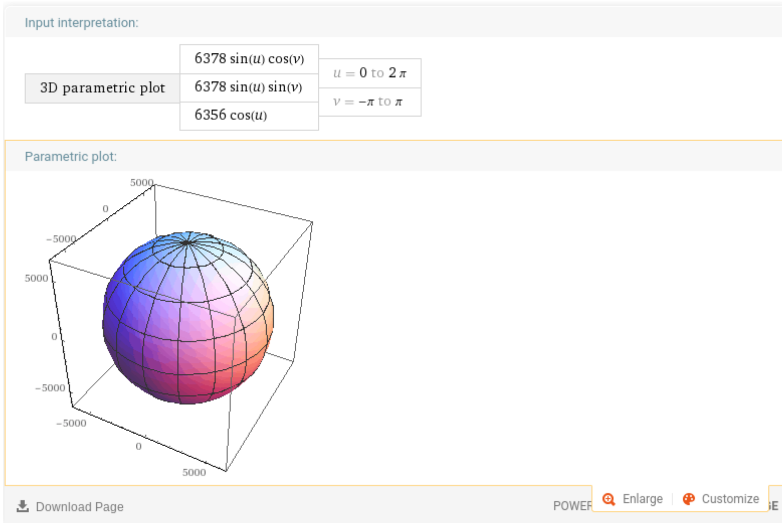

Figure 4: A 3D model of Earth

Figure 4 shows that Earth is in fact not a sphere but an oblate spheroid. This means that due to its creation, the Earth is not in the shape of a perfect sphere but is more compressed at the poles, forming a shape similar to a three dimensional oval that lacks an extremely evident shape. Hence, the following knowns are needed:

- The Equatorial Radius of Earth: 6,378 km
- The Polar Radius of Earth: 6,356 km

Definitions

The following section is intended for explaining the definitions of initial variables used in the modelling process. When necessary, new items will be defined later or old items will be updated. *Assume the reference of three dimensions unless stated otherwise.*

Definition 0:

Let the ISS be counted as a singular point in space. All of its length, width, and height attributes are negligible for the purpose of calculation. When referencing the ISS the center of mass of the ISS will be what is referenced.

Definition 1:

Let $t = 0$ equal the day November 15th 2020 starting at 10:41:54 MST. This will be used in the model testing phase specifically.

Definition 2:

Let the variable x represent the depth change in position of the ISS in kilometers over time (t) in the three dimensional plane.

Definition 3:

Let the variable y represent the horizontal change in position of the ISS in kilometers over time (t) in the three dimensional plane.

Definition 4:

Let the variable z represent the vertical change in position of the ISS in kilometers over time (t) in the three dimensional plane.

Definition 5:

Altitude can be expressed in terms of z but ranges from 6,756 km above the surface of Earth to 6,778 km above the surface of Earth.

Three-Dimensional Representation

Assume the X-Y plane is the plane that describes the cross-section cutting the Earth at the equator.

Assume the X-Z and the Z-Y planes are perpendicular to each other and form a perpendicular vertical angle with the X-Y plane.

Assume the origin of the three dimensional figure is located at the center of the Earth. The X-Y, X-Z, and Z-Y planes all intersect the origin forming perpendicular angles.

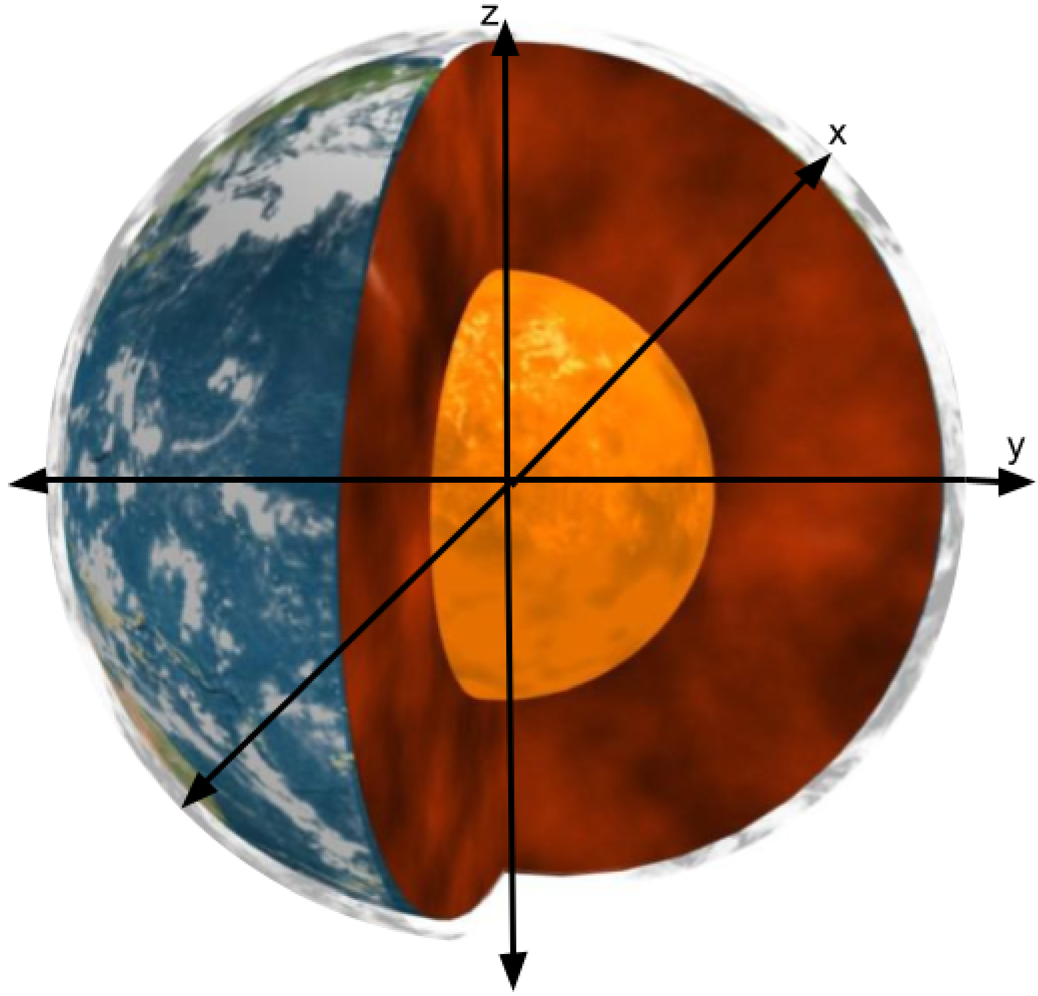

Figure 5: Combination of Definitions 2-4 to make the 3D representation.

