

# Stopping Redundancy Hierarchy Beyond the Minimum Distance

Yauhen Yakimenka<sup>1</sup>, *Student Member, IEEE*, Vitaly Skachek<sup>2</sup>, *Member, IEEE*,  
Irina E. Bocharova<sup>3</sup>, *Senior Member, IEEE*, and Boris D. Kudryashov<sup>4</sup>, *Senior Member, IEEE*

**Abstract**—Stopping sets play a crucial role in failure events of iterative decoders over a binary erasure channel (BEC). The  $\ell$ th stopping redundancy is the minimum number of rows in the parity-check matrix of a code, which contains no stopping sets of size up to  $\ell$ . In this paper, a notion of coverable stopping sets is defined. In order to achieve maximum-likelihood performance under iterative decoding over the BEC, the parity-check matrix should contain no coverable stopping sets of size  $\ell$ , for  $1 \leq \ell \leq n - k$ , where  $n$  is the code length,  $k$  is the code dimension. By estimating the number of coverable stopping sets, we obtain upper bounds on the  $\ell$ th stopping redundancy,  $1 \leq \ell \leq n - k$ . The bounds are derived for both specific codes and code ensembles. In the range  $1 \leq \ell \leq d - 1$ , for specific codes, the new bounds improve on the results in the literature. Numerical calculations are also presented.

**Index Terms**—Binary erasure channel, LDPC codes, stopping redundancy hierarchy, stopping redundancy, stopping sets.

## I. INTRODUCTION

LOW-DENSITY parity-check (LDPC) codes [2] in conjunction with iterative message-passing (MP) decoding is a popular choice for error correction in practical communications and data storage systems. They allow for very fast decoding, while the decoding performance is suboptimal.

Recently, a number of new applications, in particular, in the domain of network communications and data storage, have raised interest in communications over the binary erasure channel (BEC). LDPC codes can be used over the BEC. The maximum-likelihood (ML) decoder for codes used over the BEC is equivalent to solving a sparse system of linear equations, which is relatively time-consuming for practical applications. An alternative decoding method based on iterative procedure offers complexity linear in the code length  $n$ , yet its decoding error performance is suboptimal.

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Y. Yakimenka and V. Skachek are with the Institute of Computer Science, University of Tartu, 50090 Tartu, Estonia (e-mail: vitaly.skachek@gmail.com).

I. E. Bocharova and B. D. Kudryashov are with the Institute of Computer Science, University of Tartu, 50090 Tartu, Estonia, and also with the Department of Information Systems, ITMO University, 197101 Saint Petersburg, Russia.

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On the BEC, the MP decoding algorithm can be interpreted as an iterative process to repetitively solve a single equation with one unknown. By contrast, on this channel, ML decoding is equivalent to solving a sparse system of linear equations, which can be done in  $O(n^3)$  operations by using Gaussian elimination. More elaborate algorithms allow for solving a sparse system of equations in time approximately  $O(n^2)$  (see [3], [4]), which is still too high for many practical applications.

It was observed in [5] that *stopping sets* are combinatorial structures in a parity-check matrix of a code, which are responsible for failures of iterative decoding on the BEC. Thus, stopping sets in the parity-check matrix should be eliminated in order to decrease the failure rate of the decoder.

In [6], it was suggested to adjoin redundant rows to a parity-check matrix of a code in order to eliminate stopping sets of size less than  $d$ , where  $d$  is the minimum distance of a code. It was shown that this can be done for any code by taking a sufficiently large number of redundant rows of the parity-check matrix. Bounds on the *stopping redundancy*, the minimum number of the redundant rows, were studied in [6] and subsequently improved in [7] and [8].

In [9], the authors defined the so-called *stopping redundancy hierarchy*. In their approach, the  $\ell$ -th stopping redundancy is the minimum number of rows in the parity-check matrix, such that the parity-check matrix contains no stopping sets of size at most  $\ell$ . Algorithms for finding small stopping sets were proposed in [10] and [11].

Distribution of the number of stopping sets of different sizes in the Hamming code was studied in [12]. Other related papers include, for example, [13]–[19].

In this work, we first propose an improvement to the upper bounds in [7] and [8]. While the upper bounds in [7] and [8] are obtained by probabilistic methods, in our approach, we initially select a few first rows deterministically and then continue along the lines of probabilistic analysis in [7] and [8].

Next, we consider stopping sets of size  $\ell \geq d$ . We define *coverable* stopping sets, which do not contain a support of any nonzero codeword. We extend the definition of the stopping redundancy hierarchy beyond size  $\ell = d - 1$  of the stopping sets. By estimating the number of coverable stopping sets, we derive upper bounds on the stopping redundancy hierarchy also for  $\ell \geq d - 1$ . Finally, the values of the bounds for specific codes and ensembles are calculated, and it is shown

that theoretical finding are consistent with the experimental results.

The structure of this paper is as follows. In Section II, the necessary notations are introduced and the basic facts used in the paper are established. Section III contains improved bounds on the stopping redundancy. In Section IV, we study stopping sets of size  $d$  and higher, and the bounds on the stopping redundancy hierarchy for  $\ell \geq d$  are derived. Both specific codes and code ensembles are considered in that section. Further, we experimentally obtain numerical results in Section V. Section VI concludes the paper.

## II. PRELIMINARIES

### A. Notations

We start this section by introducing some notations used throughout the paper.

Consider a binary linear  $[n, k, d]$  code  $\mathcal{C}$ , where  $n$ ,  $k$ , and  $d$  denote length, dimension, and minimum distance, respectively. Let  $H = (h_{ji})$  be a binary  $m \times n$  parity-check matrix of this code, and  $\text{rank } H = r \triangleq n - k$  be the dimension of the dual code  $\mathcal{C}^\perp$ . We note that generally  $r \leq m$ .

We denote  $[n] \triangleq \{1, 2, \dots, n\}$ . Let  $\mathcal{S} \subseteq [n]$  be a set of column indices of a matrix  $H$ . Denote by  $H_{\mathcal{S}}$  the submatrix of  $H$  consisting of the columns of  $H$  indexed by  $\mathcal{S}$ .

A *support* of a vector  $\mathbf{x}$ , denoted by  $\text{supp}(\mathbf{x})$ , is the set of indices of nonzero entries in the vector. *Hamming weight* of a vector is a cardinality of its support.

### B. Stopping Sets and Stopping Redundancy

We start this section by defining a stopping set.

*Definition 1 [5]:* Let  $H$  be an  $m \times n$  parity-check matrix of a binary code  $\mathcal{C}$ . The set  $\mathcal{S} \subseteq [n]$  is called the *stopping set* if  $H_{\mathcal{S}}$  contains no row of Hamming weight one.

It is important to note that stopping sets are structures in a particular parity-check matrix, and not in the code.

Following the terminology of [6], we formulate the following definition.

*Definition 2:* A binary vector  $\mathbf{h}$  covers a stopping set (or any subset of columns)  $\mathcal{S}$  if  $\text{supp}(\mathbf{h})$  intersects with  $\mathcal{S}$  in exactly one position. Consequently, a matrix covers  $\mathcal{S}$  if any of its rows covers  $\mathcal{S}$ .

We note that if  $\mathcal{S}$  is a stopping set in a parity-check matrix  $H$  and  $\mathbf{h}$  covers  $\mathcal{S}$ , then after adjoining  $\mathbf{h}$  as a row to  $H$ ,  $\mathcal{S}$  is not a stopping set in the obtained extended matrix. With some abuse of notation, we say that a stopping set  $\mathcal{S}$  is *covered* in that extended matrix.<sup>1</sup>

*Definition 3:* A stopping set  $\mathcal{S}$  is *coverable* (by the code  $\mathcal{C}$ ), if there exists a (possibly extended) parity-check of  $\mathcal{C}$  that covers  $\mathcal{S}$ .

In order to reduce the failure probability of the iterative decoding algorithm over the BEC, it was proposed in [6] to adjoin redundant rows, i.e., the codewords of  $\mathcal{C}^\perp$ , to a parity-check matrix in such a way that the resulting matrix has no stopping sets of small size. Specifically, we are interested

in adjoining the minimum possible number of codewords from  $\mathcal{C}^\perp$  to a parity-check matrix  $H$  such that all the stopping sets of size less than  $d$  become covered. It was shown in [6] that this is always possible, and thus all stopping sets of size less than  $d$  are coverable.

In this work, we build on the approach in [6]. Namely, we extend parity-check matrices by choosing codewords from  $\mathcal{C}^\perp$  and adjoining them as redundant rows. The extended matrices are constructed in such a way that they do not contain stopping sets of small size. In the sequel, we provide a detailed analysis of the minimum number of additional rows in order to achieve this goal. In what follows, we use the terms “row of a parity-check matrix” and “codeword from  $\mathcal{C}^\perp$ ” interchangeably. We also note that a particular order of rows in a parity-check matrix is not important. As a matter of convenience, we will denote by  $\mathcal{C}_0^\perp$  the set of all the codewords of  $\mathcal{C}^\perp$  except the zero vector.

*Definition 4 [6]:* The size of the smallest stopping set of a parity-check matrix  $H$ , denoted by  $s(H)$ , is called the *stopping distance of the matrix*.

It is known that a maximal parity-check matrix  $H^{(2^r)}$  consisting of all  $2^r$  codewords of  $\mathcal{C}^\perp$  is an orthogonal array of strength  $d - 1$  (see [20, Ch. 5, Thm. 8]). This means that for any  $\mathcal{S} \subseteq [n]$  of size  $i$ ,  $1 \leq i \leq d - 1$ ,  $H_{\mathcal{S}}^{(2^r)}$  contains each  $i$ -tuple as its row exactly  $2^{r-i}$  times and, hence,  $\mathcal{S}$  is covered by exactly  $i \cdot 2^{r-i}$  rows of  $H^{(2^r)}$ .

The following definition was introduced in [6].

*Definition 5:* The *stopping redundancy* of  $\mathcal{C}$ , denoted by  $\rho(\mathcal{C})$ , is the smallest number of rows in any parity-check matrix (of rank  $r$ ) of  $\mathcal{C}$ , such that the corresponding stopping distance equals  $d$ .

It was shown in [6, Th. 3], that any parity-check matrix  $H$  of a binary linear code  $\mathcal{C}$  with the minimum distance  $d \leq 3$  already has  $s(H) = d$ . In what follows, we are mostly interested in the case  $d > 3$ .

## III. UPPER BOUNDS ON STOPPING REDUNDANCY

Before proceeding further, we present the following technical result. It will be used in the proof of Theorem 1.

