PDEs in Fiance - Final Project: Mean Field Games with Congestion Effects

Joseph-Marie Ngueponwouo, Emmanuel Gardin

March 6, 2025

Contents

1 Introduction to Mean Field Games

Let $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $(x, m, \gamma) \mapsto f(x, m, \gamma)$ and $\phi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $(x, m) \mapsto \phi(x, m)$ be respectively a running cost and a terminal cost, on which assumptions will be made later on.
We consider the following MFG: find a flow of probability densities $\hat{m} : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ and a feedback control $\hat{v} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying the following two conditions:
1. $\hat{v}$ minimizes

$$J_{\hat{m}} : v \mapsto J_{\hat{m}}(v) = \mathbb{E}\left[\int_0^T f(X_t^v, \hat{m}(t, X_t^v), v(t, X_t^v))dt + \phi(X_T^v, \hat{m}(T, X_T^v))\right]$$

subject to the constraint that the process $X^v = (X_t^v)_{t\geq 0}$ solves the stochastic differential equation (SDE)

$$dX_t^v = b(X_t^v, \hat{m}(t, X_t^v), v(t, X_t^v))dt + \sigma dW_t, \quad t \geq 0, \ (1)$$

where σ is the volatility, b is a given function from $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ with values in $\mathbb{R}^d$, and X_0^v is an independent random variable in $\mathbb{R}^d$, distributed according to the law m_0.
2. For all $t \in [0, T]$, $\hat{m}(t, .)$ is the law of $X_t^{\hat{v}}$
It is useful to note that for a given feedback control v, the density m_t^v of the law of X_t^v following (1) solves the Kolmogorov-Fokker-Planck (KFP) equation:

$$\begin{cases} \frac{\partial m^v}{\partial t}(t, x) - \nu \Delta m^v(t, x) + \operatorname{div}\left(m^v(t, \cdot) b(\cdot, \hat{m}(t, \cdot), v(t, \cdot))\right)(x) = 0, & \text{in } (0, T] \times \mathbb{R}^d, \\ m^v(0, x) = m_0(x), & \text{in } \mathbb{R}^d, \end{cases} \quad (2)$$

where $\nu = \sigma^2/2$.
We recall the definition of the Laplacian Δ and Divergence div operators for smooth functions $\psi : \mathbb{R}^d \to \mathbb{R}$ and $V : \mathbb{R}^d \to \mathbb{R}^d$:

$$\Delta\psi : \begin{cases} \mathbb{R}^d \to \mathbb{R} \\ x \mapsto \sum_{i=1}^d \partial_{x_i}^2 \psi(x) \end{cases} \quad \text{and} \quad \operatorname{div}(V) : \begin{cases} \mathbb{R}^d \to \mathbb{R} \\ x \mapsto \sum_{i=1}^d \partial_{x_i} V_i(x) \end{cases} \quad (3)$$

where V_i denotes the i-th coordinate of V. Indeed, we also have the following definitions for the Gradient ∇ and the Hessian ∇^2 (of which the Laplacian is the trace):

$$\nabla\psi : \begin{cases} \mathbb{R}^d \to \mathbb{R}^d \\ x \mapsto (\partial_{x_1}\psi(x), \dots, \partial_{x_d}\psi(x))^T \end{cases} \quad \text{and} \quad \nabla^2\psi : \begin{cases} \mathbb{R}^d \to \mathbb{R}^{d\times d} \\ x \mapsto (\partial^2_{x_i, x_j}\psi(x))_{1\leq i,j\leq d} \end{cases} \quad (4)$$

Let $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \ni (x, m, p) \mapsto H(x, m, p) \in \mathbb{R}$ be the Hamiltonian of the control problem faced by an infinitesimal player. It is defined by

$$H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \ni (x, m, p) \mapsto H(x, m, p) = \max_{\gamma \in \mathbb{R}^d} \left(-f(x, m, \gamma) - \langle b(x, m, \gamma), p\rangle\right) \in \mathbb{R}.$$

In the sequel, we will assume that the running cost f and the drift b are such that H is well-defined, C^1 with respect to (x, p), and strictly convex with respect to p.
From standard optimal control theory, one can characterize the best strategy through the value function u of the above optimal control problem for a typical player, which satisfies a Hamilton-Jacobi-Bellman (HJB) equation. Together with the equilibrium condition on the distribution, we obtain that the equilibrium best response $\hat{v}$ is characterized by

$$\hat{v}(t, x) = \arg\max_{a \in \mathbb{R}^d} \left\{-f(x, m(t, x), a) - \langle b(x, m(t, x), a), \nabla u(t, x)\rangle\right\},$$

and, denoting H_p the gradient of H with respect to p, that the drift at equilibrium is

$$b(x, m(t, x), \hat{v}(t, x)) = -H_p(x, m(t, x), \nabla u(t, x)),$$

where (u, m) solves the following forward-backward PDE system:

$$
\begin{cases}
-\frac{\partial u}{\partial t}(t,x) - \nu\Delta u(t,x) + H(x, m(t,x), \nabla u(t,x)) = 0, & \text{in } [0,T)\times\mathbb{R}^d, \ (5a) \\
\frac{\partial m}{\partial t}(t,x) - \nu\Delta m(t,x) - \operatorname{div}\left(m(t,\cdot)H_p(\cdot, m(t,\cdot), \nabla u(t,\cdot))\right)(x) = 0, & \text{in } (0,T]\times\mathbb{R}^d, \ (5b) \\
u(T,x) = \phi(x, m(T,x)), \quad m(0,x) = m_0(x), & \text{in } \mathbb{R}^d. \ (5c)
\end{cases}
$$

2 A Model for Congestion

Example (A model for congestion) *Consider the case where the drift is the control, i.e.* $b(x, m, \gamma) = \gamma$, *and the running cost is of the form* $f(x, m, \gamma) = L_0(\gamma, m(x)) + f_0(x, m(x))$ *where* $L_0 : \mathbb{R}^d \times \mathbb{R} \ni (\gamma, \mu) \mapsto L_0(\gamma, \mu) \in \mathbb{R}$ *is given by*

$$
L_0(\gamma, \mu) = \frac{\beta - 1}{\beta}(c_0 + c_1\mu)^{\frac{\alpha}{\beta-1}}|\gamma|^{\frac{\beta}{\beta-1}}, \ (6)
$$

where $\beta > 1$, $0 \le \alpha \le \frac{4(\beta-1)}{\beta}$, $c_0 \ge 0$, *and* $c_1 > 0$. *The term* $L_0(\gamma, m(x))$ *models congestion effects, i.e. the fact that the cost of motion grows with the density of the population (the denser the population, the higher is the cost of motion). Then*

$$
H(x, m, p) = \max_{\gamma\in\mathbb{R}^d}\{-L_0(\gamma, m(x)) - \langle\gamma, p\rangle\} - f_0(x, m(x)) = H_0(p, m(x)) - f_0(x, m(x)), \ (7)
$$

where

$$
H_0(p, \mu) = \frac{1}{\beta}\frac{|p|^\beta}{(c_0 + c_1\mu)^\alpha}. \ (6)
$$

