AST 301 Practice Questions for Final Exam

The Final Exam will be made up of questions similar—perhaps even identical—to a subset of these.

1. In frame O, a meter stick lies parallel to the x axis and moves at velocity $V\boldsymbol{e}_y$. Frame $\bar{O}$ is boosted relative to O at velocity $\beta\boldsymbol{e}_x$ but not rotated, *i.e.* the spatial axes $\bar{x}\bar{y}\bar{z}$ lie parallel to xyz. Show that in $\bar{O}$, the meter stick makes a nonzero angle θ with respect to the $\bar{x}$ axis, and derive an expression for θ in terms of V and β.

2. Observer O constructs a clock in which a photon bounces back and forth along the x axis between mirrors at $x = 0$ and $x = L$. O moves at speed $(4/5)c$ relative to $\bar{O}$ along the $x\bar{x}$ axis.
(a) What is the round trip time between the two mirrors in frames O and $\bar{O}$?
(b) What is the separation between the mirrors in $\bar{O}$?
(c) The energy of the photon is E in O. What energy does $\bar{O}$ measure for the photon while it travels from $x = 0$ to $x = L$, and what energy does she measure while it travels from $x = L$ to $x = 0$? (The mirrors are too massive to recoil appreciably.)
(d) If there are N such photons distributed uniformly between the mirrors, all moving parallel to the x axis, and with equal numbers moving in each direction as seen in O; and if the surface area of each mirror is A, what are the components of the energy-momentum tensor due to these photons in frames O and $\bar{O}$?

3. *(Sagnac effect: the principle of laser ring gyros)* An optical fiber of index of refraction $n > 1$ is bent into a circular ring of radius r (much larger than the diameter of the fiber itself). As seen from an inertial frame O, the ring rotates counterclockwise around its center at angular velocity $\Omega < c/r$; the center of the ring is at rest in O.
(a) Calculate in frame O the travel time of a light beam going once around the ring counterclockwise and returning to the same angular position *as seen in O*. Let this be called t_+. *(Hint: Note that the azimuthal 3-velocity of the light is $\bar{v} = c/n$ in the corotating frame of the fiber material; typically $n \approx 1.5$ in optical fiber. Avoid approximations based on $\Omega r \ll c$ if you can.)*
(b) Use the above to calculate the time $\bar{t}_+$ *in the local rest frame of a station along the fiber* to go counter-clockwise and return to that same station. Note that this corresponds to $\Delta\phi > 2\pi$ in the inertial frame if $\Omega > 0$.
(c) Similarly calculate the corresponding times t_- and $\bar{t}_-$ for clockwise beams, and thereby obtain the difference $\Delta\bar{t} \equiv \bar{t}_- - \bar{t}_+$ in terms of Ω and r.

4. Approximately 19 neutrinos were detected from supernova 1987A in the Large Magellanic Cloud. The neutrinos arrived over a span of about 15 seconds with energies ranging from roughly 10 to 40 MeV. Given that the distance to the LMC is approximately $50\,\mathrm{kpc} \approx 5 \times 10^{12}\,\mathrm{lt\,sec}$, and assuming that the supernova did not conspire to emit the lower-energy νs before the higher-energy ones, estimate a rough upper bound to the neutrino rest mass.

5. Two particles with 4-momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ collide. After collision, there are two (possibly different) particles with 4-momenta $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$. Let

$$\begin{aligned} s &\equiv -(\boldsymbol{p}_1 + \boldsymbol{p}_2) \cdot (\boldsymbol{p}_1 + \boldsymbol{p}_2) \\ t &\equiv -(\boldsymbol{p}_1 - \boldsymbol{p}_3) \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_3) \\ u &\equiv -(\boldsymbol{p}_1 - \boldsymbol{p}_4) \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_4). \end{aligned}$$

s, t, & u are of course Lorentz invariants.
(a) Show that $s + t + u$ is a function of the rest masses m_1, m_2, m_3, m_4 alone.
(b) In the frame where particle 2 is at rest, express the energy of particle 1 in terms of s, t, u and rest masses.
(c) Do the same for the energy of particle 1 in the center-of-mass frame.

