Polynomial Optimization Based Schemes for Solving AC Optimal Power Flow Problems

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July 14, 2021

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Overview

Introduction

ACOPF Problem and Moment Relaxations

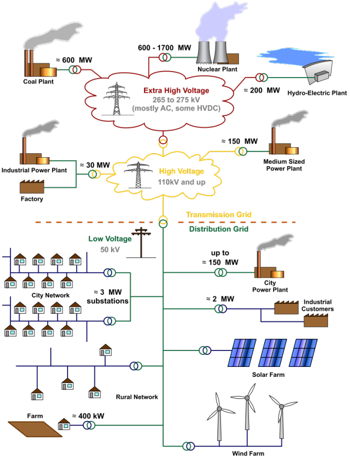

Figure: File:Electricity Grid Schematic English.svg, Author:MBizon, License: CC BY 3.0, Created: 7 March 2010

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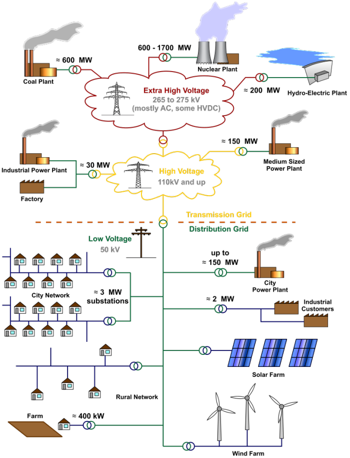

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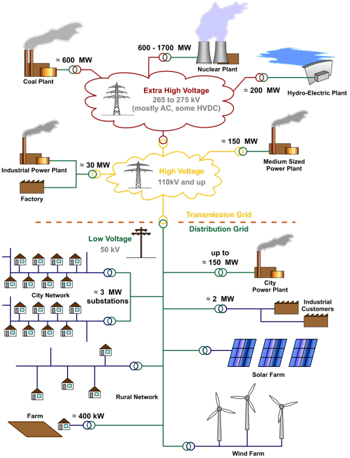

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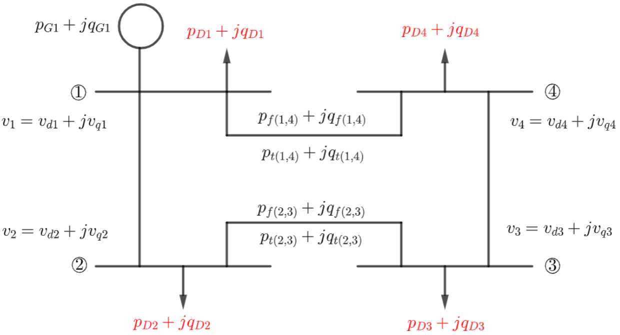
- As a nonconvex optimization program the ACOPF problem is NP-hard [LGVH15].
- As a polynomial program, global solutions can be obtained by the application of the Moment-SOS hierarchy of semidefinite relaxations.
- Even for small/medium size networks exploiting sparsity is key to reduce the computational effort.

AC Optimal Power Flow Problem

Definition of the ACOPF problem



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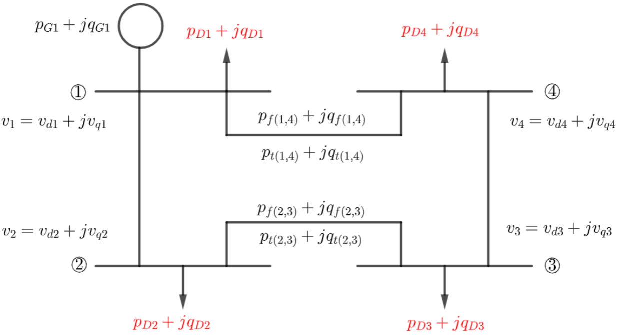

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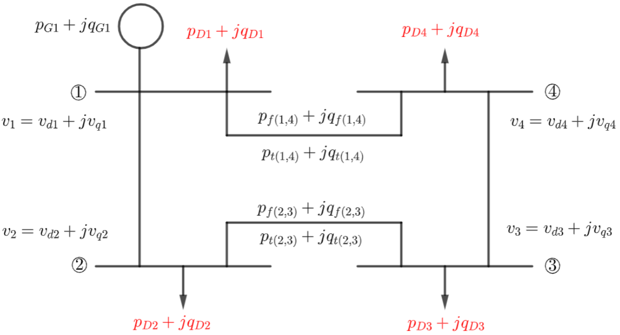

