Lecture notes: Discrete Random Variables

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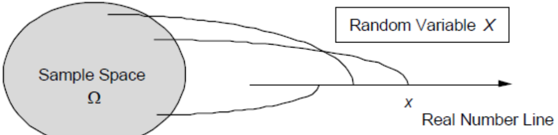
Outline

- Random variables.
- Discrete Random variables.
- Probability mass function.
- Cumulative distribution function.
- ~~Three~~ important discrete RVs.
  Four

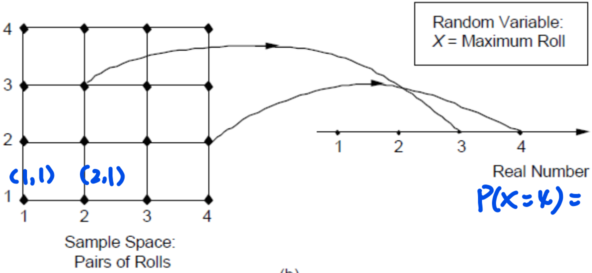
Random variable

A random variable (r.v.) associates a value (a number) to **every possible outcome.**
e.g. In a dice roll, the outcome $\{1, 2, 3, 4, 5, 6\}$ can be directly assigned to the random variable X , where $X = 1, 2, \ldots, 6$. outcome of die roll.

$P(X=1) = P(X=2) = P(X=3) \cdots = \frac{1}{6}$



(a)



(b)

$P(X=1) = P(\text{max roll} = 1) = P(\{(1,1)\})$
$= \frac{1}{16}$

$P(X=2) = P(\text{max roll} = 2) = P(\{(2,1),(1,2),(2,2)\})$
$= \frac{3}{16}$

$P(X=3) = P(\{(1,3),(2,3),(3,3),(3,2),(3,1)\}) = \frac{5}{16}$

$P(X=4) = \frac{7}{16}$

Discrete Random Variables

X: R.V. $\mathcal{X}$: possible values for X

A discrete random variable has an associated **probability mass function (PMF)**, which gives the probability of each numerical value that the random variable can take.

$$p_X : \mathcal{X} \to [0, 1]$$

where $\mathcal{X}$ is all possible values X can take.

Notation:

$$p_X(x) = P(X = x)$$

$x \in \mathcal{X}$

Properties: $p_X(x) \geq 0$, $\underbrace{\sum_x P_X(x)}$ $= 1$.

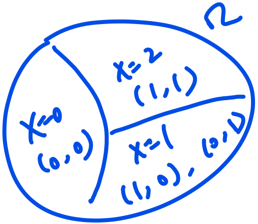
$X = 1$

$X = 2$

mutually exclusive

$P(X=1) + P(X=2) + P(X=3) + P(X=4)$ total prob.

Example



Two independent tosses of a fair coin:
Define discrete RV X: The total number of heads.
The PMF of X is

X: possible values $\{0, 1, 2\}$

$$P(X=0) = \frac{1}{4}$$

$$P(X=1) = \frac{2}{4} = \frac{1}{2}$$

$$P(X=2) = \frac{1}{4}$$

$$P(X=0) + P(X=1) + P(X=2) = P(\Omega) = 1$$

A function of one or several random variables' is also a random variable.

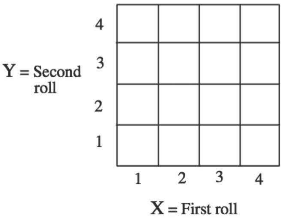
Two rolls of a 4-sided die:

$$Z = \underbrace{X + Y}_{\text{function of 2 r.v.}}$$

Z: Random variable.

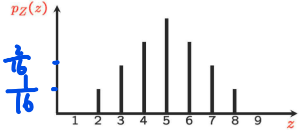
$P(Z=2) = P(X+Y=2) = P(\{(1,1)\}) = \frac{1}{16}$

$P(Z=3) = P(X+Y=3) = P(\{(1,2),(2,1)\}) = \frac{2}{16}$



Calculation of the PMF of a Random Variable For each possible value z of Z:

1. Collect all the possible outcomes that give rise to the event $\{Z = z\}$.
2. Add their probabilities to obtain $p_Z(z)$.



Question $Z_2 = X + 1(Y>3)$

$P(Z \le 5) = \sum_{k \le 5} P(Z=k) = P(Z=2) + P(Z=3) + P(Z=4) + P(Z=5)$

Cumulative Distribution Function (CDF)

$Z = X + Y$. two die roll.