Figure 5 is the best representation of the three dimensional 'field' that will be used. It is best represented as a three dimensional graph that extends from the center of the Earth, otherwise known as the origin of the axis. Figure 5 will be used throughout the rest of this Extended Essay and although it may not be referred to directly, the definitions used to create this figure are essential to the modelling process of the function. Thus, assume the following:

Assume the origin of the model is at the point of the center of the Earth.
Assume the axis follows a model similar to Figure 5.
Assume Earth's rotation on its axis are nonessential, as to focus on the pathway of the ISS itself.
Assume velocity is negligible when referring to the ISS's pathway, not it's speed.

While exploring the representation of the path, it is hard to understand what is truly being expressed. For that reason, a verbal account of the path in a three dimensional perspective will follow:

- The path acts similar to a sine function that is wrapped around a sphere.
  - Although, this sphere is flawed and is not perfectly round.
  - Thus, the path decreases in total distance from the equator as it approaches the poles.
  - For a better visualization:
    - Imagine a spaghetti noodle placed on a ball. It is placed in a curvy fashion and the end of the noodle touches the front of the noodle after being wrapped around the ball. The shape of the noodle is most similar to the letter 's' being rotated 270^{o}. All points of the noodle touch the ball so that the noodle wrapped in this 's' fashion is completely attached to the ball. The ball is then squished very slightly from the top and bottom, where the 's' opens. This is the path the ISS takes. (**Figure 6 is this written explanation of the path**)

Basic Equation Representation:

Assume the function sine will be wrapped similarly to Figure 6.
Assume all known constants will be applied at a later time (will be specified) until the search for a proper function concludes.

$$x = cos(t)$$
$$y = sin(t)$$
$$z = 1$$

These equations form the basic circle which represents the node values (where the path crosses the equator).

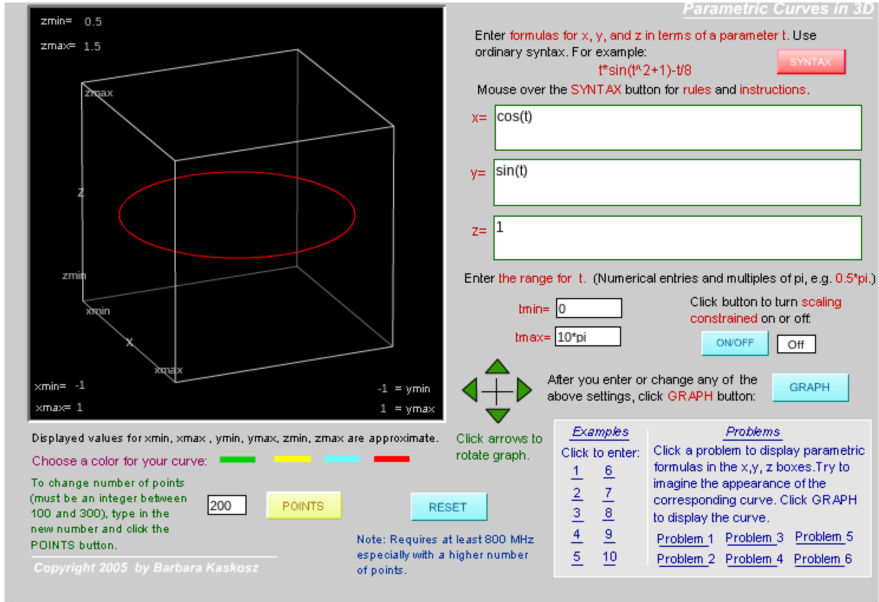

Figure 7: Basic rotation of ISS (only intersects at 2 points in a cycle).

Evidently, the path of the ISS is not this simple nor accurate enough unless applied only at a time with a known node value. As I proceeded to explore possible actions of the site, I grew frustrated as I could not understand what function keys I had to change. However, I learned that the model in Figure 7 (when changed) was not incorrect, it was simply too simple. Thus, the use of singular sine/cosine functions in each x,y,z category would not result in a more complex function like the pathway of the ISS's orbit, more complex answers require more complex input. I then came across the Examples column at the bottom of the site and looked through those until I came across Example 6.

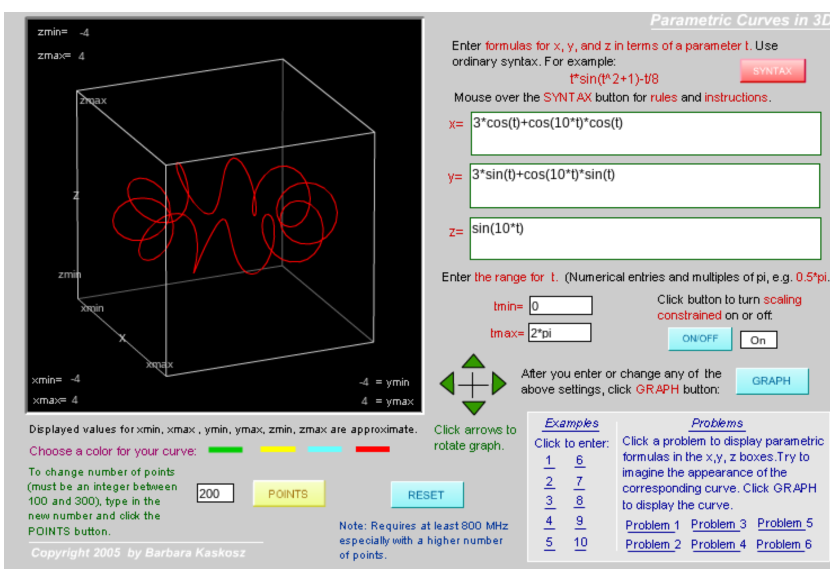

Figure 8: Example 6

Figure 8 was very similar to what I believed the path to look like; thus I explored the uses of the equation in order to make one that related directly to the path of the ISS.

More Complex Equation:

$$x = 3cos(t) + cos(2t) * cos(t)$$
$$y = 3sin(t) + cos(2t) * sin(t)$$
$$z = sin(2t)$$

