Proof of uniform convergence for a cell-centered AP discretization of the hyperbolic heat equation on general meshes

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Abstract

We prove the uniform AP convergence on unstructured meshes in 2D of a generalization, see [5], of the Gosse-Toscani 1D scheme for the hyperbolic heat equation. This scheme is also a nodal extension in 2D of the Jin-Levermore scheme described in [18] for the 1D case. In 2D, the proof is performed using a new diffusion scheme.

1 Introduction

We address the convergence analysis on unstructured meshes of diffusion asymptotic preserving schemes for the discretization of a problem with a stiff parameter denoted as $\varepsilon \le 1$. The model problem considered in this work is the hyperbolic heat equation in the domain $t \ge 0$ and $x \in \Omega \subset \mathbb{R}^n$

$$P^\varepsilon : \quad \begin{cases} \partial_t p^\varepsilon + \dfrac{1}{\varepsilon}\operatorname{div}(\mathbf{u}^\varepsilon) = 0, & p^\varepsilon \in \mathbb{R}, \\ \partial_t \mathbf{u}^\varepsilon + \dfrac{1}{\varepsilon}\nabla p^\varepsilon = -\dfrac{\sigma}{\varepsilon^2}\mathbf{u}^\varepsilon, & \mathbf{u}^\varepsilon \in \mathbb{R}^n \end{cases} \tag{1}$$

discretized with P^0 finite volume schemes. This problem is representative of many transport problem such as transfer and neutron transport, for which the small parameter ε is the ratio of two very different sound velocities and σ is the absorption or the opacity. For simplicity both ε and $\sigma > 0$ are kept constant in space in this study. The system (1) can also be introduced as a specific linearization of a pressure-velocity system of partial differential equations in the acoustic regime. In this work we will need the following well known energy estimates concerning the Cauchy problem for the partial differential equation (1):

Proposition 1.1. *If $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{T}^n$ and if the initial data $\mathbf{V}_0 = (p_0, \mathbf{u}_0) \in H^p(\Omega)$ then*

$$||\mathbf{V}||_{H^p(\Omega)} \le ||\mathbf{V}_0||_{H^p(\Omega)} \tag{2}$$

and moreover

$$\frac{\sigma}{\varepsilon^2}||\mathbf{u}||^2_{L^2([0,T];H^p(\Omega))} \le ||\mathbf{V}_0||^2_{H^p(\Omega)}. \tag{3}$$

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We will consider well prepared data in the sense that: $p^\varepsilon(t=0)$ is independent of ε and is sufficiently smooth; the initial velocity satisfies the equality in the second equation of (1) at leading order. It writes

$$p^\varepsilon(t=0) = p_0 \text{ and } \mathbf{u}^\varepsilon(0) = -\frac{\varepsilon}{\sigma}\nabla p_0. \tag{4}$$

For such well prepared data, it can be easily shown that the formal limit of P^ε for small ε is

$$P^0: \qquad \partial_t p - \frac{1}{\sigma}\Delta p = 0. \tag{5}$$

Remark 1.2. *We do not consider the regime $\sigma \to 0$, since it introduces a singularity both in the initial data of the hyperbolic heat equation and in the limit parabolic equation.*

Before addressing the main difficulty of this work which is the discretization on unstructured meshes, we briefly recall the now well known notion of an asymptotic preserving technique [16]-[17] which is illustrated in the figure 1. For the simplicity of the presentation, we will consider mainly semi-discrete numerical methods, this is why the time step does not show up. In figure 1 the parameter h designs a numerical method with characteristic length $h \le 1$: so we assume a numerical method P_h^ε for the discretization of P^ε.

Definition 1.1 (Uniform AP)**.** *If P_h^ε is consistent with P^ε uniformly with respect to ε, then we say that the scheme P_h^ε is uniformly AP (uniformly asymptotic preserving).*

However the design of such methods and the numerical proof of this property is difficult. This is why it has been proposed in [16] to rely on the simpler necessary condition, where the limit as $\varepsilon \to 0$ of P_h^ε is called the limit diffusion scheme P_h^0.

$$\begin{tikzcd}
P_h^\varepsilon \arrow[r, "\varepsilon \to 0"] \arrow[d, "h \to 0"'] & P_h^0 \arrow[d, "h \to 0"] \\
P^\varepsilon \arrow[r, "\varepsilon \to 0"'] & P^0
\end{tikzcd}$$

Figure 1: The AP (asymptotic diagram) diagram

Definition 1.2 (AP)**.** *If P_h^0 is consistent with the limit model P^0, then we say that the scheme P_h^ε is AP (asymptotic preserving).*

This property is simpler to analyze than the uniform AP. It explains why it has been very fruitful in the past. In 1D, many AP schemes have been designed for some PDE and physical problems: S. Jin, C. D. Levermore [18] or L. Gosse, G. Toscani [14] for the hyperbolic heat equation, M. Lemou, L. Mieussens, N. Crouseilles [20]-[8]-[9] for some kinetic equations, L. Gosse [15], C. Buet and co-workers [4] or S. Jin and C. D. Levermore [19] for S_N equations and C. Berthon, R. Turpault [1]-[2]-[3] for generic systems and a non linear radiative transfer model. Recently some asymptotic preserving schemes for linear systems and non linear radiative transfer model have been designed in 2D [5]-[6]-[7]. However for this type of schemes it is difficult to obtain convergence estimates due to the competition between the two parameters ε and h. To our knowledge this type of proof are only given for uniform grids [5] (consistence and stability, Lax theorem), [14] (L^1 and BV estimates), [21] (L^2 estimates). The goal of this work is to prove the uniform AP property on unstructured grids.

To this end we adapt a strategy developed in [13] in a slightly different context. It relies on the derivation of a priori estimates attached to the AP diagram in figure 1. To have a more global perspective on this strategy, let us assume some natural abstract a priori estimates for a given norm which is in our work based on $\|f\| = \|f\|_{L^2([0,T]\times\Omega)}$ where $T > 0$ is a given final time, $\Omega = \mathbb{R}$, in 1D or $\Omega = [0,1]^2$ with periodic boundary conditions in 2D. We assume four positive constants $a, b, c, d > 0$ and another universal constant C such that

$$\|P^\varepsilon - P^0\| \le C\varepsilon^a, \tag{6}$$

$$\|P_h^\varepsilon - P^\varepsilon\|_{\text{naive}} \le C\varepsilon^{-b}h^c, \tag{7}$$

$$\|P_h^0 - P^0\| \le Ch^d. \tag{8}$$

The first inequality expresses that P^0 is indeed the limit of P^ε. The second inequality is typically based on non AP error bounds. This is why we refer to it as the naive error bound. The third inequality is the AP property. A fourth inequality for $\|P_h^\varepsilon - P_h^0\|$ is of course required to close the diagram. We assume that it can be obtained in a form similar to (6)

$$\|P_h^\varepsilon - P_h^0\| \le C\varepsilon^e, \qquad e > 0. \tag{9}$$

Proposition 1.3. *Assume that all these inequalities are at hand and that $d \ge c$ and $e \ge a$. Then the uniform AP holds with a rate at least $O\left(h^{\frac{ac}{a+b}}\right)$.*

Proof. The triangular inequality writes

$$\|P_h^\varepsilon - P^\varepsilon\| \le \min\left(\|P_h^\varepsilon - P^\varepsilon\|_{\text{naive}}, \|P_h^\varepsilon - P_h^0\| + \|P_h^0 - P^0\| + \|P^\varepsilon - P^0\|\right)$$

which yields, using $\min(x, y+z) \le \min(x,y) + \min(x,z)$ and $e \ge a$.

$$\|P_h^\varepsilon - P^\varepsilon\| \le C\left(\min\left(\varepsilon^{-b}h^c, \varepsilon^a\right) + h^d + \min\left(\varepsilon^{-b}h^c, \varepsilon^e\right)\right) \le C\left(2\min\left(\varepsilon^{-b}h^c, \varepsilon^a\right) + h^d\right). \tag{10}$$

We define a threshold value $\varepsilon_{\text{thresh}}$ by $\varepsilon_{\text{thresh}}^{-b}h^c = \varepsilon_{\text{thresh}}^a$. So either $\varepsilon \le \varepsilon_{\text{thresh}}$ so that $\min\left(\varepsilon^{-b}h^c, \varepsilon^a\right) \le \varepsilon_{\text{thresh}} = h^{\frac{ac}{a+b}}$, or $\varepsilon \ge \varepsilon_{\text{thresh}}$ and the same bound is obtained by taking the other term as the minimum. And since $d \ge c$, one gets the abstract bound $\|P_h^\varepsilon - P^\varepsilon\| \le 3Ch^{\frac{ac}{a+b}}$ which ends the proof. ☐

The structure of these inequalities explains our strategy: that is we prove separately each of these inequalities (6-9) with care, so that the inequalities $d \ge c$ and $e \ge a$ are true. This part of the proof relies on specific hyperbolic and parabolic numerical methods. Even if it is technical, the first three inequalities (6-8) do not yield additional difficulties with respect to the state of the art. The proof of inequality (9) is provided in 1D, and can be probably be generalized straightforwardly on cartesian meshes in 2D and 3D. On the other hand our researches on proving (9) for $\|P_h^\varepsilon - P_h^0\|$ show a fundamental obstruction in dimension greater than one on unstructured meshes which was not expected initially. Since the main difficulty is related to P_h^0, it motivates the definition of a new diffusion scheme. To this end we remark that another diffusion scheme is naturally defined from P_h^ε by killing the derivative $\partial_t v_h$ in the discrete version of the second equation of (1). Killing at the continuous level the $\partial_t v$ is absolutely equivalent to taking the formal limit $\varepsilon \to 0^+$. But at the discrete level, it appears that it generates a new family of diffusion schemes, where both parameters h and ε are present. We call them Diffusion Asymptotic schemes, DA_h^ε. By construction $P_h^0 = \lim_{\varepsilon\to 0} DA_h^\varepsilon$. This is summarized in figure 2. Finally since the scheme DA_h^ε is still an accurate discretization of P^0, our proof of the uniform AP properties is based on the new AP diagram displayed in figure 3.

Our main theorem 3.18 in dimension 2 is based on this structure and it may be stated as follows: **The so-called JL-(b) scheme defined in [5] for the discretization of the hyperbolic heat equation (1) (the scheme is cell-centered with nodal based fluxes) is uniformly AP on unstructured meshes, with a rate of convergence at least $O(h^{\frac{1}{4}})$ for sufficiently**

$$P_h^\varepsilon \xrightarrow{\partial_t v_h = 0} DA_h^\varepsilon \xrightarrow{\varepsilon \to 0} P_h^0$$

Figure 2: Definition of the diffusion asymptotic scheme DA_h^ε.

smooth initial data. This is an improvement with respect to [5] where only AP was proven. To our knowledge this is the first time that such a result is obtained on general unstructured multidimensional meshes.

More precisely the convergence estimate can be written as

$$\textbf{error} \leq C(T) \min \left(\sqrt{\frac{h}{\varepsilon}} \;\;,\;\; \varepsilon \max \left(1, \sqrt{\frac{\varepsilon}{h}}\right) + h + \varepsilon \right)$$

where the first argument in the min function comes from the hyperbolic analysis and the second argument comes from the parabolic analysis. Some natural regularity assumptions are nevertheless imposed on the mesh in the hypothesis 2.1. This hypothesis is not very restrictive. For example meshes with angles greater than 90 degrees are allowed. If the mesh is made with triangles, the hypothesis is fulfilled if all angles are greater than 12 degrees, see [5]. It is interesting to notice that the rate of uniform convergence is $O(h^{\frac{1}{3}})$ in dimension one. The difference essentially comes from the estimate of the reconstruction of the initial velocity which is needed to rewrite a diffusion scheme as a non homogeneous hyperbolic scheme: it is much simpler in dimension one (see equation (20)) than in dimension two (see proposition (3.16)). In this work we considered only semi-discrete numerical schemes, since it simplifies a lot the notations and allow to focus on the main difficulties, but the final estimates of convergence can be generalized to fully discrete schemes, using the a priori estimates developed in [10]. For explicit schemes, these estimates add a term proportional to the square root of the maximal time step allowed by the CFL condition. Since our problem is an hyperbolic+relaxation problem, with a limit which is parabolic, this additional term can be computed and is of the order between h (for purely hyperbolic) to h^2 (for purely parabolic). We refer to [5] for the detail of CFL condition in 1D and 2D. Concerning the implicit fully discrete version of the semi-discrete scheme, the same kind of error terms can be analyzed. This is confirmed by he numerical results of section 4, which show an even better rate of convergence.

$$\begin{tikzcd}
P_h^\varepsilon \arrow[r, "\frac{d}{dt}\mathbf{v}_h^\varepsilon = 0"] \arrow[d, "h \to 0"'] & DA_h^\varepsilon \arrow[r, dashed] \arrow[d, "h \to 0"] & P_h^0 \arrow[dl, dashed] \\
P^\varepsilon \arrow[r, "\varepsilon \to 0"'] & P^0 &
\end{tikzcd}$$

Figure 3: The new AP diagram, where the previous branch is still displayed in dashed lines.

We think that some of our results can have an interest for the development and use of such methods in research or industrial codes with complex non linear physics on unstructured meshes. Indeed for such codes cell-centered Finite Volume schemes are a natural solution in terms of data structure. The point is the following: the scheme studied in this work is the only cell-centered one we know in 2D to compute the solutions of problems which admit diffusion limits in certain regimes and for which it is possible to prove the AP property. Since the structure of this cell-centered scheme is nodal based, it strongly questions the ability of standard Finite Volume methods with edge-based fluxes to recover asymptotic diffusion regimes. As demonstrated in this work, nodal based Finite Volume techniques do not suffer from this drawback.

This work is organized as follows. Section 2 is dedicated to the discretization of the model problem in dimension one on irregular grids. The convergence is proved in theorem 2.9 with order $h^{\frac{1}{3}}$ in the L^2 space-time norm. In the next section, the nodal solvers for the hyperbolic equation are defined, and the various a priori estimates proved. The main theorem of uniform AP for the JL-(b) scheme with a rate $O(h^{\frac{1}{4}})$ is proved at the end of the section. Section 4 provides numerical results that sustain the fact that the convergence order depends on the relative value of ε and h., and so is mixed hyperbolic/parabolic. Our final remarks will be gathered in a conclusion.

All our results and numerical methods in 2D can be generalized in 3D provided a convenient definition of the nodal corner vector is used as in [11].

2 Analysis in 1D

The model problem in dimension one writes

$$P^\varepsilon : \qquad \begin{cases} \partial_t p^\varepsilon + \frac{1}{\varepsilon}\partial_x u^\varepsilon = 0, \\ \partial_t u^\varepsilon + \frac{1}{\varepsilon}\partial_x p^\varepsilon = -\frac{\sigma}{\varepsilon^2}u^\varepsilon. \end{cases} \tag{11}$$

As stressed already in (4), we consider well-prepared data $p^\varepsilon(t=0) = p_0$ and $u_0^\varepsilon = -\frac{\varepsilon}{\sigma}\partial_x p_0$. The equations (11) admit the formal diffusion limit when ε tends to 0:

$$P^0 : \qquad \partial_t p - \frac{1}{\sigma}\partial_{xx} p = 0. \tag{12}$$

A useful variable will be the scaled gradient

$$v = -\frac{1}{\sigma}\partial_x p. \tag{13}$$

2.1 Notations

We denote $x_{j+1/2}$ the nodes, the cells j are the intervals $[x_{j-1/2}, x_{j+1/2}]$, thus $\Delta x_j = x_{j+1/2} - x_{j-1/2}$, x_j is the center of the cell j that is $x_j = \frac{1}{2}(x_{j+1/2} + x_{j-1/2})$, and $\Delta x_{j+1/2} = x_{j+1} - x_j = \frac{1}{2}(\Delta x_{j+1} + \Delta x_j)$. Natural assumptions on the mesh are summarized below:

Hypothesis 2.1 (Regularity of the mesh in 1D)**.** *We consider there exists a universal constant $C > 0$ independent of the mesh size $h = \sup_{j\in\mathbb{Z}} \Delta x_j$ so that one has*

$$Ch \le \Delta x_j \le h \qquad \forall j \in \mathbb{Z}.$$

The semi-discrete JL(b) scheme, derived in [5] in 2D, can also be written in 1D on irregular meshes as

$$P_h^\varepsilon : \qquad \begin{cases} \dfrac{d}{dt}p_j^\varepsilon + \dfrac{u_{j+\frac{1}{2}}^\varepsilon - u_{j-\frac{1}{2}}^\varepsilon}{\varepsilon \Delta x_j} = 0, \\ \dfrac{d}{dt}u_j^\varepsilon + \dfrac{p_{j+\frac{1}{2}}^\varepsilon - p_{j-\frac{1}{2}}^\varepsilon}{\varepsilon \Delta x_j} = -\dfrac{\sigma}{\varepsilon^2}\dfrac{u_{j+\frac{1}{2}}^\varepsilon + u_{j-\frac{1}{2}}^\varepsilon}{2}, \end{cases} \tag{14}$$

with the fluxes $p_{j+\frac{1}{2}}^\varepsilon$ and $u_{j+\frac{1}{2}}^\varepsilon$ are the solutions of the well-posed linear system

$$j \in \mathbb{Z} : \qquad \begin{cases} p_{j+\frac{1}{2}}^\varepsilon + u_{j+\frac{1}{2}}^\varepsilon + \dfrac{\sigma \Delta x_j}{2\varepsilon}u_{j+\frac{1}{2}}^\varepsilon & = p_j^\varepsilon + u_j^\varepsilon, \\ -p_{j+\frac{1}{2}}^\varepsilon + u_{j+\frac{1}{2}}^\varepsilon + \dfrac{\sigma \Delta x_{j+1}}{2\varepsilon}u_{j+\frac{1}{2}}^\varepsilon & = -p_{j+1}^\varepsilon + u_{j+1}^\varepsilon. \end{cases} \tag{15}$$

This scheme is the same as the Gosse-Toscani scheme[1]. Other equivalent forms of P_h^ε can be obtained by various manipulations, as in (25). The natural cellwise initialization is chosen

$$p_j^\varepsilon(0) = p_0(x_j) \text{ and } u_j^\varepsilon(0) = -\frac{\varepsilon}{\sigma}\partial_x p_0(x_j) \text{ for all } j \in \mathbb{Z}. \tag{16}$$

Our goal is this section is to show that this scheme is AP.

When ε tends to 0, the scheme P_h^ε admits the diffusion limit scheme P_h^0

$$P_h^0: \qquad \Delta x_j \frac{d}{dt}p_j - \frac{1}{\sigma}\left(\frac{p_{j+1}-p_j}{\Delta x_{j+\frac12}} - \frac{p_j - p_{j-1}}{\Delta x_{j-\frac12}}\right) = 0 \tag{17}$$

The natural cellwise initialization is

$$p_j(0) = p_0(x_j) \text{ for all } j \in \mathbb{Z}. \tag{18}$$

Other quantities are

$$\begin{cases} v_{j+\frac12} = -\dfrac{1}{\sigma}\dfrac{p_{j+1}-p_j}{\Delta x_{j+\frac12}}, \\ v_j = \dfrac{v_{j+\frac12}+v_{j-\frac12}}{2}. \end{cases} \tag{19}$$

Provided that the initial data is smooth, the initialization gives us

$$\|u_j^\varepsilon(0) - \varepsilon v_j(0)\|_{L^2(\mathbb{R})} \le Ch\varepsilon. \tag{20}$$

We denote by $\mathbf{V}^\varepsilon(t) = (p^\varepsilon(x_j,t), u^\varepsilon(x_j,t))_{j\in\mathbb{Z}}$ the interpolation of the solution of the hyperbolic heat equations P^ε and by $\mathbf{V}_h^\varepsilon(t) = \left(p_j^\varepsilon(t), u_j^\varepsilon(t)\right)_{j\in\mathbb{Z}}$ the solution of the JL-(b) scheme P_h^ε. Similarly we reconstruct similar quantities from the diffusion scheme: it yields $\mathbf{W}^\varepsilon(t) = (p(x_j,t), \varepsilon v(x_j,t))_{j\in\mathbb{Z}}$ which is the interpolation of the solution of the diffusion limit P^0 (12)-(13), and $\mathbf{W}_h^\varepsilon(t) = (p_j(t), \varepsilon v_j(t))_{j\in\mathbb{Z}}$ which is the solution of the diffusion scheme P_h^0 (17)-(19). For simplicity we choose a final time $T > 0$. All error estimates will be given for $t \le T$, either in the norm $\|f(t)\|_{L^\infty([0,T];L^2(\mathbb{R}))}$, or mostly in the norm $\|f\|_{L^2([0,T]\times\mathbb{R})}$

[1] A long and tedious computation shows that the scheme is strictly equivalent to the Gosse-Toscani's scheme, described in [14] but only for uniform meshes, which writes in terms of $w^\varepsilon, v^\varepsilon = p^\varepsilon \pm u^\varepsilon$

$$\begin{cases} \dfrac{dw_j}{dt} + \dfrac{M_{j-\frac12}}{\varepsilon}\dfrac{w_j^\varepsilon - w_{j-1}^\varepsilon}{\Delta x_j} = \dfrac{1}{\varepsilon\Delta x_j}(1-M_{j-\frac12})(v_j^\varepsilon - w_j^\varepsilon) = M_{j-\frac12}\dfrac{\Delta x_{j-\frac12}}{\Delta x_j}\dfrac{\sigma}{2\varepsilon^2}(v_j^\varepsilon - w_j^\varepsilon), \\ \dfrac{dv_j^\varepsilon}{dt} - \dfrac{M_{j+\frac12}}{\varepsilon}\dfrac{v_{j+1}^\varepsilon - v_j^\varepsilon}{\Delta x_j} = \dfrac{1}{\varepsilon\Delta x_j}(1-M_{j+\frac12})(w_j^\varepsilon - v_j^\varepsilon) = M_{j+\frac12}\dfrac{\Delta x_{j+\frac12}}{\Delta x_j}\dfrac{\sigma}{2\varepsilon^2}(w_j^\varepsilon - v_j^\varepsilon) \end{cases}$$

with $M_{j+\frac12} = \frac{2\varepsilon}{\sigma\Delta x_{j+\frac12}+2\varepsilon}$and $\Delta x_{j+\frac12} = \frac{\Delta x_j + \Delta x_{j+1}}{2}$. By writing

$$\begin{cases} M_{j-\frac12}(w_{j-1}^\varepsilon - w_j^\varepsilon) = M_{j-\frac12}w_{j-1} - M_{j+\frac12}w_j + (M_{j+\frac12} - M_{j-\frac12})w_j^\varepsilon \\ M_{j+\frac12}(v_{j+1}^\varepsilon - v_j^\varepsilon) = M_{j+\frac12}v_{j+1}^\varepsilon - M_{j-\frac12}v_j^\varepsilon - (M_{j+\frac12} - M_{j-\frac12})v_j^\varepsilon \end{cases}$$

then in terms of p^ε an u^ε we have evidently

$$\begin{cases} \dfrac{dp_j^\varepsilon}{dt} + \dfrac{1}{\varepsilon}\dfrac{M_{j+\frac12}u_{j+\frac12}^\varepsilon - M_{j-\frac12}u_{j-\frac12}^\varepsilon}{\Delta x_j} = 0, \\ \dfrac{du_j^\varepsilon}{dt} + \dfrac{1}{\varepsilon}\dfrac{M_{j+\frac12}p_{j+\frac12}^\varepsilon - M_{j-\frac12}p_{j-\frac12}^\varepsilon}{\Delta x_j} = -\dfrac12\left(M_{j+\frac12}\dfrac{\Delta x_{j+\frac12}}{\Delta x_j}\dfrac{\sigma}{\varepsilon^2} + M_{j-\frac12}\dfrac{\Delta x_{j-\frac12}}{\Delta x_j}\dfrac{\sigma}{\varepsilon^2}\right)u_j^\varepsilon + \dfrac{M_{j+\frac12} - M_{j-\frac12}}{\varepsilon\Delta x_j}p_j^\varepsilon \end{cases}$$

with the fluxes given by $p_{j+\frac12}^\varepsilon = \frac{p_j^\varepsilon + p_{j+1}^\varepsilon}{2} + \frac{u_j^\varepsilon - u_{j+1}^\varepsilon}{2}$ and $u_{j+\frac12} = \dfrac{u_j^\varepsilon + u_{j+1}^\varepsilon}{2} + \dfrac{p_j^\varepsilon - p_{j+1}^\varepsilon}{2}$.