6. *(Poynting-Robertson drag.)* Consider an idealized spherical dust grain of radius a_g and mass density ρ_g that absorbs optical light with cross section πa^2 and re-radiates an equivalent energy isotropically in its rest frame. Let such a grain be in a circular orbit about the Sun (mass $M_\odot = 2 \times 10^{33}\,\mathrm{g}$, luminosity $L_\odot = 4 \times 10^{33}\,\mathrm{erg\,s^{-1}}$) at an orbital radius equal to that of the Earth: $1\,\mathrm{AU} = 1.5 \times 10^{13}\,\mathrm{cm}$.
(a) What is the stress-energy tensor of the solar radiation in the local rest frame of the grain? You may neglect the angular size of the sun ($\approx 0.005\,\mathrm{rad}$ at $1\,\mathrm{AU}$), *i.e.*, assume that the photons move perfectly radially. *Hint: First find $T^{\hat{\mu}\hat{\nu}}$ in a local frame at rest with respect to the Sun using an orthonormal basis derived from spherical polars $tr\theta\phi$. Then boost along the grain's orbital motion,* $30\,\mathrm{km\,s^{-1}}$.
(b) Find expressions for the radial and azimuthal forces on the grain due to solar radiation, and evaluate these for $a_g = 1\,\mu\mathrm{m} = 10^{-4}\,\mathrm{cm}$. Compare these forces with the gravitational force, assuming $\rho_g = 2\,\mathrm{g\,cm^{-3}}$. What will happen to this grain?

7. A two-dimensional spacetime has metric

$$ds^2 = -x dv^2 + 2 dv dx. \tag{1}$$

(a) Write down the equations and constants of motion for timelike geodesics in the coordinates (x, v). Solve for $x(\tau)$ and $v(\tau)$.
(b) v is obviously a null coordinate. Find a second null coordinate—call it u—and express x as a function of (u, v).
(c) Find coordinates X, T as functions of u, v such that $ds^2 = -dT^2 + dX^2$, hence showing that (1) describes flat space.

8. Averaged over orbits about the Sun, how much slower or faster does a perfect atomic clock on the surface of Mars run compared to one on the surface of Earth? For the purpose of this problem, take the mass, radius, and orbital semi-major axis of the Earth to be $M_E \approx 6 \times 10^{27}\,\mathrm{g}$, $R_E \approx 6.4 \times 10^{8}\,\mathrm{cm}$, and $a_E = 1.5 \times 10^{15}\,\mathrm{cm}$; and the corresponding values for Mars to be $M_M \approx 0.11 M_E$, $R_M \approx 0.53 R_E$, $a_M \approx 1.52 a_E$.

9. *Fermat's Principle of Least Time* states that a light ray in a stationary medium of index of refraction $n = n(\vec{r})$ follows a trajectory that extremizes the "optical path length" ℓ defined by

$$d\ell^2 = n^2(dx^2 + dy^2 + dz^2) \equiv n^2 ds^2, \tag{2}$$

where ds is the increment in euclidean arc length. [If n is nondispersive—meaning independent of frequency—then $d\ell/c$ is the time taken to go distance ds. More generally, $\omega d\ell/c$ is the change in phase of a wave of frequency ω across distance ds.] Consider a sphere of radius R centered at $r \equiv \sqrt{x^2+y^2+z^2} = 0$ such that

$$n(\vec{r}) = \begin{cases} \sqrt{2-(r/R)^2} & \text{when} \quad r \le R, \\ 1 & \text{when} \quad r \ge R. \end{cases} \tag{3}$$

Show that parallel rays incident on one hemisphere are focused to a common point on the opposite hemisphere. Thus, if astronomers could build such a sphere, with index of refraction independent of frequency, they would have a "telescope" to image the entire sky without aberrations! *(Hint: Treat* (2) *as defining a "metric" with "arc length"* $d\ell$ *in place of* ds*. Set up the geodesic equations in cartesians, then introduce a new arc parameter* $d\sigma \equiv n^{-2}d\ell$*.)*

10. Birkhoff's Theorem says that any spherically symmetric vacuum spacetime is Schwarzschild, i.e. can be described by the Schwarzschild metric for an appropriate mass parameter M. This is the analog of the statement in newtonian gravity that a spherical distribution of total mass M attracts external bodies with the same forces as if its mass were concentrated at its center. Also, in both Newton's and Einstein's gravity, there are no gravitational forces inside a hollow spherical shell. Therefore, the line element inside such a shell describes flat space, but with a redshifted time coordinate:

$$ds^2 = -(1+z_0)^{-2}dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (0 \le r < R)\,, \tag{4}$$

where $z_0 > 0$ is a positive constant.

Consider a spacetime that is empty except for a stationary shell of mass M_S as measured from infinity, radius $R > 2M_S$, and negligible thickness.

(a) Explain why $g_{tt} \neq -1$ but $g_{rr} = 1$ in eq. (4).