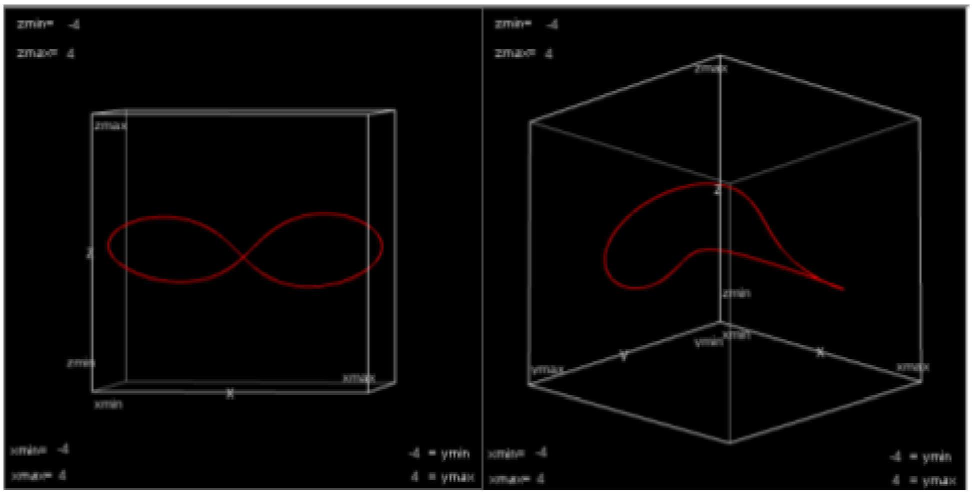

Figure 9: First edit of Figure 8

Figure 9 is similar to what I perceived the orbital to look like, by simply changing the frequency of the function (within the parenthesis of the sine/cosine aspects) the orbital was more ‘accurate’ than Figure 8 had deemed.

$$x = 100cos(t) + cos(2t) * cos(t)$$
$$y = 100sin(t) + cos(2t) * sin(t)$$
$$z = 5sin(2t)$$

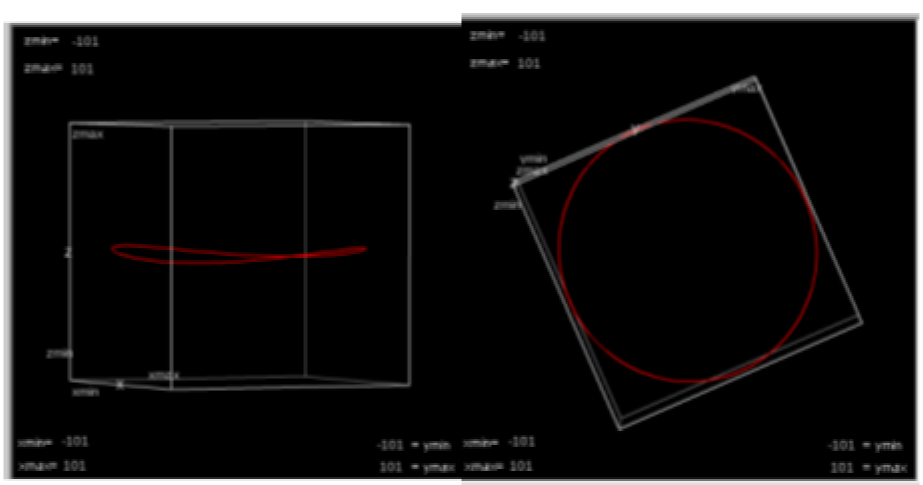

Figure 10: Second edit of Figure 8

As seen in the equations for Figure 10, by increasing the constants before the equations for x and y, the model becomes more circular from the top/bottom view. As the Earth is spherical at the equator, the model should hold this to be true.

$$x = 3389cos(t) + cos(2t) * cos(t)$$
$$y = 3389sin(t) + cos(2t) * sin(t)$$
$$z = 3378sin(2t)$$

Since the x/y components make the basis circle, they have the larger measurement of the equator (6778 km). Since z is the vertical component it has the short measurement (6756 km).

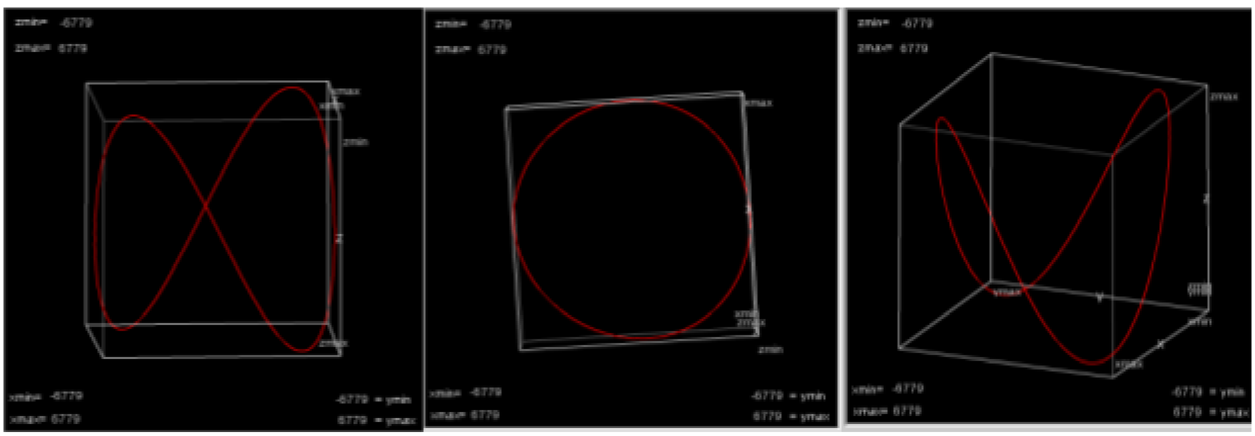

Figure 11: Third edit of Figure 8

After these attempts I realized that my evaluations of the X-Z, X-Y, and Y-Z planes were necessary in modelling the path. Therefore, the model must have some form of t^2, $cos(t)$, and a constant 2, to represent the nodes.

Final Representation:

$$x = 3389(cos(t) + cos^2(t)sin(t)) * cos^2(sin(t))$$
$$y = 3389(2sin(t) + cos^2(t)sin(t)) * cos^2(sin(t))$$
$$z = 3378sin(t)$$

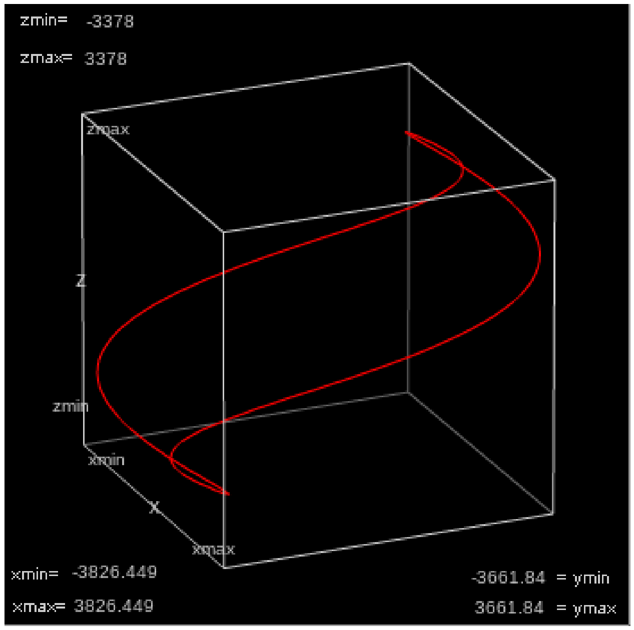

Figure 12: Final 3D Parametric Path of ISS

Figure 12 is the best possible representation I could create using three-dimensional parametric equations to simulate the ISS's path in orbit. It does so by:

- Completing one cycle in each category
- Meeting at poles
- Completing in 2pi

In order to develop this final model, I added sin(t) in the center of the cosine function to see what happened. When I added sine to the exterior as *sin(t), it created an image with more points, and when I divided it created a helix. So I tested the middle to see if anything would change in my favor and it did. I then added cosine to the end as *cos(t) as to see if it would make an 's' figure as sine/cosine make that 's' shape. When I did a singular it crossed through the center of the Earth which would be bad. I learned through previous interaction that doubling/ squaring the situation made it all positive and more "curvy". So just as a general 'test' I tried a double cosine and it created a 2 leaf compressed figure. This is how cos(sin(t)) came into the equation. The remaining additional cosine and sine functions were added to the equation in order for the parametric path to follow a curvilinear function to ensure the ISS didn't crash into Earth.