2.2 Study of $\|P^\varepsilon - P^0\|$

In this section we prove a natural error estimate [13] between the solution of the hyperbolic heat equations (11) and the solution of the diffusion limit equation (12).

Proposition 2.2. *There exists a constant $C>0$ such that*

$$\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}))} \le C\varepsilon \|\partial_{xxx} p_0\|_{L^2(\mathbb{R})}.$$

Proof. We define $u^\varepsilon = -\frac{\varepsilon}{\sigma}\partial_x p^\varepsilon$ and introduce R^ε such that the solution of the diffusion equation satisfies

$$\left\{ \begin{array}{l} \partial_t p^\varepsilon + \frac{1}{\varepsilon}\partial_x u^\varepsilon = 0, \\ \partial_t u^\varepsilon + \frac{1}{\varepsilon}\partial_x p^\varepsilon + \frac{\sigma}{\varepsilon^2} u^\varepsilon = R^\varepsilon \end{array} \right. \tag{21}$$

where $R^\varepsilon = \partial_t u^\varepsilon = -\frac{\varepsilon}{\sigma}\partial_{tx} p^\varepsilon = -\frac{\varepsilon}{\sigma^2}\partial_{xxx} p^\varepsilon$. Note that $\|R^\varepsilon(t)\|_{L^2(\mathbb{R})} \le \|R^\varepsilon(0)\|_{L^2(\mathbb{R})}$. Denoting $e^\varepsilon = p - p^\varepsilon$, $f^\varepsilon = u - u^\varepsilon$, we make the difference between the systems (11) et (21)

$$\left\{ \begin{array}{l} \partial_t e^\varepsilon + \frac{1}{\varepsilon}\partial_x f^\varepsilon = 0, \\ \partial_t f^\varepsilon + \frac{1}{\varepsilon}\partial_x e^\varepsilon + \frac{\sigma}{\varepsilon^2} f^\varepsilon = R^\varepsilon. \end{array} \right. \tag{22}$$

Since data are well-prepared, one has $e^\varepsilon(0) = f^\varepsilon(0) = 0$. Consider $\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|^2_{L^2(\mathbb{R})} = \|e^\varepsilon\|^2_{L^2(\mathbb{R})} + \|f^\varepsilon\|^2_{L^2(\mathbb{R})}$. Adding the first equation of (22) multiplied by e^ε and the second multiplied by f^ε and integrating on $\mathbb{R}$, we find out that: $\frac{1}{2}\frac{d}{dt}\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|^2_{L^2(\mathbb{R})} \le \int_{\mathbb{R}} R^\varepsilon f^\varepsilon dx \le \|R^\varepsilon\|_{L^2(\mathbb{R})}\|\mathbf{V}^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2(\mathbb{R})}$. The proof is ended by integration between 0 and T.

□

2.3 Stability estimates for P_h^ε and P_h^0

These estimates characterize the dissipation rate of both schemes.

Proposition 2.3. *The scheme P_h^ε is stable in L^2 norm. Moreover,*

$$\sqrt{\int_0^T \left(\sum \Delta x_{j+\frac{1}{2}} (u^\varepsilon_{j+\frac{1}{2}})^2\right) dt} \le \frac{\varepsilon}{\sqrt{\sigma}} \|\mathbf{V}_h^\varepsilon(0)\|_{L^2(\mathbb{R})} \tag{23}$$

and

$$\sqrt{\int_0^T \left(\sum_{j\in\mathbb{Z}} (u^\varepsilon_{j+\frac{1}{2}} - u^\varepsilon_j)^2 + \sum_{j\in\mathbb{Z}} (u^\varepsilon_{j-\frac{1}{2}} - u^\varepsilon_j)^2\right) dt} \le \sqrt{\varepsilon}\|\mathbf{V}_h^\varepsilon(0)\|_{L^2(\mathbb{R})}. \tag{24}$$

Remark 2.4. *The strategy of the proof of many estimates in this work consists to analyze the balance between the dissipation of the fluxes and the physical dissipation (all source terms like $-\frac{\sigma}{\varepsilon^2}u$) on the one hand, and some truncation errors on the other hand. This is why it is convenient to reformulate P_h^ε so that the pressure fluxes $p^\varepsilon_{j+\frac{1}{2}}$ and $p^\varepsilon_{j-\frac{1}{2}}$ are eliminated in the second equation of (14). This elimination is technically convenient since all dissipation terms are expressed using the same variable, namely u. It will simplify a lot the comparisons between all kinds of dissipation terms and other errors terms.*

Proof. According to the above remark we obtain the formulation (25) which is equivalent to P_h^ε

$$\left\{ \begin{array}{l} \Delta x_j \dfrac{d}{dt} p^\varepsilon_j + \dfrac{u^\varepsilon_{j+\frac{1}{2}} - u^\varepsilon_{j-\frac{1}{2}}}{\varepsilon} = 0, \\ \Delta x_j \dfrac{d}{dt} u^\varepsilon_j - \dfrac{u^\varepsilon_{j+\frac{1}{2}} + u^\varepsilon_{j-\frac{1}{2}}}{\varepsilon} + \dfrac{2}{\varepsilon} u^\varepsilon_j = 0, \\ \left(2 + \dfrac{\sigma \Delta x_{j+\frac{1}{2}}}{\varepsilon}\right) u^\varepsilon_{j+\frac{1}{2}} = p^\varepsilon_j - p^\varepsilon_{j+1} + u^\varepsilon_j + u^\varepsilon_{j+1}. \end{array} \right. \tag{25}$$

Consider now the discrete quadratic energy $E(t) = \frac{1}{2}\sum_j \Delta x_j((p_j^\varepsilon)^2 + (u_j^\varepsilon)^2)$. Multiplying the first equation of (25) by p_j^ε and the second equation by u_j^ε and adding on all the cells, one finds

$$E'(t) = -\sum_{j\in\mathbb{Z}} \frac{u^\varepsilon_{j+\frac12} - u^\varepsilon_{j-\frac12}}{\varepsilon} p_j^\varepsilon + \sum_{j\in\mathbb{Z}} \frac{u^\varepsilon_{j+\frac12} + u^\varepsilon_{j-\frac12}}{\varepsilon} u_j^\varepsilon - \frac{2}{\varepsilon}\sum_j (u_j^\varepsilon)^2.$$

Since $\sum_j (u^\varepsilon_{j+\frac12} - u^\varepsilon_{j-\frac12})p_j^\varepsilon = \sum_j u^\varepsilon_{j+\frac12}(p_j^\varepsilon - p^\varepsilon_{j+1})$, one has by using the third equation of (25) and rearranging the terms

$$E'(t) + \sum_{j\in\mathbb{Z}} \frac{(u^\varepsilon_{j+\frac12} - u_j^\varepsilon)^2}{\varepsilon} + \sum_{j\in\mathbb{Z}} \frac{(u^\varepsilon_{j-\frac12} - u_j^\varepsilon)^2}{\varepsilon} + \frac{\sigma}{\varepsilon^2}\sum_{j\in\mathbb{Z}} \Delta x_j \frac{(u^\varepsilon_{j+\frac12})^2 + (u^\varepsilon_{j-\frac12})^2}{2} = 0. \tag{26}$$

Integrating (26) between 0 and t, one finds $E(t) \le E(0)$, that is the L^2 stability of P_h^ε. The estimate (23) comes from $\Delta x_{j+\frac12} = \frac12(\Delta x_j + \Delta x_{j+1})$. The estimate (24) is directly deduced from (26). $\square$

Some similar bounds hold for the quantities relatives to the diffusion scheme (17). First, multiplying the diffusion scheme by p_j and adding on all the cells, one has the L^2 stability in the sense

$$\frac12 \frac{d}{dt}\sum_j \Delta x_j p_j^2 = -\frac{1}{\sigma}\sum_j \frac{(p_{j+1} - p_j)^2}{\Delta x_{j+\frac12}}.$$

Thus the following estimate holds for the function $\bar v_h = \left(v_{j+\frac12}\right)_j$ defined by (19)

$$\|\bar v_h\|_{L^2([0,T]\times\mathbb{R})} = \sqrt{\int_0^T \sum_j \Delta x_{j+\frac12}(v_{j+\frac12})^2} \le C\|p_0\|_{L^2(\mathbb{R})}, \qquad C > 0. \tag{27}$$

2.4 Study of $\|P_h^\varepsilon - P^\varepsilon\|_{\text{naive}}$

In this section we prove the convergence of P_h^ε to P^ε. We still denote $V^\varepsilon(t) = (p^\varepsilon, u^\varepsilon)$.

Proposition 2.5. *There exist a constant $C(T) > 0$ independent of ε and h such that the following estimate holds*

$$\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \le C(T)\sqrt{\frac{h}{\varepsilon}}\,\|\mathbf{V}^\varepsilon(0)\|_{H^1(\mathbb{R})}.$$

Proof. In order to establish this estimate, we will use the method introduced by C. Mazeran [22] in his PhD thesis. It starts with an estimate for the time derivative of $\mathscr{E} = \frac12\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|^2_{L^2(\mathbb{R})}$. For the sake of simplicity,for any quantity q, q' stands indifferently for $\frac{d}{dt}q$ or $\partial_t q$. One has

$$\mathscr{E}'(t) = \underbrace{\frac12\int_{\mathbb{R}}((p_h^\varepsilon)^2 + (u_h^\varepsilon)^2)'dx}_{D_1} + \underbrace{\frac12\int_{\mathbb{R}}((p^\varepsilon)^2 + (u^\varepsilon)^2)'dx}_{D_2}$$
$$+ \underbrace{\int_{\mathbb{R}}(-(p_h^\varepsilon)'p^\varepsilon - (u_h^\varepsilon)'u^\varepsilon)dx}_{D_3} + \underbrace{\int_{\mathbb{R}}(-p_h^\varepsilon(p^\varepsilon)' - u_h^\varepsilon(u^\varepsilon)'dx}_{D_4}$$

We will successively estimate each of those terms, the fundamental idea being that $D_1 \le 0$ and $D_2 \le 0$ are used to control spurious contributions in D_3 and D_4. First D_1 corresponds to the entropy production of the scheme. Thanks to the proof of the proposition 2.3, one has

$$D_1 = -\frac{1}{\varepsilon}\sum_{j\in\mathbb{Z}}(u^\varepsilon_{j+\frac12} - u_j^\varepsilon)^2 - \frac{1}{\varepsilon}\sum_{j\in\mathbb{Z}}(u^\varepsilon_{j-\frac12} - u_j^\varepsilon)^2 - \frac{\sigma}{\varepsilon^2}\sum_{j\in\mathbb{Z}}\Delta x_j \frac{(u^\varepsilon_{j+\frac12})^2 + (u^\varepsilon_{j-\frac12})^2}{2} \le 0.$$

One also directly obtains

$$D_2 = -\sum_{j\in\mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2}\left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} (u^\varepsilon)^2 dx\right) \le 0.$$

For D_4, one gets directly

$$D_4 \;=\; \sum_{j\in\mathbb{Z}} p_j^\varepsilon \frac{u^\varepsilon(x_{j+\frac12}) - u^\varepsilon(x_{j-\frac12})}{\varepsilon} + \sum_{j\in\mathbb{Z}} u_j^\varepsilon \frac{p^\varepsilon(x_{j+\frac12}) - p^\varepsilon(x_{j-\frac12})}{\varepsilon} + \sum_{j\in\mathbb{Z}} \frac{\sigma}{\varepsilon^2} u_j^\varepsilon \int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx$$

In this method the third term D_3 is more complicated to study

$$D_3 = \textstyle\sum_{j\in\mathbb{Z}} \frac{u^\varepsilon_{j+\frac12} - u^\varepsilon_{j-\frac12}}{\varepsilon}\left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} p^\varepsilon(x)dx\right) + \sum_{j\in\mathbb{Z}} \frac{p^\varepsilon_{j+\frac12} - p^\varepsilon_{j-\frac12}}{\varepsilon}\left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right)$$
$$+\textstyle\sum_{j\in\mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2} \frac{u^\varepsilon_{j+\frac12} + u^\varepsilon_{j-\frac12}}{2}\left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right).$$

It is decomposed in several pieces. We add and subtract in each fluxes the value of the unknowns in the cell. We also add and subtract to the two first integrals the value of the unknowns on the edge. Denoting by $\delta_j^\pm(g) = \frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} g(x)dx - g(x_{j\pm\frac12})$, one gets after rearrangements

$$\begin{aligned}
D_3 \;=\;& \sum_{j\in\mathbb{Z}} \tfrac{u^\varepsilon_{j+\frac12} - u^\varepsilon_j}{\varepsilon}\delta_j^+(p^\varepsilon) + \sum_{j\in\mathbb{Z}} \tfrac{u^\varepsilon_j - u^\varepsilon_{j-\frac12}}{\varepsilon}\delta_j^-(p^\varepsilon)\\
+& \sum_{j\in\mathbb{Z}} \tfrac{p^\varepsilon_{j+\frac12} - p^\varepsilon_j}{\varepsilon}\delta_j^+(u^\varepsilon) + \sum_{j\in\mathbb{Z}} \tfrac{p^\varepsilon_j - p^\varepsilon_{j-\frac12}}{\varepsilon}\delta_j^-(u^\varepsilon)\\
-& \sum_{j\in\mathbb{Z}} \tfrac{u^\varepsilon(x_{j+\frac12}) - u^\varepsilon(x_{j-\frac12})}{\varepsilon} p_j^\varepsilon - \sum_{j\in\mathbb{Z}} \tfrac{p^\varepsilon(x_{j+\frac12}) - p^\varepsilon(x_{j-\frac12})}{\varepsilon} u_j^\varepsilon\\
+& \sum_{j\in\mathbb{Z}} \Delta x_j \tfrac{\sigma}{\varepsilon^2}\tfrac{u^\varepsilon_{j+\frac12} + u^\varepsilon_{j-\frac12}}{2}\left(\tfrac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right)
\end{aligned}$$

Using the fluxes' definition (15), one can eliminate the pressure fluxes. With a Young's inequality $ab \le \alpha a^2 + \frac{1}{4\alpha}b^2$ where $\alpha > 0$, one gets

$$\sum_{j\in\mathbb{Z}} \frac{p^\varepsilon_{j+\frac12} - p^\varepsilon_j}{\varepsilon}\delta_j^+(u^\varepsilon) = \sum_{j\in\mathbb{Z}} \frac{1}{\varepsilon}(u_j^\varepsilon - u^\varepsilon_{j+\frac12})\delta_j^+(u^\varepsilon) - \frac{\sigma}{2\varepsilon^2}\sum_{j\in\mathbb{Z}} \Delta x_j u^\varepsilon_{j+\frac12}\delta_j^+(u^\varepsilon)$$

$$\le \alpha \sum_{j\in\mathbb{Z}} \frac{(u^\varepsilon_{j+\frac12} - u^\varepsilon_j)^2}{\varepsilon} + \left(\frac{1}{4\alpha\varepsilon} + \frac{\sigma}{2\varepsilon^2}\right)\sum_{j\in\mathbb{Z}} \delta_j^+(u^\varepsilon)^2 + \frac{\sigma}{8\varepsilon^2}\sum_{j\in\mathbb{Z}} \Delta x_j^2 \left(u^\varepsilon_{j+\frac12}\right)^2.$$

Using this expression in D_3 and using again Young's inequality, one gets for arbitrary $\alpha > 0$

$$\begin{aligned}
D_3 \;\le\;& \alpha \sum_{j\in\mathbb{Z}} \frac{(u^\varepsilon_{j+\frac12} - u^\varepsilon_j)^2}{\varepsilon} + \alpha \sum_{j\in\mathbb{Z}} \frac{(u^\varepsilon_{j-\frac12} - u^\varepsilon_j)^2}{\varepsilon}\\
+& \sum_{j\in\mathbb{Z}} \left(\left(\frac{1}{2\alpha\varepsilon} + \frac{\sigma}{2\varepsilon^2}\right)\left(\delta_j^+(u^\varepsilon)^2 + \delta_j^-(u^\varepsilon)^2\right) + \frac{\delta_j^+(p^\varepsilon)^2 + \delta_j^-(p^\varepsilon)^2}{2\varepsilon\alpha}\right)\\
+& \sum_{j\in\mathbb{Z}} \frac{1}{8\varepsilon}\sigma\Delta x_j^2 \frac{(u^\varepsilon_{j-\frac12})^2 + (u^\varepsilon_{j+\frac12})^2}{\varepsilon} + \sum_{j\in\mathbb{Z}} \Delta x_j \frac{\sigma}{\varepsilon^2}\frac{u^\varepsilon_{j+\frac12} + u^\varepsilon_{j-\frac12}}{2}\left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}} u^\varepsilon(x)dx\right)\\
-& \sum_{j\in\mathbb{Z}} u_j^\varepsilon \frac{p^\varepsilon(x_{j+\frac12}) - p^\varepsilon(x_{j-\frac12})}{\varepsilon} - \sum_{j\in\mathbb{Z}} p_j^\varepsilon \frac{u^\varepsilon(x_{j+\frac12}) - u^\varepsilon(x_{j-\frac12})}{\varepsilon}
\end{aligned}$$

We now sum all bounds contributing to $\mathscr{E}'(t)$ and we get:

$$\begin{aligned}
\mathscr{E}'(t) \quad &\leq \quad (-1+\alpha)\sum_{j\in\mathbb{Z}} \frac{(u^\varepsilon_{j+\frac12}-u^\varepsilon_j)^2+(u^\varepsilon_{j-\frac12}-u^\varepsilon_j)^2}{\varepsilon} \\
&+ \quad \sum_{j\in\mathbb{Z}}\left(\left(\frac{1}{2\alpha\varepsilon}+\frac{\sigma}{2\varepsilon^2}\right)\left(\delta_j^+(u^\varepsilon)^2+\delta_j^-(u^\varepsilon)^2\right)+\frac{\delta_j^+(p^\varepsilon)^2+\delta_j^-(p^\varepsilon)^2}{2\varepsilon\alpha}\right) \\
&+ \quad \sum_{j\in\mathbb{Z}}\frac{1}{8\varepsilon}\sigma\Delta x_j^2\frac{(u^\varepsilon_{j-\frac12})^2+(u^\varepsilon_{j+\frac12})^2}{\varepsilon} \\
&+ \quad \sum_{j\in\mathbb{Z}}\Delta x_j\frac{\sigma}{\varepsilon^2}\frac{u^\varepsilon_{j+\frac12}+u^\varepsilon_{j-\frac12}}{2}\left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}}u^\varepsilon(x)dx\right)-\sum_{j\in\mathbb{Z}}\Delta x_j\frac{\sigma}{2\varepsilon}\frac{(u^\varepsilon_{j-\frac12})^2+(u^\varepsilon_{j+\frac12})^2}{\varepsilon} \\
&+ \quad \sum_{j\in\mathbb{Z}}\Delta x_j\frac{\sigma}{\varepsilon^2}u_j^\varepsilon\left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}}u^\varepsilon(x)dx\right)-\sum_{j\in\mathbb{Z}}\Delta x_j\frac{\sigma}{\varepsilon^2}\left(\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}}(u^\varepsilon)^2(x)dx\right).
\end{aligned}$$

