Bisection search, also known as binary search, is an efficient method to compute the inverse cumulative distribution function (inverse CDF or quantile function), especially when the CDF does not have a closed-form inverse. Here's why and how it helps:

Why Bisection Search is Useful

1. **Monotonicity of the CDF**: The cumulative distribution function (CDF) is a non-decreasing function. This property makes it ideal for bisection search because it ensures that for any two points a and b where $a < b$, $F(a) \leq F(b)$.

2. **Inverse Problem**: To compute the inverse CDF, we need to find the value x such that $F(x) = p$ for a given probability p. This is essentially finding the root of the equation $F(x) - p = 0$.

3. **No Closed-form Solution**: Many distributions do not have an analytical expression for their inverse CDF. Numerical methods like bisection search are practical and reliable alternatives.

How Bisection Search Works

Bisection search is a root-finding method that repeatedly narrows down an interval that contains the root. Here's a step-by-step outline of how it works in the context of finding the inverse CDF:

1. **Initial Interval**: Start with an interval $[a, b]$ where a and b are initial guesses such that $F(a) \leq p \leq F(b)$.

2. **Midpoint Calculation**: Compute the midpoint $m = \frac{a+b}{2}$.

3. **Evaluation**: Evaluate the CDF at the midpoint $F(m)$.

4. **Update Interval**:

   - If $F(m) < p$, the root lies in the interval $[m, b]$. Update $a = m$.
   - If $F(m) > p$, the root lies in the interval $[a, m]$. Update $b = m$.
   - If $F(m) = p$, m is the root, and we have found the inverse CDF value.
5. **Convergence**: Repeat steps 2-4 until the interval $[a, b]$ is sufficiently small, meaning $a \approx b$, or until $F(m)$ is close enough to p within a desired tolerance.

Example

Let's consider an example using a standard normal distribution with CDF $F(x)$ and the goal of finding the 95th percentile ($p = 0.95$).

1. **Initial Interval**: Start with $[-3, 3]$ since most of the probability mass of a standard normal distribution lies within this range.

2. **Midpoint Calculation**: Compute the midpoint $m = \frac{-3+3}{2} = 0$.

3. **Evaluation**: Evaluate the CDF at m, $F(0)$. For a standard normal distribution, $F(0) = 0.5$.

4. **Update Interval**:

   - Since $F(0) < 0.95$, update $a = 0$.

5. **Repeat**:

   - Next midpoint $m = \frac{0+3}{2} = 1.5$.
   - Evaluate $F(1.5)$. Suppose $F(1.5) = 0.933$.
   - Since $F(1.5) < 0.95$, update $a = 1.5$.

6. **Continue**:

   - Next midpoint $m = \frac{1.5+3}{2} = 2.25$.
   - Evaluate $F(2.25)$. Suppose $F(2.25) = 0.987$.
   - Since $F(2.25) > 0.95$, update $b = 2.25$.

7. **Converge**:

   - Repeat the process, continually narrowing the interval $[a, b]$, until $a \approx b$.

By iteratively refining the interval where the root lies, bisection search efficiently zeroes in on the value of x such that $F(x) = p$, thus computing the inverse CDF.

The inverse cumulative distribution function (inverse CDF), also known as the quantile function, is a fundamental concept in probability and statistics. It is the inverse of the cumulative distribution function (CDF) of a random variable.

Cumulative Distribution Function (CDF)

The CDF of a random variable X, denoted as $F_X(x)$, gives the probability that X takes a value less than or equal to x:

$$F_X(x) = P(X \leq x)$$

Inverse CDF (Quantile Function)

The inverse CDF, denoted as $F_X^{-1}(p)$, is defined for a given probability $p \in [0, 1]$. It gives the value x such that the probability of the random variable X being less than or equal to x is p:

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}$$

In other words, the inverse CDF answers the question: "For a given probability p, what is the corresponding value x of the random variable X?"

Applications

1. **Random Sampling**: In simulation and Monte Carlo methods, the inverse CDF is used to generate random samples from a specified distribution. By applying the inverse CDF to uniformly distributed random numbers (between 0 and 1), we obtain samples from the target distribution.

2. **Quantile Analysis**: The inverse CDF is used to find quantiles (e.g., median, quartiles) of a distribution. For example, the median is the value at $p = 0.5$.

3. **Risk Management**: In finance and risk management, the inverse CDF is used to calculate Value at Risk (VaR) and other risk measures.

Example

For a standard normal distribution $N(0, 1)$, the CDF $F(x)$ and its inverse $F^{-1}(p)$ can be illustrated as follows:

- $F(x)$ gives the probability that a standard normal random variable is less than or equal to x.
- $F^{-1}(p)$ gives the x-value for which the cumulative probability is p.

If $p = 0.95$, $F^{-1}(0.95)$ corresponds to the 95th percentile of the standard normal distribution, which is approximately 1.645.

Visualization

A graph can help illustrate this concept. The CDF of a distribution is a curve that increases from 0 to 1 as x increases. The inverse CDF can be visualized as finding the x value on the horizontal axis corresponding to a given probability p on the vertical axis.