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Differential operators in geodetic coordinates applied to potential-field modelling

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SUMMARY

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1 INTRODUCTION

Several methods have been developed to compute the gravitational and magnetic fields produced by geological structures in Cartesian (CITAR PAPERS CLASSICOS) and spherical coordinates (CITAR PAPERS CLASSICOS).

The spherical approach is more appropriated than the Cartesian approach for modelling large-scale structures which require taking the Earth's curvature into account.

The spherical approach, however, neglects the Earth's ellipticity.

Roussel et al. (2015) is the only work presenting a potential-field modelling that takes the Earth's ellipticity into consideration. Their method is formulated in ellipsoidal coordinates and relies on the numerical solution of volume integrals by applying Gauss-Legendre quadrature.

The gravitational and magnetic fields computed by these methods cannot be directly compared to geophysical observations because the position of the last ones are commonly described in geodetic coordinates (e.g., Hotine 1969; Heiskanen & Moritz 1967; Krakiwsky & Wells 1971; Soler 1976; Vaníček & Krakiwsky 1987; Rapp 1991; Seeber 2003; Hofmann-Wellenhof & Moritz 2005; Torge & Müller 2012).

Consequently, coordinate transformations are required.

Up to now, the geoscientific community lacks of methods for computing the gravitational and magnetic fields produced by geological bodies in geodetic coordinates.

We present the differential operators gradient, divergent, Laplacian and curl in terms of the scale factors and the normal basis associated to the GCS.

We applied the differential operators presented here to define the expressions of the gravitational and magnetic fields produced by spheres.

Such expressions are can be used, for example, for producing regional magnetic anomaly maps (e.g., Von Frese et al. 1981; Mayhew 1982; Dyment & Arkani-Hamed 1998), characterizing the regional gravity field (e.g., Needham 1970; Balmino 1972; Barthelmes et al. 1991; Barthelmes & Dietrich 1991; Lehmann 1993; Antunes et al. 2003; Guspí et al. 2004; Lin et al. 2014) or approximating the gravitational and magnetic fields produced by geological structures with arbitrary shapes.

2 GEODETIC CURVILINEAR COORDINATES

2.1 Relationship between Cartesian and Geodetic coordinates

Consider a terrestrial geocentric system of Cartesian coordinates having the z-axis coincident with the mean rotational axis, the x-axis pointing to the Greenwich meridian and the y-axis directed so as to obtain a right-handed system (Fig. 1). Coordinate systems similar to this one may be found in the literature as Conventional Terrestrial Reference Coordinate System (e.g., Soler & Hothem 1988), International Terrestrial Reference System (e.g., Seeber 2003; Torge & Müller 2012) or Earth-centered Earth-fixed system (e.g., Bouman et al. 2013), for example. Here, we opted for simply using the term Geocentric Cartesian System (GCS). Let us also consider a terrestrial geocentric system of geodetic coordinates h (geometric height), φ (geodetic latitude), and λ (longitude) defined with respect to a reference ellipsoid of revolution having major semi-axis a and minor semi-axis b. For convenience, we call this system Geocentric Geodetic System (GGS).

The geodetic coordinates h, φ, and λ can be transformed into the Cartesian coordinates x, y, and z as follows (Heiskanen & Moritz 1967):

$$\begin{aligned} x &= (N+h)\,\cos\varphi\,\cos\lambda \\ y &= (N+h)\,\cos\varphi\,\sin\lambda \qquad , \\ z &= \left[N\left(1-e^2\right)+h\right]\,\sin\varphi \end{aligned} \tag{1}$$

where $e = \sqrt{a^2 - b^2}/a$ is the first eccentricity and N is the prime vertical radius of curvature given by

$$N = \frac{a}{\left(1 - e^2 \sin^2 \varphi\right)^{\frac{1}{2}}} \,. \tag{2}$$

Additionally, let us define the meridian radius of curvature M as follows:

$$M = \frac{a\left(1 - e^2\right)}{\left(1 - e^2 \sin^2 \varphi\right)^{\frac{3}{2}}} \,. \tag{3}$$

It can be easily verified that the prime vertical radius of curvature N (eq. 2), the meridian radius of curvature M (eq. 3) and their first derivatives with respect to φ satisfy the following relationships:

$$M = \frac{1 - e^2}{1 - e^2 \sin^2 \varphi} N \tag{4}$$

and

$$\frac{\partial N}{\partial \varphi} = \frac{e^2 \, \sin \varphi \, \cos \varphi}{1 - e^2 \sin^2 \varphi} N \,. \tag{5}$$

2.2 Scale factors and local basis of the GGS

Having defined the relations between the Cartesian and geodetic coordinates, we will now apply some well-established concepts of generic curvilinear coordinates. An in-depth presentation of these concepts may be found at Kellogg (1929), Hotine (1969), Arfken & Weber (2001), or Stratton (2007), for example. We shall confine ourselves to the particular case of geodetic curvilinear coordinates.

