Supplementary Material

A Proof of Proposition 1

Proposition 1 states the equivalence of tensor-pEC and Prob-EC, that is, both methods compute the same probabilities of a CE to hold at each time-point $t_k, k \in \mathcal{T}$. To keep the proof simple and easy to follow, we will prove the equivalence for a single pair of entities. The proof can easily be extended for any pair of entities.

Proof. According to Eq. (8) of the main paper, the probability of a CE to hold at a time-point $t_{k+1} \in \mathcal{T}$, for a pair of entities, is computed by:

$$P(h^{t_{k+1}}) = P(s^{t_k}) + P(h^{t_k}) \times P(d^{t_k})\,. \tag{14}$$

In contrast to Eq. (8), we have omitted $P(u^{t+1})$. $P(u^{t+1})$ has always a value of 1 (crisp fact), with the exception of the first time-point, where $P(u^1)$=0. Taking this into consideration and Eq. (14), we can express the probability of a CE to hold at any time-point for a pair of entities, as:

$$\begin{aligned}
&P(h^{t_1}) = 0, k = 0\,.\\
&P(h^{t_{k+1}}) = P(s^{t_k}) + \sum_{j=2}^{k}\Big[P(s^{t_{j-1}}) \times \prod_{j}^{k} P(d^{t_j})\Big],\\
&\forall\, k : 1 \leq k \leq \Omega{-}1\,.
\end{aligned} \tag{15}$$

Notice that at the first time-point a CE cannot hold, i.e., $P(h^{t_1}) = 0$. This agrees with the *inertia* axiom (3). For every other time-point $t_{k+1} \in \mathcal{T}$, $\forall\, k : 1 \leq k \leq \Omega{-}1$, $P(h^{t_{k+1}})$ depends on all past values up to t_k.

Next, we show that $\mathbf{G}^{-1} \in [0,1]^{N^2\Omega\times N^2\Omega}$. $\mathbf{G}^{-1}$ is the inverse of the coefficients matrix $\mathbf{G}$ in Eq. (12) of the main paper. $\mathbf{G} \in [-1,1]^{N^2\Omega\times N^2\Omega}$ is a lower unitriangular bidiagonal matrix and the elements of its principal diagonal are equal to 1. Then, it holds for the elements of $\mathbf{G}^{-1}$ (Kılıç and Stanica 2013):

$$\mathbf{G}^{-1}_{i,j} = \begin{cases} 0, & i<j,\\ 1, & i=j,\\ (-1)^{i+j}\prod_j^{i-1}\mathbf{G}^\star_j = \prod_j^{i-1}|\mathbf{G}^\star_j|, & i>j, \end{cases}$$
$$\forall\, i,j :\ 1 \leq i,j \leq N^2\Omega\,, \tag{16}$$

where $\mathbf{G}^\star_j$ denotes the j-th element of the first sub-diagonal of $\mathbf{G}$. Recall that by definition $\mathbf{G}^\star{=}-vec\big[\underline{\mathbf{D}}\big]$, that is, the vectorized tensor $\underline{\mathbf{D}}$ multiplied by -1. $\underline{\mathbf{D}}$ holds by definition the probabilities $P(d^{t_k})$, at each time-point $t_k \in \mathcal{T}$. Hence, it holds $\forall\ j\ :\ \mathbf{G}^\star_j \in [-1,0]$. Eq. (16) shows that $\mathbf{G}^{-1}$, similarly to $\mathbf{G}$, is a lower unitriangular matrix. For $i>j$, $i-j$ elements of $\mathbf{G}^\star$, with values in $[-1,0]$, are multiplied. If any of them is 0, it follows that $\mathbf{G}^{-1}_{i,j} = 0$. If none of them is 0, we have: (a) $\prod_j^{i-1}\mathbf{G}^\star_j > 0$ or (b) $\prod_j^{i-1}\mathbf{G}^\star_j < 0$. The first case (a) implies that $i-j$ is an even number. The same applies for $i+j$ in $(-1)^{i+j}$ and as a result $\mathbf{G}^{-1}_{i,j} > 0$. The second case (b) implies that $i-j$ is an odd number, and since $i+j$ would also be an odd number, $\mathbf{G}^{-1}_{i,j} > 0$ again. Therefore, the element i,j of $\mathbf{G}^{-1}$, when $i>j$, is the product of the absolute values of $\mathbf{G}^\star$, i.e., $\prod_j^{i-1}|\mathbf{G}^\star_j|$. In addition, this implies that $\mathbf{G}^{-1} \in [0,1]^{N^2\Omega\times N^2\Omega}$. Next, we show that the unique solution of Eq. (12), computed by Eq. (13), coincides with Eq. (15), as computed by Prob-EC.