In this situation, the best response at equilibrium is

$$
\hat{v}(t, x, m) = -\frac{1}{(c_0 + c_1 m(t,x))^\alpha}|\nabla u(t,x)|^{\beta-2}\nabla u(t,x), \ (8)
$$

where (u, m) *solves the PDE system*

$$
\begin{cases}
-\frac{\partial u}{\partial t}(t,x) - \nu\Delta u(t,x) + \frac{1}{\beta}\frac{|\nabla u(t,x)|^\beta}{(c_0+c_1 m(t,x))^\alpha} = f_0(x, m(t,x)), & \text{in } [0,T)\times\mathbb{R}^d, \\
\frac{\partial m}{\partial t}(t,x) - \nu\Delta m(t,x) - \operatorname{div}\left(\frac{m(t,\cdot)}{(c_0+c_1 m(t,\cdot))^\alpha}|\nabla u(t,\cdot)|^{\beta-2}\nabla u(t,\cdot)\right)(x) = 0, & \text{in } (0,T]\times\mathbb{R}^d, \\
u(T,x) = \phi(x, m(T,x)), \quad m(0,x) = m_0(x), & \text{in } \mathbb{R}^d.
\end{cases}
$$

3 Finite Difference Schemes

We present a finite-difference scheme first. We consider the special case described in Example before, where $H(x, m, p) = H_0(p, m(x)) - f_0(x, m(x))$ and H_0 is given by (6). We also suppose that

$$
f_0(x, m(x)) = \tilde{f}_0(m(x)) + g(x).
$$

For simplicity, we focus on the one-dimensional setting, i.e. $d = 1$. We also suppose that the state space is the domain $\Omega =]0, 1[$, i.e. the stochastic process involved in the dynamics of the players is reflected at $\partial\Omega$.
The boundary value problem becomes :

$$
-\frac{\partial u}{\partial t}(t,x) - \nu\frac{\partial^2 u}{\partial x^2}(t,x) + \frac{1}{\beta}\frac{|\frac{\partial u}{\partial x}(t,\cdot)|^\beta}{(c_0 + c_1 m(t,x))^\alpha} = g(x) + \tilde{f}_0(m(t,x)), \quad \text{in } [0,T)\times\Omega, \ (9)
$$

$$
\frac{\partial m}{\partial t}(t,x) - \nu\frac{\partial^2 m}{\partial x^2}(t,x) - \frac{\partial}{\partial x}\left(\frac{m(t,\cdot)}{(c_0 + c_1 m(t,\cdot))^\alpha}|\frac{\partial u}{\partial x}(t,\cdot)|^{\beta-2}\frac{\partial}{\partial x}u(t,\cdot)\right)(x) = 0, \quad \text{in } (0,T]\times\Omega. \ (10)
$$

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0, \quad \text{on } (0,T), \ (11)$$

$$\frac{\partial m}{\partial x}(t,0) = \frac{\partial m}{\partial x}(t,1) = 0, \quad \text{on } (0,T), \ (12)$$

$$u(T,x) = \phi(x, m(T,x)), \quad m(0,x) = m_0(x), \quad \text{in } \Omega. \ (13)$$

3.1 Discrete Operators

Let N_T and N_h be two positive integers. We consider $N_T + 1$ and N_h points in time and space respectively. Set $\Delta t = T/N_T$, $h = 1/(N_h - 1)$, and $t_n = n \times \Delta t$, $x_i = i \times h$ for $(n,i) \in \{0, \ldots, N_T\} \times \{0, \ldots, N_h - 1\}$.
We approximate u and m respectively by vectors U and $M \in \mathbb{R}^{(N_T+1)\times N_h}$, that is, $u(t_n, x_i) \approx U_i^n$ and $m(t_n, x_i) \approx M_i^n$ for each $(n,i) \in \{0, \ldots, N_T\} \times \{0, \ldots, N_h - 1\}$. We use a superscript and a subscript respectively for the time and space indices.
To take into account Neumann boundary conditions, we introduce ghost nodes $x_{-1} = -h$, $x_{N_h} = 1 + h$, and set

$$U_{-1}^n = U_0^n, \quad U_{N_h}^n = U_{N_h-1}^n, \quad M_{-1}^n = M_0^n, \quad M_{N_h}^n = M_{N_h-1}^n. \ (16)$$

We introduce the finite difference operators

$$(D_t W)^n = \frac{1}{\Delta t}(W^{n+1} - W^n), \quad n \in \{0, \ldots, N_T - 1\}, \quad W \in \mathbb{R}^{N_T+1},$$

$$(DW)_i = \frac{1}{h}(W_{i+1} - W_i), \quad i \in \{0, \ldots, N_h - 1\}, \quad W \in \mathbb{R}^{N_h},$$

$$(\Delta_h W)_i = \frac{1}{h^2}(2W_i - W_{i+1} - W_{i-1}), \quad i \in \{0, \ldots, N_h - 1\}, \quad W \in \mathbb{R}^{N_h},$$

$$[\nabla_h W]_i = ((DW)_i, (DW)_{i-1}) \in \mathbb{R}^2, \quad i \in \{0, \ldots, N_h - 1\}, \quad W \in \mathbb{R}^{N_h}.$$

In which the special cases $i = 0$ and $i = N_h - 1$ can be written thanks to the above mentioned discrete version of the Neumann boundary conditions, see (16).

3.2 Discrete Hamiltonian

Let $\tilde{H} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, $(p_1, p_2, \mu) \mapsto \tilde{H}(p_1, p_2, \mu)$ be a discrete Hamiltonian, assumed to satisfy the following properties: for every $\mu > 0$,
1. ($\tilde{\mathbf{H}}_1$) Monotonicity: $(p_1, p_2) \mapsto \tilde{H}(p_1, p_2, \mu)$ is nonincreasing in p_1 and nondecreasing in p_2.
2. ($\tilde{\mathbf{H}}_2$) Consistency: $\tilde{H}(p, p, m) = H_0(p, \mu)$.
3. ($\tilde{\mathbf{H}}_3$) Differentiability: $(p_1, p_2) \mapsto \tilde{H}(p_1, p_2, \mu)$ is of class C^1.
4. ($\tilde{\mathbf{H}}_4$) Convexity: $(p_1, p_2) \mapsto \tilde{H}(p_1, p_2, \mu)$ is convex.

Example 3.2.1

Since $H_0(p, \mu) = \frac{1}{\beta} \frac{|p|^\beta}{(c_0 + c_1 \mu)^\alpha}$, a sensible choice for the discrete Hamiltonian is

$$\tilde{H}(p_1, p_2, \mu) = \frac{1}{\beta} \frac{\left((p_1)_-^2 + (p_2)_+^2\right)^{\frac{\beta}{2}}}{(c_0 + c_1 \mu)^\alpha},$$

where X_+, resp. X_- stand for the positive (resp. negative) part of X: $X = X_+ - X_-$ and $|X| = X_+ + X_-$, and where we set $X_+^2 = (X_+)^2$ and $X_-^2 = (X_-)^2$.
The monotonicity of the discrete Hamiltonian guarantees uniqueness in the discrete HJB equations and discrete KFP equations below. It also guarantees that the solution of the discrete KFP equation below is nonnegative if its initial condition is nonnegative.

The consistency of the discrete Hamiltonian is a key ingredient for convergence of the discrete schemes.
The differentiability of the discrete Hamiltonian makes it possible to use Newton method for solving the discrete HJB equation below.