We now examine the sum of all terms in the two last lines of the RHS of the above inequality , which we denote S. One finds

$$\begin{aligned}
S = -\sum_{j\in\mathbb{Z}}\Delta x_j\frac{\sigma}{2\varepsilon^2}&\left[\left(u^\varepsilon_{j-\frac12}-\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}}u^\varepsilon(x)dx\right)^2+\left(u^\varepsilon_{j+\frac12}-\frac{1}{\Delta x_j}\int_{x_{j-\frac12}}^{x_{j+\frac12}}u^\varepsilon(x)dx\right)^2\right] \\
&+\frac{\sigma}{2\varepsilon^2}\sum_{j\in\mathbb{Z}}\left(\int_{x_{j-\frac12}}^{x_{j+\frac12}}u^\varepsilon(x)dx\right)\left(u_j^\varepsilon-u^\varepsilon_{j+\frac12}+u^\varepsilon_j-u^\varepsilon_{j-\frac12}\right) \\
&\leq \frac{\sigma}{2\varepsilon^2}\sum_{j\in\mathbb{Z}}\left(\int_{x_{j-\frac12}}^{x_{j+\frac12}}u^\varepsilon(x)dx\right)\left(u_j^\varepsilon-u^\varepsilon_{j+\frac12}+u^\varepsilon_j-u^\varepsilon_{j-\frac12}\right).
\end{aligned}$$

Using another Young's inequality, one has for all $\widehat{\alpha}>0$

$$S \leq \frac{\sigma^2}{8\widehat{\alpha}\varepsilon^3}\sum_{j\in\mathbb{Z}}\left(\int_{x_{j-\frac12}}^{x_{j+\frac12}}u^\varepsilon(x)dx\right)^2+\widehat{\alpha}\sum_{j\in\mathbb{Z}}\frac{(u^\varepsilon_j-u^\varepsilon_{j+\frac12})^2+(u^\varepsilon_j-u^\varepsilon_{j-\frac12})^2}{\varepsilon}.$$

For example by choosing $\alpha=\frac12$ and $\widehat{\alpha}=\frac12$, and coming back to $\mathscr{E}'(t)$ we get we obtain

$$\begin{aligned}
\mathscr{E}'(t) \quad &\leq \quad \sum_{j\in\mathbb{Z}}\left(\left(\frac{1}{\varepsilon}+\frac{\sigma}{2\varepsilon^2}\right)\left(\delta_j^+(u^\varepsilon)^2+\delta_j^-(u^\varepsilon)^2\right)+\frac{1}{\varepsilon}\left(\delta_j^+(p^\varepsilon)^2+\delta_j^-(p^\varepsilon)^2\right)\right) &(28)\\
&+ \quad \sum_{j\in\mathbb{Z}}\frac{1}{8\varepsilon}\sigma\Delta x_j^2\frac{(u^\varepsilon_{j-\frac12})^2+(u^\varepsilon_{j+\frac12})^2}{\varepsilon}+\frac{\sigma^2}{4\varepsilon^3}\sum_{j\in\mathbb{Z}}\left(\int_{x_{j-\frac12}}^{x_{j+\frac12}}u^\varepsilon(x)dx\right)^2. &(29)
\end{aligned}$$

To estimate the contributions on the first line we use the following fact: for any quantity q, one can use $q(x_{j-\frac12})=q(x)+\int_x^{x_{j-\frac12}}\frac{d}{ds}q(s)ds$ and integrate this expression in the cell Δx_j; we get $\sum_{j\in\mathbb{Z}}\delta_j^\pm(q)^2\leq h\|q\|^2_{H^1(\mathbb{R})}$. Therefore the first terms on the right hand side of (28) can be estimated as

$$\left(\frac{1}{\varepsilon}+\frac{\sigma}{2\varepsilon^2}\right)\int_0^t\sum_{j\in\mathbb{Z}}\left(\delta_j^+(u^\varepsilon)^2+\delta_j^-(u^\varepsilon)^2\right)dt\leq h\left(\frac{1}{\varepsilon}+\frac{\sigma}{2\varepsilon^2}\right)\|\mathbf{u}^\varepsilon\|^2_{L^2([0,t]:H^1(\mathbb{R})}.$$

Since $\|\mathbf{u}^\varepsilon\|^2_{L^2([0,t]:H^1(\mathbb{R})}\leq t\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\mathbb{R})}$ and also $\frac{\sigma}{\varepsilon^2}\|\mathbf{u}^\varepsilon\|^2_{L^2([0,t]:H^1(\mathbb{R})}\leq\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\mathbb{R})}$ by (2) and (3), one gets that

$$\left(\frac{1}{\varepsilon}+\frac{\sigma}{2\varepsilon^2}\right)\int_0^t\sum_{j\in\mathbb{Z}}\left(\delta_j^+(u^\varepsilon)^2+\delta_j^-(u^\varepsilon)^2\right)dt\leq h\left(\frac{t}{\varepsilon}+\frac{1}{2}\right)\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\mathbb{R})}. \qquad (30)$$

A similar and simpler formula for the next terms is

$$\frac{1}{\varepsilon}\int_0^t \sum_{j\in\mathbb{Z}} \left(\delta_j^+(p^\varepsilon)^2 + \delta_j^-(p^\varepsilon)^2\right) \le \frac{ht}{\varepsilon}\|p^\varepsilon\|^2_{H^1(\mathbb{R})} \le \frac{ht}{\varepsilon}\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\mathbb{R})}. \tag{31}$$

Next, using the assumption (2.1) on the mesh and estimate (23), one can find a constant $C \ge 0$ such that the next term can be bounded like

$$\int_0^T \sum_{j\in\mathbb{Z}} \frac{1}{8\varepsilon}\sigma\Delta x_j^2 \frac{(u^\varepsilon_{j-\frac{1}{2}})^2 + (u^\varepsilon_{j+\frac{1}{2}})^2}{\varepsilon} \le Ch\|\mathbf{V}^\varepsilon(0)\|^2_{L^2(\mathbb{R})}. \tag{32}$$

Finally the last term in (28) can be bounded using a Cauchy-Schwarz inequality

$$\frac{\sigma^2}{4\varepsilon^3}\sum_{j\in\mathbb{Z}}\left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x)dx\right)^2 \le \frac{\sigma^2}{4\varepsilon^3}h\|\mathbf{u}^\varepsilon\|^2_{L^2(\mathbb{R})}$$

so that

$$\frac{\sigma^2}{4\varepsilon^3}\int_0^t\sum_{j\in\mathbb{Z}}\left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^\varepsilon(x)dx\right)^2 \le \frac{\sigma^2}{4\varepsilon^3}h\|\mathbf{u}^\varepsilon\|^2_{L^2([0,t]\times\mathbb{R})} \le \frac{\sigma}{4\varepsilon}h\|\mathbf{V}^\varepsilon(0)\|^2_{L^2(\mathbb{R})} \tag{33}$$

by means of the energy identity. So using (30-33) we obtain for all time $t \le T$:

$$\mathscr{E}(t) \le \mathscr{E}(0) + h\left(\frac{t}{\varepsilon} + \frac{1}{2} + \frac{t}{\varepsilon} + C + \frac{\sigma}{4\varepsilon}\right)\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\mathbb{R})}.$$

The initialization stage is consistent so $\mathscr{E}(0) \le Ch^2\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\mathbb{R})}$. Making use of the initial remark 1.2, one obtains

$$\|\mathbf{V}^\varepsilon - \mathbf{V}^\varepsilon_h\|_{L^2([0,T]\times\mathbb{R})} \le C(T)\sqrt{\frac{h}{\varepsilon}}\|\mathbf{V}^\varepsilon(0)\|_{H^1(\mathbb{R})}, \qquad C(T) \ge 0.$$

The proof is ended. □

2.5 Study of $\|P_h^0 - P^0\|$

We first recall a fundamental error estimate [12] for the diffusion limit scheme, assuming smooth initial data. (17).

Proposition 2.6. *There exists a constant $C(T) > 0$ such that*

$$\|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}))} \le C(T)h\|\partial_{xxx}p_0\|_{L^2(\mathbb{R})}.$$

Proof. We use the method of Gallouet and al [12], which is based on a notion of consistency for finite volumes schemes. We set

$$s_j = \partial_{xx}p(x_j) - \frac{\partial_x p(x_{j+\frac{1}{2}}) - \partial_x p(x_{j-\frac{1}{2}})}{\Delta x_j} \text{ and } r_{j+\frac{1}{2}} = \partial_x p(x_{j+\frac{1}{2}}) - \frac{p(x_{j+1}) - p(x_j)}{\Delta x_{j+\frac{1}{2}}},$$

so that one has the identity

$$\frac{d}{dt}p(x_j) - \frac{1}{\sigma\Delta x_j}\left(\frac{p(x_{j+1}) - p(x_j)}{\Delta x_{j+\frac{1}{2}}} - \frac{p(x_j) - p(x_{j-1})}{\Delta x_{j-\frac{1}{2}}}\right) = \frac{s_j}{\sigma} + \frac{r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}}}{\sigma\Delta x_j}.$$

We next introduce the difference $e_j = p(x_j) - p_j$ which satisfies

$$\frac{d}{dt}e_j - \frac{1}{\sigma\Delta x_j}\left(\frac{e_{j+1} - e_j}{\Delta x_{j+\frac{1}{2}}} - \frac{e_j - e_{j-1}}{\Delta x_{j-\frac{1}{2}}}\right) = \frac{s_j}{\sigma} + \frac{r_{j+\frac{1}{2}} - r_{j-\frac{1}{2}}}{\sigma\Delta x_j}$$

with $e_j(0)=0$ for all j. By multiplying this equation by e_j and denoting by $\|e_h\|^2_{L^2(\mathbb{R})}=\sum_j \Delta x_j e_j^2$, one finds that

$$\frac{1}{2}\frac{d}{dt}\|e_h\|^2_{L^2(\mathbb{R})}+\frac{1}{\sigma}\sum_j \frac{(e_{j+1}-e_j)^2}{\Delta x_{j+\frac12}}=\sigma^{-1}\sum_j \Delta x_j s_j e_j+\sigma^{-1}\sum_j r_{j+\frac12}(e_j-e_{j+1}).$$

The Cauchy-Schwarz inequality yields

$$\sum_j r_{j+\frac12}(e_j-e_{j+1})\le \frac12\sum_j \frac{(e_{j+1}-e_j)^2}{\Delta x_{j+\frac12}}+\frac12\sum_j \Delta x_{j+\frac12} r^2_{j+\frac12}.$$

One finds out with natural notations

$$\frac{1}{2}\frac{d}{dt}\|e_h\|^2_{L^2(\mathbb{R})}+\frac{1}{2\sigma}\sum_j \frac{(e_{j+1}-e_j)^2}{\Delta x_{j+\frac12}}\le \sigma^{-1}\|s_h\|_{L^2(\mathbb{R})}\|e_h\|_{L^2(\mathbb{R})}+\frac12\|r_h\|^2_{L^2(\mathbb{R})}. \tag{34}$$

Using the definitions of the truncation errors s_h and r_h, these quantites can be bounded independently of the time of observation T

$$\|s_h\|_{L^2([0,T]\times\mathbb{R})}+\|r_h\|_{L^2([0,T]\times\mathbb{R})}\le Ch\Big(\|\partial_{xxx}p_0\|_{L^2(\mathbb{R})}+\|\partial_{xx}p_0\|_{L^2(\mathbb{R})}\Big). \tag{35}$$

Rescaling for convenience $\widehat{s}_h=\frac1h s_h$ for any quantity s, one gets the bound

$$\frac{1}{2}\frac{d}{dt}\|\widehat{e}_h\|^2_{L^2(\mathbb{R})}\le \|\widehat{s}_h\|_{L^2(\mathbb{R})}\|\widehat{e}_h\|_{L^2(\mathbb{R})}+\frac{1}{2\sigma}\|\widehat{r}_h\|^2_{L^2(\mathbb{R})}\le \frac12\left(\|\widehat{s}_h\|^2_{L^2(\mathbb{R})}+\frac{1}{\sigma}\|\widehat{r}_h\|^2_{L^2(\mathbb{R})}\right)+\frac12\|\widehat{e}_h\|^2_{L^2(\mathbb{R})}$$

where the term in parenthesis is bounded due to (35). The Gronwall's inequality guarantees the boundedness of $\|\widehat{e}_h\|_{L^2(\mathbb{R})}$ at any time. Therefore

$$\|e_h\|_{L^\infty([0,T];\mathbb{R})}\le C(T)h\left(\|\partial_{xx}p_0\|_{L^2(\mathbb{R})}+\|\partial_{xxx}p_0\|_{L^2(\mathbb{R})}\right). \tag{36}$$

The other term that we consider is $f_h=\varepsilon\left(v(x_j)-v_j\right)_{j\in\mathbb{Z}}=-\varepsilon\left(\frac{\partial_x p(x_j)}{\sigma}-v_j\right)_{j\in\mathbb{Z}}$. We notice that (34) and (36) yields

$$\left\|\frac{e_{j+1}-e_j}{\Delta x_{j+\frac12}}\right\|_{L^2([0,T]\times\mathbb{R})}\le C(T)h\left(\|\partial_{xx}p_0\|_{L^2(\mathbb{R})}+\|\partial_{xxx}p_0\|_{L^2(\mathbb{R})}\right)$$

which implies after some manipulations

$$\left\|\frac{p_{j+1}-p_j}{\Delta x_{j+\frac12}}-\partial_x p(x_{j+\frac12})\right\|_{L^2([0,T]\times\mathbb{R})}\le C(T)h\left(\|\partial_{xx}p_0\|_{L^2(\mathbb{R})}+\|\partial_{xxx}p_0\|_{L^2(\mathbb{R})}\right).$$

The definition (19) of v_j implies the bound

$$\|f_h\|_{L^2([0,T]\times\mathbb{R})}\le C(T)h\varepsilon\left(\|\partial_{xx}p_0\|_{L^2(\mathbb{R})}+\|\partial_{xxx}p_0\|_{L^2(\mathbb{R})}\right). \tag{37}$$

Finally, $\|\mathbf{W}^\varepsilon_h-\mathbf{W}^\varepsilon\|^2_{L^2([0,T]\times\mathbb{R})}=\|e_h\|^2_{L^2([0,T]\times\mathbb{R})}+\|f_h\|^2_{L^2([0,T]\times\mathbb{R})}$ is bounded using (36) and (37). It ends the proof. $\square$

2.6 Study of $\|P^\varepsilon_h-P^0_h\|$

In this section we prove an error estimate between the solution of the scheme (25) and the solution of the diffusion scheme (17). It is necessary to use some comparison estimates between the initial data of P^ε_h and P^0_h.

Proposition 2.7. *There exists a constant $C(T) > 0$ such that the following estimate holds*

$$\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \le C(T)\varepsilon.$$

Proof. We define R_j and $S_{j+\frac12}$ such that the solution of (17) satisfies the various relations which are generalizations of (25) and (19)

$$\begin{cases} \Delta x_j \dfrac{d}{dt} p_j + \dfrac{u_{j+\frac12} - u_{j-\frac12}}{\varepsilon} = 0, \\ \Delta x_j \dfrac{d}{dt} u_j - \dfrac{u_{j+\frac12} + u_{j-\frac12}}{\varepsilon} + \dfrac{2}{\varepsilon} u_j = \Delta x_j R_j, \\ p_j - p_{j+1} + u_j + u_{j+1} = 2u_{j+\frac12} + \sigma \Delta x_{j+\frac12} \dfrac{u_{j+\frac12}}{\varepsilon} + \Delta x_{j+\frac12} S_{j+\frac12}, \\ u_{j+\frac12} = -\dfrac{\varepsilon}{\sigma} \dfrac{p_{j+1} - p_j}{\Delta x_{j+\frac12}}, \\ u_j = \dfrac{u_{j+\frac12} + u_{j-\frac12}}{2}. \end{cases} \tag{38}$$

A simple computation using the last two identities yields

$$R_j = \frac{d}{dt} u_j \text{ and } S_{j+\frac12} = \frac{1}{\Delta x_{j+\frac12}} \left(u_j + u_{j+1} - 2u_{j+\frac12} \right).$$

Thanks to the estimate (27) and as the scheme is linear, $\frac{d}{dt} u_j$ satisfies the estimate

$$\|R\|_{L^2([0,T]\times\mathbb{R})} = \left\| \frac{d}{dt} u_j \right\|_{L^2([0,T]\times\mathbb{R})} \le C\varepsilon \left\| \frac{d}{dt} p_h(0) \right\|_{L^2(\mathbb{R})}.$$

The definition of $\partial_t p_h(0)$ is from (17-18), which implies that $\|\frac{d}{dt} p_h(0)\|_{L^2(\mathbb{R})} \le C\|p(t=0)\|_{H^2(\mathbb{R})}$. So one has the bound

$$\|R\|_{L^2([0,T]\times\mathbb{R})} \le C\varepsilon \|p(0)\|_{H^2(\mathbb{R})}. \tag{39}$$

It is seen in proposition 2.8 that

$$\|S\|_{L^2(\mathbb{R})} \le \varepsilon \|p(0)\|_{H^2(\mathbb{R})}. \tag{40}$$

We now introduce the differences

$$e_j = p_j - p_j^\varepsilon, \quad f_j = u_j - u_j^\varepsilon \text{ and } f_{j+\frac12} = u_{j+\frac12} - u_{j+\frac12}^\varepsilon.$$

Let us look at the difference between the scheme (25) and (38). We get

$$\begin{cases} \Delta x_j \frac{d}{dt} e_j + \frac{f_{j+\frac12} - f_{j-\frac12}}{\varepsilon} = 0, \\ \Delta x_j \frac{d}{dt} f_j - \frac{f_{j+\frac12} + f_{j-\frac12}}{\varepsilon} + \frac{2}{\varepsilon} f_j = \Delta x_j R_j, \\ e_j - e_{j+1} + f_j + f_{j+1} - 2f_{j+\frac12} - \sigma \Delta x_{j+\frac12} \frac{f_{j+\frac12}}{\varepsilon} = \Delta x_{j+\frac12} S_{j+\frac12}. \end{cases}$$

We use the notation $\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 = \sum_j \Delta x_j (e_j^2 + f_j^2)$. Using the same kind of proof than for the L^2 stability of proposition 2.3, one gets that

$$\frac12 \frac{d}{dt} \|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 \le \sum_j \Delta x_j R_j f_j - \sum_j \Delta x_{j+\frac12} \frac{f_{j+\frac12}}{\varepsilon} S_{j+\frac12} - \frac{\sigma}{\varepsilon^2} \sum_{j\in\mathbb{Z}} \Delta x_{j+\frac12} f_{j+\frac12}^2.$$

Using a Young's inequality on the second term of the right side of this inequality, one finds out that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 \le \sum_j \Delta x_j R_j f_j + \frac{1}{4\sigma}\sum_j \Delta x_{j+\frac{1}{2}} S_{j+\frac{1}{2}}^2.$$

Using the Cauchy-Schwarz inequality, we thus have

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 \le \frac{1}{2}\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2(\mathbb{R})}^2 + C\bigg(\|R\|_{L^2(\mathbb{R})}^2 + \|S\|_{L^2(\mathbb{R})}^2\bigg).$$

Integrating between 0 and t and using a Gronwall's inequality, one finds for $t \le T$

$$\|\mathbf{V}_h^\varepsilon(t) - \mathbf{W}_h^\varepsilon(t)\|_{L^2(\mathbb{R})}^2 \le \exp(t)\left(\|\mathbf{V}_h^\varepsilon(0) - \mathbf{W}_h^\varepsilon(0)\|_{L^2(\mathbb{R})}^2 + C(\|R\|_{L^2([0,T]\times\mathbb{R})}^2 + \|S\|_{L^2([0,T]\times\mathbb{R})}^2)\right)$$

Finally, using the previous estimates (39-40) and the well-preparedness of the data (16,18,20), one gets $\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \le C(T)\varepsilon$. The proof is ended. □

Proposition 2.8. *The bound (40) holds.*

Proof. Using the definition of u_j and $u_{j+\frac{1}{2}}$, one has

$$S_{j+\frac{1}{2}} = \frac{\varepsilon}{2}\left(\frac{\Delta x_j}{\Delta x_{j+\frac{1}{2}}}\frac{d}{dt}p_j - \frac{\Delta x_{j+1}}{\Delta x_{j+\frac{1}{2}}}\frac{d}{dt}p_{j+1}\right)$$

at all time. Moreover, $z_h = \frac{d}{dt}p_h$ is solution of P_h^0:

$$\Delta x_j \frac{d}{dt}z_j - \frac{1}{\sigma}\left(\frac{z_{j+1}-z_j}{\Delta x_{j+\frac{1}{2}}} - \frac{z_j - z_{j-1}}{\Delta x_{j-\frac{1}{2}}}\right) = 0,$$

with initial condition

$$z_j(0) = \frac{d}{dt}p_0(x_j) = \frac{1}{\Delta x_j \sigma}\left(\frac{p_0(x_{j+1}) - p_0(x_j)}{\Delta x_{j+\frac{1}{2}}} - \frac{p_0(x_j) - p_0(x_{j-1})}{\Delta x_{j-\frac{1}{2}}}\right). \tag{41}$$

One gets from a Taylor expansion with integral residue that

$$\left|\frac{p_0(x_{j+1}) - p_0(x_j)}{\Delta x_{j+\frac{1}{2}}} - \partial_x p_0(x_j)\right| \le \int_{x_j}^{x_{j+1}} |\partial_{xx}p_0(y)|\, dy.$$

Similarly one has the bound $\left|\frac{p_0(x_j)-p_0(x_{j-1})}{\Delta x_{j+\frac{1}{2}}} - \partial_x p_0(x_j)\right| \le \int_{x_{j-1}}^{x_j} |\partial_{xx}p_0(y)|\, dy$. Therefore $|z_j(0)| \le \frac{1}{\Delta x_j \sigma}\int_{x_{j-1}}^{x_{j+1}} |\partial_{xx}p_0(y)|\, dy$ from which the bound $\sqrt{\sum_j \Delta x_j z_j^2(0)} \le \sigma^{-1}\|p_0\|_{H^2(\mathbb{R})}$ is deduced. Since the scheme P_h^0 being stable in L^2, this bound is true at any time. Considering (41) the discrete second derivative attached to P_h^0 is bounded at any time, which ends the proof of the claim. □

2.7 End of the proof of uniform AP property

Theorem 2.9. *Assuming a sufficiently smooth well prepared initial data, the scheme P_h^ε converges to P^ε at order at least $\frac{1}{3}$ in $L^2([0,T]\times\mathbb{R})$, uniformly with respect to ε*

Proof. All the previous estimates show that (6-8) are true with $a = 1$, $b = c = \frac{1}{2}$ and $d = 1$. Moreover $\|P_h^\varepsilon - P_h^0\| \le C(T)\varepsilon$ by proposition (2.7) which means this term has the same scaling as $\|P^\varepsilon - P^0\| \approx C(T)\varepsilon$. So it can be incorporated in estimate (6) with $a = 1$. Using the general method described at the beginning of this work in proposition 1.3, one obtains the convergence estimate of convergence $\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \le C(T)h^q$ with the order of convergence $q = \frac{ac}{a+b} = \frac{1}{3}$. □

3 The 2D case

In this section we prove the uniform convergence of the solution of the diffusion AP scheme introduced in [5] to the solution of the hyperbolic heat equation. The structure of our proof is globally the same as in the previous section. However two major difficulties will be treated: a) the first one consists in the adaptation to our problem of a combination of specific finite volumes techniques for hyperbolic and parabolic equations; b) the second one is to derive new bounds for the scheme $\mathbf{DA}_h^\varepsilon$.