Let us first make use of the Pythagorean theorem to define the squared magnitude of the infinitesimal displacement between two neighbouring points in the GCS system is given by

$$ds^2 = dx^2 + dy^2 + dz^2 \,, \tag{6}$$

where the differentials dx, dy, and dz are obtained from eqs 1 as follows:

$$d\alpha = \frac{\partial \alpha}{\partial h} \, dh + \frac{\partial \alpha}{\partial \varphi} \, d\varphi + \frac{\partial \alpha}{\partial \lambda} \, d\lambda \,, \quad \alpha = x, y, z \,, \tag{7}$$

and the partial derivatives $\frac{\partial \alpha}{\partial \beta}$, $\alpha = x, y, z$, $\beta = h, \varphi, \lambda$, are given by (Soler 1976):

$$\begin{array}{lll} \frac{\partial x}{\partial h} = \cos\varphi \, \cos\lambda & \frac{\partial x}{\partial \varphi} = -\left(M + h\right) \, \sin\varphi \, \cos\lambda & \frac{\partial x}{\partial \lambda} = -\left(N + h\right) \, \cos\varphi \, \sin\lambda \\ \frac{\partial y}{\partial h} = \cos\varphi \, \sin\lambda & \frac{\partial y}{\partial \varphi} = -\left(M + h\right) \, \sin\varphi \, \sin\lambda & \frac{\partial y}{\partial \lambda} = \left(N + h\right) \, \cos\varphi \, \cos\lambda \\ \frac{\partial z}{\partial h} = \sin\varphi & \frac{\partial z}{\partial \varphi} = \left(M + h\right) \, \cos\varphi & \frac{\partial z}{\partial \lambda} = 0 \end{array} \,. \tag{8}$$

The partial derivatives with respect to φ were deduced from eqs 1 by using eqs 4 and 5.

From these differentials $d\alpha$ (eqs 7), we define the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial h} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \lambda} \\ \frac{\partial y}{\partial h} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \lambda} \\ \frac{\partial z}{\partial h} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \lambda} \end{bmatrix} \,. \tag{9}$$

By substituting the differentials $d\alpha$ (eqs 7) into eq. 6, we obtain a general quadratic form representing the squared distance ds^2 given by

$$ds^2 = g_{11}\,dh^2 + g_{22}\,d\varphi^2 + g_{33}\,d\lambda^2 + 2\left(g_{12}\,dh\,d\varphi + g_{13}\,dh\,d\lambda + g_{23}\,d\varphi\,d\lambda\right) \ , \tag{10}$$

where

$$\begin{aligned} g_{11} &= \left(\frac{\partial x}{\partial h}\right)^2 + \left(\frac{\partial y}{\partial h}\right)^2 + \left(\frac{\partial z}{\partial h}\right)^2 \\ g_{22} &= \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 \\ g_{33} &= \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2 \end{aligned} \quad , \tag{11}$$

and

$$\begin{aligned} g_{12} &= \left(\frac{\partial x}{\partial h}\frac{\partial x}{\partial \varphi}\right) + \left(\frac{\partial y}{\partial h}\frac{\partial y}{\partial \varphi}\right) + \left(\frac{\partial z}{\partial h}\frac{\partial z}{\partial \varphi}\right) \\ g_{13} &= \left(\frac{\partial x}{\partial h}\frac{\partial x}{\partial \lambda}\right) + \left(\frac{\partial y}{\partial h}\frac{\partial y}{\partial \lambda}\right) + \left(\frac{\partial z}{\partial h}\frac{\partial z}{\partial \lambda}\right) \\ g_{23} &= \left(\frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \lambda}\right) + \left(\frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \lambda}\right) + \left(\frac{\partial z}{\partial \varphi}\frac{\partial z}{\partial \lambda}\right) \end{aligned} \quad . \tag{12}$$

The coefficients defined by eqs 11 and 12 are commonly called metrical coefficients. It can be shown that the coefficients g_{12}, g_{13}, and g_{23} (eqs 12) are equal to zero, which implies that the GGS is an orthogonal system.