3.3 Finite Difference Operators

We introduce the following finite differences operators:

$$\partial_t\ w(t_n,x) \Longleftrightarrow (D_t\ W)^n = \frac{W^{n+1}-W^n}{\Delta t},\quad n \in 0,...,N_T-1,\ W \in \mathbb{R}^{N_T+1}$$

$$\partial_x\ w(t,x_i) \Longleftrightarrow (D_x\ W)_i = \frac{W_{i+1}-W_i}{h},\quad i \in 0,...,N_h-1,\ W \in \mathbb{R}^{N_h}$$

$$\partial_x^2\ w(t,x_i) \Longleftrightarrow (\Delta_h\ W)_i = \frac{W_{i+1}-2W_i+W_{i-1}}{h^2},\quad i \in 0,...,N_h-1,\ W \in \mathbb{R}^{N_h}$$

$$[\nabla_h W]_i = ((DW)_i,(DW)_{i-1}),\quad i \in 0,...,N_h-1,\ W \in \mathbb{R}^{N_h}$$

$$(\partial_x(t_n,x_i)) \longleftrightarrow \frac{1}{h}\begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} W_0^0 & W_0^1 & \cdots & W_0^{N_T} \\ W_1^0 & W_1^1 & \cdots & W_1^{N_T} \\ \vdots & \vdots & \ddots & \vdots \\ W_{N_h-1}^0 & W_{N_h-1}^1 & \cdots & W_{N_h-1}^{N_T} \end{pmatrix}$$

In the last row, we took into account Neumann conditions, considering the fact that $U_{N_h} = U_{N_h-1}$ and $M_{N_h} = M_{N_h-1}$. Let D_x be the matrix above.
Again, taking into account Neumann conditions:

$$\left(\partial_x^2(t_n,x_i)\right)_{\substack{0\le n\le N_T \\ 0\le i\le N_h-1}} \longleftrightarrow \frac{1}{h^2}\begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} W_0^0 & W_0^1 & \cdots & W_0^{N_T} \\ W_1^0 & W_1^1 & \cdots & W_1^{N_T} \\ \vdots & \vdots & \ddots & \vdots \\ W_{N_h-1}^0 & W_{N_h-1}^1 & \cdots & W_{N_h-1}^{N_T} \end{pmatrix}$$

Denote by D_x^2 the matrix above.
Notice that the matrix of

$$((DW)_{i-1} = \frac{1}{h}(W_i - W_{i-1}))_{0\le i<N_h}$$

is (because of Neumann conditions):

$$\frac{1}{h}\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

3.4 Solving Discrete HJB Equation

To elaborate a discrete version of the HJB equation, we need to have a discrete version of the Hamiltonian. We will model $H_0(p,\mu) = \frac{1}{\beta}\frac{|p|^\beta}{(c_0+c_1\mu)^\alpha}$ by:

$$\tilde{H}(p_1,p_2,\mu) = \frac{1}{\beta}\frac{\left((p_1)_-^2 + (p_2)_+^2\right)^{\frac{\beta}{2}}}{(c_0+c_1\mu)^\alpha}$$

where $x_+ = \max(0,x)$ and $x_- = \max(0,-x)$. Note that $\tilde{H}$ takes three arguments.
We can now consider the following discrete version of the HJB equation, supplemented with the Neumann conditions and the terminal condition:

$$\begin{cases} -(D_tU_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}([\nabla_h U^n]_i, M_i^{n+1}) = g(x_i) + \tilde{f}_0(M_i^{n+1}), & 0 \le i < N_h,\ 0 \le n < N_T \\ U_{-1}^n = U_0^n, & 0 \le n < N_T \\ U_{N_h}^n = U_{N_h-1}^n, & 0 \le n < N_T \\ U_i^{N_T} = \phi(M_i^{N_T}), & 0 \le i < N_h \end{cases} \tag{17}$$

This scheme is an implicit Euler scheme since the equation is backward in time. Given M^{n+1} and U^{n+1}, we will solve equation (17) for U^n.
We introduce

$$\mathbb{F}(U^n, U^{n+1}, M^{n+1}) := \begin{pmatrix} -(D_tU_0)^n - \nu(\Delta_h U^n)_0 + \tilde{H}([\nabla_h U^n]_0, M_0^{n+1}) - g(x_0) - \tilde{f}_0(M_0^{n+1}) \\ \vdots \\ -(D_tU_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}([\nabla_h U^n]_i, M_i^{n+1}) - g(x_i) - \tilde{f}_0(M_i^{n+1}) \\ \vdots \\ -(D_tU_{N_h-1})^n - \nu(\Delta_h U^n)_{N_h-1} + \tilde{H}([\nabla_h U^n]_{N_h-1}, M_{N_h-1}^{n+1}) - g(x_{N_h-1}) - \tilde{f}_0(M_{N_h-1}^{n+1}) \end{pmatrix}$$

When solving the HJB equation, our goal is to find U^n knowing U^{n+1} and M^{n+1}. The condition $U_i^{N_T} = \phi(M_i^{N_T})$ allows initialization for $n = N_T$. For $n < N_T$, we use Newton-Raphson iterations, which consists in estimating U^n as the limit of a sequence $(U^{n,k})_k$ defined by

$$U^{n,k+1} = U^{n,k} - \mathbb{J}^{-1}(U^{n,k}, U^{n+1}, M^{n+1})\mathbb{F}(U^{n,k}, U^{n+1}, M^{n+1}) \tag{18}$$

where $\mathbb{J}(V, U^{n+1}, M^{n+1})$ is the Jacobian of the map $V \mapsto \mathbb{F}(V, U^{n+1}, M^{n+1})$.
We may initialize $U^{n,0} = U^{n+1}$. The Newton iterations are stopped when $\|\mathbb{F}(U^{n,k}, U^{n+1}, M^{n+1})\|$ is below a given threshold, say 10^{-12}.

Explicit Formula of the Jacobian

Let $\mathbb{F}_i$ be the i-th coordinate of $\mathbb{F}(U^n, U^{n+1}, M^{n+1})$. The Jacobian of the function $V \mapsto \mathbb{F}(V, U^{n+1}, M^{n+1})$ is defined as:

$$\mathbb{J}(V, U^{n+1}, M^{n+1}) = \begin{pmatrix} \frac{\partial \mathbb{F}_0}{\partial V_0} & \frac{\partial \mathbb{F}_0}{\partial V_1} & \cdots & \frac{\partial \mathbb{F}_0}{\partial V_{N_h-1}} \\ \frac{\partial \mathbb{F}_1}{\partial V_0} & \frac{\partial \mathbb{F}_1}{\partial V_1} & \cdots & \frac{\partial \mathbb{F}_1}{\partial V_{N_h-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbb{F}_{N_h-1}}{\partial V_0} & \frac{\partial \mathbb{F}_{N_h-1}}{\partial V_1} & \cdots & \frac{\partial \mathbb{F}_{N_h-1}}{\partial V_{N_h-1}} \end{pmatrix} \tag{18}$$

Notice that (setting $A = -g(x_i) - \tilde{f}_0(M_i^{n+1})$):

$$\mathbb{F}_i = -(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}([\nabla_h U^n]_i, M_i^{n+1}) - g(x_i) - \tilde{f}_0(M_i^{n+1})$$

$$= -\frac{U_i^{n+1} - U_i^n}{\Delta t} - \nu \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} + \frac{1}{\beta(c_0 + c_1 M_i^{n+1})^\alpha h^\beta} \left(((U_{i+1}^n - U_i^n)_-)^2 + ((U_i^n - U_{i-1}^n)_+)^2\right)^{\beta/2} + A$$