*Lemma 1:* For any integers  $i, j, r \geq 1$ , and  $j < 2^r$ , define

$$\pi(r, i, j) \triangleq 1 - \frac{i \cdot 2^{r-i}}{2^r - j}.$$

Then, for any integer  $r \leq r'$ , and  $i \leq i'$ , we have  $\pi(r, i, j) \leq \pi(r', i, j)$  and  $\pi(r, i, j) \leq \pi(r, i', j)$ . In other words,  $\pi(r, i, j)$  is monotonically non-decreasing in integer variables  $r$  and  $i$ .

*Proof:* The statement of the lemma follows easily if we rewrite:

$$\pi(r, i, j) = 1 - \frac{i}{2^i} \cdot \frac{1}{1 - j \cdot 2^{-r}}.$$

■

### A. Upper Bounds for General Codes

In [6], Schwartz and Vardy presented the first upper bound on the stopping redundancy of a general binary linear  $[n, k, d]$

<sup>1</sup>In other words, we will use “the stopping set  $\mathcal{S}$  is covered by the matrix” and “ $\mathcal{S}$  is not a stopping set in the matrix” interchangeably.

code  $\mathcal{C}$ :

$$\rho(\mathcal{C}) \leq \binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{d-2}. \quad (1)$$

This bound is constructive. The other works along the same lines are [7], [9], [12], and [14]–[18], which present other constructive upper bounds, either for general linear codes, or for some specific families, or for specific codes.

On the other hand, *probabilistic* arguments gave rise to better bounds [1], [7], [8], [17], [18], yet these bounds are non-constructive. The main probabilistic technique in our paper dates back to the work of Han and Siegel [7]:

$$\rho(\mathcal{C}) \leq \min\{t \in \mathbb{N} : \mathcal{E}_{n,d}(t) < 1\} + (r - d + 1), \quad (2)$$

where

$$\mathcal{E}_{n,d}(t) \triangleq \sum_{i=1}^{d-1} \binom{n}{i} \left(1 - \frac{i}{2^t}\right)^t.$$

The bound has been improved in [8] and further refined in [1] by carefully selecting the first non-random rows, giving the smallest known values for most of codes (to the best of our knowledge).

Below we present a modified bound based on [1, Th. 1]. More precisely, we drop the burdensome requirement

$$(r-1)(d-1) \leq 2^{d-1} \quad (3)$$

thus making the bound applicable to all the binary linear codes. On the other hand, we need to add the rank deficiency term  $\Delta$  to ensure that the constructed parity-check matrix has the required rank. However, for medium and long codes, this term is negligible in comparison with the stopping redundancy.

*Theorem 1:* For an  $[n, k, d]$  linear binary code  $\mathcal{C}$  let  $H^{(\tau)}$  be any  $\tau \times n$  matrix consisting of  $\tau$  different codewords of the dual code  $\mathcal{C}^\perp$  and let  $u_i$  denote the number of stopping sets of size  $i$ ,  $i = 1, 2, \dots, d-1$ , in  $H^{(\tau)}$ . For  $t = 0, 1, \dots, 2^r - \tau$ , we introduce the following notations:

$$\begin{aligned} \mathcal{D}_t &= \sum_{i=1}^{d-1} u_i \prod_{j=\tau+1}^{\tau+t} \pi(r, i, j), \\ P_{t,0} &= \lfloor \mathcal{D}_t \rfloor, \\ P_{t,j} &= \left\lfloor \pi(r, d-1, \tau+t+j) \cdot P_{t,j-1} \right\rfloor, \quad j = 1, 2, \dots \\ \Delta &= r - \max\{\text{rank } H^{(\tau)}, d-1\}, \end{aligned}$$

and let  $\kappa_t$  be the smallest  $j$  such that  $P_{t,j} = 0$ . Then

$$\rho \leq \tau + \min_{0 \leq t < 2^r - \tau} \{t + \kappa_t\} + \Delta. \quad (4)$$

*Proof:* We prove the theorem in two steps. First, we show the existence of a  $(\tau+t) \times n$  matrix with a number of stopping sets less than or equal to  $P_{t,0}$ . Second, we show that this number further decreases when we add carefully selected rows one by one. Finally, after adding a sufficient number of rows, we obtain a matrix with no stopping sets of size less than  $d$ .

*Step 1:* By orthogonal array property, for any subset of columns  $\mathcal{S} \subseteq [n]$  of size  $i$ ,  $i = 1, 2, \dots, d-1$ , there are exactly  $i \cdot 2^{r-i}$  codewords in  $\mathcal{C}_0^\perp$  that cover  $\mathcal{S}$ . If  $\mathcal{S}$  is not covered by  $H^{(\tau)}$ , none of these  $i \cdot 2^{r-i}$  codewords is present among the rows of  $H^{(\tau)}$ .

Fix a stopping set  $\mathcal{S}$  in  $H^{(\tau)}$  and draw  $t$  codewords from the set  $\mathcal{C}_0^\perp \setminus \{\text{rows of } H^{(\tau)}\}$  at random without repetition. There are

$$\binom{2^r - \tau - 1}{t}$$

ways to do this, provided the order of selection does not matter. On the other hand, in the same set  $\mathcal{C}_0^\perp \setminus \{\text{rows of } H^{(\tau)}\}$ , there are  $(2^r - \tau - 1) - i \cdot 2^{r-i}$  codewords that do *not* cover  $\mathcal{S}$  and there are

$$\binom{(2^r - \tau - 1) - i \cdot 2^{r-i}}{t}$$

ways to draw  $t$  codewords out of them. Therefore, if we draw  $t$  codewords from the set  $\mathcal{C}_0^\perp \setminus \{\text{rows of } H^{(\tau)}\}$  at random without repetition, the probability not to cover  $\mathcal{S}$  by any one of them is (see Appendix A)

$$\binom{(2^r - \tau - 1) - i \cdot 2^{r-i}}{t} / \binom{2^r - \tau - 1}{t} \stackrel{\text{Lemma 6}}{=} \prod_{j=\tau+1}^{\tau+t} \pi(r, i, j). \quad (5)$$

This holds for each  $\mathcal{S}$  that was not originally covered by  $H^{(\tau)}$ . Since numbers of stopping sets of sizes  $1, 2, \dots, d-1$  are  $u_1, u_2, \dots, u_{d-1}$ , respectively, the average<sup>2</sup> number of stopping sets of size less than  $d$  that are left after adjoining random  $t$  rows to  $H^{(\tau)}$  is

$$\sum_{i=1}^{d-1} u_i \prod_{j=\tau+1}^{\tau+t} \pi(r, i, j) \triangleq \mathcal{D}_t. \quad (6)$$

Furthermore, since the above expression is an expected value of an integer random variable, there exists its realization (i.e., choice of  $t$  rows), such that the number of stopping sets left is not more than  $\lfloor \mathcal{D}_t \rfloor \triangleq P_{t,0}$ . Fix these  $t$  rows and further assume that we have a  $(\tau+t) \times n$  matrix  $H^{(\tau+t)}$  with not more than  $P_{t,0}$  stopping sets of size less than  $d$ .

*Step 2:* Adjoin to  $H^{(\tau+t)}$  a random codeword from  $\mathcal{C}_0^\perp \setminus \{\text{rows of } H^{(\tau+t)}\}$ . If some stopping set  $\mathcal{S}$  of size  $i$ ,  $1 \leq i \leq d-1$ , has not been covered by  $H^{(\tau+t)}$  yet, there are exactly  $i \cdot 2^{r-i}$  codewords in  $\mathcal{C}_0^\perp \setminus \{\text{rows of } H^{(\tau+t)}\}$  that cover  $\mathcal{S}$  and, thus, the probability that  $\mathcal{S}$  stays non-covered after adjoining this new row is

$$1 - \frac{i \cdot 2^{r-i}}{2^r - (\tau + t + j)} \stackrel{\text{Lemma 1}}{\leq} \pi(r, i, \tau + t + 1) \leq \pi(r, d-1, \tau + t + 1).$$

This holds for any stopping set  $\mathcal{S}$  of size  $i$ . Then, there exists a codeword in  $\mathcal{C}_0^\perp \setminus \{\text{rows of } H^{(\tau+t)}\}$  such that after adjoining it as a row to  $H^{(\tau+t)}$ , the number of non-covered stopping sets becomes less than or equal to

$$\left\lfloor \pi(r, d-1, \tau + t + 1) P_{t,0} \right\rfloor \triangleq P_{t,1}.$$

To this end, we fix this new row and further assume that we have a  $(\tau + t + 1) \times n$  matrix  $H^{(\tau+t+1)}$  with the number

<sup>2</sup>Averaging is by the choice of  $t$  rows.



TABLE I  
COMPARISON OF UPPER BOUNDS ON STOPPING REDUNDANCY  
OF DIFFERENT CODES

	[24, 12, 8]	[48, 24, 12]	[155, 64, 20]
	Golay	QR	Tanner
Equation (1), [6]	2509	4 540 385	$6.2 \cdot 10^{18}$
Equation (2), [7]	232	4440	1 526 972
[8, Th. 7]	182	3564	1 260 673
[1, Th. 1], $\tau = 1$	180	3538	1 247 888
Theorem 1, $\tau = 1$	185	3562	1 247 960
[1, Th. 1], $\tau = m$	168	2543	2573
Theorem 1, $\tau = m$	168	2543	2573

of stopping sets of size smaller than  $d$  less than or equal to  $P_{t,1}$ . After that, we iteratively repeat Step 2. We stop when the number of non-covered stopping sets is equal to zero.

Finally, we need to ensure that the rank of the resulting matrix is indeed  $r$ . We already know that it is not less than  $\text{rank } H^{(\tau)}$ . On the other hand, since we covered all the stopping sets of size less than  $d$ , the rank is at least  $d - 1$ . Hence it is enough to add  $\Delta$  additional rows to ensure the correct rank of the parity-check matrix. ■

*Note 1: The expression in (4) is monotonically non-decreasing in  $u_i$ . Sometimes, the exact values of  $u_i$  are difficult to find. In that case, upper bounds are used instead.*

*Note 2: By applying Lemma 1 to expressions for  $\mathcal{D}_t$  and  $P_{t,j}$ , we obtain that (4) is also monotonically non-decreasing in  $r$ . Sometimes, a standard parity-check matrix is redundant,<sup>3</sup> and the number of rows  $m$  in a mostly-used parity-check matrix is larger than  $r$ . It might be more convenient to use  $m$  instead of  $r$  and the bound (4) still holds.*

To give a flavor of differences between the existing bounds on stopping redundancy, we calculate the bounds (1) in [6], (2) in [7], the bound in [8, Th. 7], the bound in [1, Th. 1], and the bound in Theorem 1. The two last bounds are calculated in two modes. First, we use  $\tau = 1$  and  $H^{(\tau)}$  consists of the first row of the parity-check matrix of the corresponding code. Next, we use whole parity-check matrices of the codes as  $H^{(\tau)}$  (in Table I,  $m$  denotes the number of rows in the parity-check matrix used).