$P(Z > 5)$ or $\underbrace{P(Z \leq 5)}$

If $(\Omega, \mathcal{F}, P)$ is a probability space with X a real discrete RV on Ω, the **Cumulative Distribution Function (CDF)** is denoted as $F_X(x)$ and provides the probability $P(X \leq x)$. In particular, for every x we have

$$F_X(x) = P(X \leq x) = \sum_{k \leq x} p_X(k) = \sum_{k \leq x} P(X = k)$$

Loosely speaking, the CDF $F_X(x)$ "accumulates" probability "up to" the value x.

Example

$P(X \le 1.5) = F_X(1.5) = \sum_{k \le 1.5} P(X=k) = P(X=0) + P(X=1) = \frac{3}{4}$

I toss a coin twice. Let X be the number of observed heads. Find the CDF of X.

Recall the PMF of X:

$F_X(x) = P(X \le x) = \sum_{k \le x} P(X=k) = P(X=0) + P(X=1) + P(X=2) + \underbrace{\sum_{2 < k \le x} P(X=k)}_{=0} = 1$

$$p_X(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0, 2 \\ \frac{1}{2} & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

To find $F_X(x)$

- $x < 0$: $F_X(x) = P(X \le x) = \sum_{k \le x} P(X=k) = 0 \quad x < 0$
- $x = 0$: $F_X(0) = P(X \le 0) = \sum_{k \le 0} P(X=k) = P(X=0) = \frac{1}{4}$
- $x = 1$: $F_X(1) = P(X \le 1) = \sum_{k \le 1} P(X=k) = P(X=0) + P(X=1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$
- $x = 2$: $F_X(2) = P(X \le 2) = \sum_{k \le 2} P(X=k) = P(X=0) + P(X=1) + P(X=2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$
- $x > 2$:

Next, we will introduce some important discrete RV.

- Bernoulli Random Variable
- Binomial Random Variable.
- Geometric Random Variable.
- Poisson R.V.

Bernoulli Random Variable

e.g. Medical treatment: Two outcomes in the sample space: $x = 1$ for "success" and $x = 0$ for "failure".

Bernoulli RV: Models a trial that results in success/failure, Heads/Tails, etc.

- A **Bernoulli RV** X takes two values 0 and 1.
- The PMF for a Bernoulli RV X is defined by

$$p_X(x) = P(X = x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \\ 0, & \text{o.w.} \end{cases}$$

$$X \sim \text{Bernoulli}(p)$$

parameter of R.V.
(often used for inference)

Examples/applications:

- The coin flip is either heads (1) or tail (0).
- whether a bit is 0 or 1, whether a bit is in error, whether a component has failed, whether something has been detected.
- The state of a telephone at a given time that can be either free (0) or busy (1).
- A person who can be either healthy or sick with a certain disease.

The Binomial Random Variable

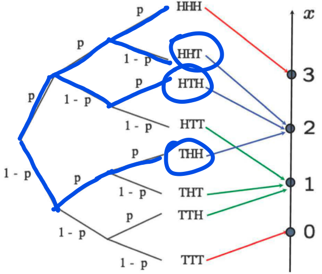
consider the following experiment:

- A biased coin is tossed **n** times.
- Each toss is independently of prior tosses: Head with probability p; Tail with probability $1 - p$.
- The number X of heads up is a binomial random variable, refer to as the **Binomial** RV **with parameters** n **and** p.

$P(X=2) = \binom{3}{2} \cdot P^2 \cdot (1-P)^1$

$n=3.$

$P(HHH) = P \cdot P \cdot P$



$P(HHT) = P \cdot P \cdot (1-P)$

$P(HTH) = P \cdot (1-P) P$

$P(THH) = (1-P) \cdot P \cdot P$

$$X \sim \text{Binomial}(n, p)$$

n choose x.

$$P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, \ldots, n \\ 0, & \text{o.w.} \end{cases}$$

where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ is the different combinations of x objects chosen from n objects.

Example: Binomial

A pharmaceutical company is testing a new medication to see if it is effective in treating a certain disease. The company wants to know the probability that the medication will work for at least 80% of patients. To test this, they give the medication to 20 patients and observe whether it works or not.

- Based on the observation that the medicine worked for 5 out of 20 patients, do you think you believe the company's claim that the effective rate is 80%?

Bernoulli: success. if medicine works for 1 patient.

X_i for i th patient $p = 80\%$.