This term only depends on $U_{i-1}^n, U_i^n, U_{i+1}^n$, so the Jacobian is a tridiagonal matrix. Moreover:

$$\frac{\partial \mathbb{F}_i}{\partial U_{i-1}^n} = -\frac{\nu}{h^2} - \frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha h^\beta} (U_i^n - U_{i-1}^n)_+ \left(((U_{i+1}^n - U_i^n)_-)^2 + ((U_i^n - U_{i-1}^n)_+)^2\right)^{\beta/2-1}$$

$$\frac{\partial \mathbb{F}_i}{\partial U_i^n} = \frac{1}{\Delta t} + \frac{k\nu}{h^2} + \frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha h^\beta} \left((U_{i+1}^n - U_i^n)_- - (U_i^n - U_{i-1}^n)_+\right) \left(((U_{i+1}^n - U_i^n)_-)^2 + ((U_i^n - U_{i-1}^n)_+)^2\right)^{\beta/2-1}$$

where $k = 1$ for boundary points ($i = 0$ or $i = N_h - 1$) due to Neumann conditions, and $k = 2$ for interior points ($0 < i < N_h - 1$).

$$\frac{\partial \mathbb{F}_i}{\partial U_{i+1}^n} = -\frac{\nu}{h^2} - \frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha h^\beta} (U_{i+1}^n - U_i^n)_- \left(((U_{i+1}^n - U_i^n)_-)^2 + ((U_i^n - U_{i-1}^n)_+)^2\right)^{\beta/2-1}$$

Let us denote J_H the Jacobian of $U^n \mapsto \left(\tilde{H}([\nabla_h U^n]_i, M_i^{n+1})\right)_{0 \le i < N_h}$ evaluated in U^n, in the equations above. It will be useful in the sequel. From what we have above, its coefficients verify:

$$(J_H)_{i,i-1} = -\frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha h^\beta} (U_i^n - U_{i-1}^n)_+ \left((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\right)^{\beta/2-1} \tag{20}$$

$$(J_H)_{i,i} = \frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha h^\beta} \left((U_{i+1}^n - U_i^n)_- + (U_i^n - U_{i-1}^n)_+\right) \left((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\right)^{\beta/2-1} \tag{21}$$

$$(J_H)_{i,i+1} = -\frac{1}{(c_0 + c_1 M_i^{n+1})^\alpha h^\beta} (U_{i+1}^n - U_i^n)_- \left((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\right)^{\beta/2-1} \tag{22}$$

3.5 Solving Discrete KFP Equation

To define an appropriate discretization of the KFP equation, we first discuss how to discretize

$$\partial_x \left(m(t,\cdot)|\partial_x u(t,\cdot)|^{\beta-2} \partial_x u(t,\cdot)\right)(x).$$

Recall that

$$\partial_x \left(m(t,x)|\partial_x u(t,\cdot)|^{\beta-2}\partial_x u(t,\cdot)\right)(x) = \partial_x \left(m(t,x)\partial_p H_0(\partial_x u(t,x), m(t,x)\right) \quad (23)$$

Let us consider a function $w \in C^\infty([0,T]\times\Omega)$. Using integration by parts and recalling Neumann boundary conditions, assuming $\partial_p H(x,0,m) = 0$, we get:

$$-\int_\Omega \partial_x\big(H_p(\partial_x u(t,x), m(t,x))m(t,x)\big)w(t,x)\,dx = \int_\Omega m(t,x)H_p(\partial_x u(t,x), m(t,x))\partial_x w(t,x)\,dx,$$

It is natural to propose the following approximation of the right-hand side above:

$$h\sum_{i=0}^{N_h-1} M_i^{n+1}\left(\partial_{p_1}\tilde{H}(\nabla_h U^n]_i, M_i^{n+1})\frac{W_{i+1}^n - W_i^n}{h} + \partial_{p_2}\tilde{H}([\nabla_h U^n]_i, M_i^{n+1})\frac{W_i^n - W_{i-1}^n}{h}\right) \quad (24)$$

Performing discrete integration by parts, we obtain the discrete counterpart of the left-hand side as

$$-h\sum_{i=0}^{N_h-1}\mathbb{T}_i(U^n, M^{n+1}, M^{n+1})W_i^n,$$

where

$$\mathbb{T}_i(U,M,\tilde{M}) = \frac{1}{h}\left(M_i\partial_{p_1}\tilde{H}([\nabla_h U^n]_i, \tilde{M}_i) - M_{i-1}\partial_{p_1}\tilde{H}([\nabla_h U^n]_{i-1}, \tilde{M}_i)\right) \quad (25)$$
$$+\frac{1}{h}\left(M_{i+1}\partial_{p_2}\tilde{H}(\nabla_h U^n]_{i+1}, \tilde{M}_{i+1}) - M_i\partial_{p_2}\tilde{H}([\nabla_h U^n]_i, \tilde{M}_i)\right) \quad (27)$$

We can now consider the following discrete version of the KFP equation, supplemented with the Neumann conditions and the terminal condition:

$$\begin{cases} (D_t M_i)^n - \nu(\Delta_h M^{n+1})_i - \mathbb{T}_i(U^n, M^{n+1}, \tilde{M}^{n+1}) = 0, & 0 \le i < N_h,\ 0 < n < N_T \\ M_{-1}^n = M_0^n, & 0 < n \le N_T \\ M_{N_h}^n = M_{N_h-1}^n, & 0 < n \le N_T \\ M_i^0 = \tilde{m}_0(x_i), & 0 \le i < N_h \end{cases} \quad (28)$$

where for example:

$$\tilde{m}_0(x_i) = \frac{1}{h}\int_{|x-x_i|\le h/2} m_0(x)\,dx \quad \text{or} \quad \tilde{m}_0(x_i) = m_0(x_i) \quad (29)$$

This scheme is also implicit, but contrary to the HJB scheme, it consists in a forward loop. Starting from time step 0, $M_i^0 = \tilde{m}_0(x_i)$ provides an explicit formula for M^0. The n-th step consists in computing M^{n+1} given U^n and M^n. The KFP system (28) being linear, it can be solved by basic linear algebra methods.
We introduce

$$\mathbb{T}(U,M,\tilde{M}) := (\mathbb{T}_0(U,M,\tilde{M}), \dots, \mathbb{T}_{N_h-1}(U,M,\tilde{M}))^T.$$

Notice that $M \mapsto \mathbb{T}(U^n, M, \tilde{M})$ is a linear map. Let A be the associated matrix. Then

$$A = (-J_H)^T.$$

with J_H computed in $U = U^n$ and $\tilde{M}^{n+1}$.
Considering the fact that

$$(D_t M)^n = \frac{1}{\Delta t}(M^{n+1} - M^n),$$

we can finally rewrite our system:

$$\frac{M^{n+1} - M^n}{\Delta t} - \nu D_x^2 M^{n+1} + (J_H)^T M^{n+1} = 0 \ (29)$$

Finding M^{n+1} then amounts to solving:

$$\left(I_{N_h} - \nu \Delta t D_x^2 + \Delta t (J_H)^T\right) M^{n+1} = M^n \ (30)$$

3.6 Solving the whole Forward-Backward system

The idea will be to use fixed points iterations to compute

$$\mathbb{M} := (M^n)_{0 \le n \le N_T} \quad \text{and} \quad \mathbb{U} := (U^n)_{0 \le n \le N_T}.$$