We calculate the aforementioned bounds for the following codes:

- the [24, 12, 8] binary extended Golay code (see Section V-A);
- the [48, 24, 12] binary quadratic residue (QR) extended code (see [20, Sec. 16]);
- the (3, 5)-regular LDPC [155, 64, 20] Tanner code in [21].

Table I presents numerical results. The original bound by Schwartz and Vardy (1) is the only constructive bound here, but it is the weakest one. Note that the bound in Theorem 1 is

only slightly worse than [1, Th. 1] but it is applicable to any code. Often, a code that does not satisfy (3) has its stopping distance equal to the minimum distance. Yet the new bound is useful for calculation of the stopping redundancy hierarchy (see Section III-B).

The bounds in [1, Th. 1] and Theorem 1 with  $\tau = m$  give the tightest results. However, they require knowledge of the stopping sets spectrum of a parity-check matrix. For the Golay and the QR codes, we calculate their spectra by exhaustive brute-force checking. For the Tanner code, we use the spectrum obtained in [10, Tab. 1]. For longer codes, calculating a stopping sets spectrum can be infeasible even for the method in [10] and similar works. We suggest a way to overcome this obstacle in Section IV-C.

### B. Stopping Redundancy Hierarchy

In Definition 5, it is required that the stopping distance of a parity-check matrix is exactly  $d$ . However, a more general requirement can be imposed. Thus, in [9], it was required that the parity-check matrix does not contain stopping sets of size up to  $\ell$ , for some  $\ell < d$ . This can be achieved by adjoining a smaller number of rows to a parity-check matrix.

It is important to notice that stopping sets of size  $d$  or larger can also cause failures of the iterative decoder on the BEC (see, for example, [12]). Thus, in order to approach the ML performance with the iterative decoder, we should also cover the stopping sets of size  $d$  or larger which, if erased, do not cause the ML decoder to fail. We can achieve this by adjoining redundant rows. The following definition is a generalization of [9, Definition 2.4].

*Definition 6: The  $\ell$ -th stopping redundancy of  $\mathcal{C}$ ,  $1 \leq \ell \leq r$ ,  $r = n - k$ , is the smallest nonnegative integer  $\rho_\ell(\mathcal{C})$  such that there exists a (possibly redundant) parity-check matrix of  $\mathcal{C}$  with  $\rho_\ell(\mathcal{C})$  rows and no stopping sets of size less than or equal to  $\ell$  which are coverable by  $\mathcal{C}$ . The ordered set of integers  $(\rho_1(\mathcal{C}), \rho_2(\mathcal{C}), \dots, \rho_r(\mathcal{C}))$  is called the stopping redundancy hierarchy of  $\mathcal{C}$ .*

The difference with [9, Definition 2.4] is that, first, in Definition 6,  $\ell$  can be as large as  $r$  (while in [9, Definition 2.4],  $\ell \leq d - 1$ , which is a more limiting condition). Second, in Definition 6, only *coverable* stopping sets are eliminated. However, as it was already mentioned, all stopping sets of size  $\ell \leq d - 1$  are coverable, and therefore Definition 6 contains [9, Definition 2.4] as a special case. Specifically, from Definition 6, we have that  $\rho(\mathcal{C}) = \rho_{d-1}(\mathcal{C})$ .

*Note 3: For  $\ell \leq d - 1$ , an upper bound on the  $\ell$ -th stopping redundancy can be formulated as in Theorem 1, where  $d$  is replaced by  $\ell + 1$ . We omit the details.*

### C. Choice of $H^{(\tau)}$

Theorem 1 does not suggest how one should choose the initial  $\tau \times n$  matrix. In general, it is a difficult question, as it strongly depends on the particular code. Below, we propose some simple heuristics.

Fix  $\tau = 1$ . Then, Lemma 7 in Appendix B gives two values for a weight  $w$  of the initial row of  $H^{(\tau)}$ , one of which is

<sup>3</sup>For instance, for Gallager  $(J, K)$ -regular codes (see Section V-D).

guaranteed to cover the maximum number of stopping sets of size not more than  $\ell$ :

$$w_{\text{opt}} \in \left\{ \left\lfloor \frac{n+1}{\ell} \right\rfloor, \left\lceil \frac{n}{\ell} \right\rceil \right\}.$$

However, a codeword of such weight does not necessarily exist in  $C^\perp$ . Hence one needs to consider the closest alternatives. After the dual codeword of weight  $w$  is fixed, the number of stopping sets of size less than  $d$  in  $H^{(\tau)}$  is expressed as

$$u_i = \binom{n}{i} - w \binom{n-w}{i-1},$$

and these values can be further used with the bound (4).

The situation becomes more complicated for  $\tau = 2$ , as in that case the optimal choice depends not only on the weights of the first two rows of  $H^{(\tau)}$  but also on the size of the intersection of their supports. For simplicity, we can take two different rows of the same weight and obtain the corresponding estimate on the number of stopping sets. More precisely, if  $\tau = 2$ ,  $H^{(\tau)}$  consists of two dual codewords  $\mathbf{h}_1$  and  $\mathbf{h}_2$  of weight  $w$  each with an intersection of supports of size  $|\text{supp}(\mathbf{h}_1) \cap \text{supp}(\mathbf{h}_2)| = \delta$ , then the total number of stopping sets of size less than  $d$  in  $H^{(\tau)}$  equals (see [1, Corollary 2])

$$u_i = \binom{n}{i} - 2w \binom{n-w}{i-1} + \delta \binom{n-2w+\delta}{i-1} + (w-\delta)^2 \binom{n-2w+\delta}{i-2}.$$

We can generalize this approach for  $\tau > 2$  rows in  $H^{(\tau)}$  by using the principle of inclusion-exclusion. However, this leads to explosion of terms in the formula for  $u_i$ . We do not continue in that direction.

#### IV. ACHIEVING ML PERFORMANCE

The ML decoder for the BEC is equivalent to solving a system of linear equations. More precisely, assume that we have a code with a parity-check matrix  $H$ , and that the received word is  $\mathbf{c}$ . Let the positions of erasures be  $\mathcal{E} \subseteq [n]$ . Denote by  $H_{\mathcal{E}}$  the matrix formed by the columns of  $H$  indexed by  $\mathcal{E}$ , and by  $\mathbf{c}_{\mathcal{E}}$  denote the vector formed by the entries of  $\mathbf{c}$  indexed by  $\mathcal{E}$ . Denote  $\bar{\mathcal{E}} = [n] \setminus \mathcal{E}$ , and, similarly, define  $H_{\bar{\mathcal{E}}}$  and  $\mathbf{c}_{\bar{\mathcal{E}}}$ . Then the parity-check equations can be written as

$$H_{\mathcal{E}}\mathbf{c}_{\mathcal{E}} + H_{\bar{\mathcal{E}}}\mathbf{c}_{\bar{\mathcal{E}}} = \mathbf{0},$$

where  $\mathbf{0}$  is the all-zero vector of the corresponding length. Since  $\mathbf{c}_{\bar{\mathcal{E}}}$ ,  $H_{\bar{\mathcal{E}}}$ , and  $H_{\mathcal{E}}$  are known, we can rewrite the equations in the following form:

$$H_{\mathcal{E}}\mathbf{c}_{\mathcal{E}} = H_{\bar{\mathcal{E}}}\mathbf{c}_{\bar{\mathcal{E}}}. \quad (7)$$

It is a system of linear equations with the vector of unknowns  $\mathbf{c}_{\mathcal{E}}$  and a matrix of coefficients  $H_{\mathcal{E}}$ . This system always has at least one solution, the originally transmitted codeword. If this solution is not unique, we say that the ML decoder fails.

The reason for the difference in performance of the ML and the iterative decoders is existence of stopping sets in a parity-check matrix used for iterative decoding. In the following sections, we aim at making the iterative decoding performance closer to that of the ML performance.

##### A. Coverable Stopping Sets

In Section III, we analyzed techniques for removal of all stopping sets of size up to  $d-1$ . However, stopping sets of size  $d$  or larger can also cause failures of the iterative decoder on the BEC. As it is mentioned above, in order to approach the ML performance with iterative decoding, one should also cover stopping sets of size  $d$  or larger which, if erased, do not cause the ML decoder to fail. This can be achieved by adjoining redundant rows to a parity-check matrix. The following two lemmas will be instrumental in the analysis that follows.

As before, let  $H$  be the parity-check matrix of the code  $C$ . By  $H^{(2^r)}$  we denote the matrix whose rows are all  $2^r$  codewords of  $C^\perp$ , and  $H_{\mathcal{E}}^{(2^r)}$  denotes the matrix formed from columns of  $H^{(2^r)}$  indexed by  $\mathcal{E}$ . The next lemma consists of well-known results, see for example [22, Sec. 3.2.1].

*Lemma 2: The following statements are equivalent:*

- i) columns of  $H_{\mathcal{E}}$  are linearly dependent;
- ii) there exists a non-zero codeword  $\mathbf{c}$ , such that  $\text{supp}(\mathbf{c}) \subseteq \mathcal{E}$ ;
- iii) if all positions in  $\mathcal{E}$  have been erased then the ML decoder fails.

Next, consider the case when the columns of  $H_{\mathcal{E}}$  are linearly independent.

*Lemma 3: The following statements are equivalent:*

- i) columns of  $H_{\mathcal{E}}$  are linearly independent;
- ii)  $H_{\mathcal{E}}^{(2^r)}$  is an orthogonal array of strength  $|\mathcal{E}|$ .

And if any of them holds then

- iii)  $\mathcal{E}$  is not a stopping set in  $H^{(2^r)}$ .

*Proof:* Both statements i) and iii) follow from ii) in a straightforward manner.

We prove next that ii) follows from i). First of all, if there are redundant rows in  $H$ , we can ignore them and assume that  $m = r$ . Owing to the fact that columns of  $H_{\mathcal{E}}$  are linearly independent, there exist  $|\mathcal{E}|$  rows in  $H_{\mathcal{E}}$  that form a full-rank square matrix. Then, each of the remaining  $r - |\mathcal{E}|$  rows of  $H_{\mathcal{E}}$  can be represented as a linear combination of these  $|\mathcal{E}|$  rows. Without loss of generality assume that

$$H_{\mathcal{E}} = \begin{pmatrix} B \\ TB \end{pmatrix},$$

where  $B$  is an  $|\mathcal{E}| \times |\mathcal{E}|$  full-rank matrix, and  $T$  is a  $(r - |\mathcal{E}|) \times |\mathcal{E}|$  matrix of coefficients.