The model problem is the hyperbolic heat equation in the domain $\Omega =]0,1[^2$ with periodic boundary conditions and well-prepared data

$$\mathbf{P}^\varepsilon : \qquad \begin{cases} \partial_t p^\varepsilon + \dfrac{1}{\varepsilon}\operatorname{div}(\mathbf{u}^\varepsilon) = 0, \\[2ex] \partial_t \mathbf{u}^\varepsilon + \dfrac{1}{\varepsilon}\nabla p^\varepsilon = -\dfrac{\sigma}{\varepsilon^2}\mathbf{u}^\varepsilon, \\[2ex] p^\varepsilon(t=0) = p_0,\ \mathbf{u}^\varepsilon(t=0) = \mathbf{u}_0^\varepsilon = -\frac{\varepsilon}{\sigma}\nabla p_0. \end{cases}$$

When ε tends to zero, this problem admits the following diffusion limit

$$\mathbf{P}^0 : \qquad \partial_t p - \frac{1}{\sigma}\operatorname{div}(\nabla p) = 0, \qquad p(t=0) = p_0.$$

The rescaled gradient is $\mathbf{v} = -\frac{1}{\sigma}\nabla p$. We will admit the following proposition, the proof of which can be easily obtained by a method similar to the one of proposition 2.2.

Proposition 3.1. *The error between the two solutions can be upper bounded by*

$$|p^\epsilon - p\|_{L^\infty([0,T];H^n(\Omega))} + \|\mathbf{u}^\varepsilon\|_{L^\infty([0,T];H^n(\Omega))} \le C\varepsilon\|p_0\|_{H^{3+n}(\Omega)}, \qquad n \in \mathbb{N}. \tag{42}$$

Proof. The structure of the proof in the $L^\infty([0,T];L^2(\Omega))$ norm is the same as the one of proposition 2.2. Since the coefficients of the problem are constant, similar bounds are obtained at any order of derivation which proves the estimate for any $n > 0$. □

3.1 Definition of $\mathbf{P}_h^\varepsilon$

Let us consider an unstructured mesh in dimension 2. The mesh is defined by a finite number of vertices $\mathbf{x}_r$ and cells Ω_j. We denote $\mathbf{x}_j$ a point chosen arbitrarily inside Ω_j. For simplicity we will call this point the center of the cell. By convention the vertices are listed counter-clockwise $\mathbf{x}_{r-1}, \mathbf{x}_r, \mathbf{x}_{r+1}$ with coordinates $\mathbf{x}_r = (x_r, y_r)$. We note $l_{jr}\mathbf{n}_{jr}$ the vector as follows

$$l_{jr} = \frac{1}{2}\operatorname{dist}(\mathbf{x}_{r-1}, \mathbf{x}_{r+1}) \text{ and } \mathbf{n}_{jr} = \frac{1}{2l_{jr}}(\mathbf{x}_{r+1} - \mathbf{x}_{r-1})^\perp. \tag{43}$$

This notion of a corner vector can be done also in any dimension using the abstract definition [11]. The scalar product of two vectors is denoted as $(\mathbf{x}, \mathbf{y})$.

The numerical approximation of the problem $\mathbf{P}^\varepsilon$ that we study is the JL-(b) scheme defined in [5]

$$\mathbf{P}_h^\varepsilon : \qquad \begin{cases} |\ \Omega_j\ | \dfrac{d}{dt}p_j^\varepsilon + \dfrac{1}{\varepsilon}\displaystyle\sum_r (l_{jr}\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) = 0 \\ |\ \Omega_j\ | \dfrac{d}{dt}\mathbf{u}_j^\varepsilon + \dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr}\mathbf{n}_{jr}p_{jr}^\varepsilon = -\dfrac{\sigma}{\varepsilon^2}\displaystyle\sum_r \widehat{\beta}_{jr}\mathbf{u}_r^\varepsilon, \end{cases} \tag{44}$$

with initial data $p_j^\varepsilon(0) = p_0(\mathbf{x}_j)$ and $\mathbf{u}_j^\varepsilon(0) = -\varepsilon\sigma^{-1}\nabla p_0(\mathbf{x}_j)$. The fluxes are defined by the so-called corner problem

$$\begin{cases} p_{jr}^\varepsilon - p_j^\varepsilon = \mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon) - \dfrac{\sigma}{\varepsilon}(\mathbf{x}_r - \mathbf{x}_j, \mathbf{u}_r^\varepsilon), \\ \sum_j l_{jr}p_{jr}^\varepsilon\mathbf{n}_{jr} = 0. \end{cases} \tag{45}$$

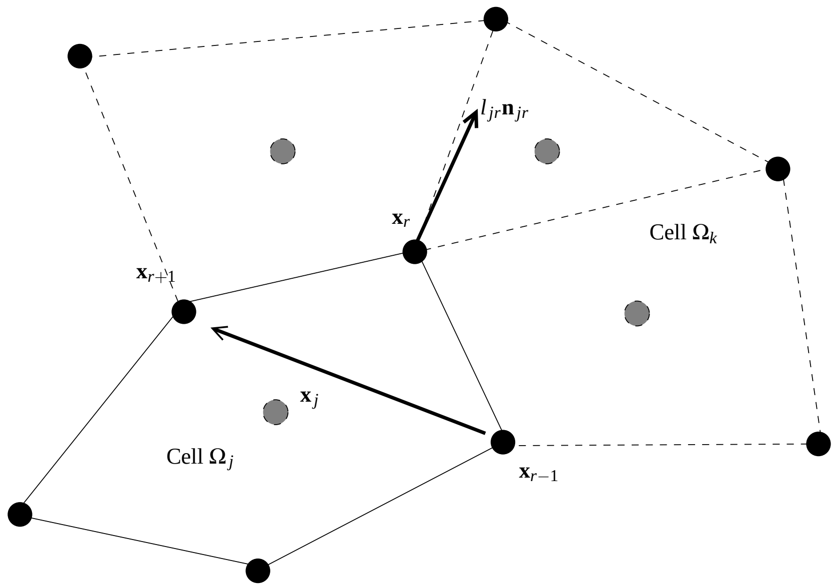

Figure 4: Notation for node formulation. The corner length l_{jr} and the corner normal $\mathbf{n}_{jr}$ are defined in equation (43). The point $\mathbf{x}_j$ is an arbitrary point inside the cell, typically the centroid of the cell or an averaged of the corners.

This corner problem has been introduced in [5] as a multidimensional version of the 1D Jin-Levermore technique [18]. Its solution is provided by the solution of the linear system

$$\left(\sum_j \widehat{\alpha}_{jr} + \sum_j \frac{\sigma}{\varepsilon}\widehat{\beta}_{jr}\right)\mathbf{u}_r^\varepsilon = \sum_j l_{jr}p_j^\varepsilon \mathbf{n}_{jr} + \sum_j \widehat{\alpha}_{jr}\mathbf{u}_j^\varepsilon,$$

where the geometry of the mesh serves to define the matrices $\widehat{\alpha}_{jr}$ and $\widehat{\beta}_{jr}$

$$\widehat{\alpha}_{jr} = l_{jr}\mathbf{n}_{jr}\otimes\mathbf{n}_{jr}, \text{ and } \widehat{\beta}_{jr} = l_{jr}\mathbf{n}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j). \tag{46}$$

We will use the notations $A_j = \sum_r \widehat{\alpha}_{jr}$, $A_r = \sum_j \widehat{\alpha}_{jr}$ and $B_r = \sum_j \widehat{\beta}_{jr}$. By comparison with the scheme P_h^ε in dimension one, one see at once that the multi-dimensional scheme (44-46) is more tricky than the 1D scheme (14-15).

Starting from (44) and taking into account of the definitions of the fluxes (45) and also the identity $\sum_r l_{jr}\mathbf{n}_{jr} = 0$, the scheme $\mathbf{P}_h^\varepsilon$ can also be rewritten as

$$\mathbf{P}_h^\varepsilon : \quad \begin{cases} \mid \Omega_j \mid \dfrac{d}{dt}p_j^\varepsilon + \dfrac{1}{\varepsilon}\displaystyle\sum_r (l_{jr}\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) = 0 \\ \mid \Omega_j \mid \dfrac{d}{dt}\mathbf{u}_j^\varepsilon + \dfrac{1}{\varepsilon}\displaystyle\sum_r l_{jr}(\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon)\mathbf{n}_{jr} = 0 \end{cases} \tag{47}$$

When $\varepsilon \to 0$ the scheme $\mathbf{P}_h^\varepsilon$ admits the limit diffusion scheme $\mathbf{P}_h^0$

$$\mathbf{P}_h^0 : \quad \begin{cases} |\Omega_j| \dfrac{d}{dt}p_j + \displaystyle\sum_r l_{jr}\Big(\mathbf{v}_r, \mathbf{n}_{jr}\Big) = 0, \\ \mathbf{v}_r = \dfrac{1}{\sigma}B_r^{-1}\displaystyle\sum_j l_{jr}p_j\mathbf{n}_{jr}, \end{cases} \tag{48}$$

with $B_r = \sum_j l_{jr} \mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$. We define additionally $\mathbf{v}_j$ by a kind of mean

$$\left(\sum_r \widehat{\alpha}_{jr}\right) \mathbf{v}_j = \sum_r \widehat{\alpha}_{jr} \mathbf{v}_r.$$

This is well defined since the matrix $\sum_r \widehat{\alpha}_{jr}$ is symetric positive by definition of the $\widehat{\alpha}_{jr}$.

3.2 Definition of $\mathbf{DA}_h^\varepsilon$

We define now that is call thereafter the "diffusion approximation" scheme. We just neglect the time derivative in the second equation, that we make $\partial_t \mathbf{u}_j^\varepsilon = 0$ for (47). It leads to the scheme

$$\mathbf{DA}_h^\varepsilon : \begin{cases} \mid \Omega_j \mid \dfrac{d}{dt} p_j^\varepsilon + \dfrac{1}{\varepsilon} \displaystyle\sum_r (l_{jr} \mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) = 0 \\ \frac{1}{\varepsilon} \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon) \mathbf{n}_{jr} = 0 \\ \left(\displaystyle\sum_j \widehat{\alpha}_{jr} + \sum_j \frac{\sigma}{\varepsilon} \widehat{\beta}_{jr} \right) \mathbf{u}_r^\varepsilon = \sum_j l_{jr} p_j^\varepsilon \mathbf{n}_{jr} + \sum_j \widehat{\alpha}_{jr} \mathbf{u}_j^\varepsilon \end{cases} \tag{49}$$

This scheme depends of two parameters, the size of the mesh h and the small parameter ε. We notice that $\mathbf{DA}_h^\varepsilon \neq \mathbf{P}_h^0$ for $\varepsilon > 0$, and that $\lim_{\varepsilon \to 0^+} \mathbf{DA}_h^\varepsilon = \mathbf{P}_h^0$. The initial data for (49) is $p_j^\varepsilon(0) = p_0(\mathbf{x}_j)$. There is no need of initial data for $(u_j^\varepsilon(0))$, which will be obtained as a function of $(p_j^\varepsilon(0))$ by solving a linear system.

3.3 Mesh assumptions

The characteristic length of the mesh is

$$h = \max_j \left(\mathrm{diam}(\Omega_j)\right).$$

By definition there exists a constant $C > 0$ such that

$$l_{jr} \leq Ch, \qquad \forall j, r. \tag{50}$$

The control volume V_r around the vertex $\mathbf{x}_r$ is defined by the closed loop $\ldots, \mathbf{x}_{j-\frac{1}{2}}, \mathbf{x}_j, \mathbf{x}_{j+\frac{1}{2}}, \ldots$. Here the $\mathbf{x}_j$'s are the center of the cells, and the $\mathbf{x}_{j+\frac{1}{2}}$'s are the middle of the edges around the vertices $\mathbf{x}_r$. A typical example is depicted in figure 5.

Additional geometrical assumptions are always necessary in dimension greater than one to guarantee some minimal regularity of the mesh. We make the usual assumptions listed below from 1 to 3. The two last items are more specific.

Hypothesis 3.2. *Our geometrical assumptions will be the following*

1. *The mesh is regular in the sense that there are two constants $C_1, C_2 > 0$ such that*

$$C_1 h^2 \leq |\Omega_j| \leq C_2 h^2, \quad \forall j \qquad \textit{uniformly with respect to } h. \tag{51}$$

   and that

$$C_1 h^2 \leq |V_r| \leq C_2 h^2, \quad \forall r \qquad \textit{uniformly with respect to } h. \tag{52}$$

   We recall that V_r is the volume control (centered on $\mathbf{x}_r$) and Ω_j is the cell j.

2. *The numbers of cells which share a node r is bounded independently of h, which means there exists $P \in \mathbb{N}$ independent of h such that*

$$\sum_j \delta_{jr} \leq P. \tag{53}$$

   For example, for a structured mesh of quadrangular cells $P = 4$.

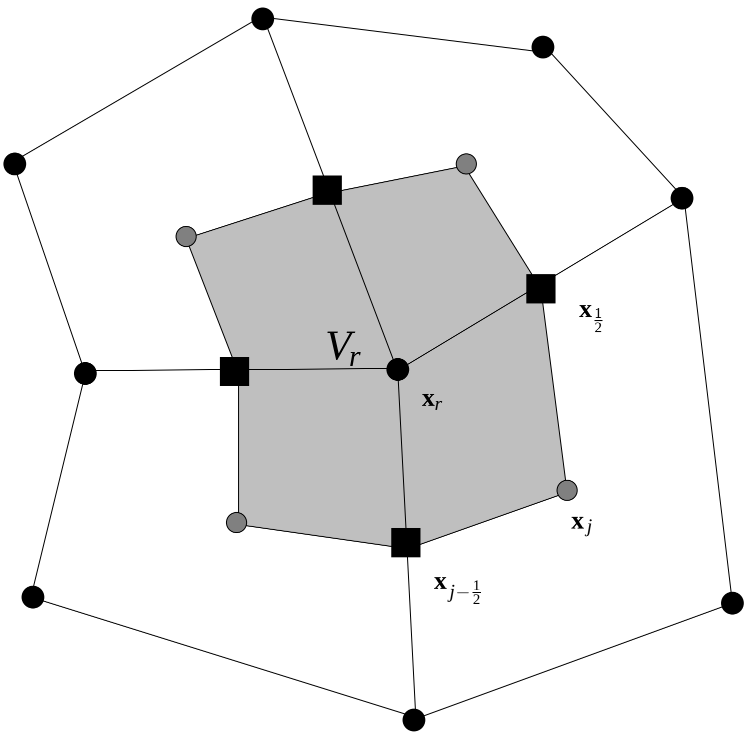

Figure 5: Definition of the control volume V_r around vertex $\mathbf{x}_r$. The control volume around the vertex $\mathbf{x}_r$ is defined by the closed loop that joins the center of the cells ($\mathbf{x}_j$'s) and the middle of the edges ($\mathbf{x}_{j+\frac{1}{2}}$'s).

3. *For each cell of the mesh, the number of edges is bounded independently of h, or equivalently the numbers of vertices for a cell is bounded independently of h.*

4. *A consequence of the items 1-3 is that there exists a constant $\beta > 0$ such that*

$$(A_j \mathbf{u}, \mathbf{u}) \geq \beta h(\mathbf{u}, \mathbf{u}). \tag{54}$$

   It can be proved with a geometrical identity that we borrow from [11] (proposition 8).

5. *The matrix B_r is positive and that there exists a constant $\alpha > 0$ independent of r such that*

$$(B_r \mathbf{u}, \mathbf{u}) = (B_r^s \mathbf{u}, \mathbf{u}) \geq \alpha |V_r| (\mathbf{u}, \mathbf{u}) \tag{55}$$

   where $B_r^s = \frac{1}{2}(B_r + B_r^t)$ is the symmetric part of B_r. Square meshes satisfy (55). This assumption is however not trivial to check in the general case. We point out [5] where sufficient conditions such that (55) is satisfied can be found; in particular it is shown that triangular meshes with all angles greater than 12 degrees satisfy it.

We use the convention that the quadratic norm of any cell centered quantity $f = (f_j)_{j \in \text{Cells}}$ is $\|f\|_{L^2(\Omega)} = \sqrt{\sum_j |\Omega_j||f_j|^2}$, while the quadratic norm of any vertex based quantity $g = (g_r)_{r \in \text{Vertices}}$ is $\|g\|_{L^2(\Omega)} = \sqrt{\sum_r |V_r||g_r|^2}$. Useful quantities are

- $\mathbf{V}_h^\varepsilon = \left(p_j^\varepsilon, \mathbf{u}_j^\varepsilon\right)_{j \in \text{Cells}}$ is the solution of $\mathbf{P}_h^\varepsilon$.
- $\mathbf{V}^\varepsilon = \left(p^\varepsilon(\mathbf{x}_j), \mathbf{u}^\varepsilon(\mathbf{x}_j)\right)_{j \in \text{Cells}}$ is the solution of $\mathbf{P}^\varepsilon$,

- $\mathbf{W}_h^\varepsilon = \left(p_j^\varepsilon, \mathbf{u}_j^\varepsilon\right)_{j\in\text{Cells}}$ is the solution of $\mathbf{DA}_h^\varepsilon$. Notice that an abuse of notations is been made with the solution of $\mathbf{P}_h^\varepsilon$.
- $\mathbf{W}^\varepsilon = \left(p(\mathbf{x}_j), -\frac{\varepsilon}{\sigma}\nabla p(\mathbf{x}_j)\right)_{j\in\text{Cells}}$ is the solution of $\mathbf{P}^0$.

With these notations, (42) is rewritten as

$$\|\mathbf{W}^\varepsilon - \mathbf{V}^\varepsilon\|_{L^2([0,T]\times L^2(\Omega))} \leq C\varepsilon. \tag{56}$$

3.4 Study of $\|\mathbf{P}_h^\varepsilon - \mathbf{P}^\varepsilon\|_{\text{naive}}$

In this part, we exploit the hyperbolic nature of both $\mathbf{P}^\varepsilon$ and $\mathbf{P}_h^\varepsilon$. We first prove the L^2 stability of the scheme JL-(b) defined in (44,45).