Let $0 < \theta < 1$ be a parameter (for instance, $\theta = 0.01$). Let

$$(\mathbb{M}^{(k)}, \mathbb{U}^{(k)})$$

be the running approximation of $(\mathbb{M}, \mathbb{U})$. The next approximation

$$(\mathbb{M}^{(k+1)}, \mathbb{U}^{(k+1)})$$

is computed as follows:

1. Solve the discrete HJB equation given $(\mathbb{M}^{(k)}, \mathbb{U}^{(k)})$. The solution is named $\hat{U}^{(k+1)}$.
2. Solve the discrete KFP equation given $(\mathbb{M}^{(k)}, \hat{U}^{(k+1)})$. The solution is named $\hat{\mathbb{M}}^{(k+1)}$.
3. Set
$$(\mathbb{M}^{(k+1)}, \mathbb{U}^{(k+1)}) = (1-\theta)(\mathbb{M}^{(k)}, \mathbb{U}^{(k)}) + \theta(\hat{\mathbb{M}}^{(k+1)}, \hat{U}^{(k+1)}).$$

The iterations are stopped when the norm of the increment

$$(\mathbb{M}^{(k+1)}, \mathbb{U}^{(k+1)}) - (\mathbb{M}^{(k)}, \mathbb{U}^{(k)})$$

becomes smaller than a given threshold, say 10^{-7}.
To initialize the loop, we set

$$M_i^{n,(0)} = \tilde{m}_0(x_i)$$

for all $0 \le i < N_h$ and $0 \le n \le N_T$. The matrix $U^{(0)}$ initial value has minimum consequence on the convergence of the algorithm. We set

$$U_i^{n,(0)} = 0$$

for all i, n.

4 Theoretical Questions

1. **Understand why the mass of M^n, i.e. $\sum_{i=0}^{Nh-1} M_i^n$, does not depend on n**
We consider that M^n is a vector column. We recall that:

$$M^n = (I_{Nh} - \nu \Delta t D_x^2 + \Delta t (J_H)^T) M^{n+1} := B\ M^{n+1}$$

Let $v = (1\ 1\ ...\ 1) \in \mathbb{R}^{Nh}$. If we can prove that $B^T\ v = v$, then we are done as this would imply that $v^T\ B = v^T$ and then $v^T\ B\ M^{n+1} = v^T\ M^n \implies v^T\ M^{n+1} = v^T\ M^n$, which is basically what we are trying to show.
We have

$$B^T\ v = (I_{Nh} - \nu \Delta t D_x^2 + \Delta t (J_H)^T)\ v$$

The definition of the Jacobian matrix J_H implies that the sum of the coefficients of J_H over each row is zero and the same goes for matrix D_x^2 so that $(B^T\ v)_i = v_i$
Hence $B^T\ v = v$ and we can conclude.
2. **Try to prove uniqueness in the discrete HJB equation, i.e. that given $(M^n)_n$, $(U^n)_n$ is unique. Hint: take two solutions $(U^n)_n$ and $(V^n)_n$ and consider n_0, i_0 such that $max_{(n,i)}\ (U_i^n - V_i^n)$ is achieved at (n_0, i_0), and use the monotonicity of the discrete Hamiltonian**
Let $(U^n)_n$ and $(V^n)_n$ be two solutions of the HJB equation (given $(M^n)_n$). We consider (n_0, i_0) such that $U_{i_0}^{n_0} - V_{i_0}^{n_0} = \max_{(n,i)}(U_i^n - V_i^n)$. Thus:

$$-(D_t U_{i_0})^{n_0} - \nu(\Delta_h U^{n_0})_{i_0} + \tilde{H}([\nabla_h U^{n_0}]_{i_0}, M_{i_0}^{n_0+1}) = -(D_t V_{i_0})^{n_0} - \nu(\Delta_h V^{n_0})_{i_0} + \tilde{H}([\nabla_h V^{n_0}]_{i_0}, M_{i_0}^{n_0+1})$$

We can rearrange the terms:

$$\tilde{H}([\nabla_h U^{n_0}]_{i_0}, M_{i_0}^{n_0+1}) - \tilde{H}([\nabla_h V^{n_0}]_{i_0}, M_{i_0}^{n_0+1}) = (D_t U_{i_0})^{n_0} - (D_t V_{i_0})^{n_0} + \nu(\Delta_h U^{n_0})_{i_0} - \nu(\Delta_h V^{n_0})_{i_0}$$

Knowing that $U_{i_0}^{n_0} - V_{i_0}^{n_0}$ is the largest difference possible, we have:

$$\begin{cases} (D_t U_{i_0})^{n_0} - (D_t V_{i_0})^{n_0} = \dfrac{1}{\Delta t}\left(-(U_{i_0}^{n_0} - V_{i_0}^{n_0}) + (U_{i_0}^{n_0+1} - V_{i_0}^{n_0+1})\right) \leq 0 \\ \nu(\Delta_h U^{n_0})_{i_0} - \nu(\Delta_h V^{n_0})_{i_0} = \dfrac{\nu}{h^2}\left((U_{i_0+1}^{n_0} - V_{i_0+1}^{n_0}) - 2(U_{i_0}^{n_0} - V_{i_0}^{n_0}) + (U_{i_0-1}^{n_0} - V_{i_0-1}^{n_0})\right) \leq 0 \end{cases}$$

Hence,

$$\tilde{H}((DU^{n_0})_{i_0}, (DU^{n_0})_{i_0-1}, M_{i_0}^{n_0+1}) \leq \tilde{H}((DV^{n_0})_{i_0}, (DV^{n_0})_{i_0-1}, M_{i_0}^{n_0+1}).$$

However:

$$\begin{cases} (DU^{n_0})_{i_0} - (DV^{n_0})_{i_0} = \dfrac{1}{h}\left((U_{i_0+1}^{n_0} - V_{i_0+1}^{n_0}) - (U_{i_0}^{n_0} - V_{i_0}^{n_0})\right) \leq 0 \\ (DU^{n_0})_{i_0-1} - (DV^{n_0})_{i_0-1} = \dfrac{1}{h}\left((U_{i_0}^{n_0} - V_{i_0}^{n_0}) - (U_{i_0-1}^{n_0} - V_{i_0-1}^{n_0})\right) \geq 0 \end{cases}$$

As $\tilde{H}$ does not increase in its first argument p_1 and does not decrease in its second argument p_2 (**monotonicity of the discrete hamiltonian**), it follows that:

$$\begin{aligned} \tilde{H}((DU^{n_0})_{i_0}, (DU^{n_0})_{i_0-1}, M_{i_0}^{n_0+1}) &\leq \tilde{H}((DV^{n_0})_{i_0}, (DV^{n_0})_{i_0-1}, M_{i_0}^{n_0+1}) \\ &\leq \tilde{H}((DU^{n_0})_{i_0}, (DV^{n_0})_{i_0-1}, M_{i_0}^{n_0+1}) \quad \text{by non-increasing property} \\ &\leq \tilde{H}((DU^{n_0})_{i_0}, (DU^{n_0})_{i_0-1}, M_{i_0}^{n_0+1}) \quad \text{by non-decreasing property} \end{aligned}$$

We conclude that

$$\tilde{H}((DU^{n_0})_{i_0}, (DU^{n_0})_{i_0-1}, M_{i_0}^{n_0+1}) = \tilde{H}((DV^{n_0})_{i_0}, (DV^{n_0})_{i_0-1}, M_{i_0}^{n_0+1}).$$

A sum of negative numbers is equal to 0 whenever all of the terms are identically null. It then translates to

$$\begin{cases} (D_t U_{i_0})^{n_0} - (D_t V_{i_0})^{n_0} = 0 \\ \nu(\Delta_h U^{n_0})_{i_0} - \nu(\Delta_h V^{n_0})_{i_0} = 0 \end{cases}$$

From what we said above, we conclude that the maximum of $(U_i^n - V_i^n)$ is also reached on the adjacent points of $((i_0, n_0))$. The arguments we developed then propagate until we reach the boundary condition

$$U_i^{N_T} = V_i^{N_T} = \phi(M_i^{N_T}) \quad \text{for all } 0 \leq i < N_h,$$

which implies that

$$max_{(i,n)}(U_i^n - V_i^n) = 0.$$

Hence $(U = V)$ and the solution to the HJB equation is unique.