Each row of  $H_{\mathcal{E}}^{(2^r)}$  is bijectively mapped onto  $r$  coefficients of linear combination  $\boldsymbol{\alpha} = (\boldsymbol{\alpha}' \mid \boldsymbol{\alpha}'')$ , where  $\boldsymbol{\alpha}' \in \mathbb{F}_2^{|\mathcal{E}|}$ , and  $\boldsymbol{\alpha}'' \in \mathbb{F}_2^{r-|\mathcal{E}|}$ , as follows:

$$\boldsymbol{\alpha} \begin{pmatrix} B \\ TB \end{pmatrix} = \boldsymbol{\alpha}'B + \boldsymbol{\alpha}''TB = (\boldsymbol{\alpha}' + \boldsymbol{\alpha}''T)B.$$

Fix the vector  $\boldsymbol{\alpha}''$  (and therefore the vector  $\boldsymbol{\alpha}''T$  of size  $|\mathcal{E}|$  is fixed). Then, the transformation

$$\boldsymbol{\alpha}' \mapsto \boldsymbol{\alpha}' + \boldsymbol{\alpha}''T$$

is a bijection of  $\mathbb{F}_2^{|\mathcal{E}|}$ . Since  $B$  is a full-rank matrix,

$$\boldsymbol{\alpha}' \mapsto (\boldsymbol{\alpha}' + \boldsymbol{\alpha}''T)B$$

is a bijection too. Hence, for a fixed  $\alpha''$ , if we iterate over all  $\alpha'$ , each of the rows in  $\mathbb{F}_2^{|\mathcal{E}|}$  is generated exactly once. This holds for each of  $2^{r-|\mathcal{E}|}$  possible choices for  $\alpha''$ . Hence, each vector of  $\mathbb{F}_2^{|\mathcal{E}|}$  appears as a row in  $H_{\mathcal{E}}^{(2^r)}$  exactly  $2^{r-|\mathcal{E}|}$  times. Thus,  $H_{\mathcal{E}}^{(2^r)}$  is an orthogonal array of strength  $|\mathcal{E}|$ . ■

All things considered, an erasure pattern  $\mathcal{E}$  can be tackled by the ML decoder if and only if there exists a redundant row that covers  $\mathcal{E}$ . Moreover, in this case  $H_{\mathcal{E}}^{(2^r)}$  is an orthogonal array of strength  $|\mathcal{E}|$  and, therefore, the techniques for calculating probability of being covered in the proof of Theorem 1 are still applicable. In the sequel, we reformulate the upper bound (4).

Recall that the stopping set  $\mathcal{S}$  is coverable if there is a parity-check matrix in which  $\mathcal{S}$  is not a stopping set. As we see, coverable stopping sets are exactly those that, if erased, can be recovered by the ML decoder.

We note that the  $r$ -th stopping redundancy  $\rho_r(\mathcal{C})$  of  $\mathcal{C}$  is the smallest number of rows in a parity-check matrix of  $\mathcal{C}$  such that the iterative decoder achieves the ML decoding performance. Next, we formulate an upper bound on the  $\ell$ -th stopping redundancy, as defined in Definition 6, for  $\ell \leq r$ ,  $r = n - k$ .

**Theorem 2:** For an  $[n, k, d]$  linear code  $\mathcal{C}$  let  $H^{(\tau)}$  be any  $\tau \times n$  matrix consisting of  $\tau$  different non-zero codewords of the dual code  $\mathcal{C}^\perp$  and let  $u_i$  denote the number of not covered stopping sets of size  $i$ ,  $i = 1, 2, \dots, \ell$  ( $\ell \leq r$ ), in  $H^{(\tau)}$  that are coverable by  $\mathcal{C}$ . Then the  $\ell$ -th stopping redundancy is

$$\rho_\ell(\mathcal{C}) \leq \Xi_\ell^{(I)}(u_1, u_2, \dots, u_\ell) \triangleq \tau + \min_{0 \leq t < 2^r - \tau} \{t + \kappa_t\} + \Delta,$$

where

$$\mathcal{D}_t = \sum_{i=1}^{\ell} u_i \prod_{j=\tau+1}^{\tau+t} \pi(r, i, j),$$

$$P_{t,0} = \lfloor \mathcal{D}_t \rfloor,$$

$$P_{t,j} = \left\lfloor \pi(r, \ell, \tau + t + j) P_{t,j-1} \right\rfloor, \quad j = 1, 2, \dots$$

$$\Delta = r - \max \left\{ \text{rank } H^{(\tau)}, \ell \right\},$$

and  $\kappa_t$  is the smallest  $j$  such that  $P_{t,j} = 0$ .

We remark that the difference between the statements of Theorem 2 and of Theorem 1 is that the value  $d-1$  is replaced by  $\ell$ .

*Proof:* The proof follows the lines of that in Theorem 1 with the only difference that now for each coverable stopping set  $\mathcal{S}$ , the corresponding matrix  $H_{\mathcal{S}}^{(2^r)}$  contains all the tuples of size  $|\mathcal{S}|$  equal number of times, as it was shown in Lemma 3.

We also observe that if, after adjoining rows, the matrix  $H^{(\tau)}$  does not contain coverable stopping sets of size at most  $\ell$ ,  $\ell \leq r$ , then the rank of this matrix is at least  $\ell$ . This follows from the fact that there exist  $\ell$  linearly independent columns in the resulting matrix, as otherwise the corresponding set of coordinates would contain a stopping set that is not a support of a codeword. ■

**Corollary 1:** There exists an extended parity-check matrix with no more than  $\Xi_r^{(I)}(u_1, u_2, \dots, u_r)$  rows, such that the iterative decoder fails if and only if the ML decoder fails. It follows that the decoding error probability of these two decoders is equal.

Computing the number  $u_i$  of stopping sets of size  $i$ —or even finding the corresponding upper bound—might be a difficult task for general codes, except for trivial cases. In what follows, we suggest two approaches:

- ensemble-average approach (see Section IV-B);
- finding estimates on  $u_i$  numerically (see Section IV-C).

### B. Exact Ensemble-Average ML Stopping Redundancy

In order to apply the upper bounds on the stopping redundancy to the ensemble-average values, we formulate a weaker bound inspired by [8].

**Theorem 3:** Assume that  $\mathcal{C}$  is a linear  $[n, k]$ -code and  $H$  is a parity-check matrix consisting of  $m$  different rows being codewords of the dual code  $\mathcal{C}^\perp$ , such that there are  $u_i$  stopping sets of size  $i = 1, 2, \dots, \ell$  ( $\ell \leq r$ ), in  $H$  coverable by  $\mathcal{C}$ . Then the  $\ell$ -th stopping redundancy is bounded from above as follows:

$$\rho_\ell(\mathcal{C}) \leq \Xi_\ell^{(II)}(u_1, u_2, \dots, u_\ell) \triangleq m + \min_{0 \leq t < 2^m - m} \left\{ t + \sum_{i=1}^{\ell} u_i \prod_{j=m+1}^{m+t} \pi(m, i, j) \right\}.$$

*Proof:* Analogous to Step 1 in Theorem 1, choose  $t$ ,  $t = 0, 1, \dots, 2^r - m - 1$ , codewords from  $\mathcal{C}_0^\perp \setminus \{\text{rows of } H\}$  uniformly at random and without repetitions, and adjoin them to  $H$ . Then, the average number of coverable but not covered stopping sets in this extended matrix becomes equal to

$$\sum_{i=1}^{\ell} u_i \prod_{j=m+1}^{m+t} \pi(r, i, j).$$

For each of these stopping sets, we add one row from  $\mathcal{C}_0^\perp$  to cover it, and thus the total number of rows in the parity-check matrix becomes

$$m + t + \sum_{i=1}^{\ell} u_i \prod_{j=m+1}^{m+t} \pi(r, i, j) \leq m + t + \sum_{i=1}^{\ell} u_i \prod_{j=m+1}^{m+t} \pi(m, i, j).$$

By minimizing this expression over the choice of  $t$ , we obtain the required upper bound. We note that minimizing over  $t$  up to  $2^m - m$  is just a matter of further convenience, as the true minimum value is obtained for  $t < 2^r - m \leq 2^m - m$ . ■

Now we can formulate the ensemble-average result.

**Corollary 2:** Consider an ensemble  $\mathfrak{C}$  of codes, where the probability distribution of the codes is determined by the probability distribution on  $m \times n$  parity-check matrices. Moreover, assume that the parity-check matrix  $H$  of rank  $r = n - k$  corresponding to the  $[n, k]$  code  $\mathcal{C} \in \mathfrak{C}$  has  $u_i^{(H)}$  size- $i$  stopping sets coverable by  $\mathcal{C}$ , where  $i = 1, 2, \dots, \ell$ . Denote the ensemble-average number of such stopping sets:

$$\bar{u}_i = \mathbb{E}_{\mathfrak{C}} \left\{ u_i^{(H)} \right\}.$$

Then, the average  $\ell$ -th stopping redundancy over the ensemble  $\mathfrak{C}$  is bounded from above as follows:

$$\mathbb{E}_{\mathfrak{C}} \{ \rho_\ell(\mathcal{C}) \} \leq \Xi_\ell^{(II)}(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_\ell).$$

*Proof:* First, we observe that Theorem 3 yields an upper bound on  $\rho_\ell(C)$  for every integer  $0 \leq t < 2^m - m$ :

$$\rho_\ell(C) \leq m + t + \sum_{i=1}^{\ell} u_i \prod_{j=m+1}^{m+t} \pi(m, i, j).$$

Then,  $\Xi_\ell^{(I)}$  is a minimum of these upper bounds over the values of  $t$ .