Proposition 3.3 (Stability)**.** *Under the geometrical assumption (55), the semi-discrete general JL-(b) scheme defined by (44,45) is stable in the L^2 norm in the sense that $\frac{d}{dt}||\mathbf{V}_h^\varepsilon(t)|| \leq 0$. Moreover we have the bounds*

$$\frac{\sigma}{\varepsilon^2}||\mathbf{u}_r^\varepsilon||_{L^2([0,T]\times\Omega)} \leq C||V_h^\varepsilon(0)||_{L^2(\Omega)}, \qquad C>0, \tag{57}$$

$$\int_0^T \sum_j\sum_r l_{jr}(\mathbf{n}_{jr}, (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon))^2 dt \leq C\varepsilon||\mathbf{V}_h^\varepsilon(0)||^2_{L^2(\Omega)}. \qquad C>0. \tag{58}$$

Proof. We define the functions p_h^ε and $\mathbf{u}_h^\varepsilon$ by $p_h^\varepsilon = p_j$ and $\mathbf{u}_h^\varepsilon = \mathbf{u}_j$ on Ω_j. We set for convenience $E(t) = ||\mathbf{V}_h^\varepsilon(t)||^2$. One has

$$E^{'}(t) = \frac{1}{2}\int_\Omega \frac{d}{dt}(\mid p_h^\varepsilon\mid^2 + (\mathbf{u}_h^\varepsilon, \mathbf{u}_h^\varepsilon)) = \int_\Omega p_h^\varepsilon \frac{d}{dt}p_h^\varepsilon + (\mathbf{u}_h^\varepsilon, \frac{d}{dt}\mathbf{u}_h^\varepsilon) = \sum_j |\Omega_j|\, p_j^\varepsilon \frac{d}{dt}p_j^\varepsilon + (\mathbf{u}_j^\varepsilon, \frac{d}{dt}\mathbf{u}_j^\varepsilon).$$

Using the definition of scheme

$$E^{'}(t) = -\frac{1}{\varepsilon}\sum_j\sum_r l_{jr}p_j^\varepsilon(\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) - \frac{1}{\varepsilon}\sum_j\sum_r (l_{jr}p_{j,r}^\varepsilon \mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon) - \frac{\sigma}{\varepsilon^2}\sum_j\sum_r (\widehat{\beta}_{jr}\mathbf{u}_r^\varepsilon, \mathbf{u}_j^\varepsilon). \tag{59}$$

Using (45) we expand the second term of the previous equation

$$\sum_j\sum_r (l_{jr}p_{j,r}^\varepsilon \mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon) = \sum_j\sum_r l_{jr}p_j^\varepsilon(\mathbf{u}_j^\varepsilon, \mathbf{n}_{jr}) + \sum_j\sum_r (\widehat{\alpha}_{jr}(\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon), \mathbf{u}_j^\varepsilon) - \frac{\sigma}{\varepsilon}\sum_j\sum_r (\widehat{\beta}_{jr}\mathbf{u}_r^\varepsilon, \mathbf{u}_j^\varepsilon). \tag{60}$$

Since $\sum_r l_{jr}\mathbf{n}_{jr} = 0$ the first term of (60) is zero. Summing on r the second equation of (45) and permuting the sums, we show that $0 = \sum_j\sum_r l_{jr}p_{jr}(\mathbf{u}_r, \mathbf{n}_{jr})$ which yields that

$$0 = \sum_j\sum_r l_{jr}p_j^\varepsilon(\mathbf{u}_r^\varepsilon, \mathbf{n}_{jr}) - \sum_j\sum_r ((\widehat{\alpha}_{jr} + \frac{\sigma}{\varepsilon}\widehat{\beta}_{jr})\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) + \sum_j\sum_r (\widehat{\alpha}_{jr}\mathbf{u}_j^\varepsilon, \mathbf{u}_r^\varepsilon). \tag{61}$$

Plugging (60) and (61) in (59) and permuting the sums in $E'(t)$ gives

$$E^{'}(t) = -\frac{1}{\varepsilon}\sum_j\sum_r (\widehat{\alpha}_{jr}(\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon), \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon) - \frac{\sigma}{\varepsilon^2}\sum_r\sum_j (\widehat{\beta}_{jr}\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon)$$

which gives

$$E^{'}(t) + \frac{1}{\varepsilon}\sum_r\sum_j l_{jr}(\mathbf{n}_{jr}, (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon))^2 + \frac{\sigma}{\varepsilon^2}\sum_r (B_r\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) = 0. \tag{62}$$

By geometrical assumption (55) we have $E^{'}(t) \leq 0$, that is the L^2 stability, and by integrating this equality on $[0, T]$ we obtain

$$E(T) + \int_0^T \frac{1}{\varepsilon} \sum_r \sum_j l_{jr} (\mathbf{n}_{jr}, (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon))^2 + \int_0^T \frac{\sigma}{\varepsilon^2} \sum_r (B_r \mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) = E(0)$$

Using again the geometrical assumption (55) for the terms $(B_r \mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon)$ we have

$$E(T) + \int_0^T \frac{1}{\varepsilon} \sum_r \sum_j l_{jr} (\mathbf{n}_{jr}, (\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon))^2 + \alpha \int_0^T \frac{\sigma}{\varepsilon^2} \sum_r |V_r| \ |\mathbf{u}_r^\varepsilon|^2 \leq E(0)$$

which gives (57) and (58). The proof is ended. □

Our goal now is to prove the following result which will be the consequence of propositions 3.5 to 3.9. This part is the more technical one of the paper, but is essential to be able to use the general strategy of proposition 1.3 with convenient exponents. As one will see below, the convergence estimate (63) is not trivial. It indicates that, for a problem with $O(\varepsilon^{-2})$ terms, a scheme converges, with h, with at rate $O(\varepsilon^{-\frac{1}{2}})$ with respect to ε.

Proposition 3.4 (Convergence)**.** *There exist a constant $C(T, \sigma) > 0$ such that the following estimate holds*

$$\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{L^\infty([0,T]\times L^2(\Omega))} \leq C(T, \sigma) \|\mathbf{V}^\varepsilon(0)\|_{H^3(\Omega)} \sqrt{\frac{h}{\varepsilon}}. \tag{63}$$

In the whole proof, we will use a constant $C > 0$ large enough. Like in 1D, we use the method introduced by Mazeran [22]. We introduce $\mathscr{E}(t) = \frac{1}{2}\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|_{L^2(\Omega)}^2$. As for the 1D proof and for the sake of simplicity, for any quantity q, q' stands indifferently for $\frac{d}{dt}q$ or $\partial_t q$.

Proposition 3.5. *One has the formula*

$$\begin{aligned}
\mathscr{E}'(t) = & -\frac{1}{\varepsilon} \sum_{j,r} l_{j,r} (\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 \\
& + \frac{1}{\varepsilon} \sum_j \sum_r \Big(l_{j,r}(\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r} \Big) \delta_{j,r}(p^\varepsilon) + \frac{1}{\varepsilon} \sum_j \sum_r \Big(l_{j,r} \mathbf{n}_{j,r} (p_{jr}^\varepsilon - p_j^\varepsilon) \ , \ \delta_{j,r}(\mathbf{u}^\varepsilon) \Big) \\
& + \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| p_j^\varepsilon (\mathbf{n}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon)) + \frac{1}{\varepsilon} \sum_j \sum_r |\Gamma_{j,r}| \Big(\mathbf{u}_j^\varepsilon, \mathbf{n}_{j,r} \tilde{\delta}_{j,r}(p^\varepsilon) \Big) \\
& + \frac{\sigma}{\varepsilon^2} \sum_r \sum_j \Big(\widehat{\beta}_{j,r} \mathbf{u}_r^\varepsilon \ , \ \frac{1}{|\Omega_j|} \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \Big) + \frac{\sigma}{\varepsilon^2} \sum_j \Big(\mathbf{u}_j^\varepsilon, \int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x} \Big) \\
& - \frac{\sigma}{\varepsilon^2} \sum_j \int_{\Omega_j} (\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) d\mathbf{x} - \frac{\sigma}{\varepsilon^2} \sum_r (B_r \mathbf{u}_r^\varepsilon, \boldsymbol{u}_r^\varepsilon)
\end{aligned} \tag{64}$$

where the extra notations are: $\delta_{j,r}(f) = \frac{1}{|\Omega_j|} \int_{\Omega_j} f d\mathbf{x} - f(\mathbf{x}_r)$ is an interpolation error term that compares mean values and point values; $\Gamma_{j,r} = [\mathbf{x}_r, \mathbf{x}_{r+1}]$ is the edge oriented toward the outside of the cell j, with length $|\Gamma_{j,r}|$; and $\tilde{\delta}_{j,r}(h) = \frac{1}{|\Gamma_{j,r}|} \int_{\Gamma_{j,r}} h ds - \frac{h(\boldsymbol{x}_r) + h(\boldsymbol{x}_{r+1})}{2}$ is another interpolation error contribution that compares the mean value and the mid sum, on the edge.

Proof. We first consider the time derivative

$$\begin{aligned}
\mathscr{E}'(t) = & \underbrace{\int_\Omega (p_h^\varepsilon (p_h^\varepsilon)' + (\mathbf{u}_h^\varepsilon, (\mathbf{u}_h^\varepsilon)')) d\mathbf{x}}_{D_1} + \underbrace{\int_\Omega (p^\varepsilon (p^\varepsilon)' + (\mathbf{u}^\varepsilon, (\mathbf{u}^\varepsilon)')) d\mathbf{x}}_{D_2} \\
& + \underbrace{\int_\Omega (-(p_h^\varepsilon)' p^\varepsilon - ((\mathbf{u}_h^\varepsilon)', \mathbf{u}^\varepsilon)) d\mathbf{x}}_{D_3} + \underbrace{\int_\Omega (-p_h^\varepsilon (p^\varepsilon)' - (\mathbf{u}_h^\varepsilon, (\mathbf{u}^\varepsilon)')) d\mathbf{x}}_{D_4}.
\end{aligned}$$

One has thanks to (62)

$$D_1 = -\frac{1}{\varepsilon}\sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 - \frac{\sigma}{\varepsilon^2}\sum_r (B_r\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon).$$

One also directly has

$$D_2 = -\frac{\sigma}{\varepsilon^2}\int_\Omega (\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) d\mathbf{x} = -\frac{\sigma}{\varepsilon^2}\sum_j \int_{\Omega_j} (\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon) d\mathbf{x}.$$

Then, using the definition (44,45) of the scheme we have

$$\begin{aligned} D_3 = &\tfrac{1}{\varepsilon}\textstyle\sum_j\sum_r \Big(l_{j,r}\mathbf{u}_r^\varepsilon, \mathbf{n}_{j,r}\Big) \tfrac{1}{|\Omega_j|}\textstyle\int_{\Omega_j} p^\varepsilon dx \\ &+\tfrac{1}{\varepsilon}\textstyle\sum_j \Big(\textstyle\sum_r l_{jr}\mathbf{n}_{jr} p_{j,r}^\varepsilon + \tfrac{\sigma}{\varepsilon}\textstyle\sum_r \widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon \ , \ \tfrac{1}{|\Omega_j|}\textstyle\int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}\Big) \end{aligned}$$

Since $\sum_r l_{jr}\mathbf{n}_{jr} = 0$, we can write

$$\begin{aligned} D_3 = \ &\tfrac{1}{\varepsilon}\textstyle\sum_j\sum_r \Big(l_{j,r}(\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r}\Big) \tfrac{1}{|\Omega_j|}\textstyle\int_{\Omega_j} p^\varepsilon dx \\ &+\tfrac{1}{\varepsilon}\textstyle\sum_j \Big(\textstyle\sum_r l_{jr}\mathbf{n}_{jr}(p_{j,r}^\varepsilon - p_j^\varepsilon) \ , \ \tfrac{1}{|\Omega_j|}\textstyle\int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}\Big) \\ &+\tfrac{\sigma}{\varepsilon^2}\Big(\textstyle\sum_r\sum_j \widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon \ , \ \tfrac{1}{|\Omega_j|}\textstyle\int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}\Big). \end{aligned}$$

One gets

$$\begin{aligned} D_3 = &\tfrac{1}{\varepsilon}\textstyle\sum_j\sum_r \Big(l_{j,r}(\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r}\Big)\delta_{j,r}(p^\varepsilon) + \tfrac{1}{\varepsilon}\textstyle\sum_j\sum_r \Big(l_{jr}\mathbf{n}_{jr}(p_{j,r}^\varepsilon - p_j^\varepsilon) \ , \ \delta_{j,r}(\mathbf{u}^\varepsilon)\Big) \\ &+\tfrac{1}{\varepsilon}\textstyle\sum_j\sum_r \Big(l_{j,r}(\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r}\Big) p^\varepsilon(\mathbf{x}_r) + \tfrac{1}{\varepsilon}\textstyle\sum_j\sum_r \Big(l_{jr}\mathbf{n}_{jr}(p_{j,r}^\varepsilon - p_j^\varepsilon) \ , \ \mathbf{u}^\varepsilon(\mathbf{x}_r)\Big) \\ &+\tfrac{\sigma}{\varepsilon^2}\Big(\textstyle\sum_r\sum_j \widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon \ , \ \tfrac{1}{|\Omega_j|}\textstyle\int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}\Big). \end{aligned}$$

We have the identities $\sum_{j,r} l_{jr}\mathbf{n}_{jr} = 0$ and $\sum_j l_{jr}\mathbf{n}_{jr}p_{j,r}^\varepsilon = 0$ by definition (45). Therefore one can simplify the third and fourth term in the previous expression and get

$$\begin{aligned} D_3 = &\frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}(\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r}\Big)\delta_{j,r}(p^\varepsilon) + \frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{jr}\mathbf{n}_{jr}(p_{j,r}^\varepsilon - p_j^\varepsilon) \ , \ \delta_{j,r}(\mathbf{u}^\varepsilon)\Big) \\ &-\frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}\mathbf{u}_j^\varepsilon, \mathbf{n}_{j,r}\Big) p^\varepsilon(\mathbf{x}_r) - \frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}p_j^\varepsilon\mathbf{n}_{j,r} \ , \ \mathbf{u}^\varepsilon(\mathbf{x}_r)\Big) \\ &+\frac{\sigma}{\varepsilon^2}\Big(\sum_r\sum_j \widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon \ , \ \frac{1}{|\Omega_j|}\int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}\Big). \end{aligned}$$

We now look at D_4. By definition, one has

$$D_4 = \frac{1}{\varepsilon}\sum_j p_j^\varepsilon \sum_r \int_{\Gamma_{j,r}} (\mathbf{u}^\varepsilon, \tilde{\mathbf{n}}_{j,r}) d\sigma + \frac{1}{\varepsilon}\sum_j \Big(\mathbf{u}_j^\varepsilon, \Big(\sum_r \int_{\Gamma_{j,r}} p^\varepsilon \tilde{\mathbf{n}}_{j,r} d\sigma + \frac{\sigma}{\varepsilon}\int_{\Omega_j} \mathbf{u}^\varepsilon d\mathbf{x}\Big)\Big)$$

where $\tilde{\mathbf{n}}_{j,r}$ is the normal to the edge $\Gamma_{j,r} = [\mathbf{x}_r, \mathbf{x}_{r+1}]$ oriented toward the outside of the cell j. This expression needs an important manipulation which is to approximate the integral on edges by corner values. This necessary manipulation is one of the ideas that was introduced in [22] in order to proceed to the numerical analysis of such corner based finite volume schemes. This is why interpolation terms $\tilde{\delta}_{j,r}(h) = \frac{1}{|\Gamma_{jr}|}\int_{\Gamma_{j,r}} h - \frac{h(\mathbf{x}_r)+h(\mathbf{x}_{r+1})}{2}$ are introduced. One gets after an algebraic manipulation

$$D_4 = \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}| p_j^\varepsilon \Big(\tilde{\mathbf{n}}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon)\Big) + \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}| \Big(\mathbf{u}_j^\varepsilon, \tilde{\mathbf{n}}_{j,r}\tilde{\delta}_{j,r}(p^\varepsilon)\Big) + \frac{\sigma}{\varepsilon^2}\sum_j \Big(\mathbf{u}_j^\varepsilon, \int_{\Omega_j}\mathbf{u}^\varepsilon\Big)$$
$$+ \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}| p_j^\varepsilon \Big(\tilde{\mathbf{n}}_{j,r}, \frac{\mathbf{u}^\varepsilon(\mathbf{x}_r)+\mathbf{u}^\varepsilon(\mathbf{x}_{r+1})}{2}\Big) + \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}| \Big(\mathbf{u}_j^\varepsilon, \tilde{\mathbf{n}}_{j,r}\frac{p^\varepsilon(\mathbf{x}_r)+p^\varepsilon(\mathbf{x}_{r+1})}{2}\Big)$$

By definition (43), $\mathbf{n}_{jr}l_{jr} = \frac{\tilde{n}_{j,r}|\Gamma_{j,r}|+\tilde{n}_{j,r-1}|\Gamma_{j,r-1}|}{2}$, so one can see that

$$\sum_j\sum_r |\Gamma_{j,r}| p_j^\varepsilon \Big(\tilde{\mathbf{n}}_{j,r}, \frac{\mathbf{u}^\varepsilon(\mathbf{x}_r)+\mathbf{u}^\varepsilon(\mathbf{x}_{r+1})}{2}\Big) = \sum_j\sum_r l_{jr} p_j^\varepsilon \big(\mathbf{n}_{jr}, \mathbf{u}^\varepsilon(\mathbf{x}_r)\big).$$

It yields a slightly simpler expression

$$D_4 = \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}| p_j^\varepsilon \Big(\tilde{\mathbf{n}}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon)\Big) + \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}| \Big(\mathbf{u}_j^\varepsilon, \tilde{\mathbf{n}}_{j,r}\tilde{\delta}_{j,r}(p^\varepsilon)\Big) + \frac{\sigma}{\varepsilon^2}\sum_j \Big(\mathbf{u}_j^\varepsilon, \int_{\Omega_j}\mathbf{u}^\varepsilon\Big)$$
$$+ \frac{1}{\varepsilon}\sum_j\sum_r l_{jr} p_j^\varepsilon \big(\mathbf{n}_{jr}, \mathbf{u}^\varepsilon(\mathbf{x}_r)\big) + \frac{1}{\varepsilon}\sum_j\sum_r l_{jr} p^\varepsilon(\mathbf{x}_r) \big(\mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon\big)$$

One can now compute the sum $D_3 + D_4$

$$D_3 + D_4 = \frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}(\mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon), \mathbf{n}_{j,r}\Big)\delta_{j,r}(p^\varepsilon) + \frac{1}{\varepsilon}\sum_j\sum_r \Big(l_{j,r}\mathbf{n}_{j,r}(p_{j,r}^\varepsilon - p_j^\varepsilon)\ ,\ \delta_{j,r}(\mathbf{u}^\varepsilon)\Big)$$
$$+ \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{jr}| p_j^\varepsilon \Big(\mathbf{n}_{j,r}, \tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon)\Big) + \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{jr}| \Big(\mathbf{u}_j^\varepsilon, \mathbf{n}_{j,r}\tilde{\delta}_{j,r}(p^\varepsilon)\Big)$$
$$+ \frac{\sigma}{\varepsilon^2}\Big(\sum_r\sum_j \widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\ ,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\Big) + \frac{\sigma}{\varepsilon^2}\sum_j \Big(\mathbf{u}_j^\varepsilon, \int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\Big).$$

One finally gets after rearrangement the final result (64) for $\mathscr{E}'(t) = D_1 + D_2 + D_3 + D_4$. $\square$

Before going further in the examination of each term in the right hand side of (64), it is worthwhile to notice that terms like $\delta_{jr}(\dots)$ and $\widetilde{\delta}_{jr}(\dots)$ are small in some sense. For this we recall the results taken from [22], chapter 4:

Proposition 3.6. *For any funtion q in $H^3(\Omega)$, using Sobolev embeddings, one has the inequalities*

$$|\delta_{jr}(q)| \leq C\|q\|_{H^1(\Omega_j)} \tag{65}$$

and

$$|\widetilde{\delta}_{jr}(q)| \leq Ch^{3/2}\|q\|_{H^3(\Omega_j)} \tag{66}$$

Our aim is to now examine each term in the right hand side of (64). Its first line is already non positive. We look at the second line of (64) which we call E_1.