For $k=0$ and time-point $t_1 \in \mathcal{T}$, Eq. (13) of the main paper states that the element of tensor $\underline{\mathbf{H}}$, encoding the probability of a CE to hold at the first time-point, is computed as:

$$\mathbf{h}_1 = \sum_{j=1}^{k+1}\mathbf{G}^{-1}_{1,j}\,\mathbf{b}_j = \mathbf{G}^{-1}_{1,1}\,\mathbf{b}_1 = \mathbf{b}_1 = 0\,. \tag{17}$$

In Eq. (17), $\mathbf{h}_1$ is equal to the dot product of the first line of $\mathbf{G}^{-1}_{1,:}$ and vector $\mathbf{b}$. According to Eq.(16), when $i<j$, $\mathbf{G}^{-1}_{i,j} = 0$. This leads to the second equality of Eq. (17), that is the product of elements $\mathbf{G}^{-1}_{1,1}$ and $\mathbf{b}_1$. Since $\mathbf{G}^{-1}_{1,1} = 1$ (Eq. (16)), the value of element $\mathbf{b}_1$ is the result of Eq.(17). Recall from the main paper, that vector $\mathbf{b}$, by definition, is the vectorization of tensor $\underline{\mathbf{S}}$, shifted (mode-(3,1) product) by matrix $\mathbf{U}$, i.e., $\mathbf{b} = vec\bigg[\Big(\underline{\mathbf{S}} \times_{3,1} \mathbf{U}\Big)\bigg] \in [0,1]^{N^2\Omega}$. $\underline{\mathbf{S}}$ represents the probabilities $P(s^{t_k})$ of Eq. (8) of the main paper, at each time-point $t_k \in \mathcal{T}$. Multiplying a tensor with a shift matrix results in the shifting of elements along the temporal dimension, i.e., the first temporal slice is a matrix of zeros. Hence, the first element of $\mathbf{b}_1 = 0$, and tensor-pEC computes the same probability for t_1 as Prob-EC. Next, we proceed with time-point $t_{k+1}, \forall\, k : 1 \leq k \leq \Omega{-}1$.

According to Eq. (13) of the main paper, the probability at $t_{k+1} \in \mathcal{T}, \forall\, k : 1 \leq k \leq \Omega{-}1$, is computed as:

$$
\begin{aligned}
\mathbf{h}_{k+1} = & \sum_{j=1}^{k+1} \mathbf{G}^{-1}_{k+1,j}\, \mathbf{b}_j \overset{\mathbf{b}_1=0}{=} \\
& \sum_{j=2}^{k+1} \mathbf{G}^{-1}_{k+1,j}\, \mathbf{b}_j = \\
& \sum_{j=2}^{k} \mathbf{G}^{-1}_{k+1,j}\, \mathbf{b}_j \,+ \mathbf{G}^{-1}_{k+1,k+1}\, \mathbf{b}_{k+1} \overset{\substack{\mathbf{b}_{k+1}=P(s^{t_k}),\\ \mathbf{G}^{-1}_{k+1,k+1}=1}}{=} \\
& P(s^{t_k}) + \sum_{j=2}^{k} \mathbf{G}^{-1}_{k+1,j}\, \mathbf{b}_j \overset{\mathbf{G}^{-1}_{i>j}}{=} \\
& P(s^{t_k}) + \sum_{j=2}^{k} \mathbf{b}_j \prod_{j}^{k} |\mathbf{G}^{\star}_{j}| \overset{\mathbf{G}^{\star}_{j}=-P(d^{t_j})}{=} \\
& P(s^{t_k}) + \sum_{j=2}^{k} \mathbf{b}_j \prod_{j}^{k} P(d^{t_j}) \overset{\mathbf{b}_j\,=\,P(s^{t_{j-1}})}{=} \\
& P(s^{t_k}) + \sum_{j=2}^{k} P(s^{t_{j-1}}) \prod_{j}^{k} P(d^{t_j})\,.
\end{aligned} \tag{17}
$$

In Eq. (17), we annotate the equal sign, when it is not clear, to denote which operation leads to the next line. In what follows, we discuss in detail each line derivation of Eq. (17).