$n = 20$ independent trials $\Rightarrow$ the number of successes. $\to$ Binomial

$X = \sum_{i=1}^{n} X_i \sim \text{Binom}(n, p)$

P-value: $P(X=5) = \binom{20}{5} \cdot (0.8)^5 \cdot (1-0.8)^{20-5} \approx 1.66 \times 10^{-7} < 0.05$

Reject the null hypothesis

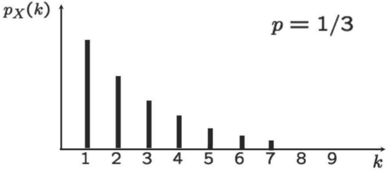
Geometric Random Variable

k-1 fail kth is success

- Experiment: infinitely many independent tosses of a coin $P(Heads) = p$.
- Sample space: Set of infinitte sequences of H and T.
- Random variable X: number of tosses until the first Heads.
- Models of: waiting times; number of trials until a success.

CDF: $F_X(k) = P(X \le k)$

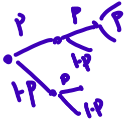
$= 1 - \underbrace{P(X > k)}_{?} = 1 - (1-p)^k$



decreases as a **geometric progression** with parameter $1 - p$.

$$p_X(k) = (1-p)^{k-1}p$$

what is the probability of no heads ever? $= 0.$



$$\lim_{k\to+\infty} p_X(k) = \lim_{k\to+\infty} \underbrace{(1-p)}_{p=0.0001}{}^{k-1}\cdot p = 0$$

Example

A customer calls a tech support line, and each agent has a 20% chance of successfully resolving the issue. The customer may need to speak to multiple agents before the problem is fixed. Assuming all agents are not sharing the knowledge/experience learned from talking with the customer, how many agents the customer need to talk to before his issue is solved with probability $> 95\%$?

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Bernoulli: talk to one customer support. $P = 20\%$.

wait until first success:

X: number of call until success

CDF

? $P(X=k) < P(X=1) = 0.2$

? $P(X \le k) > 95\%$

$F_X(k) = 1 - (1-P)^k > 95\%$

$1 - 0.8^k > 0.95 \Rightarrow 0.8^k < 0.05 \Rightarrow k > \frac{\log(0.05)}{\log(0.8)} = 13.4$

Poisson random variable

Consider the random variable X: The number of typos in a book of n words: - each word can be misspelled with a probability p.

Binom: $X=k$: k typos from n words.

each typo happens with prob. p ... Bernoulli

$$P(X=k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

Each book. $n \approx 50,000 \sim 100,000$

$p = 0.002 \sim 0.005$.

Poisson random variable

Consider the random variable X: The number of typos in a book of n words: - each word can be misspelled with a probability p.

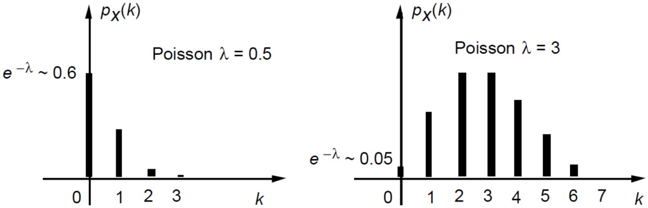
$$X \sim \text{Poisson}(\lambda)$$

$$P_X(x) = \begin{cases} \frac{\lambda^x}{x!} e^{-\lambda}, & x = 0, 1, \ldots \\ 0, & \text{o.w.} \end{cases}$$

where $\boxed{\lambda = n \cdot p,}$ the expected number of occurrences in the interval.
— A Poisson RV with parameter λ is an approximation of binomial RV with parameter $n \gg 0$ and $p \ll 1$.

Poisson random variable

$\lambda = n \cdot p$ relate to Binom.



Figure

- $\lambda \leq 1$, monotonically decreasing.
- $\lambda > 1$, first increase and then decrease.

Example

suppose the book has 50,000 words, and each word can be mistyped with a probability $p = 0.2\%$. What is the probability that the book has five typos?

Binom: X_1

$$P(X_1 = 5) = \binom{50{,}000}{5} \cdot (0.002)^5 \cdot (1 - 0.002)^{\cancel{5}\,49995} \approx 3.1 \times 10^{-36}$$

Poisson X_2

$$P(X_2 = 5) = e^{-\lambda} \cdot \frac{\lambda^5}{5!} = \frac{e^{-100} \cdot 100^5}{5!} \approx 3.1 \times 10^{-36}$$

$\lambda = n \cdot p$

$= 50{,}000 \times 0.002$

$= 100$

Summary

- Bernoulli: A single trial with two possible outcomes: success (1) with probability p and failure (0) with probability $1 - p$.
- Binomial: The number of successes in n independent Bernoulli trials, each with success probability p.
- Geometric: The number of trials until the first success in repeated independent Bernoulli trials with success probability p.
- Poisson: Counting rare events (e.g., defective items in manufacturing, earthquakes per year). Approximate Binomial with $n \gg 1$ and $p \ll 1$.