Two-Dimensional Relationship

Y-Z Plane

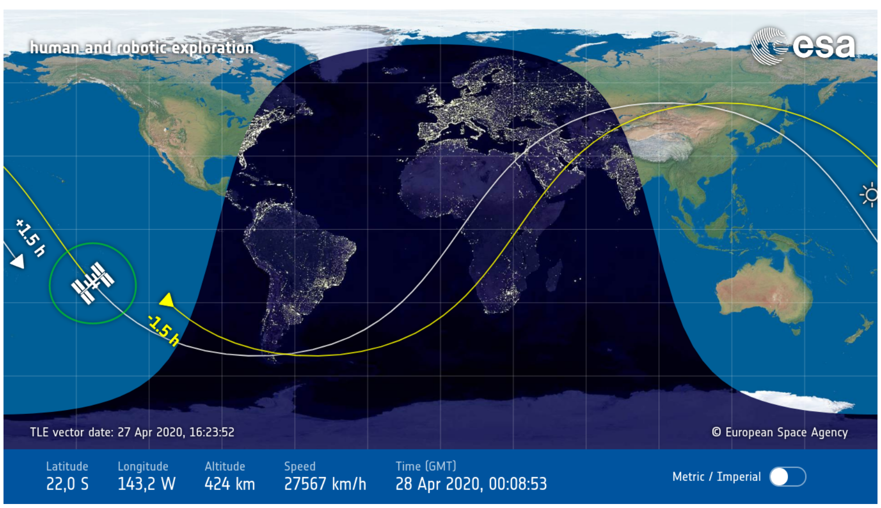

Figure 13

As seen in Figure 13, this is a representation of Earth placed in two dimensions, when leaving x as a constant. However, this image is only applicable for the full Earth, not a singular graph. For the purposes of simplicity, only the section right of the prime meridian will be used to express the occurances of the Y-Z plane.

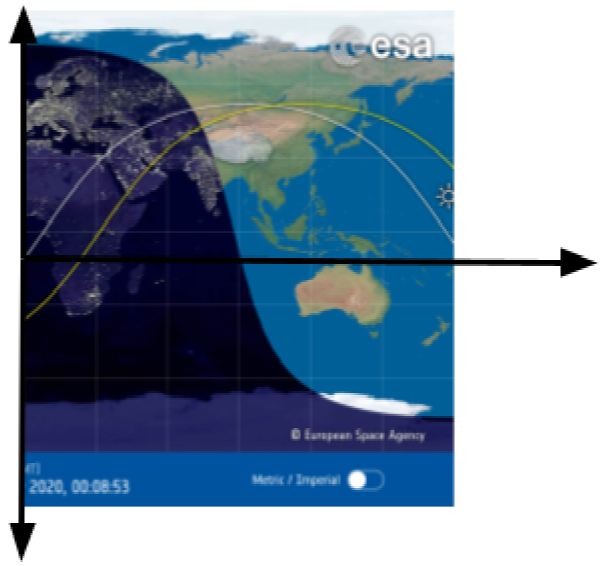

Figure 14: Image of ISS pathway on Quadrants 1 and 4 of Earth.

As seen in Figure 14 above, the ISS completes half a cycle for half of the Earth. Given this attitude the following is capable:

Assume the function is accurate enough to be represented with a sine function with negligible error.

Let P(y) represent the function of the path of the satellite to find any position P(y) in latitude at any y position (longitude). Let it be defined as such in both two and three dimensional space.

$$P(y) = sin(y)$$

Parametric representation:

$$z = sin(t)$$
$$y = t$$

For the purposes of representation, Figure 14 is extended to create Figure 15.

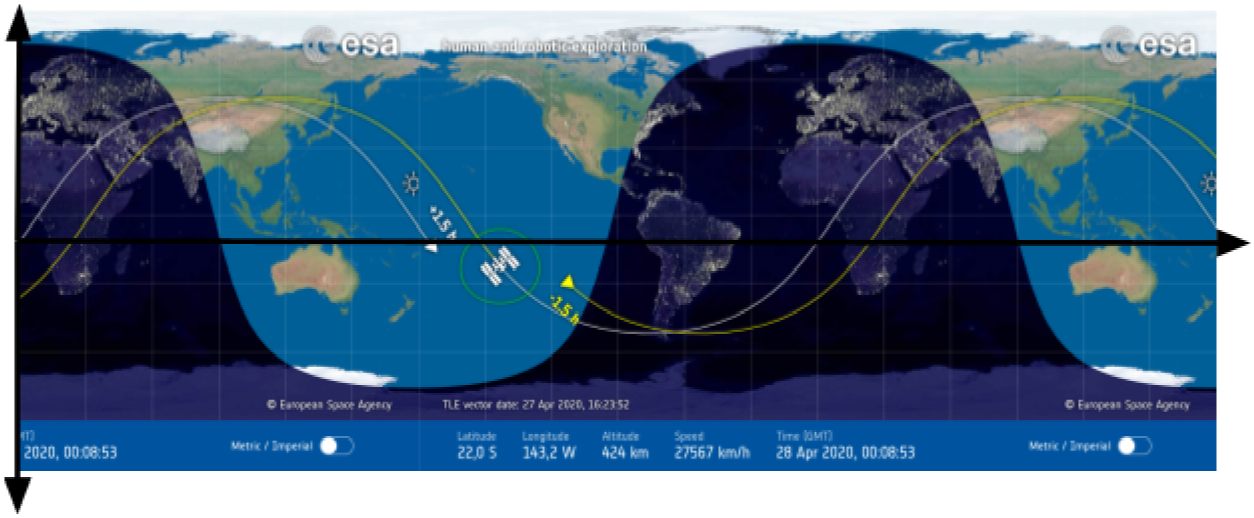

Figure 15: Figure 14 extended to depict 1 ½ cycles

If Figure 12 (the final model) is placed in a similar perspective in relation to solely the Y-Z plane, it creates a similar figure to that of Figure 15. Therefore, for the 2D modelling of the function in the Y-Z plane it is accurate enough of a descriptor.