3. **Try to prove uniqueness in the discrete KFP equation, i.e. that given $(U^n)_n$ and $(M^n)_n$, $(M^n)_n$ is unique. Hint: prove that the matrices of the linear systems arising in the discrete KFP are the conjugate of M-matrices**

We know that finding (M^{n+1}) given (M^n) and $((U^n)_n)$ amounts to solving equation (29). We recall it:

$$\left(I_{N_h} - \nu\Delta t D_x^2 + \Delta t (J_H)^T\right) M^{n+1} := B\ M^{n+1} = M^n$$

We will show that (B^T) has the (M)-property.

- **First case**: $(B_{i,i}^T = B_{i,i} > 0)$ for all $(0 \leq i < N_h)$. Indeed:

$$B_{i,i} = 1 + \nu\Delta t \cdot \frac{1}{h^2} + \Delta t \cdot (J_H)_{i,i}^T \geq 1 + \nu\Delta t \cdot \frac{1}{h^2} + \Delta t \cdot (J_H)_{i,i}^T$$

Or,

$$B_{i,i} = 1 + \nu\Delta t \cdot \frac{2}{h^2} + \Delta t \cdot (J_H)_{i,i}^T \geq 1 + \nu\Delta t \cdot \frac{1}{h^2} + \Delta t \cdot (J_H)_{i,i}^T$$

which is strictly greater than 0, recalling the form of (J_H) in equations $20 - 22$.

- **Second case**: for $(j \neq i)$, we have on the other hand $(B_{i,j}^T \leq 0)$. Indeed:

$$B_{i,j}^T = -\nu\Delta t \cdot \frac{1}{h^2} + \Delta t \cdot (J_H)_{i,j} \leq 0$$

for the same reason.

- **Third case**: $(\sum_{j=0}^{N_h-1} B_{i,j}^T > 0)$ comes from the fact that the sum of coefficients over any row of (J_H) is null (this also holds for (D_x^2)), which leads to

$$\sum_{j=0}^{N_h-1} B_{i,j}^T = 1 - \nu\Delta t \cdot 0 + \Delta t \cdot 0 > 0$$

As a result, (B^T) has the (M)-property, so (B^T) is invertible and (B) is also invertible. Hence, the system $(BM^{n+1} = M^n)$ has a unique solution $(M^{n+1} \in \mathbb{R}^{N_h})$. The KFP equation therefore has a unique solution.

4. **Try to prove that if M_0 is positive, then M_n is positive for all n.**

From the previous question, we know that $(M^n)_n$ satisfies

$$BM^{n+1} = M^n,$$

with B^T an M-matrix. It follows that the entries of $(B^T)^{-1}$ are all nonnegative, so do the entries of B^{-1}. As a result,

$$M^{n+1} = B^{-1}M^n$$

implies that whenever $M^n \geq 0$, we also have $M^{n+1} \geq 0$.

Finally, from $M^0 \geq 0$, we get $M^1 \geq 0$, and then $M^2 \geq 0$, and so on. Hence,

$$M^0 \geq 0$$

does imply that M^n is positive for all n.

5 Results

5.1 Simulate the MFG corresponding to the following data:

- $\Omega =]0, 1[$, $T = 1$
- $g(x) = 0$
- Try the following sets of parameters:
  (a) $\beta = 2$, $c_0 = 0.1$, $c_1 = 1$, $\alpha = 0.5$, $\sigma = 0.02$
  (b) $\beta = 2$, $c_0 = 0.1$, $c_1 = 5$, $\alpha = 1$, $\sigma = 0.02$
  (c) $\beta = 2$, $c_0 = 0.01$, $c_1 = 2$, $\alpha = 1.2$, $\sigma = 0.1$
  (d) $\beta = 2$, $c_0 = 0.01$, $c_1 = 2$, $\alpha = 1.5$, $\sigma = 0.2$
  (e) $\beta = 2$, $c_0 = 1$, $c_1 = 3$, $\alpha = 2$, $\sigma = 0.002$
- $\tilde{f}(m(x)) = \frac{m(x)}{10}$
- $\phi(x, m) = -\exp(-40(x - 0.7)^2)$
- $m_0(x) = \sqrt{300/\pi}\exp(-300(x - 0.2)^2)$.
- Choose for example $N_h = 201$, $N_T = 100$.
- The parameter θ should be chosen in such a way that the fixed point method converges. Depending on the case of interest, the good values of θ may lie in the interval $[0.001, 0.2]$. If the fixed point iterations fail to converge, then try to decrease θ.
- Stopping criteria in the Newton method: 10^{-12}
- Stopping criteria in the fixed point method: not larger than 2.10^{-5}

5.2 Contour lines of m and u in the plane (x, t)

5.3 Interpretation of the Numerical Results

We analyze five parameter sets to understand how congestion sensitivity (α), congestion cost (c_1), direct movement cost (c_0), and noise level (σ) affect agent behavior in mean field games:

1. **Parameter set (a):** $\alpha = 0.5$, $c_1 = 1$, $c_0 = 0.1$, $\sigma = 0.02$
   - **Moderate congestion effects** producing an elliptical symmetric structure with smooth density evolution.
   - **Interpretation:**
     - The density distribution $M(t, x)$ shows balanced concentration between bottom left and top right regions.
     - Agents follow relatively direct paths while still avoiding the highest-density areas.
     - The symmetric behavior suggests evolution toward a stable equilibrium with predictable agent movement patterns.
2. **Parameter set (b):** $\alpha = 1$, $c_1 = 5$, $c_0 = 0.1$, $\sigma = 0.02$
   - **Stronger congestion avoidance** with noticeable asymmetry in contour concentration and increased spreading.