First, to compute the probability at t_{k+1}, we take the dot product of row $k+1$ of $\mathbf{G}^{-1}$ and vector $\mathbf{b}$. Since, by definition, $\mathbf{b}_1 = 0$, the product $\mathbf{G}^{-1}_{k+1,1} \times \mathbf{b}_1 = 0$, and it is removed from the sum by increasing the start of the j-counter in the sum sign to 2 (second line in Eq. (17)). In the third line, we take out of the sum the product $\mathbf{G}^{-1}_{k+1,k+1} \times \mathbf{b}_{k+1}$ and decrease the end value of the j-counter in the sum sign to k. Then, we consider that (a) $\mathbf{b}_{k+1} = P(s^{t_k})$ and (b) $\mathbf{G}^{-1}_{k+1,k+1} = 1$. (a) holds by definition, since the elements of $\mathbf{b}$ are the elements of the shifted tensor $\underline{\mathbf{S}}$, and thus, it follows that $\mathbf{b}_{k+1} = P(s^{t_k})$. (b) holds by definition of the inverse in Eq. (16). By performing a reordering of the terms, we result in the fourth line of Eq. (17). Considering again Eq. (16) for elements $i > j$ of $\mathbf{G}^{-1}$, and the fact that by definition $\mathbf{G}^{\star} = P(d^{t_j})$, we result in the sixth line of Eq. (17). Finally, by taking once more into account that $\mathbf{b}$ holds the elements of the shifted tensor $\underline{\mathbf{S}}$, i.e., $\mathbf{b}_j = P(s^{t_{j-1}})$, we result in the computation of the last line of Eq. (17), which is identical to the one of $P(h^{t_{k+1}})$ in Eq.(15), i.e., $\mathbf{h}_{k+1} = P(h^{t_{k+1}})$.

In conclusion, Eq. (17) and Eq. (15) produce the same probabilities at each time-point $t_k \in \mathcal{T}$, and thus, tensor-pEC and Prob-EC are equivalent. By following the same procedure, we can prove the equivalence of tensor-pEC and Prob-EC for any pair of entities. □

B Code & Data

In this section, we provide the instructions for running the source code of Prob-EC and tensor-pEC. Both methods are implemented in Python. Prob-EC depends on ProbLog (Fierens et al. 2015), which is also mainly written in Python. The experiments presented in the main paper were performed with Python 3.12.2, NumPy 1.26.4 and SciPy 1.13.1. All the datasets are provided as csv files. Regarding the maritime application, we provide a subset of the public dataset of Brest, due to its size.

To run the source code of each method, you have to enter the corresponding directory, that is, './code/Prob-EC' for Prob-EC and './code/Tensor-pEC' for the tensor method. Then, open a terminal and type the following command:

```
$ python3 run-exps.py <app>
```

where `<app>` corresponds to one of the following:

- `m`, for the Brest dataset.
- `c`, for the CAVIAR dataset.

For example, assume you want to execute the source code of tensor-pEC for the maritime monitoring application. Then, you will type in the command line the following:

```
$ python3 run-exps.py m
```

When the probabilistic CER process terminates, the results are saved in the folder of the executed method. Assume, for example, you have performed probabilistic CER with tensor-pEC in the CAVIAR dataset. The results will be saved in the directory './code/Tensor-pEC/examples/caviar/results/'.

References

Fierens, D.; Van den Broeck, G.; Renkens, J.; Shterionov, D.; Gutmann, B.; Thon, I.; Janssens, G.; and De Raedt, L. 2015. Inference and learning in probabilistic logic programs using weighted Boolean formulas. *Theory and Practice of Logic Programming* 15:358–401.

Kılıç, E., and Stanica, P. 2013. The inverse of banded matrices. *Journal of Computational and Applied Mathematics* 237(1):126–135.