X-Z Plane

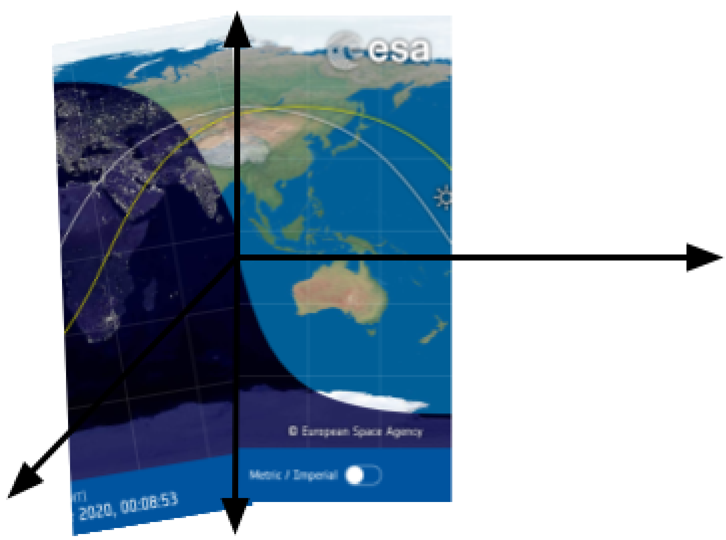

Figure 16: Depiction of Earth including x, y, and z planes using 2D map

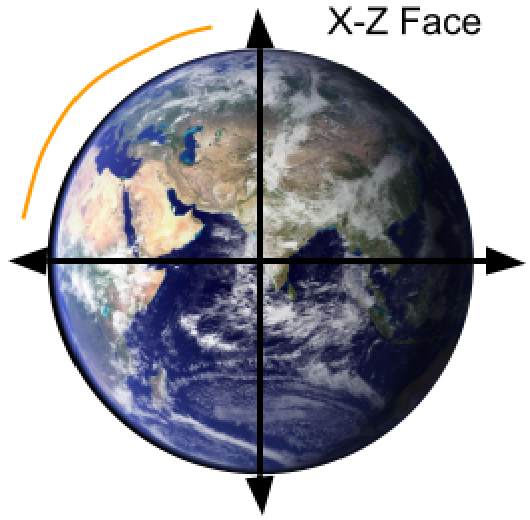

Figure 17: Pathway of ISS from side view (theoretical due to rotation and only half visible portion of Earth)

Let Figure 17 be a quarter cycle of the ISS pathway from t = 0 to the peak of Figure 5.

As seen in Figure 17, the path of the ISS as it approaches the poles loses radial length from the center of the Earth to its location. From here generalizations can be drawn:

As the ISS approaches the pole in the interval of time for ¼ cycle, it approaches 6756 km in total altitude from the specified origin.

As the ISS approaches the equator in the interval of time for ¼ cycle, it approaches 6778 km in total altitude from the specified origin.

The shape of the path of the ISS in the X-Z plane more closely represents an ellipse (ovular shape) as it is not a perfect sphere/circle.

If Figure 12 (the final model) is placed in a similar perspective in relation to solely the X-Z plane, it creates a similar figure to that of Figure 16 and Figure 17. Therefore, for the 2D modelling of the function in the X-Z plane it is accurate enough of a descriptor.

X-Y Plane

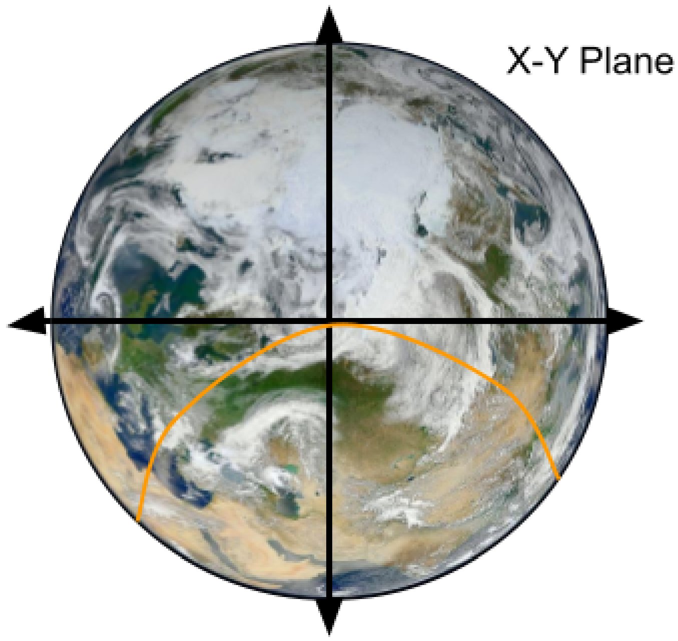

Figure 18: Pathway of the ISS from Top View (relationship)

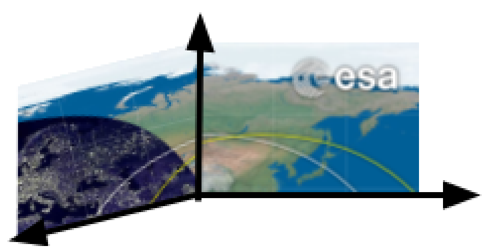

Figure 19: Pathway of ISS in X-Y plane during ½ cycle

The hardest plane to identify the relationship between the variables is the X-Y plane due to the odd shape the ISS makes in orbit. My best description of this relationship is a function that forms an ellipse like pattern but is inverted from a usual x^2 equation. For instance, in an x^2

equation it goes from close values and extrapolates the larger the x-value. In the case of Figure 19, it goes from extrapolated values to closer values as x-increases (within the ¼ cycle frame). Thus, the shape, if it were graphed, would appear as follows:

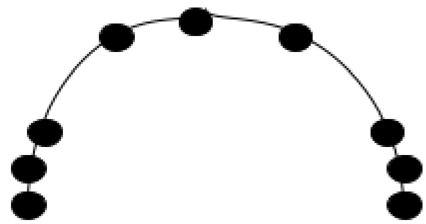

Figure 20: Best possible representation of ISS path in X-Y plane

Similar to previous planes, this path has the option of being a sine/cosine, an ellipse, or a parabola. However, given that a sine/cosine 'snakes' over the course of time (s shape) it is likely that it models similarly to a sine/cosine function.

If Figure 12 (the final model) is placed in a similar perspective in relation to solely the X-Y plane, it creates a similar figure to that of Figure 20. Therefore, for the 2D modelling of the function in the X-Y plane it is accurate enough of a descriptor.