Fix some integer  $0 \leq t < 2^m - m$  and take the average over  $\mathfrak{C}$ :

$$\mathbb{E}_{\mathfrak{C}}\{\rho_\ell(C)\} \leq m + t + \sum_{i=1}^{\ell} \bar{u}_i \prod_{j=m+1}^{m+t} \pi(m, i, j).$$

As it holds for each  $t$ , it should also hold for their minimum:

$$\mathbb{E}_{\mathfrak{C}}\{\rho_\ell(C)\} \leq m + \min_{0 \leq t < 2^m - m} \left\{ t + \sum_{i=1}^{\ell} \bar{u}_i \prod_{j=m+1}^{m+t} \pi(m, i, j) \right\}.$$

■

### C. Statistical Estimation of the Number of Coverable Stopping Sets

In this section, we aim at finding statistical estimates on the number of coverable stopping sets and further apply them to the upper bounds on the stopping redundancy hierarchy. In what follows, we use the cumulative distribution function of the standard normal distribution  $\mathcal{N}(0, 1)$  given by:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt.$$

*Lemma 4:* Consider a parity-check matrix  $H$  of an  $[n, k]$ -code  $C$ . For  $1 \leq i \leq r$ , fix a large number  $N_i$  and generate  $N_i$  random subsets of  $[n]$  uniformly at random (with repetitions), namely  $\mathcal{S}_1^{(i)}, \mathcal{S}_2^{(i)}, \dots, \mathcal{S}_{N_i}^{(i)}$ , each subset consisting of  $i$  elements. For  $j = 1, 2, \dots, N_i$ , we define the following events:

$$x_j^{(i)} = \begin{cases} 1, & \text{if } \mathcal{S}_j^{(i)} \text{ is a coverable stopping set in } H, \\ 0, & \text{otherwise.} \end{cases}$$

If  $u_i$  is a number of size- $i$  stopping sets in  $H$  coverable by  $C$ , and  $\epsilon_i$  is some small fixed number, then

$$\mathbb{P}\{u_i < \hat{u}_i\} = 1 - \epsilon_i,$$

where

$$\hat{u}_i = \binom{n}{i} \left( \bar{x}^{(i)} + \kappa \sqrt{\frac{\hat{V}}{N_i} + \frac{\gamma_1 \hat{V} + \gamma_2}{N_i^2}} \right), \quad (8)$$

$$\kappa = \Phi^{-1}(1 - \epsilon_i), \quad \eta = \frac{\kappa^2}{3} + \frac{1}{6}, \quad (9)$$

$$\bar{x}^{(i)} = \frac{\sum_{j=1}^{N_i} x_j^{(i)}}{N_i}, \quad \tilde{x}^{(i)} = \frac{N_i \bar{x}^{(i)} + \eta}{N_i + 2\eta}, \quad (10)$$

$$\gamma_1 = \frac{-13}{18} \cdot \kappa^2 - \frac{17}{18}, \quad \gamma_2 = \frac{\kappa^2}{18} + \frac{7}{36}, \quad (11)$$

$$\hat{V} = \bar{x}^{(i)}(1 - \bar{x}^{(i)}). \quad (12)$$

*Proof:* Random variables  $\{x_j^{(i)}\}$  are independent and identically distributed according to the Bernoulli distribution with success probability

$$\theta_i = \frac{u_i}{\binom{n}{i}}.$$

Here  $\theta_i$  is unknown because  $u_i$  is unknown.

We further apply the  $1 - \epsilon_i$  upper limit second-order corrected one-sided confidence interval, constructed in [23, eq. (10)] and based on Edgeworth expansion. In our notation, it states that

$$\mathbb{P} \left\{ \theta_i < \tilde{x}^{(i)} + \kappa \sqrt{\frac{\hat{V}}{N_i} + \frac{\gamma_1 \hat{V} + \gamma_2}{N_i^2}} \right\} = 1 - \epsilon_i. \quad (13)$$

From this we obtain the required result. ■

This estimate can be used in conjunction with the upper bounds in Theorem 2 and Theorem 3. More specifically, we fix  $N_1, N_2, \dots, N_\ell$ , and  $\epsilon_1, \epsilon_2, \dots, \epsilon_\ell$ , and then we obtain that

$$\rho_\ell(C) \leq \Xi_\ell^{(I)}(\hat{u}_1, \hat{u}_2, \dots, \hat{u}_\ell),$$

which holds with probability

$$\prod_{i=1}^{\ell} (1 - \epsilon_i).$$

Furthermore, this approach can be extended to estimating the ensemble-average  $\ell$ -th stopping redundancy,  $\mathbb{E}_{\mathfrak{C}}\{\rho_\ell(C)\}$ .

*Lemma 5:* In the settings of Corollary 2, for  $1 \leq i \leq m$ , fix a large number  $N_i$  and generate  $N_i$  random pairs  $(H_j^{(i)}, \mathcal{S}_j^{(i)})$ ,  $j = 1, 2, \dots, N_i$ , where  $H_j^{(i)}$  is a parity-check matrix of a code from  $\mathfrak{C}$  and  $\mathcal{S}_j^{(i)}$  is a random subset of  $[n]$  consisting of  $i$  elements,  $H_j^{(i)}$  and  $\mathcal{S}_j^{(i)}$  being independent.

For  $j = 1, 2, \dots, N_i$ , we define the following events:

$$y_j^{(i)} = \begin{cases} 1, & \text{if } \mathcal{S}_j^{(i)} \text{ is a coverable stopping set in } H_j^{(i)}, \\ 0, & \text{otherwise.} \end{cases}$$

For a fixed small  $\epsilon_i$ ,

$$\mathbb{P}\{\bar{u}_i < \hat{u}_i\} = 1 - \epsilon_i,$$

where  $\hat{u}_i$  is defined similar to  $\hat{u}_i$  in (8) to (12) with  $x_j^{(i)}$ ,  $\bar{x}^{(i)}$  and  $\tilde{x}^{(i)}$  replaced by  $y_j^{(i)}$ ,  $\bar{y}^{(i)}$  and  $\tilde{y}^{(i)}$ , respectively.

*Proof:* Analogous to the proof of Lemma 4. ■

If we fix  $N_1, N_2, \dots, N_\ell$  and  $\epsilon_1, \epsilon_2, \dots, \epsilon_\ell$ , we obtain that

$$\mathbb{E}_{\mathfrak{C}}\{\rho_\ell(C)\} \leq \Xi_\ell^{(II)}(\hat{u}_1, \hat{u}_2, \dots, \hat{u}_\ell) \quad (14)$$

with probability  $\prod_{i=1}^{\ell} (1 - \epsilon_i)$ .

### D. Case Study: Standard Random Ensemble

In this section, we demonstrate application of the aforementioned bounds to the standard random ensemble (SRE)  $\mathfrak{S}(n, m)$ . This ensemble is defined by a means of its  $m \times n$  parity-check matrices  $H$ , where each entry of  $H$  is an independent and identically distributed (i.i.d.) Bernoulli random variable with parameter  $1/2$ .



[illegible]



TABLE III  
STOPPING REDUNDANCY HIERARCHIES. THE [24, 12, 8]  
EXTENDED GOLAY CODE

	coverable stopping sets		Theorem 2		Theorem 3	
	$u_1$	$u_2$	$\rho_1$	$\rho_2$	$\rho_1$	$\rho_2$
Exact $u_i$	0	0	12	12	12	12
	0	0	12	12	12	12
	0	0	12	12	12	12
	110	110	25	25	27	27
	1837	1837	49	49	51	51
	14 795	14 795	91	91	95	95
	74 349	74 349	168	168	174	174
	257 796	257 796	304	304	316	316
	649 275	649 275	540	540	560	560
	1 206 755	1 206 755	927	927	960	960
	1 585 794	1 585 794	1507	1507	1558	1558
	1 189 574	1 189 574	2241	2241	2309	2309
Estimates $\hat{u}_i$ ( $N_i = 10^3$ )	0	0	12	12	12	12
	1	1	13	13	13	13
	12	12	17	17	17	17
	247	247	28	28	30	30
	2596	2596	51	51	53	53
	21 061	21 061	94	94	98	98
	90 406	90 406	171	171	178	178
	288 582	288 582	307	307	319	319
	700 573	700 573	544	544	564	564
	1 309 119	1 309 119	933	933	967	967
	1 740 882	1 740 882	1519	1519	1570	1570
	1 384 130	1 384 130	2265	2265	2333	2333
Estimates $\hat{u}_i$ ( $N_i = 10^6$ )	0	0	12	12	12	12
	0	0	12	12	12	12
	0	0	12	12	12	12
	112	112	25	25	27	27
	1853	1853	49	49	51	51
	14 930	14 930	91	91	95	95
	74 656	74 656	168	168	174	174
	259 204	259 204	304	304	316	316
	651 167	651 167	540	540	561	561
	1 211 318	1 211 318	927	927	961	961
	1 590 393	1 590 393	1508	1508	1559	1559
	1 194 310	1 194 310	2241	2241	2310	2310

Based on the ideas discussed in Section IV, we can apply the algorithm akin to that of Schwartz and Vardy beyond the code minimum distance. In that case, the algorithm works with the full list of *coverable* stopping sets of the code. We now describe the algorithm in more details. We use the systematic double-circulant matrix  $H$  given in [20, p. 65] as a means to define the Golay code. Its stopping distance  $s(H) = 4$ .

Fix  $\ell$ ,  $4 \leq \ell \leq 12$ , and generate the list

$$\mathcal{L} = \{\mathcal{S} \subseteq [n] : |\mathcal{S}| \leq \ell, \text{rank } H_{\mathcal{S}} = |\mathcal{S}|\},$$

i.e., the list of stopping sets of size up to  $\ell$  (incl.) coverable by the Golay code. Next, we iteratively construct the extended parity-check matrix, starting with the empty matrix. At each iteration, we find one of the 4095 non-zero dual codewords<sup>4</sup>

<sup>4</sup>Although in fact the [24, 12, 8] Golay code is self-dual.

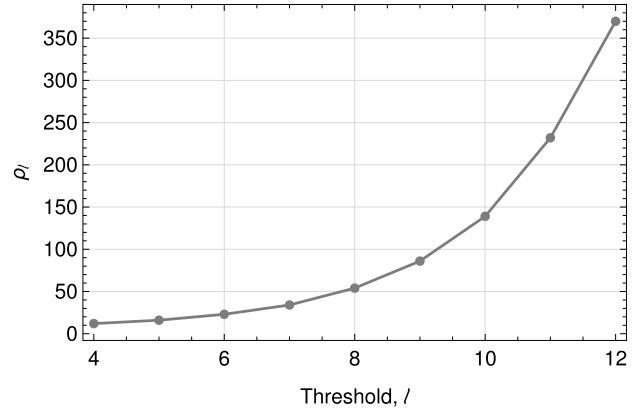


Fig. 2. Estimate on stopping redundancy hierarchy obtained by greedy search.

with the highest score. The score is of heuristic nature and for a dual codeword  $\mathbf{h}$  it is calculated as follows:

$$\text{score}(\mathbf{h}) = \sum_{\mathcal{S} \in \mathcal{L}} |\mathcal{S}| \cdot \mathbb{I}\{\mathbf{h} \text{ covers } \mathcal{S}\},$$

where  $\mathbb{I}\{\cdot\}$  is the indicator function. The row  $\mathbf{h}^*$  with the maximum score is added to the matrix we build, and the stopping sets covered by  $\mathbf{h}^*$  are removed from  $\mathcal{L}$ . Iterations continue until  $\mathcal{L}$  is empty. As we have only coverable sets in  $\mathcal{L}$ , the algorithm will stop before we add all the 4095 rows. To this end, we verify that the obtained parity-check matrix has rank 12.