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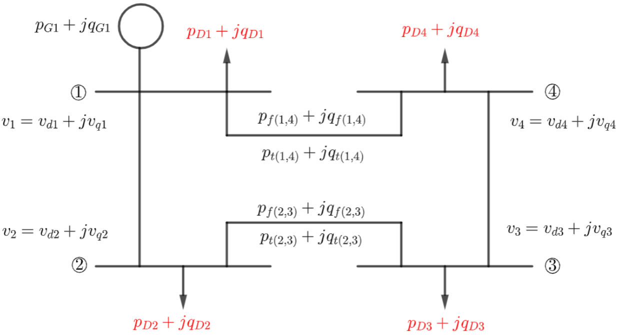

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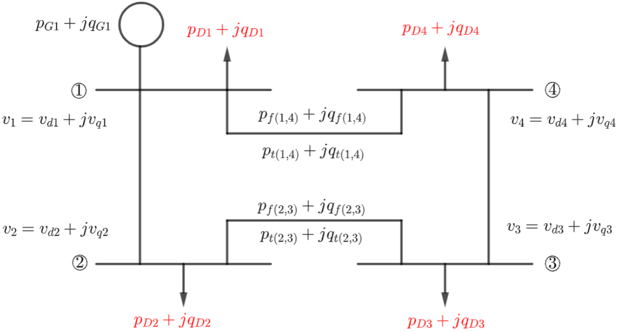

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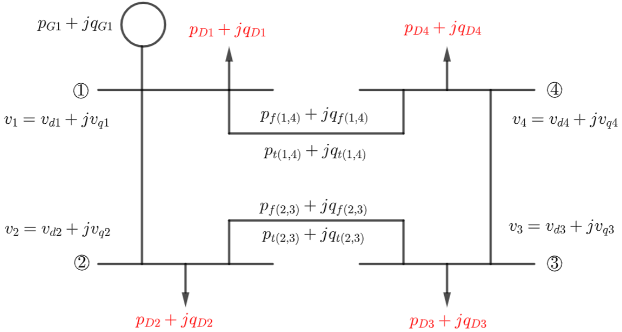

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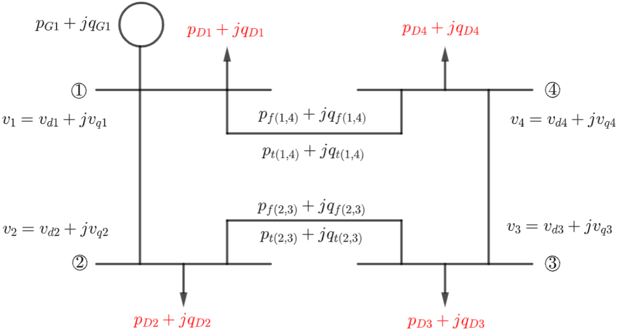

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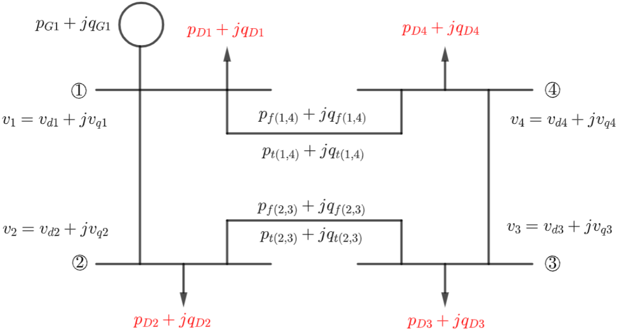

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- Voltage variables: $v \in \mathbb{C}^{|\mathcal{N}|}$

AC Optimal Power Flow Problem

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$$\sum_{g \in \mathcal{G}_k} p_{G_g} - p_{Dk} - g'_k |v_k|^2 = \sum_{l=(k,m) \in \mathcal{L}} p_{fl} + \sum_{l=(m,k) \in \mathcal{L}} p_{tl} \quad \forall k \in \mathcal{N},$$
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- Branch flow:

$$\frac{v_k}{t_l}\left[\left(j\frac{b'_l}{2} + y_l\right)\frac{v_k}{t_l} - y_l v_m\right]^* = p_{fl} + jq_{fl} \quad \forall l = (k,m) \in \mathcal{L},$$
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- Ref. bus: $\mathrm{Im}(v_1) = 0$

Moment-SOS Hierarchy for Polynomial Optimization

(Real) Moment-SOS Hierarchy [Las01, Las09]