(b) Write down the metric for $r > R$ and determine the constant z_0 in eq. (4). *(Hint: Assume that stationary observers on the inner and outer surfaces of the shell have the same gravitational redshift relative to infinity.)*

(c) An advanced civilization builds such a spherical shell and lives inside it. The mass of the civilization is negligible compared to that of the sphere itself. To support its lifestyle, the civilization converts a small fraction, ΔM, of the sphere's mass to energy, eventually radiating this energy from the sphere as heat. For stationary observers at infinity, the energy radiated is ΔM, and the sphere's mass decreases from M to $M - \Delta M$. Inside the sphere, the energy used is larger than ΔM—**by what factor?**

(d) The total mass-energy of the sphere measured by observers inside it—call this $\widehat{M}$—is larger than M by the same factor you identified in part **b**. To compensate for the loss ΔM, the civilization decreases R so as to keep $\widehat{M}$ constant. Show that the civilization can keep "spending" mass in this way indefinitely.

(e) If the radiated power is constant as measured inside the sphere, express M as a function of (i) proper time inside the sphere; (ii) proper time at infinity.

11. Supermassive black holes reside in the centers of most galaxies. Every $\sim 10^4$ yr, typically a star approaches the black hole on a marginally bound, extremely eccentric orbit with a pericenter (r_p) small enough that the star is *tidally disrupted* if the $\hat{r}\hat{r}$ component of the the tidal field at pericenter is $\gtrsim GM_*/R_*^3$, where M_* & R_* are the mass and radius of the star.
(a) Estimate the maximum r_p for tidal disruption of the Sun in terms of $M_{\rm bh}$ [$M_\odot \approx 2 \times 10^{33}$ g, $R_\odot \approx 7 \times 10^{10}$ cm].
(b) Just as disrupted/evaporated comets make meteor streams, the debris of a disrupted star will spread out along the star's original orbit. However, this leads to observable consequences only if that orbit does not plunge into the black hole. Estimate the maximum black-hole mass for which a sunlike star can be disrupted on a non-plunging, marginally-bound orbit. *Hint: "Marginally bound" means $E = m$ in the notation of our lectures ($e = 1$ in the book).*
(c) Assuming $r_p \gg M$, estimate the radius $r_i \gg r_p$ at which the outgoing part of the orbit intersects the incoming part. You may use the following approximation for the change in periapse angle per period of a nearly keplerian ($r_p \gg M$) orbit of semi-major axis a and eccentricity ϵ:

$$\delta\phi_p \approx \frac{6\pi M}{a(1-\epsilon^2)} \qquad \text{[eq. (9.57) in Hartle]}.$$

Hint: Recall $r_p = a(1-\epsilon)$ and $r(\phi) = a(1-\epsilon^2)/[1+\epsilon\cos(\phi-\phi_p)]$ for keplerian orbits. If you get two solutions for r_i, use the one farther from pericenter.

12. The star known as S2 orbits the black hole at Sgr A* in the Galactic Center with a period of 15.9 yr. (Slightly more than one full period has been recorded using IR adaptive optics.) The eccentricity of the orbit is $e = 0.8831 \pm 0.0034$, and the mass of the black hole $M_{\rm bh} = 4.30 \pm 0.27 \times 10^6\, M_\odot$ [Gillesen et al., `arXiv:0910.3069`].
(a) Estimate the relativistic precession of periapse angle per orbit.
(b) The latest estimate of the periapse angle (defined with respect to the line of intersection of the orbital and sky planes) is $\omega = 64.98^\circ \pm 0.81^\circ$. In view of the quoted 1σ error, how long will it be before the precession is detected at 3σ, assuming that the cadence and accuracy of the astrometric measurements remain constant? (Actually, accuracy improved a lot over the first orbit.)
(c) Estimate the difference between the orbital period measured by astronomers—defined as the time between periapses—and that experienced by the proper time of S2 itself. *(Hint: The time averages of Newtonian kinetic and potential energies are rather simple for a Keplerian orbit.)*

13. In its equatorial plane ($\theta = \pi/2$), the metric of a rotating black hole reduces to

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 - \frac{4aM}{r}dt\,d\phi + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)d\phi^2 + \left(1 - \frac{2M}{r} + \frac{a^2}{r^2}\right)^{-1} dr^2\,. \tag{5}$$