Proposition 3.7. *One has the bound with a constant Cproportional to $\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\Omega)}$*

$$\int_0^T E_1(t)dt \leq \frac{\gamma}{\varepsilon}\int_0^T \sum_{j,r} l_{j,r}(\boldsymbol{n}_{j,r}, \boldsymbol{u}_j^\varepsilon - \boldsymbol{u}_r^\varepsilon)^2 + Ch(\frac{T}{\gamma} + 1). \tag{67}$$

Proof. We use a Young's inequality $ab \leq \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2$,with some positive constantγ which will be defined later, for the second term and the definition of the fluxes (45) for the third term: we get

$$\begin{aligned} E_1 \leq & \frac{\gamma}{2\varepsilon}\sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \frac{1}{2\gamma\varepsilon}\sum_j\sum_r l_{j,r}\delta_{j,r}(p^\varepsilon)^2 \\ & + \frac{1}{\varepsilon}\sum_j\sum_r \left(\widehat{\alpha}_{j,r}(\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)\,,\,\delta_{j,r}(\mathbf{u}^\varepsilon)\right) - \frac{1}{\varepsilon}\sum_j\sum_r\left(\frac{\sigma}{\varepsilon}\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\,,\,\delta_{j,r}(\mathbf{u}^\varepsilon)\right)\end{aligned}$$

By definition of $\widehat{\alpha}_{j,r}$, one rewrites

$$\sum_j\sum_r \left(\widehat{\alpha}_{j,r}(\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)\,,\,\delta_{j,r}(\mathbf{u}^\varepsilon)\right) = \sum_j\sum_r l_{jr}(\mathbf{n}_{jr}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)(\mathbf{n}_{jr}, \delta_{j,r}(\mathbf{u}^\varepsilon)).$$

Another use of Young's inequality with the same coefficient γ for this term yields

$$\begin{aligned} E_1 \leq & \frac{\gamma}{\varepsilon}\sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \frac{1}{2\gamma\varepsilon}\sum_j\sum_r l_{j,r}\delta_{j,r}(p^\varepsilon)^2 \\ & + \frac{1}{2\gamma\varepsilon}\sum_j\sum_r \ \delta_{j,r}(\mathbf{u}^\varepsilon)^2 - \frac{1}{\varepsilon}\sum_j\sum_r\left(\frac{\sigma}{\varepsilon}\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\,,\,\delta_{j,r}(\mathbf{u}^\varepsilon)\right)\end{aligned}$$

We now look at the last term of (67), which we call W. One has, by definition of $\widehat{\beta}_{j,r}$

$$W = -\frac{1}{\varepsilon}\sum_j\sum_r\left(\frac{\sigma}{\varepsilon}\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\,,\,\delta_{j,r}(\mathbf{u}^\varepsilon)\right) = -\frac{\sigma}{\varepsilon^2}\sum_j\sum_r\left((l_{jr})^{\frac{1}{2}}\mathbf{n}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j)\mathbf{u}_r^\varepsilon\,,\,(l_{jr})^{\frac{1}{2}}\delta_{j,r}(\mathbf{u}^\varepsilon)\right)$$

Using the Cauchy-Schwarz inequality, we get

$$|W| \leq \frac{\sigma}{\varepsilon^2}\left(\sum_j\sum_r l_{jr}\left|\mathbf{n}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j)\mathbf{u}_r^\varepsilon\right|^2\right)^{\frac{1}{2}}\left(\sum_j\sum_r l_{jr}|\delta_{j,r}(\mathbf{u}^\varepsilon)|^2\right)^{\frac{1}{2}}$$

Using the assumptions of the mesh, there exist a constant $C > 0$ such that $|\mathbf{x}_r - \mathbf{x}_j| \leq Ch$. Using successively assumptions (52),(53) and (55), we get

$$|W| \leq C\frac{\sigma h}{\varepsilon^2}\left(\sum_r (B_r\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon)\right)^{\frac{1}{2}}\left(\sum_j\sum_r|\delta_{j,r}(\mathbf{u}^\varepsilon)|^2\right)^{\frac{1}{2}}.$$

and therefore

$$|W| \leq C\frac{\sigma h}{2\varepsilon^2}\left(\sum_r (B_r\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) + \sum_j\sum_r|\delta_{j,r}(\mathbf{u}^\varepsilon)|^2\right). \tag{68}$$

so that

$$\begin{aligned} E_1 \leq & \frac{\gamma}{\varepsilon}\sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \frac{1}{2\gamma\varepsilon}\sum_j\sum_r l_{j,r}\delta_{j,r}(p^\varepsilon)^2 \\ & + \frac{1}{2\gamma\varepsilon}\sum_j\sum_r l_{j,r}\left(\mathbf{n}_{j,r}\,,\,\delta_{j,r}(\mathbf{u}^\varepsilon)\right)^2 + C\frac{\sigma h}{2\varepsilon^2}\left(\sum_r (B_r\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) + \sum_j\sum_r|\delta_{j,r}(\mathbf{u}^\varepsilon)|^2\right)\end{aligned} \tag{69}$$

Furthermore, using the classical interpolation results of proposition 3.6 , there exists another constant $C \geq 0$ such that

$$\sum_j\sum_r(\delta_{j,r}\mathbf{u}^\varepsilon)^2 \leq C||\mathbf{u}^\varepsilon||^2_{H^1(\Omega)}.$$

and

$$\sum_j \sum_r l_{j,r}(\delta_{j,r}p^\varepsilon)^2 \le Ch||p^\varepsilon||^2_{H^1(\Omega)}.$$

So we obtain, after redefinition of all the constants C

$$\int_0^T E_1 dt \le \int_0^T \frac{\gamma}{\varepsilon}\sum_{j,r} l_{j,r}(\mathbf{n}_{j,r},\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \frac{Ch}{2\gamma\varepsilon}\Big(||\mathbf{u}^\varepsilon||^2_{H^1(\Omega)} + ||p^\varepsilon||^2_{H^1(\Omega)}\Big) + C\frac{h\sigma}{2\varepsilon^2}\Big(\sum_r (B_r\mathbf{u}_r^\varepsilon, \mathbf{u}_r^\varepsilon) + ||\mathbf{u}^\varepsilon||^2_{H^1(\Omega)}\Big)dt.$$

Using energy estimate (2) for the the second term of the rhs of the above inequality, (57) for the third term and (3) for the last term, one gets finally after simplifications and up to another redefinition of the constant C(which is proportional to the $H^1(\Omega)$ norm of the initial data $\mathbf{V}_0$ and is independent of the parameter σ)

$$\int_0^T E_1(t)dt \le \frac{\gamma}{\varepsilon}\int_0^T \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r},\mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + Ch(\frac{1}{\gamma}+1). \tag{70}$$

□

Now we consider the third line of $\mathscr{E}'(t)$, which we call E_2.

Proposition 3.8. *One has the bound*

$$\int_0^T E_2(t)dt \le \frac{CTh}{\varepsilon}. \tag{71}$$

where the constant C depends on $\|\mathbf{V}^\varepsilon(0)\|^2_{H^3(\Omega)}$

Proof. $E_2 = A + B$ is made of two contributions. Making use of the second set of inequalities of proposition 3.6, one gets concerning the first one

$$|A| = \left|\frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}|p_j^\varepsilon(\mathbf{n}_{j,r},\tilde{\delta}_{j,r}(\mathbf{u}^\varepsilon))\right| \le \frac{C}{\varepsilon}\sum_j h^{5/2}|p_j^\varepsilon|\|\mathbf{V}^\varepsilon(t)\|_{H^3(\Omega_j)}.$$

And using the inequality $ab \le \frac{1}{2}(a^2+b^2)$

$$|A| \le \frac{C_1}{\varepsilon}\sum_j h^3|p_j^\varepsilon|^2 + \frac{C_2}{\varepsilon}\sum_j h^2\|\mathbf{V}^\varepsilon(t)\|^2_{H^3(\Omega_j)} \le \frac{hC_3}{\varepsilon}\|\mathbf{V}_h^\varepsilon(t)\|^2_{L^2(\Omega)} + \frac{C_2h^2}{\varepsilon}\|\mathbf{V}^\varepsilon(t)\|^2_{H^3(\Omega)}.$$

The L^2 stability (57) of the scheme $\mathbf{P}_h^\varepsilon$ yields that $\|\mathbf{V}_h^\varepsilon(t)\|^2_{L^2(\Omega)} \le \|\mathbf{V}_h^\varepsilon(0)\|^2_{L^2(\Omega)} \le \|\mathbf{V}^\varepsilon(0)\|^2_{L^2(\Omega)} + C\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\Omega)}$. And with the basic energy estimate (2) , and since h is bounded, we obtain with a constant C_4 proportional to $\|\mathbf{V}^\varepsilon(0)\|^2_{H^3(\Omega)}$

$$\int_0^T |A|dt \le C_4\frac{hT}{\varepsilon}.$$

For the second contribution

$$B = \frac{1}{\varepsilon}\sum_j\sum_r |\Gamma_{j,r}|\Big(\mathbf{u}_j^\varepsilon, \mathbf{n}_{j,r}\tilde{\delta}_{j,r}(p^\varepsilon)\Big),$$

by similar calculations we obtain

$$\int_0^T |B|dt \le C_5\frac{hT}{\varepsilon},$$

with a constant C_5 proportional to $\|\mathbf{V}^\varepsilon(0)\|^2_{H^3(\Omega)}$. Taking $C = 2\max(C_4, C_5)$ yields (71). □

We now study the two last lines of $\mathscr{E}'(t)$, which we call S.

Proposition 3.9. *There exists a constant $C > 0$ proportional to $\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\Omega)}$ such that one has for all $\hat{\gamma} > 0$*

$$\int_0^T Sdt \le C\left(h^2 + \frac{h}{\widehat{\gamma}\varepsilon}\right) + \frac{\sigma\hat{\gamma}}{2\varepsilon}\int_0^T \sum_r \sum_j l_{jr}\left(\mathbf{n}_{jr}\ ,\ \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right)^2 dt \tag{72}$$

Proof. These two last lines write

$$\begin{aligned} S =& \frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\ ,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) + \frac{\sigma}{\varepsilon^2}\sum_j\left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j}\mathbf{u}^\varepsilon\right) \\ &- \frac{\sigma}{\varepsilon^2}\sum_j\int_{\Omega_j}(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon)d\mathbf{x} - \frac{\sigma}{\varepsilon^2}\sum_j\sum_r(\widehat{\beta}_j\mathbf{u}_r^\varepsilon,\mathbf{u}_r^\varepsilon). \end{aligned}$$

Using the Cauchy-Schwarz inequality on the third term $\int(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon)$, one gets

$$\begin{aligned} S \le& \frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\ ,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) + \frac{\sigma}{\varepsilon^2}\sum_j\left(\mathbf{u}_j^\varepsilon, \int_{\Omega_j}\mathbf{u}^\varepsilon\right) \\ &- \frac{\sigma}{\varepsilon^2}\sum_j\frac{1}{|\Omega_j|}\left(\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)^2 - \frac{\sigma}{\varepsilon^2}\sum_j\sum_r(\widehat{\beta}_j\mathbf{u}_r^\varepsilon,\mathbf{u}_r^\varepsilon), \end{aligned}$$

which can be written

$$S \le -\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\mathbf{u}_r^\varepsilon\ ,\ \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) - \frac{\sigma}{\varepsilon^2}\sum_j\left(\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}, \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x} - \mathbf{u}_j^\varepsilon\right).$$

We can rewrite this inequality on the form

$$\begin{aligned} S \le& -\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\left(\mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)\ ,\ \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) \\ &- \frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\ ,\ \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) \\ &- \frac{\sigma}{\varepsilon^2}\sum_j\left(\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}, \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x} - \mathbf{u}_j^\varepsilon\right). \end{aligned}$$

One has, using the geometric identity $\sum_r \widehat{\beta}_{jr} = |\Omega_j| I_d$ which can be found in [5, 11],

$$\begin{aligned} \sum_r\sum_j\left(\widehat{\beta}_{j,r}\frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\ ,\ \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) =& \sum_r\sum_j\left(\widehat{\beta}_{j,r}\frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\ ,\ \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right) \\ &- \sum_j\left(\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\ ,\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x} - \mathbf{u}_j^\varepsilon\right). \end{aligned}$$

We thus get

$$S \le \left.\begin{array}{l} -\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\left(\mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)\ ,\ \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right) \\ -\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\ ,\ \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right). \end{array}\right| \begin{array}{l} = S_1 \\ = S_2 \end{array} \tag{73}$$

We add and subtract at each average on the cell the nodal value. We recall the notation $\delta_{j,r}(\mathbf{u}^\varepsilon) = \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x} - \mathbf{u}^\varepsilon(\mathbf{x}_r)$. We get for the term under the first sum in (73)

$$\left(\widehat{\beta}_{j,r}\left(\mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)\ ,\ \mathbf{u}_r^\varepsilon - \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)$$

$$= \left(\widehat{\beta}_{j,r}\left(\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right),\ \mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right) - \left(\widehat{\beta}_{j,r}\left(\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right),\ \delta_{j,r}(\mathbf{u}^\varepsilon)\right)$$

$$-\left(\widehat{\beta}_{j,r}\delta_{j,r}(\mathbf{u}^\varepsilon),\ \mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right) + \left(\widehat{\beta}_{j,r}\delta_{j,r}(\mathbf{u}^\varepsilon),\ \delta_{j,r}(\mathbf{u}^\varepsilon)\right). \tag{74}$$

The first of these quantities is purely nodal, so one has

$$\sum_j \sum_r \left(\widehat{\beta}_{j,r}\left(\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right),\ \mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right)$$

$$= \sum_r \left(B_r\left(\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right),\ \mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)\right) \geq \alpha \sum_r |V_r||\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)|^2 \tag{75}$$

with the help of (55).

The second and third term in the identity (74) can be bounded by a Young's inequality with a convenient constant so that all terms containing $\mathbf{u}_r^\varepsilon - u^\varepsilon(\mathbf{x}_r)$ are controlled by (75). So we obtain concerning S_1 defined in (73)

$$S_1 \leq C\frac{h^2\sigma}{\varepsilon^2}\sum_r\sum_j |\delta_{j,r}(\mathbf{u}^\varepsilon)|^2 .$$

Using the standard interpolation result stressed in proposition 3.6, one has in dimension two $|\delta_{j,r}(\mathbf{u}^\varepsilon)| \leq C_1\|\mathbf{u}^\varepsilon(t)\|^2_{H^1(\Omega_j)}$. So, taking into account energy estimate (3) we have for the first term

$$\int_0^T S_1 dt \leq C_2 h^2 \|\mathbf{V}^\varepsilon(0\|^2_{H^1(\Omega)}.$$

We now consider the second term called S_2 in (73)

$$S_2 = -\frac{\sigma}{\varepsilon^2}\sum_r\sum_j\left(\widehat{\beta}_{j,r}\frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x},\ \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right).$$

Using $(\vec{a}\otimes\vec{b}\,\vec{c},\vec{d}) = (\vec{b},\vec{c})(\vec{a},\vec{d})$, one has

$$S_2 = -\frac{\sigma}{\varepsilon^2}\sum_r\sum_j l_{jr}\left((\mathbf{x}_r - \mathbf{x}_j),\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)\left(\mathbf{n}_{jr},\ \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right)$$

Using the Young's inequality $ab \leq \frac{\widehat{\gamma}\varepsilon}{2}a^2 + \frac{1}{2\widehat{\gamma}\varepsilon}b^2$, we get

$$\int_0^T S_2 dt \leq \frac{\widehat{\gamma}\sigma}{2\varepsilon}\int_0^T\sum_r\sum_j l_{jr}\left(\mathbf{n}_{jr}, \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right)^2 dt + \int_0^T\frac{\sigma}{2\widehat{\gamma}\varepsilon^3}\sum_r\sum_j l_{jr}\left((\mathbf{x}_r - \mathbf{x}_j),\ \frac{1}{|\Omega_j|}\int_{\Omega_j}\mathbf{u}^\varepsilon d\mathbf{x}\right)^2 dt$$

Using one more time the energy estimate (3) the second term in the right hand side of the above inequality is bounded by $\frac{C_3 h}{\widehat{\gamma}\varepsilon}\|\mathbf{V}^\varepsilon(0)\|^2_{L^2(\Omega)}$.

Thus

$$\int_0^T S dt \leq C_2 h^2\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\Omega)} + \frac{\widehat{\gamma}\sigma}{2\varepsilon}\int_0^T\sum_r\sum_j l_{jr}\left(\mathbf{n}_{jr},\ \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right)^2 dt + \frac{C_3 h}{\widehat{\gamma}\varepsilon}\|\mathbf{V}^\varepsilon(0)\|^2_{L^2(\Omega)}$$

One finally obtains the claim with a new constant $C \geq 0$ proportional to $\|\mathbf{V}^\varepsilon(0)\|^2_{H^1(\Omega)}$. $\square$

End of the proof of the proposition (3.4). One gets

$$\mathscr{E}(T) \le \mathscr{E}(0) - \frac{1}{\varepsilon}\int_0^T \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + \int_0^T E_1(t)dt + \int_0^T E_2(t)dt + \int_0^T S(t)dt$$

where integrals are estimated in (67), (71) and (72). Using $\mathscr{E}(0) = O(h)$, it yields with some new constant C

$$\begin{aligned}\mathscr{E}(T) \;& \le Ch \\ & -\frac{1}{\varepsilon}\int_0^T \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 \\ & +\frac{\gamma}{\varepsilon}\int_0^T \sum_{j,r} l_{j,r}(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon)^2 + Ch(\frac{1}{\gamma}+1) \\ & +CT\frac{h}{\varepsilon} \\ & +C(h^2 + \frac{h}{\widehat{\gamma}\varepsilon}) + \frac{\sigma\widehat{\gamma}}{2\varepsilon}\int_0^T \sum_r\sum_j l_{jr}\left(\mathbf{n}_{jr}\ ,\ \mathbf{u}_r^\varepsilon - \mathbf{u}_j^\varepsilon\right)^2\end{aligned}.$$

Therefore

$$\mathscr{E}(T) \le Ch\left(2 + \frac{1}{\gamma} + \frac{T}{\varepsilon} + h + \frac{1}{\varepsilon\widehat{\gamma}}\right) - \frac{1}{\varepsilon}\left(1 - \frac{1}{\gamma} - \frac{\sigma\widehat{\gamma}}{2}\right)\int_0^T \sum_{j,r} l_{j,r}\left(\mathbf{n}_{j,r}, \mathbf{u}_j^\varepsilon - \mathbf{u}_r^\varepsilon\right)^2 dt.$$

This estimate is fundamental, since it shows the competition between different kind of error terms and the dissipation of the fluxes. Choosing by example $\widehat{\gamma} < \frac{1}{\sigma}$ and $\gamma < \frac{1}{2}$, the last term is non positive, which means that the dissipation of the fluxes is dominant in some sense. We choose $\widehat{\gamma} = \frac{1}{\sigma}$ and $\gamma = \frac{1}{2}$, the last term vanishes thus

$$\mathscr{E}(T) \le Ch\left(3 + \frac{T}{\varepsilon} + h + \frac{\sigma}{\varepsilon}\right)$$

Rereading the proof one can check that the constant C is proportional to $\|\mathbf{V}^\varepsilon(0)\|^2_{H^3(\Omega)}$, and if $\sigma = 0$ one recovers the result of Mazeran [22], chapter 4. The proof was more difficult to obtain due to the non standard discretization of the source term and its incorporation in the approximate nodal Riemann solver.

Since h and ε can be taken less than 1, elementary comparison principles yield $\mathscr{E}(t) \le C'(T,\sigma)\|\mathbf{V}^\varepsilon(0)\|_{H^3(\Omega)}\dfrac{h}{\varepsilon}$. The proof is ended. □

3.5 Study of $\|\mathbf{DA}_h^\varepsilon - \mathbf{P}^0\|$

We consider the semi-discrete scheme (49) wherein for convenience we made the following change of unknowns

$$\bar{\mathbf{u}}_r^\varepsilon = \frac{\mathbf{u}_r^\varepsilon}{\varepsilon} \text{ and } \bar{\mathbf{u}}_j^\varepsilon = \frac{\mathbf{u}_j^\varepsilon}{\varepsilon}. \tag{76}$$

But in order to keep a simple notation we dropped the superscript ε and the bars. Thus the scheme (49) is now written as:

$$\begin{cases} \mid \Omega_j \mid \dfrac{d}{dt}p_j + \sum_r (l_{jr}\mathbf{u}_r, \mathbf{n}_{jr}) = 0 \\ \sum_r l_{jr}(\mathbf{n}_{jr}, \mathbf{u}_r - \mathbf{u}_j)\mathbf{n}_{jr} = 0 \\ \left(\varepsilon\sum_j \widehat{\alpha}_{jr} + \sigma B_r\right)\mathbf{u}_r = \sum_j l_{jr}p_j\mathbf{n}_{jr} + \varepsilon\sum_j \widehat{\alpha}_{jr}\mathbf{u}_j \end{cases} \tag{77}$$

Remark 3.10. *If wet set $\varepsilon = 0$ we naturally recover the limit diffusion scheme (48).*

Well-posedness

What we mean about well-posedness is the following: if we are able to write the last two relations of (77) as a non singular linear system with the $\mathbf{u}_r$'s and $\mathbf{u}_j$'s as unknowns, then we have a unique solution in terms of the p_j's. This notion is the relevant one for numerical discretization.

Let us denote $Y = (\{\mathbf{u}_j\}, \{\mathbf{u}_r\})$ the vector of unknowns. We can write the last two relations of (77) as $MY = b$ where M is a $(J+R)^2$ square matrix, J is the number of cells and R. One can observe that unless $\varepsilon = 0$, M is not a blockwise triangular matrix. One has

$$(MY, Y) = \sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) + \varepsilon \sum_j \sum_r l_{jr} \left(\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr}\right)^2$$

Assume $(MY, Y) = 0$: in this case the geometrical assumption (55) implies that all the $\mathbf{u}_r$ are null and therefore it remains to study $\sum_j \sum_r l_{jr} \left(\mathbf{u}_j, \mathbf{n}_{jr}\right)^2 = 0$ that is $\sum_j (\mathbf{u}_j, C_j \mathbf{u}_j) = 0$ where $C_j = \sum_r l_{jr} \mathbf{n}_{jr} \otimes \mathbf{n}_{jr}$. Since the C_j are all invertible unless the mesh is degenerate, all the $\mathbf{u}_j$ are null: we have proved the invertibility of the matrix M and thus the scheme (77) exists and is uniquely defined.

Stability

The L^2 norm of the solution is defined as $E(t) = \frac{1}{2} \sum_j |\Omega_j| p_j^2$.