Evaluation of Planes

The separation of each individual plane of three-dimensional reality was needed for me to have a firmer grasp of the properties of the ISS's pathway. Of course, a similar and possibly more efficient process would be graph, then test and repeat; however, I wanted to have substantial knowledge of the path after I stopped modelling the graph in real-time. Through the process of evaluation done above, I now know the following of the function:

- In the X-Z plane, the function more closely resembles an ellipse than a sine/cosine function.
    - Theories:
        - The property of the X-axis enforces an elliptical path around Earth.
        - The X variable is what makes the elliptical pattern, not the Y or Z variables. Hence, what occurs with the Y-Z plane as without the X variable the function takes a new appearance.
- In the Y-Z plane, the function is exemplary of a sine function.
- In the X-Y plane, the function resembles a parabola

I also know that the pathway of the ISS resembles the following (approximate):

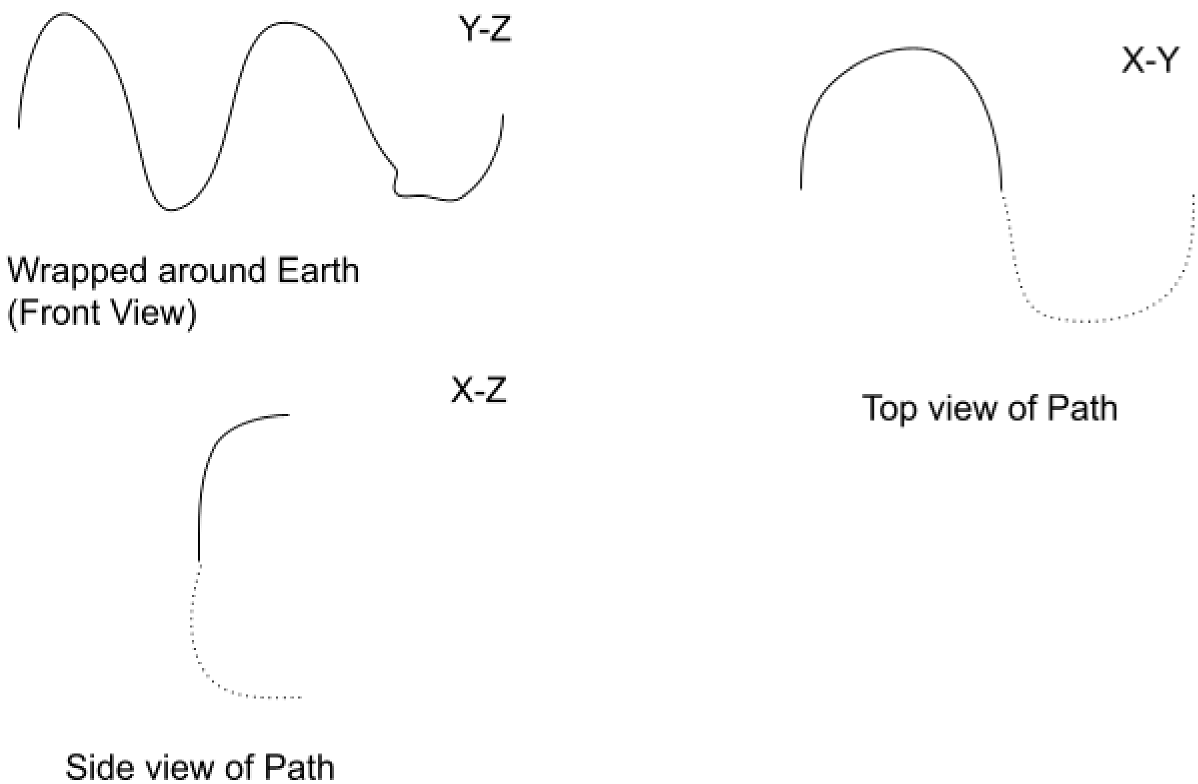

Figure 21: Rudimentary sketches of path in different perspectives.

By applying this knowledge of the components to the final path I was better able to express the accuracy of the path in each dimension. The path modelled by the 3D model I developed in the first section of the investigation very closely resembles each dimensional pathway of the ISS in real time. It is because of their close similarity that it is right to say the function model of the ISS path <u>does</u> model the actual path of the ISS.

3D Final Model Test

$$x = 3389(cos(t) + cos^2(t)sin(t)) * cos^2(sin(t))$$
$$y = 3389(2sin(t) + cos^2(t)sin(t)) * cos^2(sin(t))$$
$$z = 3378sin(t)$$

(Corresponding equation to Figure 12)

*Assume **Figure 12** and its corresponding equation are repeated every 1.5 hours. So,* $(0 \le t \le 2\pi) = 1.5$*hours*

To extrapolate:
$(0 \le t \le 2k\pi) = 1.5$ *where k = the number of cycles*

Additionally: t = (4pi/3)T where T is the real time in hours. (In order to make to scale)