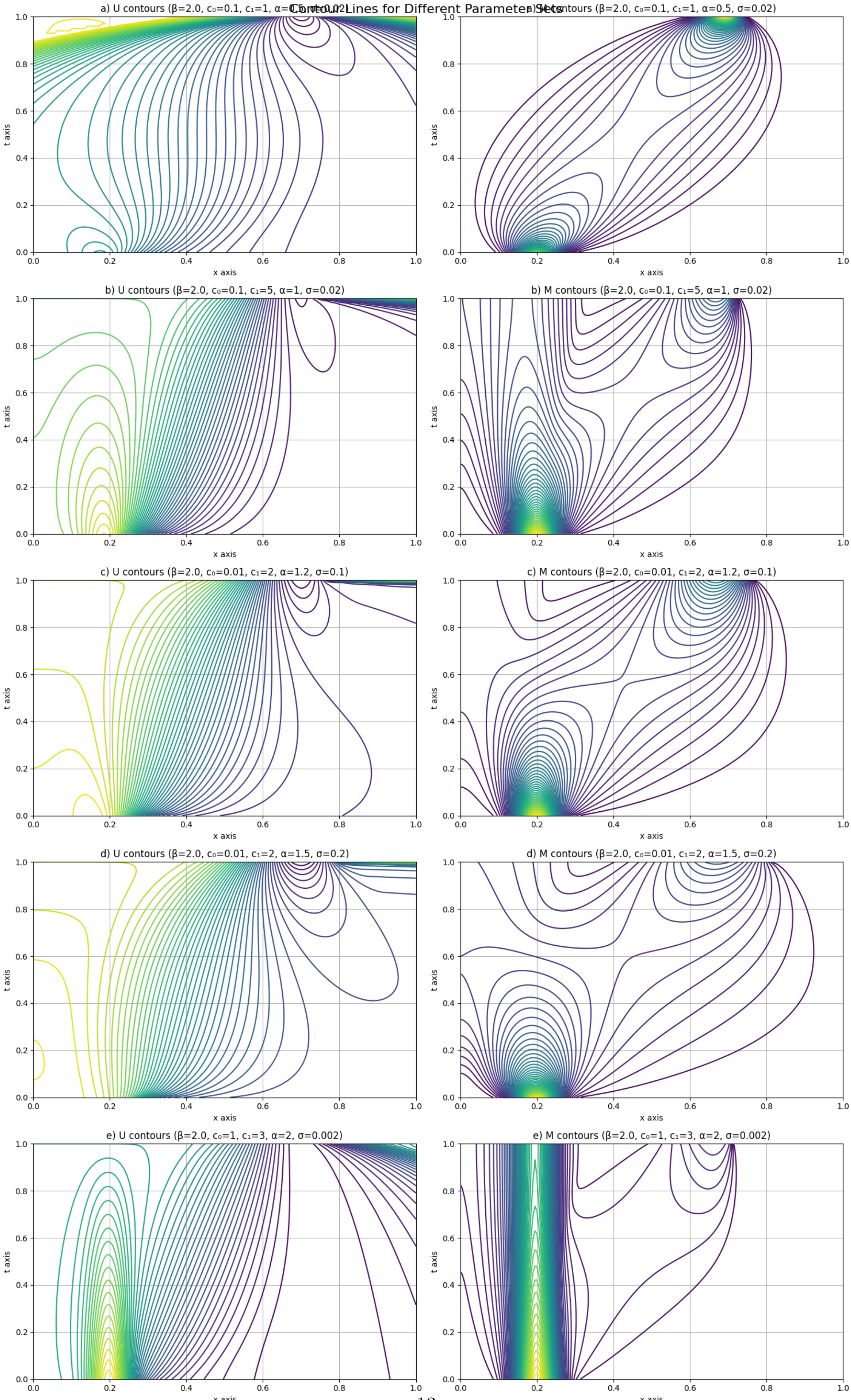

Figure 1: Contour lines of m and u

- **Interpretation:**
  - Higher congestion sensitivity ($c_1 = 5$) leads to more uniform density distribution as agents actively avoid crowded regions.
  - Agents clearly deviate from direct routes, creating a sharper transition in dynamic behavior.
  - Initial accumulation gradually disperses as agents prioritize less congested paths, even at the cost of longer routes.

3. **Parameter set (c):** $\alpha = 1.2$, $c_1 = 2$, $c_0 = 0.01$, $\sigma = 0.1$
   - **Increased diffusion and congestion sensitivity** with progressive spreading patterns.
   - **Interpretation:**
     - Stronger congestion effects ($\alpha = 1.2$) cause earlier splitting of density profiles.
     - Higher noise value ($\sigma = 0.1$) introduces more randomness in agent movement, creating wider dispersion.
     - The propagation follows a nonlinear pattern where densities spread more gradually compared to parameter set (b).

4. **Parameter set (d):** $\alpha = 1.5$, $c_1 = 2$, $c_0 = 0.01$, $\sigma = 0.2$
   - **More regular and evenly spaced contours** with widespread distribution.
   - **Interpretation:**
     - Higher noise ($\sigma = 0.2$) results in significantly more spread-out distributions and smoother transitions.
     - Multiple distinct pathways emerge as agents strongly avoid congestion zones.
     - System exhibits controlled diffusion with moderate dissipation, creating a more fluid spreading of the solution.

5. **Parameter set (e):** $\alpha = 2$, $c_1 = 3$, $c_0 = 1$, $\sigma = 0.002$
   - **Extreme congestion with minimal diffusion** causing highly localized concentration.
   - **Interpretation:**
     - Very low noise ($\sigma = 0.002$) prevents density diffusion, creating rigid barriers or "gridlock."
     - High congestion penalty ($\alpha = 2$) makes moving through dense regions prohibitively costly.
     - Vertical "tunnels" form as agents become trapped, unable to reach terminal condition efficiently.
     - This singular behavior represents a Nash equilibrium: no single agent can improve by deviating.
     - Demonstrates a Dirac delta-type behavior where density remains trapped in narrow corridors without transversal spreading.

6 Mean Field Control

6.1 Mean Field Control Formulation

Mean Field Control (MFC) represents a centralized optimization approach, where a social planner optimizes the aggregate welfare of all agents, in contrast to Mean Field Games (MFG) where agents individually optimize their own costs. The key difference is that MFC internalizes the congestion externalities that individual agents in MFG ignore.
The MFC problem can be formulated with the following set of equations:

$$-\frac{\partial u}{\partial t}(t,x) - \nu \frac{\partial^2 u}{\partial x^2}(t,x) + \frac{1}{\beta} \frac{|\frac{\partial u}{\partial x}(t,x)|^{\beta}}{(c_0 + c_1 m(t,x))^{\alpha}} - \frac{c_1 \alpha}{\beta} \frac{m(t,x)|\frac{\partial u}{\partial x}(t,x)|^{\beta}}{(c_0 + c_1 m(t,x))^{\alpha+1}}$$

$$= g(x) + \tilde{f}_0(m(t,x)) + m(t,x)\tilde{f}_0'(m(t,x)), \quad \text{in } [0,T) \times \Omega,$$

$$\frac{\partial m}{\partial t}(t,x) - \nu \frac{\partial^2 m}{\partial x^2}(t,x) - \frac{\partial}{\partial x}\left(\frac{m(t,\cdot)}{(c_0 + c_1 m(t,\cdot))^\alpha}\left|\frac{\partial u}{\partial x}(t,\cdot)\right|^{\beta-2}\frac{\partial}{\partial x}u(t,\cdot)\right)(x) = 0, \quad \text{in } (0,T] \times \Omega,$$