Proposition 3.11. *Under the geometrical assumption (55), the diffusion approximation scheme (77) is stable in the L^2 norm, in the sense that $E'(t) \leq 0$. Moreover we have*

$$||\mathbf{u}_r||_{L^2([0,T]\times\Omega)} \leq C \left\| (p_j(0))_j \right\|_{L^2(\Omega)} \tag{78}$$

and

$$\varepsilon \int_{[0,T]} \sum_j \sum_r l_{jr} (\mathbf{n}_{jr}, (\mathbf{u}_j - \mathbf{u}_r))^2 \leq \left\| (p_j(0))_j \right\|_{L^2(\Omega)}. \tag{79}$$

Proof. One has

$$E^{'}(t) = \sum_j |\Omega_j| p_j \frac{d}{dt} p_j = -\sum_j p_j \sum_r (l_{jr} \mathbf{u}_r, \mathbf{n}_{jr}) = \sum_r \left(\mathbf{u}_r, \sum_j l_{jr} \mathbf{n}_{jr} p_j \right).$$

Now using the last equation of (77), one finds

$$E^{'}(t) = -\sum_r \left(\mathbf{u}_r, \left(\varepsilon \sum_j \widehat{\alpha}_{jr} + \sigma B_r \right) \mathbf{u}_r - \varepsilon \sum_j \widehat{\alpha}_{jr} \mathbf{u}_j \right)$$

We expand the right hand side of the previous equation

$$E^{'}(t) = -\sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) - \varepsilon \sum_r \left(\mathbf{u}_r, \sum_j \widehat{\alpha}_{jr} (\mathbf{u}_r - \mathbf{u}_j \right).$$

Permuting the sums in the second term of the right hand side , we show that

$$E^{'}(t) = -\sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) - \varepsilon \sum_j \sum_r (\mathbf{u}_r, \widehat{\alpha}_{jr} (\mathbf{u}_r - \mathbf{u}_j)). \tag{80}$$

Using the definition of the $\mathbf{u}_j$, second line of (77), one has

$$\sum_j \left(\mathbf{u}_j, \sum_r \widehat{\alpha}_{jr} (\mathbf{u}_r - \mathbf{u}_j) \right) = 0. \tag{81}$$

Combining (81)×ε with (80) and using the definition of the matrices $\widehat{\alpha}_{jr}$ one has finally

$$E^{'}(t) = -\sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) - \varepsilon \sum_j \sum_r l_{jr} (\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr})^2 .$$

By the geometrical assumption (55) we have $E^{'}(t) \leq 0$, that is the L^2 stability. By integrating this equality on $[0, T]$ we obtain

$$E(T) + \int_{[0,T]} \sum_r (\sigma B_r \mathbf{u}_r, \mathbf{u}_r) + \int_{[0,T]} \varepsilon \sum_j \sum_r l_{jr} (\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr})^2 = E(0)$$

Using again the geometrical assumption (55) for the terms $(B_r \mathbf{u}_r, \mathbf{u}_r)$ we have

$$E(T) + \alpha \int_{[0,T]} \sum_r |V_r| \, ||\mathbf{u}_r||^2 + \int_{[0,T]} \varepsilon \sum_j \sum_r l_{jr} (\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr})^2 \leq E(0)$$

which gives (78) and (79). □

Consistency

For convenience we set

$$\bar{p}_j = p(\mathbf{x}_j, t) \quad \bar{\mathbf{u}}_j = -\frac{1}{\sigma} \nabla p(\mathbf{x}_j, t) \quad \bar{\mathbf{u}}_r = -\frac{1}{\sigma} \nabla p(\mathbf{x}_r, t)$$

where $p(x,t)$ is the solution of the diffusion equation. We define the consistency error by inserting these quantities into the three equations of (77). It yields

$$\begin{cases} a_j = \dfrac{d}{dt} \bar{p}_j + \dfrac{1}{| \Omega_j |} \displaystyle\sum_r (l_{jr} \bar{\mathbf{u}}_r, \mathbf{n}_{jr}) \\ \mathbf{b}_r = \frac{1}{|V_r|} \left(\sigma B_r \bar{\mathbf{u}}_r - \sum_j l_{jr} \bar{p}_j \mathbf{n}_{jr} + \varepsilon \sum_j \widehat{\alpha}_{jr} (\bar{\mathbf{u}}_r - \bar{\mathbf{u}}_j) \right), \\ \mathbf{c}_j = \frac{1}{h} \sum_r l_{jr} (\mathbf{n}_{jr}, \bar{\mathbf{u}}_r - \bar{\mathbf{u}}_j) \mathbf{n}_{jr} = 0. \end{cases}$$

Lemma 3.12. *Assume the geometrical assumptions (3.2). Assume $p_0 \in H^4(\Omega)$. Then there exists a constant $C > 0$ such that the following estimates hold*

$$|a_j| \leq Ch \quad \textit{for all } j, \tag{82}$$

$$|\mathbf{b}_r| \leq C(h + \varepsilon), \quad \textit{for all } r. \tag{83}$$

and

$$|\mathbf{c}_j| \leq Ch, \quad \textit{for all } j. \tag{84}$$

Proof. Since $p_0 \in H^4(\Omega)$, one has that $p(t) \in H^4(\Omega)$ which turns into the fact that $\nabla p(t) \in L^\infty(\Omega)$ and $\nabla^2 p(t) \in L^\infty(\Omega)$ by means of Sobolev embeddings. It is sufficient to justify the Taylor expansions done hereafter. By construction

$$\partial_t p(\mathbf{x}_j, t) = \frac{1}{\sigma} \frac{\int_{\Omega_j} \partial_t p(x,t) dx}{|\Omega_j|} + O(h) = \frac{1}{\sigma} \frac{\int_{\Omega_j} \Delta p dx}{|\Omega_j|} + O(h) = \frac{1}{\sigma |\Omega_j|} \int_{\partial \Omega_j} \partial_n p d\sigma + O(h).$$

By definition of $l_{jr} \mathbf{n}_{jr}$ one has

$$\sum_r l_{jr} (\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r, t)) = \sum_k \int_{\partial \Omega_{jk}} \left(\frac{\nabla p \left(x_{jk}^+ \right) + \nabla p \left(x_{jk}^- \right)}{2}, \mathbf{n}_j \right) d\sigma$$

where $n_j = \tilde{n}_{j,r}$ defined in the previous part and the nodes x_{jk}^+ and x_{jk}^- are the end of the edge $\partial\Omega_{jk} = \Omega_j \bigcap \Omega_k$. Note that $\partial\Omega_j = \bigcup \partial\Omega_{jk}$. Therefore

$$a_j = O(h) + \frac{1}{\sigma}\frac{1}{|\Omega_j|}\sum_k \int_{\partial\Omega_{jk}} \left(\nabla p - \frac{\nabla p\left(x_{jk}^+\right) + \nabla p\left(x_{jk}^-\right)}{2}, \mathbf{n}_j \right) d\sigma$$

Since the function under the integral is approximated by the trapezoidal rule, the error of integration is $O(h^2)$

$$\left| \int_{\partial\Omega_{jk}} \left(\nabla p - \frac{\nabla p\left(x_{jk}^+\right) + \nabla p\left(x_{jk}^-\right)}{2}, \mathbf{n}_j \right) d\sigma \right| \leq Ch^2 \left|\partial\Omega_{jk}\right| \leq Ch^3.$$

After division by $|\ \Omega_j\ |$ and using the lower bound of the regularity hypothesis (51), one gets that $a_j = O(h)$. Now we write $\mathbf{b}_r = \mathbf{b}_r^a + \mathbf{b}_r^b$ with

$$\begin{aligned}\mathbf{b}_r^a &= \frac{1}{|V_r|}\left(\left(\sigma \sum_j l_{jr}\mathbf{n}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j) \right) \left(-\frac{\nabla p(\mathbf{x}_r, t)}{\sigma} \right) - \sum_j l_{jr}\mathbf{n}_{jr} p(\mathbf{x}_j, t) \right) \\ &= \frac{1}{|V_r|}\sum_j \Big((\mathbf{x}_j - \mathbf{x}_r, \nabla p(\mathbf{x}_r, t)) - p(\mathbf{x}_j, t) \Big) l_{jr}\mathbf{n}_{jr}\end{aligned}$$

and

$$\mathbf{b}_r^b = \frac{\varepsilon}{\sigma |V_r|}\left(\sum_j l_{jr}\mathbf{n}_{jr} \otimes \mathbf{n}_{jr} \left(\nabla p(\mathbf{x}_j, t) - \nabla p(\mathbf{x}_r, t)\right) \right)$$

A simple Taylor expansion shows that

$$p(\mathbf{x}_j, t) + (\mathbf{x}_r - \mathbf{x}_j, \nabla p(\mathbf{x}_r, t)) = p(\mathbf{x}_r, t) + O_{jr}(h^2).$$

So $\mathbf{b}_r^a = \frac{1}{|V_r|}\sum_j O_{jr}(h^2) l_{jr}\mathbf{n}_{jr} = O(h)$ due to (50), (52). Another Taylor expansion shows that $\nabla p(\mathbf{x}_j, t) - \nabla p(\mathbf{x}_r, t) = O_{jr}(h)$ so that $||\mathbf{b}_r^b|| = O(\varepsilon)$ and $|\mathbf{b}_r| = O(h + \varepsilon)$. We finally obtain with similar arguments $|\mathbf{c}_j| = O(h)$. It ends the proof of the lemma. □

Convergence

Let us define three error variables

$$e_j = p_j - \bar{p}_j \ ,\ \mathbf{f}_r = \mathbf{u}_r - \bar{\mathbf{u}}_r \text{ and } \mathbf{f}_j = \mathbf{u}_j - \bar{\mathbf{u}}_j$$

The numerical error is $E(t) = \frac{1}{2}\|e\|_{L^2(\Omega)}^2 = \sum_j |\Omega_j| \left(p_j - p(\mathbf{x}_j, t)\right)^2$. We will also consider $F(t) = \|\mathbf{f}\|_{L^2([0,t]\times\Omega)}^2 = \int_0^t \sum_r |V_r|\,|\mathbf{f}_r|^2$, and $\|\mathbf{g}\|_{L^2([0,t]\times\Omega)}^2 = \int_0^t \sum_j |\Omega_j|\ \ |\mathbf{f}_j|^2$.

Theorem 3.13. *Assume $p \in W^{3,\infty}(\Omega)$ and assume that geometrical conditions (3.2) are verified. There exists a constant $C(T) > 0$ such that*

$$\|e\|_{L^\infty([0,T]:L^2(\Omega))} \leq C(T)(h + \varepsilon), \tag{85}$$

$$\|\mathbf{f}\|_{L^2([0,T]\times\Omega)} \leq C(T)(h + \varepsilon), \tag{86}$$

$$\varepsilon \int_0^T \sum_j \sum_r l_{jr}\left(\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr}\right)^2 \leq C(T)(h + \varepsilon)^2, \tag{87}$$

and

$$\|\mathbf{g}\|_{L^2([0,T]\times\Omega)} \leq C(T)(h + \varepsilon)\sqrt{(1 + \frac{h}{\varepsilon})}. \tag{88}$$

Proof. By construction

$$\begin{cases} \mid \Omega_j \mid e_j' + \sum_r (l_{jr}\mathbf{f}_r, \mathbf{n}_{jr}) = - \mid \Omega_j \mid a_j \\ \left(\varepsilon \sum_j \widehat{\alpha}_{jr} + \sigma B_r\right) \mathbf{f}_r - \sum_j l_{jr} e_j \mathbf{n}_{jr} - \varepsilon \sum_j \widehat{\alpha}_{jr} \mathbf{f}_j = - \mid V_r \mid \mathbf{b}_r, \\ \sum_r l_{jr}(\mathbf{n}_{jr}, \mathbf{f}_r - \mathbf{f}_j)\mathbf{n}_{jr} = -h\mathbf{c}_j. \end{cases}$$

By proceeding as for the results of stability one has the identity

$$E'(t) = \sum_j |\Omega_j| e_j e_j' = \sum_j e_j \left(- \left(\sum_r l_{jr} \left(\mathbf{n}_{jr}, \mathbf{f}_r\right) \right) - |\Omega_j| \, a_j \right)$$

$$= - \sum_r \sum_j (l_{jr} \mathbf{n}_{jr} e_j, \mathbf{f}_r) - \sum_j |\Omega_j| \, e_j a_j$$

$$= - \sum_r \left(\mathbf{f}_r, \left(\varepsilon \sum_j \widehat{\alpha}_{jr} + \sigma B_r \right) \mathbf{f}_r - \varepsilon \sum_j \widehat{\alpha}_{jr} \mathbf{f}_j \right) - \sum_j |\Omega_j| \, a_j e_j - \sum_r |V_r| \, \mathbf{b}_r \mathbf{f}_r$$

$$= - \sum_r \left(\sigma B_r \mathbf{f}_r, \mathbf{f}_r\right) - \varepsilon \sum_r \left(\mathbf{f}_r, \sum_j \widehat{\alpha}_{jr} (\mathbf{f}_r - \mathbf{f}_j \right) - \sum_j |\Omega_j| \, a_j e_j - \sum_r |V_r| \, \mathbf{b}_r \mathbf{f}_r$$

$$= - \sum_r \left(\sigma B_r \mathbf{f}_r, \mathbf{f}_r\right) - \varepsilon \sum_j \sum_r l_{jr} \left(\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr}\right)^2 - \sum_j |\Omega_j| \, a_j e_j - \sum_r |V_r| \, \mathbf{b}_r \mathbf{f}_r + \varepsilon \sum_j h \mathbf{f}_j \mathbf{c}_j.$$

Using a Young's inequality and the Cauchy-Schwarz inequality, one gets

$$\begin{aligned} E'(t) \leq & ||e||_{L^2(\Omega)} \, ||a||_{L^2(\Omega)} + \left(\frac{\mu}{2} ||\mathbf{f}||^2_{L^2(\Omega)} + \frac{1}{2\mu} ||\mathbf{b}||^2_{L^2(\Omega)} \right) - \alpha \|\mathbf{f}\|^2_{L^2(\Omega)} \\ & - \varepsilon \sum_j \sum_r l_{jr} \left(\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr}\right)^2 + \frac{\varepsilon}{2h} \left(\eta ||\mathbf{g}||^2_{L^2(\Omega)} + \frac{1}{\eta} ||\mathbf{c}||^2_{L^2(\Omega)} \right) \end{aligned} \tag{89}$$

where $\mu, \eta > 0$ are two arbitrary coefficients that will be specified later. Now using (54) we have

$$|\Omega_j| \; |\mathbf{f}_j|^2 \leq Ch \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{f}_j)^2.$$

Therefore

$$|\Omega_j| \; |\mathbf{f}_j|^2 \leq Ch \left(2 \sum_r l_{jr} (\mathbf{n}_{jr}, \mathbf{f}_j - \mathbf{f}_r)^2 + 2 \sum_r l_{jr} |\mathbf{f}_r|^2 \right).$$

So there exists two constant C_1 and C_2 such that

$$||\mathbf{g}||^2_{L^2(\Omega)} \leq C_1 h \sum_{jr} l_{jr} (\mathbf{n}_{jr}, \mathbf{f}_j - \mathbf{f}_r)^2 + C_2 ||\mathbf{f}||^2_{L^2(\Omega)}. \tag{90}$$

So from (89) we obtain

$$E'(t) \leq ||e||_{L^2(\Omega)} \, ||a||_{L^2(\Omega)} + \frac{1}{2\mu} ||\mathbf{b}||^2_{L^2(\Omega)} + (\frac{\mu}{2} + \frac{C_2 \varepsilon \eta}{2h} - \alpha) \|\mathbf{f}\|^2_{L^2(\Omega)}$$

$$+ (\frac{C_1}{2} \eta - 1) \varepsilon \sum_j \sum_r l_{jr} \left(\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr}\right)^2 + \frac{\varepsilon}{2h\eta} ||\mathbf{c}||^2, \qquad \forall \mu, \eta > 0.$$

Let us choose $\mu = \frac{\alpha}{2}$ and $\eta = \min(\frac{1}{C_1}, \frac{\alpha h}{C_2 \varepsilon})$, so that

$$E'(t) \leq \sqrt{2}\sqrt{E(t)}\ ||a||_{L^2(\Omega)} - \frac{\alpha}{4}F'(t) - \frac{\varepsilon}{2}\sum_j \sum_r l_{jr}\left(\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr}\right)^2 + \frac{1}{2\alpha}||\mathbf{b}||^2_{L^2(\Omega)} + \max(\frac{\varepsilon C_1}{2h}, \frac{C_2\varepsilon^2}{\alpha h^2})||\mathbf{c}||^2_{L^2(\Omega)}.$$

where we have used $\frac{1}{\min(a,b)} = \max(\frac{1}{a}, \frac{1}{b})$. By the consistency estimates (82-83-84) and since the domain Ω is bounded, one finds a constant C such that

$$E'(t) \leq E(t) - \frac{\alpha}{4}F'(t) - \frac{\varepsilon}{2}\sum_j \sum_r l_{jr}\left(\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr}\right)^2 + C\left(h+\varepsilon\right)^2. \tag{91}$$

Thus

$$E'(t) \leq E(t) + C\left(h+\varepsilon\right)^2.$$

By construction $E(0) = O(h^2)$ for a smooth initial data. So by the Grönwall lemma $E(t) \leq C(t)(h+\varepsilon)^2$ which gives (85). Integrating (91) in the time interval $[0,t]$, we find that for any for $t \leq T$

$$E(t) + \frac{\alpha}{4}F(t) + \int_0^t \frac{\varepsilon}{2}\sum_j \sum_r l_{jr}\left(\mathbf{f}_r - \mathbf{f}_j, \mathbf{n}_{jr}\right)^2 \leq E(0) + \int_0^t \left(E(t) + C(h+\varepsilon)^2\right)dt \leq C'(T)(h+\varepsilon)^2.$$

It shows the estimates (86) and (87). Using (90) one gets

$$\int_0^T ||\mathbf{g}||^2_{L^2(\Omega)} \leq C_1 h \int_0^T \sum_{jr} l_{jr}(\mathbf{n}_{jr}, \mathbf{f}_j - \mathbf{f}_r)^2 + C_2 \int_0^T ||\mathbf{f}||^2_{L^2(\Omega)} \leq C"(T)(h+\varepsilon)^2(1+\frac{h}{\varepsilon})$$

from which (88) follows. The proof is finished. □

Corollary 3.14. *From (85) and (86) we deduce*

$$\|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\|_{L^2([0,T]\times\Omega))} \leq C(T)(h+\varepsilon). \tag{92}$$

L^∞ stability of the derivative

This estimate is needed in the next section.

Proposition 3.15. *Consider a smooth enough initial p_0. One has*

$$\left\|\frac{d}{dt}p_h\right\|_{L^\infty([0,T];L^2(\Omega))} \leq C\max\left(1, \sqrt{\frac{\varepsilon}{h}}\right),$$

and

$$\left\|(\frac{d}{dt}\mathbf{u}_r)_r\right\|_{L^2([0,T]\times\Omega)} \leq C\max\left(1, \sqrt{\frac{\varepsilon}{h}}\right). \tag{93}$$

Proof. Let us denote the time derivative of any f as $\tilde{f} = \partial_t f$. By linearity of the system (77), one has

$$\begin{cases} \mid \Omega_j \mid \dfrac{d}{dt}\tilde{p}_j + \sum_r (l_{jr}\tilde{\mathbf{u}}_r, \mathbf{n}_{jr}) = 0 \\ \sum_r l_{jr}(\mathbf{n}_{jr}, \tilde{\mathbf{u}}_r - \tilde{\mathbf{u}}_j)\mathbf{n}_{jr} = 0 \\ \left(\varepsilon \sum_j \widehat{\alpha}_{jr} + \sigma B_r\right)\tilde{\mathbf{u}}_r = \sum_j l_{jr}\tilde{p}_j \mathbf{n}_{jr} + \varepsilon \sum_j \widehat{\alpha}_{jr}\tilde{\mathbf{u}}_j \end{cases}$$

The L^2 stability property yields

$$\|\tilde{p}_h(t)\|^2_{L^\infty([0,T];L^2(\Omega))} + \int_0^t \sum_r (B_r\tilde{\mathbf{u}}_r, \tilde{\mathbf{u}}_r)dt \leq \|\tilde{p}(0)\|_{L^2(\Omega)}$$

where this last quantity can be estimated with the first equation of (77): the square of the norm in (93) is also bounded by the same quantity. It remains to bound $\|\tilde{p}(0)\|_{L^2(\Omega))}$. At $t=0$ one has

$$\frac{d}{dt}p_j = \frac{1}{\sigma}\frac{1}{|\Omega_j|}\sum_r l_{jr}(\nabla p(\mathbf{x}_r), \mathbf{n}_{jr}) - \frac{1}{\sigma}\frac{1}{|\Omega_j|}\sum_r l_{jr}(\mathbf{q}_r, \mathbf{n}_{jr}).$$

Using the same consistency arguments as before, the first term in the right hand side is bounded in L^2 uniformly with respect to h and ε. The second one is bounded in L^2 by a term of order $C\frac{h\sqrt{\max(h^2,\varepsilon h)}}{h^2} = C\max\left(1, \sqrt{\frac{\varepsilon}{h}}\right)$. The proof is ended. □

Proposition 3.16. *One has that*

$$\left\| \left(\mathbf{u}_r + \frac{1}{\sigma}\nabla p(\mathbf{x}_r)\right)(t=0)\right\|_{L^2(\Omega)} \le C\sqrt{h\max(h,\varepsilon)}$$

and

$$\left\| \left(\mathbf{u}_j + \frac{1}{\sigma}\nabla p(\mathbf{x}_j)\right)(t=0)\right\|_{L^2(\Omega)} \le C\sqrt{\frac{h}{\varepsilon}}\max(h,\varepsilon). \tag{94}$$

Proof. Let us write $\mathbf{q}_r = \mathbf{u}_r + \frac{1}{\sigma}\nabla p(\mathbf{x}_r)$ and $\mathbf{s}_j = \mathbf{u}_j + \frac{1}{\sigma}\nabla p(\mathbf{x}_j)$. These quantities are solution of the system

$$\begin{cases} \left(\varepsilon\sum_j l_{jr}\mathbf{n}_{jr}\otimes\mathbf{n}_{jr} + \sigma B_r\right)\mathbf{q}_r & -\varepsilon\sum_j l_{jr}\mathbf{n}_{jr}\otimes\mathbf{n}_{jr}\mathbf{s}_j & = \mathbf{d}_r^1 + \mathbf{d}_r^2, & \forall r, \\ -\varepsilon\sum_r l_{jr}(\mathbf{n}_{jr},\mathbf{q}_r)\mathbf{n}_{jr} & +\varepsilon\sum_r l_{jr}(\mathbf{n}_{jr},\mathbf{s}_j)\mathbf{n}_{jr} & = \mathbf{d}_j, & \forall j, \end{cases}$$

where the right hand sides are

$$\mathbf{d}_r^1 = \sum_j l_{jr}p(\mathbf{x}_j)\mathbf{n}_{jr} + \sum_j l_{jr}(\mathbf{x}_r - \mathbf{x}_j, \nabla p(\mathbf{x}_r))\mathbf{n}_{jr},$$

$$\mathbf{d}_r^2 = \varepsilon\sum_j l_{jr}\left(\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)\right)\mathbf{n}_{jr}$$

and

$$\mathbf{d}_j = -\varepsilon\sum_r l_{jr}\left(\mathbf{n}_{jr}, \nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)\right).$$

Taking the scalar product of the first line by $\mathbf{q}_r$ and of the second line by $\mathbf{s}_j$, one gets the identity

$$\sigma\sum_r(B_r\mathbf{q}_r,\mathbf{q}_r) + \varepsilon\sum_{jr} l_{jr}\left(\mathbf{n}_{jr},\mathbf{q}_r - \mathbf{s}_j\right)^2$$

$$= \sum_r\left(\mathbf{d}_r^1,\mathbf{q}_r\right) + \varepsilon\sum_{jr} l_{jr}\left(\mathbf{n}_{jr},\mathbf{q}_r - \mathbf{s}_j\right)\left(\mathbf{n}_{jr},\nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)\right).$$

A Young's inequality yields

$$\sigma\sum_r(B_r\mathbf{q}_r,\mathbf{q}_r) + \frac{\varepsilon}{2}\sum_{jr} l_{jr}\left(\mathbf{n}_{jr},\mathbf{q}_r - \mathbf{s}_j\right)^2 \le \sum_r\left(\mathbf{d}_r^1,\mathbf{q}_r\right) + \frac{\varepsilon}{2}\sum_{jr} l_{jr}\left(\mathbf{n}_{jr},\nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)\right)^2. \tag{95}$$

One gets

$$\left|\mathbf{d}_r^1\right| \le Ch^2 \text{ and } \sum_{jr} l_{jr}\left(\mathbf{n}_{jr},\nabla p(\mathbf{x}_r) - \nabla p(\mathbf{x}_j)\right)^2 \le Ch.$$

So (95) implies

$$\|\mathbf{q}\|^2_{L^2(\Omega)} \le Ch\|\mathbf{q}\|_{L^2(\Omega)} + C\varepsilon h.$$

It means that $\|\mathbf{q}\|_{L^2(\Omega)}$ is below the maximal root of the polynomial $p(x) = x^2 - Chx - C\varepsilon h$, that is for some constant $K > 0$

$$\|\mathbf{q}\|_{L^2(\Omega)} \leq x^+ = \frac{Ch + \sqrt{C^2h^2 + 4C\varepsilon h}}{2} \leq K\sqrt{\max(h^2, h\varepsilon)}. \tag{96}$$

It yields the proof of the first inequality.