Initial Point:
15 Nov 2020 at 10:41:54 MST → t = 0

(x,y,z) → (3389, 0, 0) → this checks out with Figure 12 when translated into 3D

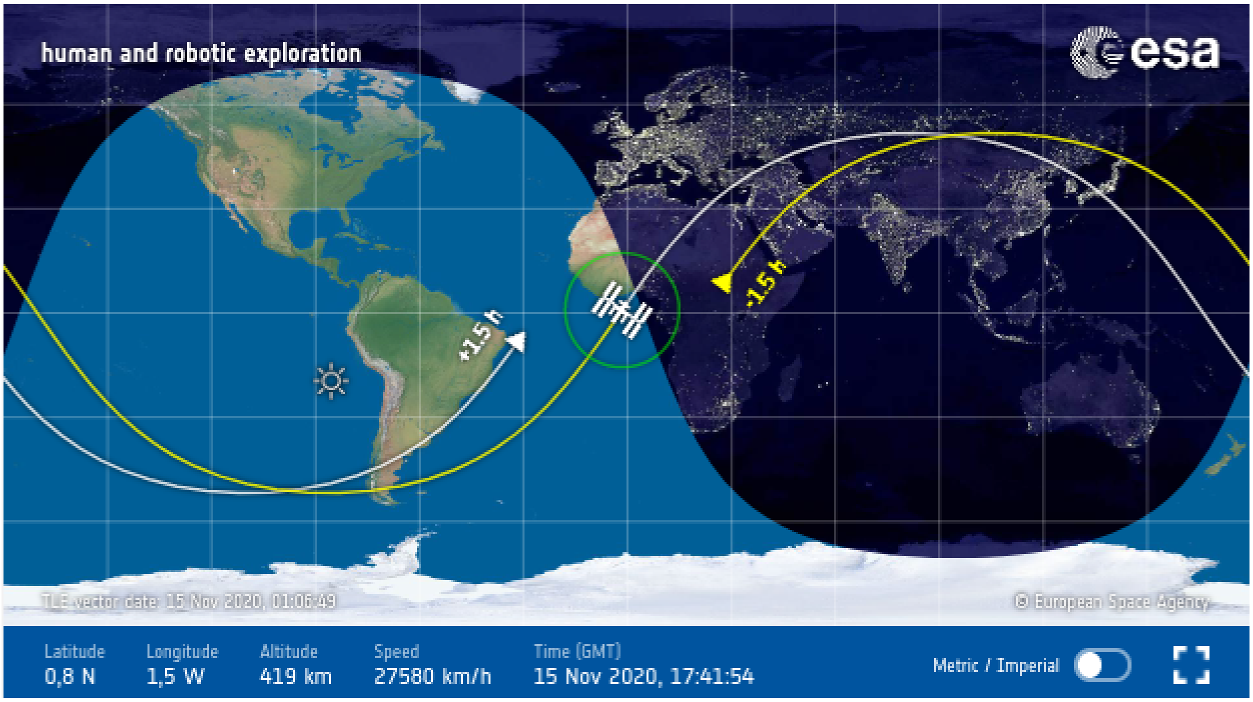

Figure 22: t = 0 *actual* ISS Location

Projection 1:
15 Nov 2020 at 10:51:45 → T = ⅙ → t = 0.7

(x,y,z) → (2482.3, 3610.4, 2171.3)

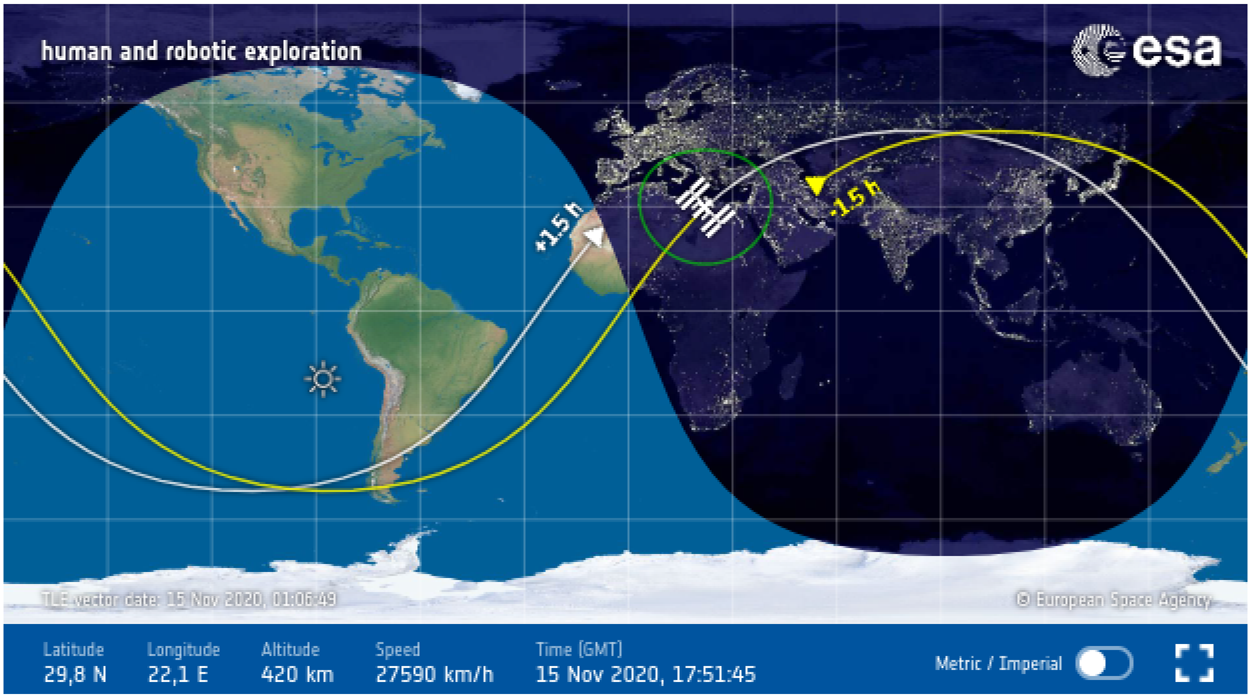

Figure 23: t = 0.7 *actual* ISS Location

If the center of the 2D map is (0,0,0) *then* every 'cube' is ~1063 km tall by ~530 km wide

So… (x,y,z) → (~Width, Altitude + Radius of Earth, ~Height)

~Coordinates: (500, 3609, 1063)

ERROR:

X = ~334%
Y = ~0.04%
Z = ~104%

Projection 2:
15 Nov 2020 at 11:11:45 → T = ½ → t = 2.09

(x,y,z) → (-403.3, 2771.7, 2925.4)

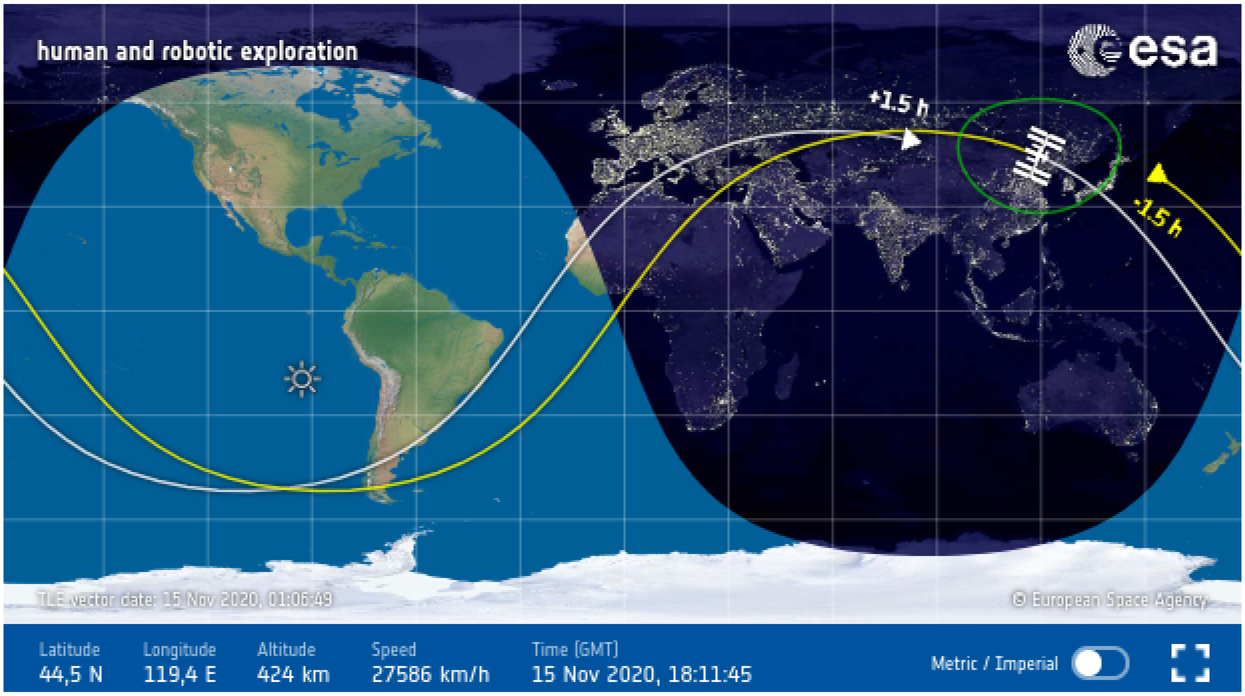

Figure 24: t = 2.09 *actual* ISS location

~Coordinates: (2120, 3613, 1771.7)