A small difference with [6] in the proposed approach is a random choice of  $\mathbf{h}^*$  when several dual codewords have the same score. In that case, we run the algorithm several times and choose the matrix with the least number of rows. Fig. 2 illustrates the number of rows in the best obtained matrices for  $\ell = 4, 5, \dots, 12$ . We further refer to these matrices as  $H^{(12)}$ ,  $H^{(16)}$ ,  $H^{(23)}$ ,  $H^{(34)}$ ,  $H^{(54)}$ ,  $H^{(86)}$ ,  $H^{(139)}$ ,  $H^{(232)}$ , and  $H^{(370)}$ , according to the number of rows they have. The notation  $\Psi_H$  is used to denote the number of undecodable erasure patterns in the parity-check matrix  $H$ .

Table IV shows the numbers of undecodable patterns for the aforementioned extended parity-check matrices. Note that the number of such patterns for the BP decoder with  $H^{(370)}$  is *exactly* the same as for the ML decoder. This is in accordance with the discussion in Section IV.

Further, let  $\Psi(w)$  be a number of erasure patterns of weight  $w$ ,  $0 \leq w \leq n$ , in a code of length  $n$ , that cannot be decoded by some decoding method. Then, the frame error rate (also known as the block error rate) is a function of the bit erasure probability  $p$ , as follows:

$$\text{FER}(p) = \sum_{w=0}^n \Psi(w) p^w (1-p)^{n-w}.$$

Based on the number of undecodable erasure patterns, we plot the performance curves in Fig. 3. We note that plots for  $H^{(54)}$  and larger matrices are almost visually indistinguishable from the plot for  $H^{(370)}$ .

TABLE IV  
NUMBER OF UNDECODABLE ERASURE PATTERNS FOR DIFFERENT PARITY-CHECK MATRICES OF THE [24, 12, 8] GOLAY CODE

	$w$										
	0–3	4	5	6	7	8	9	10	11	12	$\geq 13$
Total patterns		10 626	42 504	134 596	346 104	735 471	1 307 504	1 961 256	2 496 144	2 704 156	$\binom{24}{w}$
$\Psi_H$	0	110	2 277	19 723	100 397	343 035	844 459	1 568 875	2 274 130	2 637 506	$\binom{23}{w}$
$\Psi_{H^{(34)}}$	0	0	0	0	0	3 598	82 138	585 157	1 717 082	2 556 402	$\binom{24}{w}$
$\Psi_{H^{(54)}}$	0	0	0	0	0	759	16 424	195 190	1 027 002	2 242 956	$\binom{24}{w}$
$\Psi_{H^{(86)}}$	0	0	0	0	0	759	12 144	98 822	570 567	1 774 724	$\binom{24}{w}$
$\Psi_{H^{(139)}}$	0	0	0	0	0	759	12 144	91 080	437 744	1 438 874	$\binom{24}{w}$
$\Psi_{H^{(232)}}$	0	0	0	0	0	759	12 144	91 080	425 040	1 324 074	$\binom{24}{w}$
$\Psi_{H^{(370)}} = \Psi_{ML}$	0	0	0	0	0	759	12 144	91 080	425 040	1 313 116	$\binom{24}{w}$

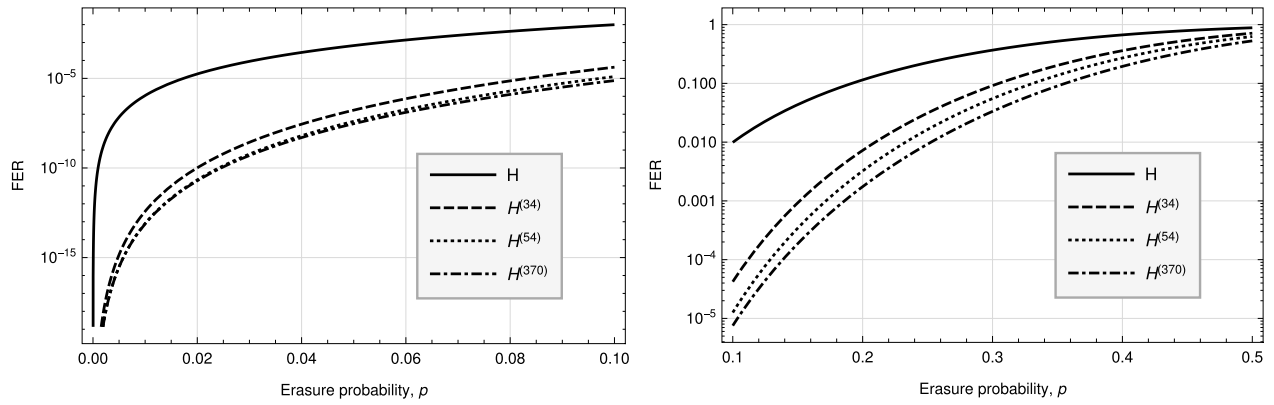


Fig. 3. Frame error rates for different parity-check matrices of the extended Golay code, obtained by a randomized greedy algorithm. There are no coverable stopping sets of size up to 3, 7, 8, and 12 for  $H$ ,  $H^{(34)}$ ,  $H^{(54)}$ , and  $H^{(370)}$ , respectively.

TABLE V  
ML STOPPING REDUNDANCIES AVERAGE OVER  $\mathfrak{S}(n, m)$ . ESTIMATES HOLD WITH PROBABILITY 95%

$n$	$R = 1/3$			$R = 1/2$			$R = 2/3$		
	$\rho_m$	$\hat{\rho}_m$	$\epsilon^{(m)}, \%$	$\rho_m$	$\hat{\rho}_m$	$\epsilon^{(m)}, \%$	$\rho_m$	$\hat{\rho}_m$	$\epsilon^{(m)}, \%$
6	6	6	1.27	3	3	1.7	2	2	2.53
12	84.99	85	0.64	34.75	34.77	0.85	10.55	10.55	1.27
18	1223.92	1224.18	0.43	281.32	281.37	0.57	46.11	46.12	0.85
24	18 557	18 557.6	0.32	2234.5	2234.82	0.43	189.07	189.08	0.64
30	288 386	288 422	0.26	17 715.6	17 717.1	0.34	758.87	758.9	0.51
36	$4.5288 \cdot 10^6$	$4.5301 \cdot 10^6$	0.21	140 636	140 645	0.28	3027.58	3027.7	0.43
42	$7.1464 \cdot 10^7$	$7.1467 \cdot 10^7$	0.18	$1.1180 \cdot 10^6$	$1.1181 \cdot 10^6$	0.24	12 064.5	12 065.1	0.37
48	$1.1308 \cdot 10^9$	$1.1310 \cdot 10^9$	0.16	$8.8982 \cdot 10^6$	$8.8987 \cdot 10^6$	0.21	48 084	48 085.4	0.32
54	$1.7926 \cdot 10^{10}$	$1.7928 \cdot 10^{10}$	0.14	$7.0879 \cdot 10^7$	$7.0883 \cdot 10^7$	0.19	191 731	191 734	0.28

### C. Standard Random Ensemble

In this section, we apply the results of Lemma 5 to the standard random ensemble  $\mathfrak{S}(n, m)$ . We calculate estimates on  $\mathbb{E}_{\mathfrak{S}(n, m)} \{\rho_\ell(C)\}$  for different  $n$  and  $m = (1 - R)n$  for “design” code rates  $R \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$ . For each pair  $(n, m)$  and each size  $i = 1, 2, \dots, m$ , we generate  $N = 10^7$  pairs  $(H^{(i)}, \mathcal{S}^{(i)})$  and register the frequencies of  $\mathcal{S}^{(i)}$  being a coverable stopping set in  $H^{(i)}$ .

Based on the frequencies, we obtain estimates  $\hat{u}_i$  on the ensemble-average sizes  $\bar{u}_i$ . For each size of the stopping sets  $i$ , we use  $\epsilon_i = 1 - 0.95^{1/m}$ , which gives a confidence of 95% that the estimates on  $\bar{u}_i$  hold.

After that, we apply Corollary 2 in order to obtain bounds on  $\mathbb{E}_{\mathcal{C}} \{\rho_m(C)\}$ , for selected values of  $m$ . These bounds are denoted by  $\hat{\rho}_m$ . Table V presents the resulting values. They are compared to the values  $\Xi_\ell^{(II)}(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_\ell)$  (obtained

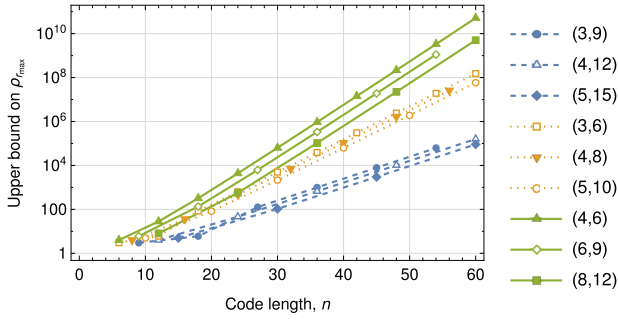


Fig. 4. Upper bounds on  $\mathcal{G}\mathcal{a}\mathcal{l}(n, J, K)$ -average  $r_{\max}$ -th stopping redundancy.

analytically, and denoted by  $\rho_m$ ). We observe that the numerical results are a very good approximation to the theoretical values.

#### D. Gallager Ensemble

We repeat the experiments of the previous subsection on the Gallager ensemble  $\mathcal{G}\mathcal{a}\mathcal{l}(n, J, K)$  of  $(J, K)$ -regular LDPC codes [2] for different choices of  $(J, K)$  and different lengths  $n$ . The ensemble is defined by parity-check matrices of a special form. An  $\left(\frac{nJ}{K}\right) \times n$  parity-check matrix consists of  $J$  strips of width  $M = n/K$  rows each. In the first strip, the  $j$ th row contains  $K$  ones in positions  $(j-1)K+1, (j-1)K+2, \dots, jK$  for  $j = 1, 2, \dots, M$ . Each of the other strips is a random column permutation of the first strip.

It is known that the rank of a parity-check matrix in  $\mathcal{G}\mathcal{a}\mathcal{l}(n, J, K)$  cannot be larger than  $r_{\max} = \frac{nJ}{K} - (J-1)$  due to the presence of redundant rows in any such matrix. Therefore, the ML decoding performance is achieved when all the coverable stopping sets of size up to  $r_{\max}$  are covered.