The squared infinitesimal distance ds^2 (eq. 10) can be rewritten as follows:

$$ds^2 = (h_1\,dh)^2 + (h_2\,d\varphi)^2 + (h_3\,d\lambda)^2 \ , \tag{13}$$

where h_i, $i = 1, 2, 3$, are called scale factors. These coefficients are defined as the square root of the metrical coefficients g_{ii} (eqs 11) as follows:

$$\begin{aligned} h_1 &= 1 \\ h_2 &= M + h \\ h_3 &= (N + h)\cos\varphi \end{aligned} \quad , \tag{14}$$

where N and M are defined by eqs 2 and 3, respectively. Notice that, in eq. 13, the terms $h_1\,dh$, $h_2\,d\varphi$, and $h_3\,d\lambda$ represent infinitesimal displacement components along the geodetic coordinates h, φ, and λ, respectively. By using the scale factors h_1, h_2, and h_3 (eqs 14), we may define the metric matrix $\mathbf{H}$ (Soler 1976) as follows:

$$\mathbf{H} = \begin{bmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{bmatrix} . \tag{15}$$

It is also necessary to define a set of mutually orthogonal vectors at a given fixed point. To do this, let us first define the *position vector* $\mathbf{r}$ with elements defined by the Cartesian coordinates x, y and z (eq. 1) as follows:

$$\mathbf{r} = \begin{bmatrix} x(h, \varphi, \lambda) \\ y(h, \varphi, \lambda) \\ z(h, \varphi, \lambda) \end{bmatrix} . \tag{16}$$

Then, the set of mutually orthogonal vectors can be given by

$$\begin{aligned} \mathbf{e}_1 &= \frac{\partial \mathbf{r}}{\partial h} \\ \mathbf{e}_2 &= \frac{\partial \mathbf{r}}{\partial \varphi} \\ \mathbf{e}_3 &= \frac{\partial \mathbf{r}}{\partial \lambda} \end{aligned} . \tag{17}$$

Such vectors are called *unitary vectors* (Stratton 2007). Notice that the unitary vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are defined, respectively, in the direction of the increasing geodetic coordinates h, φ, and λ. Besides, the magnitude of a unitary vector $\mathbf{e}_i$ is equal to the scale factor h_i, $i = 1, 2, 3$, which shows that it is not necessarily a unit vector. By multiplying the unitary vectors $\mathbf{e}_i$ (eq. 17) by the reciprocal of the scale factors h_i (eq. 14) and using the partial derivatives defines by eqs 8, we obtain a set of three mutually orthogonal unit vectors given by (Soler 1976)

$$\begin{aligned} \hat{\mathbf{e}}_1 &= \begin{bmatrix} \cos\varphi \, \cos\lambda \\ \cos\varphi \, \sin\lambda \\ \sin\varphi \end{bmatrix} \\ \hat{\mathbf{e}}_2 &= \begin{bmatrix} -\sin\varphi \, \cos\lambda \\ -\sin\varphi \, \sin\lambda \\ \cos\varphi \end{bmatrix} \\ \hat{\mathbf{e}}_3 &= \begin{bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix} \end{aligned} , \tag{18}$$

where N and M are defined by eqs 2 and 3, respectively. These vectors form a local orthonormal basis in the space of Cartesian coordinates. By using these vectors (eqs 18), we can define the matrix equation transforming a vector $\mathbf{v}_C$, defined at a point in the GCS, into a vector $\mathbf{v}_G$, defined at the same point in the GGS, as follows:

$$\mathbf{v}_G = \mathbf{R}^\top \mathbf{v}_C , \tag{19}$$

where $\mathbf{R}$ is a rotation matrix (Soler 1976) given by

$$\mathbf{R} = \begin{bmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \end{bmatrix} . \tag{20}$$

