Newton-Raphson's Method

To use the slope of the tangent to a function $f(x)$ at $x = t$ to derive Newton's method for finding a root of the function

1. **Find the equation of the tangent line:**

- The slope of the tangent line to $f(x)$ at $x = t$ is given by $f'(t)$.
- The point of tangency is $(t, f(t))$.

The equation of a line with slope m passing through the point (x_0, y_0) is given by:

$$y - y_0 = m(x - x_0)$$

In this case, $m = f'(t)$ and $(x_0, y_0) = (t, f(t))$.

Therefore, the equation of the tangent line is:

$$y - f(t) = f'(t)(x - t)$$

2. **Rearranging to solve for y, we get:**

$$y = f(t) + f'(t) \cdot (x - t)$$

To find where this tangent line intersects the x-axis, set $y = 0$:

$$0 = f(t) + f'(t) \cdot (x - t)$$

Solving for x, gives us the x-coordinate of the intersection point:

$$f'(t) \cdot (x - t) = -f(t)$$

$$x - t = \frac{-f(t)}{f'(t)}$$

Newton's method can be used to find the root of any differentiable function $f(x)$. The method iteratively updates the estimate using the formula derived from the tangent line intersection with the x-axis:

$$t_{i+1} = t_i - \frac{f(t_i)}{f'(t_i)}$$

Example

Consider the function $f(x) = x^2 - 2$ (to find the square root of 2):

Function and its derivative: $f(x) = x^2 - 2, f'(x) = 2x$

Newton's iteration formula: $t_{i+1} = t_i - \frac{t_1^2 - 2}{2t_i}$

Starting with an initial guess $t_0 = 1$:

$$t_1 = 1 - \frac{1^2-2}{2\cdot 1} = 1 - (-\frac{1}{2}) = 1 + \frac{1}{2} = 1.5$$

Next iteration:

$$t_2 = 1.5 - \frac{1.5^2-2}{2\cdot 1.5} = 1.5 - (\frac{2.25-2}{3}) = 1.5 - \frac{0.25}{3} = 1.4167$$

By continuing this process, we get increasingly better approximations of the square root of 2.

Applying Newton's Method to Find the k-th Root

To find the k-th root of n (i.e., a number x such that $x^k = n$), we need to transform this problem into a root-finding problem. We can do this by defining a function $f(x)$ whose root is the k-th root of n.

Function Definition

Consider the function:

$$f(x) = x^k - n$$

A root of this function $f(x)$ is a value x such that:

$$x^k - n = 0 \quad \text{or} \quad x^k = n$$

This is exactly the condition we want to satisfy to find the k-th root of n.

Derivative of the Function

To apply Newton's method, we also need the derivative of $f(x)$. The derivative is:

$$f'(x) = kx^{k-1}$$

Newton's Method Formula

Substituting $f(x) = x^k - n$ and $f'(x) = kx^{k-1}$ into the Newton's method formula, we get:

$$t_{n+1} = t_n - \frac{t_n^k - n}{kt_n^{k-1}}$$

Simplifying the Formula

Simplifying the above expression:

$$t_{n+1} = t_n - \frac{t_n^k - n}{kt_n^{k-1}} = t_n - \frac{t_n \cdot t_n^{k-1} - n}{kt_n^{k-1}} = t_n - \frac{t_n^k - n}{kt_n^{k-1}} = t_n - \frac{t_n^k}{kt_n^{k-1}} + \frac{n}{kt_n^{k-1}}$$

$$t_{n+1} = t_n - \frac{t_n}{k} + \frac{n}{kt_n^{k-1}}$$

$$t_{n+1} = \left(t_n - \frac{t_n}{k}\right) + \frac{n}{kt_n^{k-1}} = \frac{k-1}{k}t_n + \frac{n}{kt_n^{k-1}}$$

Thus, the Newton's method iteration formula for finding the k-th root of n becomes:

$$t_{n+1} = \frac{(k-1)t_n + \frac{n}{t_n^{k-1}}}{k}$$

This formula is more suited for our specific problem of finding the k-th root because it directly uses the definition of the k-th root and ensures that each iteration moves closer to the actual root.

Summary

The adaptation from the original Newton's method formula to the specific form for finding k-th roots is necessary to tailor the general method to our particular problem. By defining $f(x) = x^k - n$, we transform the problem of finding the k-th root into a root-finding problem, allowing us to apply Newton's method effectively. This results in a specialized iteration formula that converges to the k-th root of n.