ERROR:

X = ~120%
Y = ~23%
Z = ~65%

Projection 3:
15 Nov 2020 at 12:24:56 → T = 1.5 → t = 6.3

(x,y,z) → (1484.1, 3121.7, 2690.7)

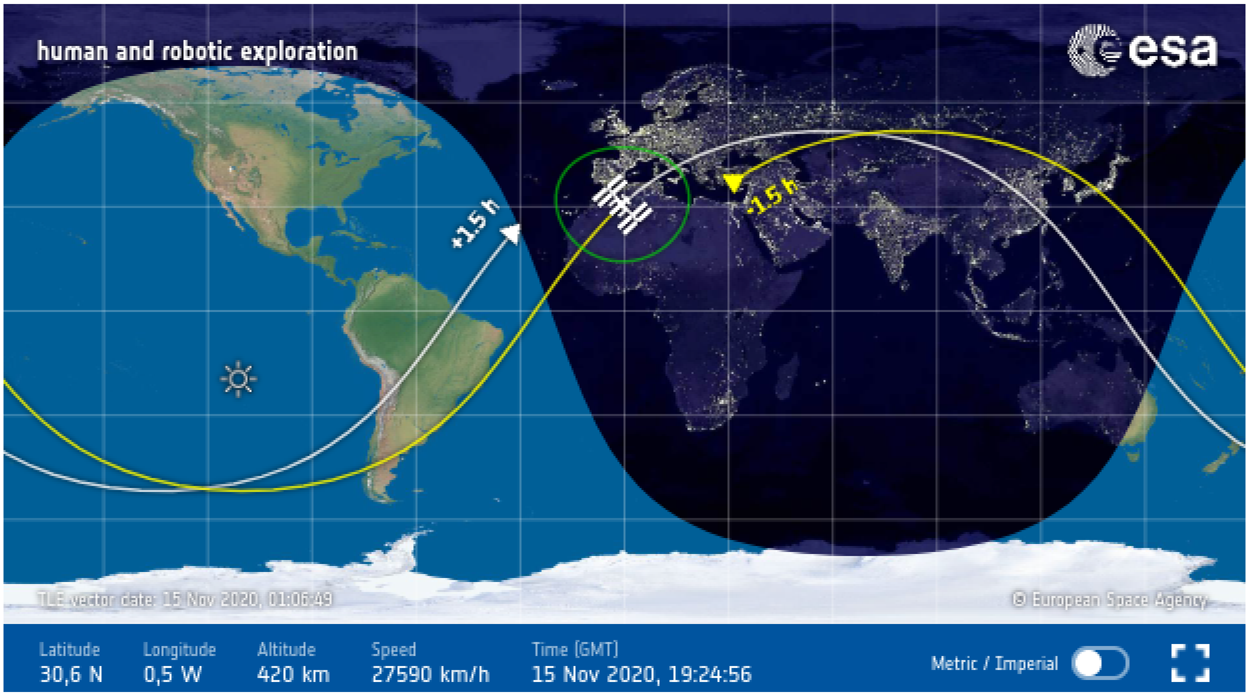

Figure 25: t = 1.7 *actual* ISS Location

~Coordinates: (0, 3609, 1063)

ERROR:

X = N/A
Y = ~14%
Z = ~153%

Reflection:

3D Parametric Equations are not very successful in predicting a satellite's orbit in Earth's atmosphere. In the case of this study, the variables defining the x and z coordinates for the orbital pattern of the satellite averaged over 90% error which concludes that the use of parametric modelling to predict the motion of a satellite is insufficient and results in high amounts of inaccuracies. When applying this conclusion to a real life situation such as the International Space Station (ISS) orbiting our planet at this very moment, the use of normal three-dimensional parametric equations would result in catastrophic failure for the station; the ISS would re enter Earth's lower orbit and fall to Earth or it would be more likely to encounter other satellites in it's varying orbit, damaging the ship. According to my model, 3D parametric equations are only useful in approximating the height of the ISS over Earth.

First, the error within the y-coordinate of the model was below 20% throughout all predictions, and approximated the relationship between ***a*** and ***t*** quite well (***a*** meaning altitude). This result indicates that for my specified model, the best use of parametric equations in defining the orbital locations of a satellite would be to find the approximate altitude of the object. However, this can easily be found with Newton's Law of Gravitation, and it is not necessarily useful for aerospace engineers/scientists to use a long sequence of inaccurate parametric equations to find a value they can approximate easily. Additionally, satellites such as the ISS and newer technologies use thruster technology to achieve ideal orbit and the use of such thrusters in orbit complicates the simplicity of the parameters, especially because this EE removed all other variables of the station. For this reason, the use of parametric equations for orbital calculations is unhelpful and I would recommend the use of regular and advanced astrophysics as a better approximation of orbital patterns.

Equally important, my model (Figure 20) for predicting the satellite's orbit seemed accurate enough to approximate the correct orbits of the ISS. I know after experimentation that it was incorrect; however, the use of parametric equations to model the shape of an orbit was successful. The verbal representation of my model (Figure 14) matched my final model with the exception of pure circular orbits, if the model had more curvature as to create a circular side view, the parametric model would have been highly successful in showing the orbit of the satellite. Though not fully successful in this criteria, it was successful as a representation for orbital patterns of a satellite in orbit around Earth.

Further research and insight into this topic may yield potential error in my experimentation. The potential error being inaccuracies in my calculations, the screenshots of the ISS and my interpretation thereof, as well as error in accuracy for identifying the shape of the orbit. However, these limitations created by error can easily be removed if the researcher utilizes a professor or expert in the field for questions involving the math, as well as the use of a more accurate tracking website for the ISS. Future research might include development of the model itself in order to prose more accurate results (limit error) and/or development of the model in order to procure a visual representation of an orbit without literal calculative value. The use of models that are ‘not to scale’ are useful in presenting ideas, and though not accurate, they do a good job in representing complications, and potential orbital patterns.

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