Fig. 4 demonstrates the values of the ML stopping redundancy,  $\rho_{r_{\max}}$ , for different lengths and different choices of  $J$  and  $K$ . We observe three clusters of plots according to the design rates of the codes.

## VI. CONCLUSIONS

In this work, we observed that stopping sets of size  $\ell$ ,  $d \leq \ell \leq r$ , are important for analysis of ML decoding over the BEC. We generalized the analytical approach in [6] by estimating the number of stopping sets of such sizes. This novel approach led to estimates on the number of redundant rows in the parity-check matrix needed in order to make the performance of the iterative decoder similar to that of ML decoding. The results were tested numerically, and the experimental results were compared to the theoretical counterparts.

## APPENDIX A

This appendix explains in details the equality (5) used in the proof of Theorem 1.

*Lemma 6:* For all non-negative integers  $r, i, \tau$ , and  $t$  such that  $0 \leq t \leq (2^r - \tau - 1) - i \cdot 2^{r-i}$ ,

$$\binom{(2^r - \tau - 1) - i \cdot 2^{r-i}}{t} \bigg/ \binom{2^r - \tau - 1}{t} = \prod_{j=\tau+1}^{\tau+t} \pi(r, i, j).$$

*Proof:* Indeed,

$$\begin{aligned} & \binom{(2^r - \tau - 1) - i \cdot 2^{r-i}}{t} \bigg/ \binom{2^r - \tau - 1}{t} \\ &= \frac{(2^r - \tau - 1 - i \cdot 2^{r-i})!}{t! (2^r - \tau - 1 - i \cdot 2^{r-i} - t)!} \cdot \frac{t! (2^r - \tau - 1 - t)!}{(2^r - \tau - 1)!} \\ &= \prod_{j=\tau+1}^{\tau+t} \frac{2^r - j - i \cdot 2^{r-i}}{2^r - j} = \prod_{j=\tau+1}^{\tau+t} \pi(r, i, j). \end{aligned}$$

## APPENDIX B

In this appendix, we aim to find a weight  $w$  of a row in a parity-check matrix, which covers the maximum number of stopping sets of size up to  $\ell$ , provided that  $n$  is fixed. It is easy to see that any row of length  $n$  and weight  $w$  covers exactly

$$w \sum_{i=1}^{\ell} \binom{n-w}{i-1}$$

stopping sets of weight up to  $\ell$ . Lemma 7 provides an answer to this maximization question.

*Lemma 7:* Fix two positive integers  $n$  and  $2 \leq \ell \leq n$  and define a discrete function  $F: \{1, 2, \dots, n - \ell + 1\} \rightarrow \mathbb{N}$  in the following way:

$$F(w) = F_{n,\ell}(w) = w \sum_{i=0}^{\ell-1} \binom{n-w}{i}.$$

Then

$$\arg \max_w F(w) \in \left\{ \left\lfloor \frac{n+1}{\ell} \right\rfloor, \left\lceil \frac{n}{\ell} \right\rceil \right\}.$$

*Proof:* In order to prove the statement of the lemma, it is sufficient to show that  $F(w)$  increases for  $w \leq \lfloor \frac{n+1}{\ell} \rfloor$  and decreases for  $w \geq \lceil \frac{n}{\ell} \rceil$ .

Consider the finite difference:

$$\Delta F(w) = F(w+1) - F(w).$$

It can be expanded as follows:

$$\begin{aligned} \Delta F(w) &= F(w+1) - F(w) \\ &= (w+1) \sum_{i=0}^{\ell-1} \binom{n-w-1}{i} - w \sum_{i=0}^{\ell-1} \binom{n-w}{i}. \end{aligned}$$

Consequently,

$$\begin{aligned} \Delta F(w) &= (w+1) \sum_{i=0}^{\ell-1} \binom{n-w-1}{i} \\ &\quad - w \sum_{i=0}^{\ell-1} \left( \binom{n-w-1}{i} + \binom{n-w-1}{i-1} \right) \\ &= \sum_{i=0}^{\ell-1} \binom{n-w-1}{i} - w \sum_{i=0}^{\ell-1} \binom{n-w-1}{i-1}. \end{aligned}$$

We have:

$$\begin{aligned}\Delta F(w) &= \sum_{i=0}^{\ell-1} \left( \binom{n-w-1}{i} - w \binom{n-w-1}{i-1} \right) \\ &= \sum_{i=0}^{\ell-1} \frac{(n-w-1)!}{i!(n-w-i)!} (n-i-w(i+1)).\end{aligned}$$

If we require that

$$w \leq \frac{n-\ell+1}{\ell},$$

then it follows also that

$$w < \frac{n-i}{i+1} \quad \text{for all } i < \ell-1.$$

Hence, each of the terms  $(n-i-w(i+1))$  is positive for  $i < \ell-1$  and  $(n-\ell+1-w\ell) \geq 0$ . Therefore,

$$F(1) < F(2) < \dots < F\left(\left\lfloor \frac{n+1}{\ell} \right\rfloor\right).$$

On the other hand, we can write:

$$\begin{aligned}\Delta F(w) &= \sum_{i=0}^{\ell-1} \binom{n-w-1}{i} - w \sum_{i=0}^{\ell-1} \binom{n-w-1}{i-1} \\ &= \sum_{i=0}^{\ell-1} \binom{n-w-1}{i} - w \sum_{i=0}^{\ell-2} \binom{n-w-1}{i} \\ &= \binom{n-w-1}{\ell-1} + (1-w) \sum_{i=0}^{\ell-2} \binom{n-w-1}{i}.\end{aligned}$$

And, if  $w > 1$ , we have:

$$\begin{aligned}\Delta F(w) &< \binom{n-w-1}{\ell-1} + (1-w) \binom{n-w-1}{\ell-2} \\ &= \frac{(n-w-1)!}{(\ell-1)!(n-\ell-w+1)!} (n-w\ell).\end{aligned}$$

If we further require  $w \geq \frac{n}{\ell}$ , then  $\Delta F(w) < 0$  and

$$F\left(\left\lceil \frac{n}{\ell} \right\rceil\right) > F\left(\left\lceil \frac{n}{\ell} \right\rceil + 1\right) > \dots > F(n-\ell+1).$$

## APPENDIX C

In this appendix, we compute the number of full-rank binary matrices with no rows of weight one. The results in this appendix are based on [26].

*Lemma 8:* Let  $m \geq i$  and denote by  $\mathcal{N}(m, i)$  the number of full-rank binary  $m \times i$  matrices with no rows of Hamming weight one. Then

$$\begin{aligned}\mathcal{N}(m, i) &= \sum_{k=0}^i \binom{i}{k} \cdot k! \sum_{p=0}^m (-1)^{m-p} \cdot \binom{m}{p} \\ &\quad \cdot 2^{kp} \cdot S(m-p, k) \prod_{t=0}^{i-k-1} (2^p - 2^t), \quad (16)\end{aligned}$$

where  $S(x, y)$  is a Stirling number of the second kind (the number of ways to partition a set of  $x$  labelled objects into  $y$  nonempty unlabelled subsets).

*Proof:* First, we calculate the number of  $m \times i$  binary matrices of full rank, which we denote by  $\mathcal{M}(m, i)$ . As  $m \geq i$ , all the columns in such matrices are linearly independent. We have  $2^m - 1$  choices for the first column (any nonzero vector in  $\mathbb{F}_2^m$ ),  $2^m - 2$  choices for the second column (any vector in  $\mathbb{F}_2^m$  except the all-zero vector and the first column),  $2^m - 2^2$  choices for the third column (any vector in  $\mathbb{F}_2^m$  except for the vectors in the subspace spanned by the first two columns), etc. Altogether, we have

$$\mathcal{M}(m, i) = \prod_{t=0}^{i-1} (2^m - 2^t).$$

Next, the number of full-rank  $m \times i$  matrices with exactly  $z$  zero rows can be obtained by using the inclusion-exclusion principle, as follows:

$$\binom{m}{z} \sum_{p=0}^{m-z} (-1)^{m-z-p} \binom{m-z}{p} \prod_{t=0}^{i-1} (2^p - 2^t). \quad (17)$$

Now, let us consider the requirement not to have rows of weight one. We use the inclusion-exclusion principle.

Let  $P_i$  ( $i = 1, 2, \dots, i$ ) be the property that there is a row with a single 1 at  $i$ 'th coordinate. Suppose that an  $m \times i$  matrix satisfies properties with indices from a set  $R \subseteq [i]$  with  $|R| = k$ . Then the set of row indices is partitioned as

$$[m] = J \sqcup \bar{J},$$

where  $J$  consists of indices corresponding to rows with a single 1 at a coordinate from  $R$ , and  $\bar{J} = [m] \setminus J$ . Let  $|J| = j$  (we have  $j \geq k$ ).

To enumerate possible submatrices, whose rows are indexed by  $J$  and columns by  $[i]$ , we notice that their columns essentially define an ordered partition of their rows into  $k$  nonempty sets. Hence, the number of such submatrices equals to  $k! \cdot S(j, k)$ .

The number of submatrices whose rows and columns are indexed by  $\bar{J}$  and  $\bar{R}$ , respectively, with exactly  $z$  zero rows can be calculated from (17). They can be extended to all submatrices with rows indexed by  $\bar{J}$  in  $(2^k - k)^z (2^k)^{m-j-z}$  ways because each zero row can be extended by anything except for  $k$ -vectors of weight 1 (as we already collected them in rows  $J$ ), and others can be extended by anything.