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0, \quad \text{on } (0,T),$$

$$\frac{\partial m}{\partial x}(t,0) = \frac{\partial m}{\partial x}(t,1) = 0, \quad \text{on } (0,T),$$

$$u(T,x) = \phi(x, m(T,x)), \quad m(0,x) = m_0(x), \quad \text{in } \Omega.$$

It can be proved that if $\beta > 1$, $0 \leq \alpha < 1$ and $\tilde{f}_0$ is convex, then there is a unique solution.
Solution: The only difference between thus problem and the previous one relies on the first equation (21). We just have to define a new Hamiltonian given by :

$$H_0(t,x) = \frac{1}{\beta}\frac{|\frac{\partial u}{\partial x}(t,x)|^\beta}{(c_0 + c_1 m(t,x))^\alpha}\left(1 - \frac{c_1 \alpha m(t,x)}{c_0 + c_1 m(t,x)}\right)$$

The discrete Hamiltonian is thus given by:

$$\tilde{H}(p_1, p_2, \mu) = \frac{1}{\beta}\frac{\left((p_1)_-^2 + (p_2)_+^2\right)^{\frac{\beta}{2}}}{(c_0 + c_1\mu)^\alpha}\left(1 - \frac{c_1 \alpha \mu}{c_0 + c_1 \mu}\right)$$

6.2 Comparison between MFG and MFC

MFG and MFC solutions are compared for $\beta = 2,\ c_0 = 0.1,\ c_1 = 1,\ \alpha = 0.5,\ \tilde{f}_0 = 0,\ g = 0,\ \sigma = 0.02.$ Figure 2 presents the contour lines of density distribution M and value function U for both Mean Field Control and Mean Field Game approaches.

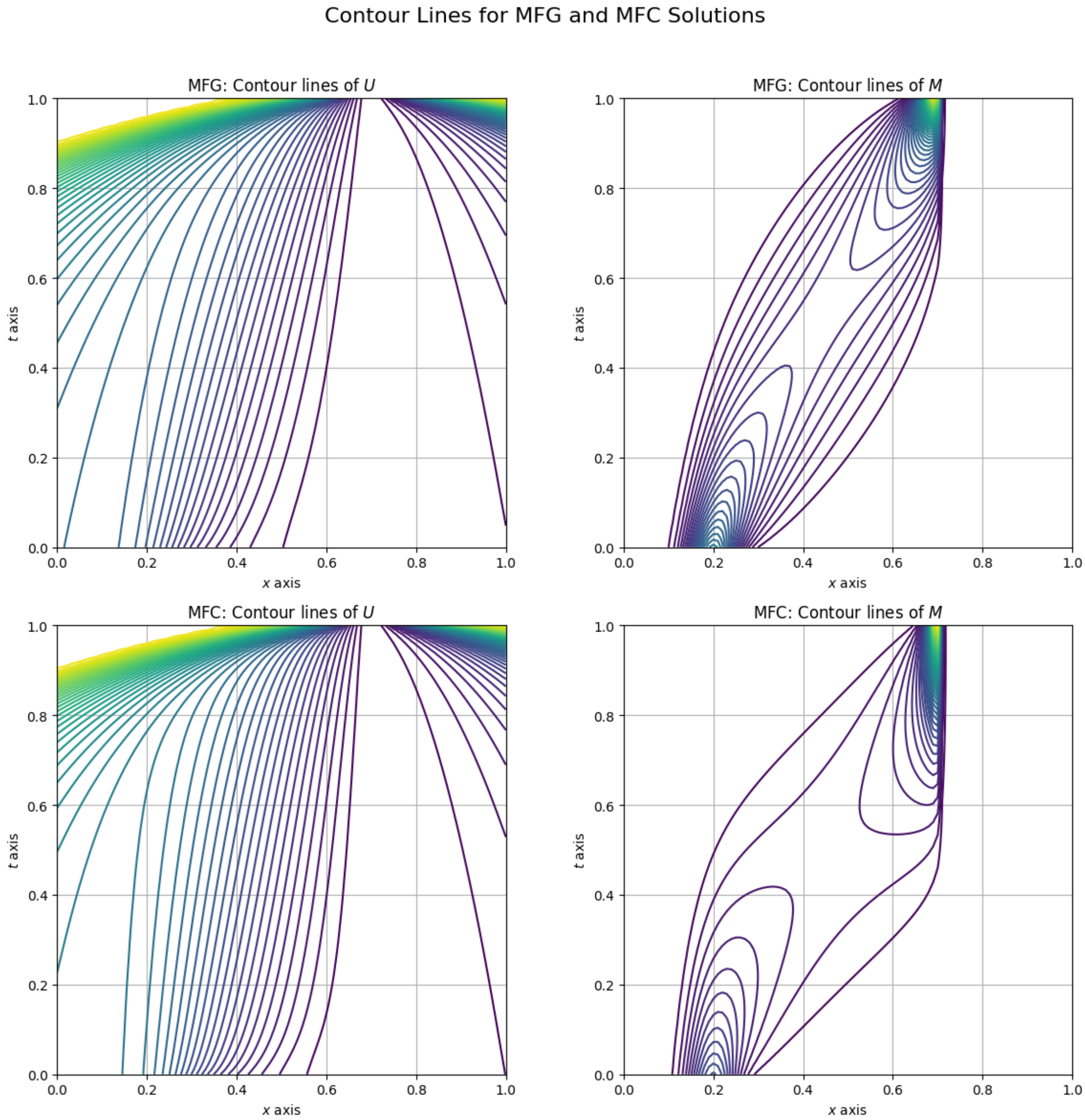

Figure 2: Contour lines of m and u in Mean Field Control (left) and Mean Field Game (right)

6.2.1 Key Differences in Solutions

- **Broader spatial distribution:** MFC solutions exhibit significantly more spread-out density patterns compared to MFG solutions. This occurs because the central planner optimizes for the collective by distributing agents to minimize aggregate congestion costs, whereas individual agents in MFG tend to cluster along what they perceive as optimal routes without considering their contribution to others' costs.

- **Congestion externality internalization:** MFC explicitly accounts for the externality impact of congestion through the modified Hamiltonian term $H_0(t, x)$ that includes the factor $\left(1 - \frac{c_1 \alpha m(t,x)}{c_0 + c_1 m(t,x)}\right)$. This mathematical adjustment causes agents to consider how their presence affects others, leading to more socially efficient distributions visible in Figure 2.

- **Faster terminal convergence:** MFC agents demonstrate more efficient progression towards the terminal condition. The central coordination enables better collective pathing strategies, evident in the more organized contour lines of the value function u.

- **Reduced density peaks:** The MFC solution consistently shows lower maximum density values compared to MFG. This characteristic directly translates to reduced social congestion cost, as the nonlinear congestion term $(c_0 + c_1 m)^\alpha$ grows superlinearly with density when $\alpha > 1$. By avoiding high concentration areas, the central planner significantly reduces the total social cost.

These differences highlight the fundamental distinction between decentralized Nash equilibria (MFG) and centralized social optima (MFC).

6.3 Social Cost Analysis

The social cost calculation involves computing the optimal control γ, the Lagrangian $L_0(\gamma, m)$, the running cost, and terminal cost. We compare the costs between MFG and MFC solutions.

	Running Cost	Terminal Cost	Total Cost
MFG	0.24427	-0.96882	-0.72455
MFC	0.24479	-0.98913	-0.74434

Table 1: Cost comparison between MFG and MFC solutions (lower is better)

As shown in Table 1, MFC achieves approximately 2.73% improvement in total social cost compared to MFG. The improvement is primarily driven by better terminal cost optimization, even though MFC incurs a slightly higher running cost. This demonstrates that centralized coordination can achieve better overall outcomes by making trade-offs between immediate costs and long-term benefits.