Concerning the second inequality, we note that

$$\|\mathbf{s}\|^2_{L^2(\Omega)} \leq 2\|\mathbf{q}\|^2_{L^2(\Omega)} + Ch \sum_{jr} l_{jr} \left(\mathbf{n}_{jr}, \mathbf{q}_r - \mathbf{s}_j\right)^2$$

which can be upper bounded using (95) and (96). We obtain

$$\|\mathbf{s}\|^2_{L^2(\Omega)} \leq C\max(h^2, h\varepsilon) + C\frac{h}{\varepsilon} h\sqrt{\max(h^2, h\varepsilon)}$$

The numbers h and ε can be considered less than 1. There are two cases:

- Either $h < \varepsilon$: so $\|\mathbf{s}\|^2_{L^2(\Omega)} \leq \widetilde{C} h\varepsilon$ for another constant $\widetilde{C}$.
- Or $\varepsilon \leq h$: so $\|\mathbf{s}\|^2_{L^2(\Omega)} \leq \widetilde{C}\frac{h^3}{\varepsilon}$ for another constant $\widetilde{C}$.

So we can writes

$$\|\mathbf{s}\|_{L^2(\Omega)} \leq C\sqrt{\frac{h}{\varepsilon}} \max(h, \varepsilon), \qquad C > 0.$$

The proof of (94) is ended. □

3.6 Study of $\|\mathbf{P}^\varepsilon_h - \mathbf{D}\mathbf{A}^\varepsilon_h\|$

In this section we prove that for fixed step mesh, the discrete solution of the heat equation scheme tends to the discrete solution of the diffusion equation scheme when ε tends to zero. Indeed, we will prove the following result:

Proposition 3.17. *Assume a smooth initial data. There exists a constant $C(T) > 0$ such that the following estimate holds:*

$$\|\mathbf{V}^\varepsilon_h - \mathbf{W}^\varepsilon_h\|_{L^2([0,T]\times\Omega))} \leq C(T) \left(\varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) + \left(h^2 + \varepsilon^2\right)\right). \tag{97}$$

Proof. We introduce $\mathbf{R}_j = \frac{d}{dt}\mathbf{u}_j$ such that the solution $\mathbf{V}_h$ of the diffusion scheme (49) satisfies

$$\begin{cases} \mid \Omega_j \mid \dfrac{d}{dt} p_j + \dfrac{1}{\varepsilon} \displaystyle\sum_r (l_{jr}\mathbf{n}_{jr}, \mathbf{u}_r) = 0, \\ \mid \Omega_j \mid \dfrac{d}{dt} \mathbf{u}_j + \dfrac{1}{\varepsilon} \displaystyle\sum_r (l_{jr} p_j \mathbf{n}_{jr} + \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r)) = |\Omega_j| \mathbf{R}_j, \\ \left(A_r + \dfrac{\sigma}{\varepsilon} B_r\right) \mathbf{u}_r - \displaystyle\sum_j l_{jr} p_j \mathbf{n}_{jr} - \sum_j \widehat{\alpha}_{jr} \mathbf{u}_j = 0. \end{cases} \tag{98}$$

By definition $\|\mathbf{R}\|_{L^2(\Omega)} = \|\frac{d}{dt}\mathbf{u}_j\|_{L^2(\Omega)}$. Using the third line of (49), one has $\mathbf{u}_j = A_j^{-1} \sum_r \widehat{\alpha}_{jr} \mathbf{u}_r$ and thus $\|\frac{d}{dt}\mathbf{u}_j\|_{L^2(\Omega)} \leq C\|\frac{d}{dt}\mathbf{u}_r\|_{L^2(\Omega)}$. Using (93) (and taking care that a rescaling by a factor ε was systematically used in the previous section, see (76)), one gets for a smooth initial data

$$\|\mathbf{R}\|_{L^2([0,T]\times\Omega)} \leq C\varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right).$$

We denote by $e_j = p_j - p_j^\varepsilon$, $\mathbf{f}_j = \mathbf{u}_j - \mathbf{u}_j^\varepsilon$ and $\mathbf{f}_r = \mathbf{u}_r - \mathbf{u}_r^\varepsilon$. One finds, making the difference between the schemes (98) and (44):

$$\begin{cases} \mid \Omega_j \mid \dfrac{d}{dt} e_j + \dfrac{1}{\varepsilon} \displaystyle\sum_r (l_{jr} \mathbf{n}_{jr}, \mathbf{f}_r) = 0, \\ \mid \Omega_j \mid \dfrac{d}{dt} \mathbf{f}_j + \dfrac{1}{\varepsilon} \displaystyle\sum_r (l_{jr} e_j \mathbf{n}_{jr} + \widehat{\alpha}_{jr}(\mathbf{f}_j - \mathbf{f}_r)) = |\Omega_j| \mathbf{R}_j, \\ \left(A_r + \dfrac{\sigma}{\varepsilon} B_r \right) \mathbf{f}_r - \displaystyle\sum_j l_{jr} e_j \mathbf{n}_{jr} - \sum_j \widehat{\alpha}_{jr} \mathbf{f}_j = 0. \end{cases}$$

We are going to write an inequality satisfied by $E(t) = \|e(t)\|^2_{L^2(\Omega)} + \|\mathbf{f}(t)\|^2_{L^2(\Omega)}$, knowing that $e(0) = 0$. Using the same kind of proof than for the L^2 stability of the JL-(b) scheme (proposition 3.3), one can show that

$$\frac{1}{2} \frac{d}{dt} E(t) \leq \sum_j |\Omega_j| (\mathbf{R}_j, \mathbf{f}_j) \leq \|\mathbf{f}\|_{L^2(\Omega)} \|\mathbf{R}\|_{L^2(\Omega)} \leq \sqrt{E(t)} \|\mathbf{R}\|_{L^2(\Omega)}.$$

So one has for $t \leq T$

$$\sqrt{E(t)} \leq \sqrt{E(0)} + Ct \|\mathbf{R}\|_{L^2([0,T]\times\Omega)} = \|\mathbf{f}(0)\|_{L^2(\Omega)} + Ct \|\mathbf{R}\|_{L^2([0,T]\times\Omega)}.$$

One has $\|\mathbf{f}(0)\|_{L^2(\Omega)} = O(\sqrt{h\varepsilon} \max(h, \varepsilon))$ by virtue of (94) (taking care that there is a rescaling by a factor ε, see (76)). In any case, one has $\|\mathbf{f}(0)\|_{L^2(\Omega)} \leq C(h^2 + \varepsilon^2)$. Since $\|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\|_{L^2([0,T]\times\Omega))} = \sqrt{\int_0^T E(t) dt}$, the proof is ended. ☐

3.7 End of the proof of the uniform AP property in 2D

We have the following result of uniform convergence.

Theorem 3.18. *We assume the initial data is smooth. Under the geometrical assumptions (3.2) in 2D, there exists a constant C(T)>0 independent of ε, such that the following estimate holds:*

$$\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|_{L^2([0,T]\times\Omega)} \leq C(T) h^{\frac{1}{4}}.$$

Proof. The proof is a slight adaptation of our initial proposition 1.3, where we use the norm $\|\cdot\| = \|\cdot\|_{L^2([0,T]\times\Omega))}$. Indeed one has

$$\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\| \leq \min \left(\|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\|_{\text{naive}}, \|\mathbf{V}_h^\varepsilon - \mathbf{W}_h^\varepsilon\| + \|\mathbf{W}_h^\varepsilon - \mathbf{W}^\varepsilon\| + \|\mathbf{W}^\varepsilon - \mathbf{V}^\varepsilon\| \right)$$

where each estimates are given in order in (63), (97), (92) and (56). Therefore

$$\begin{aligned} \|\mathbf{V}_h^\varepsilon - \mathbf{V}^\varepsilon\| &\leq C(T) \min \left(\sqrt{\frac{h}{\varepsilon}} \ , \ \varepsilon \max \left(1, \sqrt{\frac{\varepsilon}{h}} \right) + h + (h+\varepsilon) + \varepsilon \right) \\ &\leq C(T) \min \left(\sqrt{\frac{h}{\varepsilon}} \ , \ \varepsilon \right) + C(T) \min \left(\sqrt{\frac{h}{\varepsilon}} \ , \ \sqrt{\frac{\varepsilon^3}{h}} \right) \\ &\leq C(T) \min \left(\varepsilon^{-\frac{1}{2}} h^{\frac{1}{2}} \ , \ \varepsilon \right) + C(T) h^{-\frac{1}{2}} \min \left(\varepsilon^{-\frac{1}{2}} h \ , \ \varepsilon^{\frac{3}{2}} \right). \end{aligned}$$

The first contribution is estimated with $a = 1$ and $b = c = \frac{1}{2}$ as in 1D and as in the inequality (10) of proposition 1.3: it yields a contribution $O(h^{\frac{1}{3}})$. The second contribution can also be estimated in the context of proposition 1.3. One gets a convergence rate $O\left(\frac{ac}{a+b} - \frac{1}{2}\right)$ where now $a = \frac{3}{2}$, $b = \frac{1}{2}$ and $c = 1$: that is $O(h^{\frac{1}{4}})$. The worst case is $O(h^{\frac{1}{4}})$ which ends the proof. ☐

4 Numerical illustration

To illustrate the theory and have a more quantitative version of the error estimates studied in this work, we consider the academic square $\Omega = [0,1]^2$ and discretize the hyperbolic heat equation of a mesh made with random quads. A random quad mesh is made of quads where the vertices are moved randomly around their initial position, by a factor between 10% and 30%. We use the fully implicit time discretization version of the 2D scheme detailed in this work. The solution of the linear systems is computed via an iterative GMRES algorithm, which converges smoothly in our numerical experiments. The reference analytical solution used in our tests is designed by separation of variables. A solution of (1) is

$$p = f + \frac{\varepsilon^2}{\sigma}\partial_t f \text{ and } \mathbf{u} = -\frac{\varepsilon}{\sigma}\nabla f,$$

with f solution of $\partial_t f + \frac{\varepsilon^2}{\sigma}\partial_t^2 f - \frac{1}{\sigma}\Delta f = 0$. We add periodic boundary conditions and consider

$$f(t,x,y) = \alpha(t)\cos(\pi x))\cos(\pi y).$$

The function α is determined as the solution of

$$\alpha'(t) + \frac{\varepsilon^2}{\sigma}\alpha''(t) + \frac{2\pi}{\sigma}\alpha = 0$$

whith $\alpha'(0) = 0$ and $\alpha(0) = 1$. For small ε, which is the case we are interested in, the solution is computed as follows. First determine

$$\lambda_1 = -\frac{\sigma\left(\sqrt{1 - \frac{\varepsilon^2}{\sigma^2}8\pi^2} + 1\right)}{2\varepsilon^2} \text{ and } \lambda_2 = \frac{\sigma\left(\sqrt{1 - \frac{\varepsilon^2}{\sigma^2}8\pi^2} - 1\right)}{2\varepsilon^2}.$$

Then

$$\alpha(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1}e^{\lambda_2 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1}e^{\lambda_1 t}$$

from which $p(t)$ and $\mathbf{u}(t)$ are easily recovered.

We decide that an exact relation is enforced between ε and $h = \frac{1}{N}$, so that the error can be expressed as a function of h solely. The relation between ε and h writes $\varepsilon = 0.01(40h)^\gamma$ for $\gamma \in \{0, \frac{1}{4}, \frac{1}{2}, 1, 2\}$. The error between the exact solution and the numerical solution is computed numerically in function of $h = \frac{1}{N}$, for different values of γ, and the results of some of these numerical experiments is displayed in figure 6. The results correspond to the time $T = 0.02$ using the time step $\Delta t = 0.2h^2$.

As predicted by the theory, the scheme is uniformly AP and the error behavior is a continuous function of γ between the hyperbolic and parabolic limits. However the results are much better, in the sense the order is greater than the theoretical prediction since the order is approximatively 1 for $\gamma = 0$ (hyperbolic limit) and 2 for $\gamma = 2$ (parabolic regime). The reason is probably that the theory is based on worst case estimates, as it is often the case for the numerical analysis of finite volume schemes [12].

5 Conclusion

The proof that was given of the uniform AP property is quite technical. It relies on specific hyperbolic and parabolic estimates for linear nodal finite volume schemes on general meshes. We observe that the multidimensional case yields an additional contribution in the error that ultimately slightly degrades the convergence rate. It is an open problem to determine if these inequalities are optimal. The numerical results indicate that it is probably not the case.

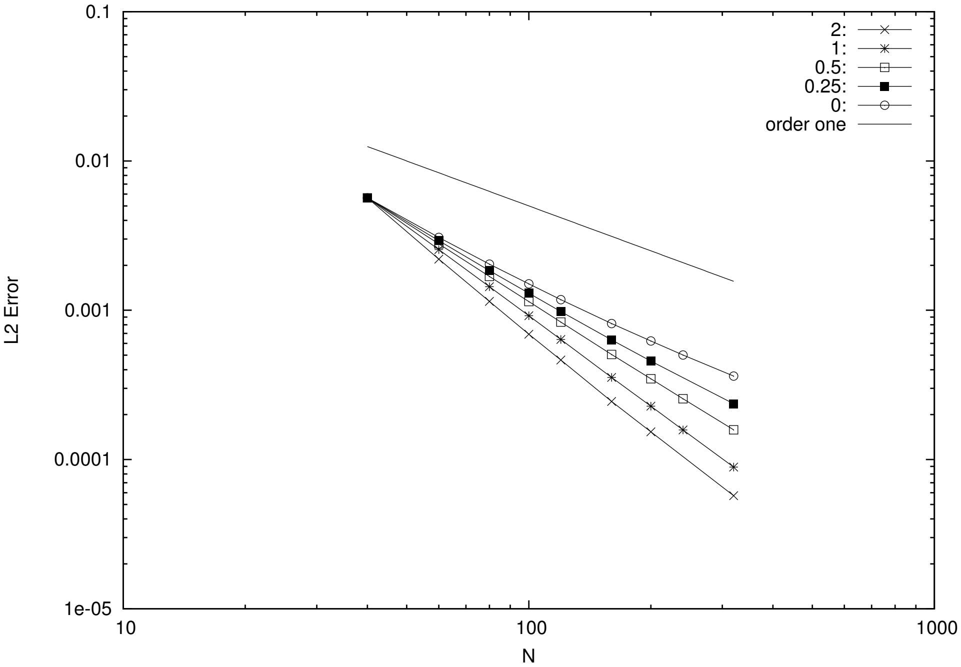

Figure 6: The error is plotted in log scale versus the number of cells per direction for the test problem described in section 4. Each curve corresponds to a value of $\gamma \in \{0, \frac{1}{4}, \frac{1}{2}, 1, 2\}$, plus a reference line for order one. One sees that the order of convergence is an increasing function of γ.

References

[1] C. Berthon, P. Charrier, R. Turpault *An HLLC scheme to solve the M^1 model of radiative transfer in two dimensions* J. Scie. Comput., J. Sci. Comput., 31, no. 3, pp 347-389, 2007.

[2] C. Berthon, P. LeFloch, R. Turpault *Late-time relaxation limits of nonlinear hyperbolic systems. A general framework*, Math. of Comput., 82 , pp. 831-860, 2013.

[3] C. Berthon, R. Turpault *Asymptotic preserving HLL schemes*, Numerical Methods for Partial Differential Equations, 27 (6) pp 1396-1422, 2011.

[4] C. Buet, S. Cordier, B. Lucquin-Desreux, S. Mancini, *Diffusion limit of the lorentz model: asymptotic preserving schemes.* ESAIM: M2AN vol. 36, 4, pp 631-655, 2002.

[5] C. Buet, B. Després, E. Franck *Design of asymptotic preserving schemes for the hyperbolic heat equation on unstructured meshes* Numerish Mathematik, Volume 122, Issue 2, pp 227-278, 2012.

[6] C. Buet, B. Després, E. Franck, *An asymptotic preserving scheme with the maximum principle for the M_1 model on distorted meshes*, C.R. Acad. Sci., Paris, Sér. I, Math., Vol 350, 11-12 pp 633-638, 2012.

[7] C. Buet, B. Després, E. Franck, *Asymptotic preserving finite volumes discretization for non-linear moment model on unstructured meshes*, Finite Volumes for Complex Applications VI Problems, Perspectives, Springer Proceedings in Mathematics Volume 4, pp 467-474, 2011.

[8] N. Crouseilles, M. Lemou *An asymptotic preserving scheme based on a micro-macro decomposition for collisional Vlasov equations: diffusion and high-field scaling limits*, Kinetic and related models 4, 2 pp 441-477, 2011.

[9] N. Crouseilles, M. Roger *A dynamic multi-scale model for transient radiative transfer calculations* Journal of Quantitative Spectroscopy and Radiative Transfer , pp 110-121, 2013.

[10] B. Després, *Lax theorem and finite volume schemes*, Math. Comput 0, 73:1203-1234, 2004.

[11] B. Després, *Weak consistency of the cell-centered Lagrangian GLACE scheme on general meshes in any dimension*, Comput. Methods Appl. Mech. Engrg. 199, pp 2669-2679, 2010.

[12] R. Eymard, T. Gallouet, R. Herbin, *The finite volume method*, Handbook for Numerical Analysis, Ph. Ciarlet J.L. Lions eds, North Holland, 715-1022, 2000.

[13] F. Golse, S. Jin, C. D. Levermore *The convergence of numerical transfer schemes in diffusive regimes I: discrete-ordinate method*, SIAM J. Numer. Anal. 36, pp 1333-1369, 1999.

[14] L. Gosse, G. Toscani *An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations* C. R. Acad. Sci Paris,Ser. I 334, pp 337-342, , 2000.

[15] L. Gosse *Transient radiative transfer in the grey case: Well-balanced and asymptotic-preserving schemes built on Cases's elementary solutions* Journal of Quantitative Spectroscopy and Radiative Transfer 112, pp 1995-2012, 2011.

[16] S. Jin, *Efficient Asymptotic-Preserving (AP) Schemes for Some Multiscale Kinetic Equations*, SIAM J. Sci. Comp. 21, 441-454, 1999.

[17] S. Jin, *Asymptotic preserving (AP) schemes for multiscale kinetic and hyperbolic equations: a review*, Lecture Notes for Summer School on "Methods and Models of Kinetic Theory" (M and MKT), Porto Ercole (Grosseto, Italy), 2010.

[18] S. Jin, D. Levermore *Numerical schemes for hyperbolic conservation laws with stiff relaxation terms.* JCP 126, pp 449-467, 1996.

[19] S. Jin, D. Levermore *The discrete-ordinate method in diffusive regime.* Transport Theory and Statistical Physics, vol. 20, issue 5, pp 413-439, 1991.

[20] M. Lemou, L.Mieussens *A new asymptotic preserving scheme based on micro-macro formulation for linear kinetic equations in the diffusion limit.* SIAM J. Sci. COMPUT. Vol. 31, 1, pp 334-368, 2008.

[21] J. G. Liu, L. Mieussens *Analysis of an Asymptotic Preserving Scheme for Linear Kinetic Equations in the Diffusion Limit* SIAM J. Numer. Anal, 48(4), pp 1474-1491, 2010.

[22] C. Mazeran *Sur la structure mathematique et l'approximation numerique de l'hydrodynamique lagrangienne bidimentionnelle* PhD, University Bordeaux 1, 2007.