Moment relaxation of degree d:

$$
\begin{aligned}
\inf L_y(f) & \\
\text{s.t.} & \\
M_d(y) \succeq 0, & \\
M_{d-d_j}(g_j y) \succeq 0, & \quad j = 1, ..., m, \\
& \quad d_j = \deg g_j \\
y_0 = 1. &
\end{aligned}
$$

$d \geq \max(\lceil \text{deg}(f)/2 \rceil, \{\lceil \text{deg}(g_j)/2\}_j \rceil)$

- $L_y(f) = \sum_\alpha f_\alpha y_\alpha$

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Correlative Sparsity [WKKM06]:

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M_d(y, I_l) \succeq 0, & \quad l = 1, ..., p, \\
M_{d-d_j}(g_j y, I_l) \succeq 0, & \quad j = 1, ..., m, \\
& \quad l = 1, ..., p, \\
& \quad \text{vars}(g_j) \subseteq I_l, \\
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\end{aligned}
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- Using the maximal cliques I_l of a chordal sparsity graph

A Relaxation with Moments up to degree 3

Tightening of the First-order Relaxation

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- Solving a sparse second-order relaxation is already expensive, it could be intractable for power networks with a hundred buses.
- Therefore, in this work we focus on building a tightening of the first-order relaxation as described next.

Auxiliary Constraints

- We choose the sign of v_1:

$$\underline{v}_1 \leq |v_1| \leq \overline{v}_1, \quad \mathsf{Im}(v_1) = 0 \quad \Rightarrow \quad \underline{v}_1 \leq \mathsf{Re}(v_1) \leq \overline{v}_1. \qquad (1)$$

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- For the rest of the buses, $k \geq 2, k \in \mathcal{N}$, we consider the constraint $\underline{v_k} \leq |v_k| \leq \overline{v}_k$ which yields

$$-\overline{v_k} \leq \text{Re}(v_k) \leq \overline{v}_k, \quad k \geq 2, k \in \mathcal{N} \tag{2}$$

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- Original OPF constraints: g_j, $j = 1, ..., m$.
- For simplicity, we will refer to constraints (1)-(3) as constraints $h_r \geq 0$, for $r = 1, ..., 4|\mathcal{N}| - 2$.

A Relaxation with Moments up to degree 3

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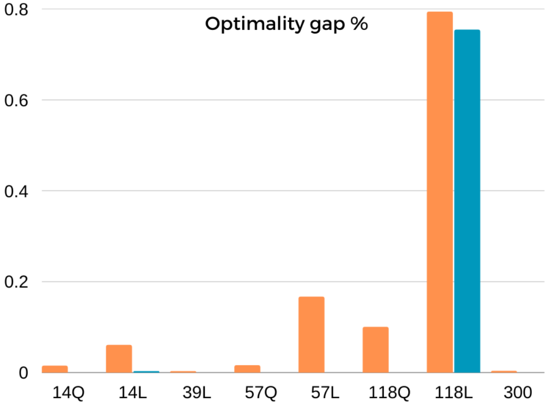
\begin{aligned}
\inf L_y(f) & \\
\text{s.t.} & \\
M_1(y, I_l) \succeq 0, &\quad l = 1, \ldots, p, \\
M_0(g_j y, I_l) \succeq 0, &\quad \mathsf{vars}(g_j) \subseteq I_l, \\
&\quad l = 1, \ldots, p, \quad j = 1, \ldots, m, \\
y_0 = 1, & \\
L_y(\mathsf{Re}(v_1)) \geq \frac{L_y(\mathsf{Re}(v_1)^2) + \underline{v}_1 \overline{v}_1}{\underline{v}_1 + \overline{v}_1}, &\quad \text{[BALD19]} \\
M_1(h_r y, Nb(k)) \succeq 0, &\quad \mathsf{var}(h_r) \in Nb(k), \\
&\quad r = 1, \ldots, 4|\mathcal{N}| - 2, \; k = 1, \ldots, |\mathcal{N}| \\
L_y(h_r \cdot g_j) \geq 0, &\quad \mathsf{var}(h_r) \in Nb(k), \quad \mathsf{vars}(g_j) \subseteq Nb(k) \\
&\quad r = 1, \ldots, 4|\mathcal{N}| - 2, \; j = 1, \ldots, m. \\
&\quad k = 1, \ldots, |\mathcal{N}|
\end{aligned}
$$

Numerical Experiments

- We choose the benchmark of [MH14] to test our tightening of the first-order relaxation.
  The optimality gap for the 1st-ord relaxation and our tightening:



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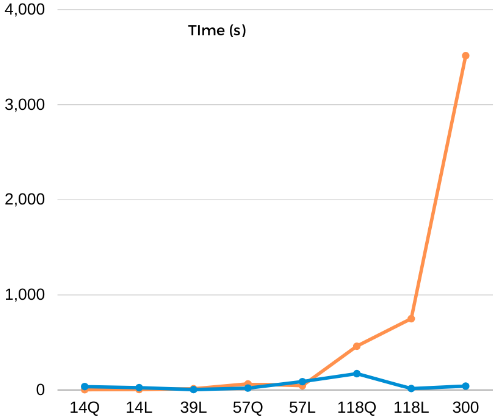

Conclusion and Future Work

- We built a relaxation that allowed us to reduce the optimality gap for OPF problems where a first-order relaxation is not exact. Extraction of minimizers is possible in most cases.
- The key features of our relaxations are the use of localizing matrices for auxiliary constraints, avoiding large moment matrices (i.e. SD constraints on big matrices).
- Possible improvement:
  - Use of *term sparsity* instead of correlative sparsity.
  - Selective/iterative schemes like [MH14] that increases the order of the relaxation on selected regions of the network.

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Appendix A: Moment Relaxations

- Consider the polynomial program

$$\inf x_1 x_2 \quad \text{s.t.} \quad 0.9^2 \leq x_1^2 + x_2^2 \leq 1.1^2$$

1st-order relaxation

$$\inf_y y_{11}$$

s.t.

$$0.9^2 \leq y_{20} + y_{02} \leq 1.1^2$$

$$\begin{bmatrix} y_{00} & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{bmatrix} \succeq 0$$

$$y_{00} = 1$$

Example: $\inf x_1 x_2$ s.t. $0.9^2 \le x_1^2 + x_2^2 \le 1.1^2$

- Localizing matrix for $1.1^2 - x_1^2 - x_2^2$:

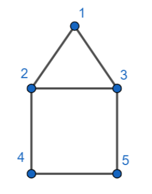
$$\begin{bmatrix} 1.1^2 y_{00} - y_{20} - y_{02} & 1.1^2 y_{10} - y_{30} - y_{12} & 1.1^2 y_{01} - y_{21} - y_{03} \\ 1.1^2 y_{10} - y_{30} - y_{12} & 1.1^2 y_{20} - y_{40} - y_{22} & 1.1^2 y_{11} - y_{31} - y_{13} \\ 1.1^2 y_{01} - y_{21} - y_{03} & 1.1^2 y_{11} - y_{31} - y_{13} & 1.1^2 y_{02} - y_{22} - y_{04} \end{bmatrix} \succeq 0$$

- Second-order Moment Matrix:

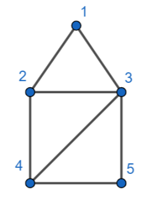
$$\begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succeq 0$$

Appendix B: Maximal Clique Decomposition

Sparsity graph of X



Chordal extension



$$\begin{bmatrix} X_{11} & X_{12} & X_{13} & & \\ X_{21} & X_{22} & X_{23} & X_{24} & \\ X_{31} & X_{32} & X_{33} & & X_{35} \\ & X_{42} & & X_{44} & X_{45} \\ & & X_{53} & X_{54} & X_{55} \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \succeq 0, \quad C_1 = \{1,2,3\}$$

$$\begin{bmatrix} X_{22} & X_{23} & X_{24} \\ X_{32} & X_{33} & X_{34} \\ X_{42} & X_{43} & X_{44} \end{bmatrix} \succeq 0, \quad C_2 = \{2,3,4\}$$

$$\begin{bmatrix} X_{33} & X_{34} & X_{35} \\ X_{43} & X_{44} & X_{45} \\ X_{53} & X_{54} & X_{55} \end{bmatrix} \succeq 0, \quad C_3 = \{3,4,5\}$$

Numerical Experiments

- We choose the benchmark of [MH14] to test our tightening of the first-order relaxation.

case	Opt. 1stOrd	Opt. Tight.	Time (s)	Max. 2nd Eigval
14Q	3301.35	3301.80	3.45	1.06e-5
14L	9353.58	9358.92	5.72	2.3e-3
39L	41906.85	41907.47	13.62	6.64e-8
57Q	7350.74	7351.82	64.63	2.37e-5
57L	43909.43	43982.17	45.46	1.30e-6
118Q	81427.75	81508.47	461.29	1.26e-5
118L	133833.67	133887.13	750.60	7.4e-2
300	719707.50	719724.78	3516	6.5e-4