Putting all together, we have

$$\begin{aligned}\mathcal{N}(m, i) &= \sum_{k=0}^i (-1)^k \binom{i}{k} \sum_{j=k}^m \binom{m}{j} \cdot k! \cdot S(j, k) \cdot \sum_{z=0}^{m-j} \binom{m-j}{z} \\ &\quad \cdot (2^k - k)^z \cdot (2^k)^{m-j-z} \\ &\quad \cdot \sum_{p=0}^{m-j-z} (-1)^{m-j-z-p} \binom{m-j-z}{p} \cdot \prod_{t=0}^{i-k-1} (2^p - 2^t).\end{aligned}$$



This can be further rewritten as

$$\begin{aligned} \mathcal{N}(m, i) &= \sum_{k=0}^i (-1)^k \binom{i}{k} \sum_{j=k}^m \binom{m}{j} \cdot k! \cdot S(j, k) \\ &\quad \cdot \sum_{p=0}^{m-j} (-1)^{m-j-p} \binom{m-j}{p} \cdot 2^{kp} \cdot k^{m-j-p} \prod_{t=0}^{i-k-1} (2^p - 2^t), \end{aligned}$$

which is in turn equivalent to

$$\begin{aligned} \mathcal{N}(m, i) &= \sum_{k=0}^i \binom{i}{k} \sum_{\ell=0}^k (-1)^{k-\ell} \cdot \binom{k}{\ell} \cdot \sum_{p=0}^m \binom{m}{p} \\ &\quad \cdot 2^{kp} \cdot (-\ell)^{m-p} \prod_{t=0}^{i-k-1} (2^p - 2^t) \\ &= \sum_{k=0}^i \binom{i}{k} \cdot k! \sum_{p=0}^m (-1)^{m-p} \cdot \binom{m}{p} \\ &\quad \cdot 2^{kp} \cdot S(m-p, k) \prod_{t=0}^{i-k-1} (2^p - 2^t). \end{aligned}$$

We note that for the medium and large values of  $m$  and  $i$ , the ratio of the number of full-rank binary  $m \times i$  matrices without rows of weight one to the number of all full-rank binary matrices is very close to 1, and hence the relative error becomes close to 0. For example, for  $m = 50$  and  $i = 30$  we have

$$\frac{\mathcal{M}(50, 30) - \mathcal{N}(50, 30)}{\mathcal{N}(50, 30)} \approx 1.40 \times 10^{-6}.$$

Since obviously  $\mathcal{M}(m, i) \geq \mathcal{N}(m, i)$ , the former is a correct upper bound, which is rather tight for the medium and large values of  $m$  and  $i$ . For practical purposes, it is much easier to calculate and analyze  $\mathcal{M}(m, i)$  than  $\mathcal{N}(m, i)$ .

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#### REFERENCES

- [1] Y. Yakimenka and V. Skachek, "Refined upper bounds on stopping redundancy of binary linear codes," in *Proc. IEEE Inf. Theory Workshop (ITW)*, Apr./May 2015, pp. 1–5.
- [2] R. G. Gallager, "Low-density parity-check codes," *IRE Trans. Inf. Theory*, vol. 8, no. 1, pp. 21–28, Jan. 1962.
- [3] D. H. Wiedemann, "Solving sparse linear equations over finite fields," *IEEE Trans. Inf. Theory*, vol. IT-32, no. 1, pp. 54–62, Jan. 1986.
- [4] A. A. Mofrad, M.-R. Sadeghi, and D. Panario, "Solving sparse linear systems of equations over finite fields using bit-flipping algorithm," *Linear Algebra Appl.*, vol. 439, no. 7, pp. 1815–1824, 2013.
- [5] C. Di, D. Proietti, I. E. Telatar, T. J. Richardson, and R. L. Urbanke, "Finite-length analysis of low-density parity-check codes on the binary erasure channel," *IEEE Trans. Inf. Theory*, vol. 48, no. 6, pp. 1570–1579, Jun. 2002.

- [6] M. Schwartz and A. Vardy, "On the stopping distance and the stopping redundancy of codes," *IEEE Trans. Inf. Theory*, vol. 52, no. 3, pp. 922–932, Mar. 2006.
- [7] J. Han and P. H. Siegel, "Improved upper bounds on stopping redundancy," *IEEE Trans. Inf. Theory*, vol. 53, no. 1, pp. 90–104, Jan. 2007.
- [8] J. Han, P. H. Siegel, and A. Vardy, "Improved probabilistic bounds on stopping redundancy," *IEEE Trans. Inf. Theory*, vol. 54, no. 4, pp. 1749–1753, Apr. 2008.
- [9] T. Hehn, O. Milenkovic, S. Laendner, and J. B. Huber, "Permutation decoding and the stopping redundancy hierarchy of cyclic and extended cyclic codes," *IEEE Trans. Inf. Theory*, vol. 54, no. 12, pp. 5308–5331, Dec. 2008.
- [10] E. Rosnes and Ø. Ytrehus, "An efficient algorithm to find all small-size stopping sets of low-density parity-check matrices," *IEEE Trans. Inf. Theory*, vol. 55, no. 9, pp. 4167–4178, Sep. 2009.
- [11] M. Karimi and A. H. Banihashemi, "Efficient algorithm for finding dominant trapping sets of LDPC codes," *IEEE Trans. Inf. Theory*, vol. 58, no. 11, pp. 6942–6958, Nov. 2012.
- [12] J. H. Weber and K. A. S. Abdel-Ghaffar, "Stopping set analysis for Hamming codes," in *Proc. Inf. Theory Workshop (ITW)*, 2005, pp. 244–247.
- [13] N. Kashyap and A. Vardy, "Stopping sets in codes from designs," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jun./Jul. 2003, p. 122.
- [14] T. Etzion, "On the stopping redundancy of Reed-Muller codes," *IEEE Trans. Inf. Theory*, vol. 52, no. 11, pp. 4867–4879, Nov. 2006.
- [15] H. D. L. Hollmann and L. M. G. M. Tolhuizen, "Generic erasure correcting sets: Bounds and constructions," *J. Combinat. Theory A*, vol. 113, no. 8, pp. 1746–1759, 2006.
- [16] H. D. L. Hollmann and L. M. G. M. Tolhuizen, "Generating parity check equations for bounded-distance iterative erasure decoding," in *Proc. 27th Symp. Inf. Theory Benelux*, 2006, pp. 17–24.
- [17] H. D. L. Hollmann and L. M. G. M. Tolhuizen, "Generating parity check equations for bounded-distance iterative erasure decoding," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jul. 2006, pp. 514–517.
- [18] H. D. L. Hollmann and L. M. G. M. Tolhuizen, "On parity-check collections for iterative erasure decoding that correct all correctable erasure patterns of a given size," *IEEE Trans. Inf. Theory*, vol. 53, no. 2, pp. 823–828, Feb. 2007.
- [19] J. Zumbärgel, V. Skachek, and M. F. Flanagan, "On the pseudocodeword redundancy of binary linear codes," *IEEE Trans. Inf. Theory*, vol. 58, no. 7, pp. 4848–4861, Jul. 2012.
- [20] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*. Amsterdam, The Netherlands: Elsevier, 1977.
- [21] R. M. Tanner, D. Sridhara, and T. Fuja, "A class of group-structured LDPC codes," in *Proc. Int. Symp. Commun. Theory Appl. (ISCTA)*, 2011, pp. 365–370.
- [22] T. Richardson and R. Urbanke, *Modern Coding Theory*. Cambridge, U.K.: Cambridge Univ. Press, 2008.
- [23] T. T. Cai, "One-sided confidence intervals in discrete distributions," *J. Statist. Planning Inference*, vol. 131, no. 1, pp. 63–88, 2005.
- [24] I. E. Bocharova, B. D. Kudryashov, V. Skachek, and Y. Yakimenka, "Distance properties of short LDPC codes and their impact on the BP, ML and near-ML decoding performance," in *Proc. 5th Int. Castle Meeting Coding Theory Appl. (ICMCTA)*, 2017, pp. 48–61.
- [25] I. E. Bocharova, B. D. Kudryashov, V. Skachek, and Y. Yakimenka, "Improved redundant parity-check based BP decoding of LDPC codes," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jun. 2018, pp. 1161–1165.
- [26] M. Alekseyev. (2017). Number of Full-Rank Binary Matrices With no Rows of Weight 1. MathOverflow. Accessed: Mar. 30, 2017. [Online]. Available: <https://mathoverflow.net/q/264835>

**Yauhen Yakimenka** was born in Baranavichy, Belarus, in 1986. He received the B.Sc. degree in computer security from Belarusian State University, Minsk, Belarus, in 2008, and M.Sc. joint degree in cyber security from Tallinn University of Technology, Estonia, and University of Tartu, Estonia, in 2014. Since then, he has been working on his Ph.D. degree at the Institute of Computer Science, University of Tartu.

**Vitaly Skachek** received the B.A. (*cum laude*), M.Sc. and Ph.D. degrees in computer science from the Technion—Israel Institute of Technology, in 1994, 1998 and 2007, respectively.

Since 2012, Dr. Skachek is employed by the Institute of Computer Science, University of Tartu, where he is an Associate Professor. He held visiting positions with the Mathematics of Communications Department, Bell Laboratories, Murray Hill, with the Claude Shannon Institute, University College Dublin, Dublin, with the School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore, with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, and with the Department of Electrical and Computer Engineering, McGill University, Montreal. He is a member of the editorial board for the *Advances in Mathematics of Communications* and a guest editor for the special issue of the *Cryptography and Communications—Discrete Structures, Boolean Functions and Sequences*. He served as a guest editor for the special issue on network coding of the *EURASIP Journal on Advances in Signal Processing*.

Dr. Skachek is a recipient of the Permanent Excellent Faculty Instructor award, given by Technion.

**Irina E. Bocharova** received the Diploma in Electrical Engineering in 1978 from the Leningrad Electro-technical Institute and the Ph.D. degree in technical sciences in 1986 from Leningrad Institute of Aircraft Instrumentation. Since 1986 until 2007, she has been Senior Researcher, Assistant Professor, and then Associate Professor at the Leningrad Institute of Aircraft Instrumentation (now State University of Aerospace Instrumentation, St. Petersburg, Russia). Since 2007 she has been Associate Professor at the St. Petersburg State University of Information Technologies, Mechanics and Optics. Additionally, since 2016 she has been Senior Research Fellow of Coding Theory at the Institute of Computer Science of the University of Tartu.

Her research interests include convolutional codes, communication systems, source coding and its applications to speech, audio and image coding. She has more than 70 papers published in journals and proceedings of international conferences, and seven U.S. patents in speech, audio and video coding. She has authored the textbook *Compression for Multimedia* (Cambridge University Press, 2010). Professor Bocharova was awarded the Lise Meitner Visiting Chair in engineering at Lund University, Lund, Sweden twice (January–June 2005 and 2011). In 2014 she received Scopus Russia Award.

**Boris D. Kudryashov** received the Diploma in Electrical Engineering in 1974 and the Ph.D. in technical sciences degree in 1978 both from the Leningrad Institute of Aerospace Instrumentation, and the Doctor of Science degree in 2005 from Institute of Problems of Information Transmission, Moscow. In 1978, he became Assistant Professor, in 1983 Associate Professor, and in 2005 Professor at Leningrad Institute of Aerospace Instrumentation (now the State University on Aerospace Instrumentation, St.-Petersburg, Russia). Since November 2007, he has been Professor at the State University on Information Technologies, Mechanics and Optics, St. Petersburg, Russia. Additionally, since 2017 he has been Visiting Associate Professor at the University of Tartu.

His research interests include coding theory, information theory and applications to speech, audio and image coding. He has authored two textbooks on information theory and coding theory (both in Russian), and has more than 80 papers published in journals and proceedings of international conferences, 20 US patents in image, speech and audio coding. Professor Kudryashov served as a member of the Organizing Committees of ACCT International Workshops.