It can be easily verified that the Jacobian $\mathbf{J}$ (eq. 9), metric $\mathbf{H}$ (eq. 15), and rotation $\mathbf{R}$ (eq. 20) matrices satisfy the following relationship (Soler 1976):

$$\mathbf{RJ} = \mathbf{H} . \tag{21}$$

2.3 Differential operators

The *gradient*, *divergent*, *Laplacian*, and *curl* differential operators in terms of the scale factors h_i (eq. 14) and the unit vectors $\hat{\mathbf{e}}_i$ (eq 18) as follows (e.g., Kellogg 1929; Hotine 1969; Arfken & Weber 2001; Stratton 2007):

$$\nabla \psi = \frac{\partial \psi}{\partial h} \hat{\mathbf{e}}_1 + \frac{1}{(M+h)} \frac{\partial \psi}{\partial \varphi} \hat{\mathbf{e}}_2 + \frac{1}{(N+h)\cos\varphi} \frac{\partial \psi}{\partial \lambda} \hat{\mathbf{e}}_3 \quad , \tag{22}$$

$$\begin{aligned} \nabla \cdot \mathbf{V} = & \left[\frac{1}{(N+h)} + \frac{1}{(M+h)} \right] V_1 + \frac{\partial V_1}{\partial h} \\ & + \frac{\tan\varphi}{(N+h)} V_2 + \frac{1}{(M+h)} \frac{\partial V_2}{\partial \varphi} + \frac{1}{(N+h)\cos\varphi} \frac{\partial V_3}{\partial \lambda} \end{aligned} \quad , \tag{23}$$

$$\begin{aligned} \nabla^2 \psi = & \left[\frac{1}{(N+h)} + \frac{1}{(M+h)} \right] \frac{\partial \psi}{\partial h} + \frac{\partial^2 \psi}{\partial h^2} \\ & - \left[\frac{\tan\varphi}{(N+h)(M+h)} + \frac{1}{(M+h)^3} \frac{\partial M}{\partial \varphi} \right] \frac{\partial \psi}{\partial \varphi} + \frac{1}{(M+h)^2} \frac{\partial^2 \psi}{\partial \varphi^2} \\ & + \frac{1}{(N+h)^2 \cos^2\varphi} \frac{\partial^2 \psi}{\partial \lambda^2} \end{aligned} \tag{24}$$

and

$$\begin{aligned} \nabla \times \mathbf{V} = & \left[-\frac{\tan\varphi}{(N+h)} V_3 + \frac{1}{(M+h)} \frac{\partial V_3}{\partial \varphi} - \frac{1}{(N+h)\cos\varphi} \frac{\partial V_2}{\partial \lambda} \right] \hat{\mathbf{e}}_1 \\ & + \left[-\frac{1}{(N+h)} V_3 + \frac{1}{(N+h)\cos\varphi} \frac{\partial V_1}{\partial \lambda} - \frac{\partial V_3}{\partial h} \right] \hat{\mathbf{e}}_2 \\ & + \left[\frac{1}{(M+h)} V_2 + \frac{\partial V_2}{\partial h} - \frac{1}{(M+h)} \frac{\partial V_1}{\partial \varphi} \right] \hat{\mathbf{e}}_3 \end{aligned} \quad , \tag{25}$$

where ψ is a scalar field which is invariant to a rotation of the coordinate system and $\mathbf{V}$ is a vector field with components V_1, V_2, and V_3 along the unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ (eqs 18), respectively.

3 APPLICATION TO POTENTIAL-FIELD MODELLING

Potential fields are commonly represented by functions of the form $f(x - x_0, y - y_0, z - z_0)$, which tends to zero as the differences $x - x_0$, $y - y_0$, and $z - z_0$ increase.

3.1 Gravitational modelling

3.2 Magnetic modelling

4 CONCLUSIONS

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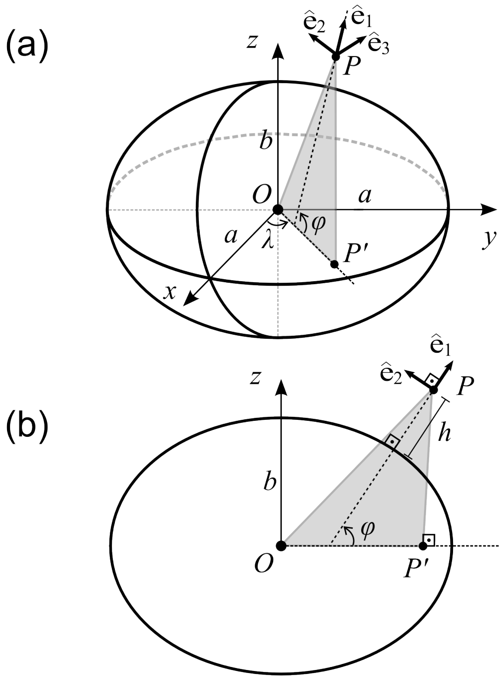

Figure 1. Schematic representation of the Cartesian and Geodetic systems.

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APPENDIX A: APPENDIX

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