The spin angular momentum of the black hole is aM, $|a| < M$.
(a) Find three constants of motion for timelike and null geodesics in the equatorial plane. *(Hint: Minimize the algebra by using an affine parameter λ for both*

timelike and null geodesics, such that $dx^\mu/d\lambda = p^\mu$, the coordinate components of 4-momentum. Be careful of a factor of 2 in $g_{t\phi}$.)
(b) Show that $g_{tt}g_{\phi\phi} - g_{t\phi}^2 = -r^2(g_{rr})^{-1}$, and use this to solve for $dt/d\lambda$ and $d\phi/d\lambda$ in terms of $(-E, L)$, where E and L are the constants of motion that reduce to $dt/d\lambda$ and to $r^2 d\phi/d\lambda$ as $r \to \infty$, respectively (i.e., the energy and angular momentum).
(c) The horizon—the radius within which no future-directed timelike or null geodesic can escape to $r = \infty$—actually occurs not where $g_{tt} = 0$ but where $g_{rr} = \infty$. Show that there are actually two roots $\{r_+, r_-\}$ for the horizon radius if $0 < |a/M| < 1$. Show further that there is a radius r_E such that there can be no timelike stationary observers [i.e., observers with $dr/d\tau = 0 = d\phi/d\tau$) in the region $\max(r_+, r_-) \le r \le r_E$], and express r_E in terms of a and M. *(This region is called the ergosphere.)*
(d) The horizon is generated by "frozen" null geodesics that neither fall into the black hole nor escape to $r = \infty$. In these coordinates they are described by constant radii, $r = r_H \in \{r_+, r_-\}$, so that $dr/d\lambda = 0$ if λ is an affine parameter. Show that

$$\Omega_H \equiv \left(\frac{d\phi/d\lambda}{dt/d\lambda} \right)_{r=r_H}$$

must be constant for such geodesics, and express Ω_H in terms of a, M, and $r_\pm$. This is called the angular velocity of the horizon.

14. It can be shown that in three dimensions, i.e., where the indices range from 0 to 2 (Lorentzian) or from 1 to 3 (Riemannian), the Riemann tensor can be expressed in terms of the Ricci tensor and Ricci scalar by

$$R_{\lambda\mu\nu\kappa} = A\left(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}\right) + BR\left(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}\right), \qquad (6)$$

for certain dimensionless constants A and B.
(a) Determine A and B from the condition $g^{\lambda\nu}R_{\lambda\mu\nu\kappa} = R_{\mu\kappa}$.
(b) Show that in 3D, Einstein's equations imply that vacuum regions are locally flat.

15. An astronaut finds herself in free fall in outer space, and not rotating. She is equipped with some small useless objects, and to combat boredom while she awaits rescue, she flings these away from herself. (Assume that she does this in such a way that she does not begin to rotate.) Regardless of the magnitude or direction of the initial velocity she imparts to them, the objects return to her after about one hour with minus their initial velocity, each along the same line as that by which it left her.
(a) She deduces that the space around her, although transparent and offering no resistance, is not a true vacuum. Why not? *(You may neglect the gravitational field that her own body creates.)*
(b) Estimate the local matter density in $\mathrm{g\,cm^{-3}}$, not including the contribution of her own body.

16. Consider cosmological perfect fluid with $p = w\rho$, where w is a constant factor. Assume the universe is spatially homogeneous and isotropic and that the fluid is comoving, $i.e$ its energy-momentum tensor is $T^{\hat{0}\hat{0}} = \rho$, $T^{\hat{i}\hat{j}} = p\eta^{\hat{i}\hat{j}}$ in an orthonormal basis based on the usual cosmological coordinates $tr\theta\phi$.

(a) Show that $\rho \propto a^q$ as the scale factor a in the cosmological metric changes with time, and relate the exponent q to w.
(b) Suppose $q = -1$. (This could be approximated, for example, by a "foam" of domain walls consisting of false vacuum—an unlikely model for the present universe. Such walls would have constant surface tension κ and mass per unit area σ related by $\sigma = -\kappa > 0$.) Supposing that this fluid makes the only significant contribution to the mass and energy density, discuss the evolution of $a(t)$ for universes of positive, negative, and zero spatial curvature. In which cases can there be a "big bang," *i.e.* a starting from zero? In which cases can $\dot{a}$ change sign?

17. Consider a positively curved universe consisting of pure radiation (i.e. $\Omega_m = \Omega_\Lambda = 0$, $p = \rho/3$). For convenience, scale the comoving coordinates so that $k = 1$, i.e.

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2)\right].$$

A noninteracting massless particle is emitted from $\chi = 0$ at $t = 0$ (the Big Bang) and follows a radial null geodesic. How far does it travel (maximum χ) before the Big Crunch (the time at which $a(t)$ returns to zero)?

18. As you will learn or have already learned, in Special Relativity an isolated charge q undergoing proper acceleration $\boldsymbol{a}$ radiates electromagnetic waves with total power (in gaussian electromagnetic units)

$$P = \frac{2}{3c^3}q^2|\boldsymbol{a}|^2.$$

Consider a charge at rest on the surface of a stationary gravitating body such as a neutron star. Does the Principle of Equivalence demand that this charge radiate? If so, where does the energy come from? If not, why not? *(Hint: Use dimensional analysis to estimate the wavelength and period of the radiation in terms of the mass and radius of the body.)*