To derive the formula for the **average path length** L in a k**-ring graph** with V vertices, as a function of both V and k, let's proceed step by step.

1. Structure of a k-Ring Graph

A k**-ring graph** is a circular graph where each vertex is connected to its k nearest neighbors. This means that:

- Every node is connected to $2k$ other nodes (i.e., k neighbors on either side).
- The graph is periodic, with the first and last vertices connected in a cycle, meaning you can "wrap around" the ends of the graph.

2. Path Length Definition

The **path length** between two nodes u and v is defined as the minimum number of edges traversed to get from u to v. In a k-ring graph, the distance between any two nodes depends on how far apart they are on the ring.

Nearest Neighbors:

- If two vertices u and v are within k steps of each other on the ring, the shortest path between them is just the number of edges directly connecting them (this is 1 edge if they are immediate neighbors, 2 edges if they are 2 steps away, etc.).

Farther Vertices:

- If two vertices are farther than k steps apart, the shortest path may involve wrapping around the ring, as it could be shorter to go the other way around.

3. Key Insights for Path Length in a k-Ring Graph

- **Direct Path**: For two vertices that are close to each other (within k neighbors), the path length is just the direct distance.
- **Wrap-Around Path**: For vertices farther apart, you might need to wrap around the ring, and the path length is the smaller of the two possible routes.

The key observation here is that the **path length between any two vertices** depends on their **circular distance** on the ring, with the maximum distance capped by k due to the k-nearest neighbor structure.

4. Average Path Length Formula

To calculate the **average path length** L, we need to consider the average shortest path between all pairs of vertices in the k-ring graph. Steps to Derive the Formula:

Distance Calculation:

The distance between two vertices i and j on this ring can be thought of as the minimum of the clockwise and counterclockwise path lengths.

The distance $d(i, j)$ is given by:

$$d(i, j) = \min(|i - j|, V - |i - j|)$$

Average Path Length:

To find the average path length, sum the distances between all pairs of vertices and divide by the total number of unique pairs.

Total Distances:

For a specific vertex i, consider distances to vertices $i + 1, i + 2, \ldots, i + \lfloor V/2 \rfloor$ (due to symmetry, we only need to calculate up to half the graph).

The sum of distances for one vertex can be calculated, and then multiplied by V since the graph is symmetric.

Formula:

You can derive a closed formula by calculating this sum and then generalizing it.

Here is a rough outline of the calculation:

For each vertex, calculate the sum of distances:

$\sum_{j=1}^{\lfloor V/2 \rfloor} j$

Multiply by V (number of vertices) and divide by the total number of pairs

$\binom{V}{2} = \frac{V(V-1)}{2}$.

For a vertex i, we calculate the sum of distances to vertices $i + 1, i + 2, \ldots, i + \lfloor V/2 \rfloor$

$\sum_{j=1}^{\lfloor V/2 \rfloor} j = \frac{\lfloor V/2 \rfloor (\lfloor V/2 \rfloor + 1)}{2}$

2. Total Distance for All Pairs:

Since the graph is symmetric, multiply the sum for one vertex by V (the number of vertices):

$V \cdot \frac{\lfloor V/2 \rfloor (\lfloor V/2 \rfloor + 1)}{2}$

3. Average Path Length:

Divide the total distance by the number of unique pairs of vertices, which is (2V)=2V(V−1):
$$\text{Average Path Length} = \frac{V \cdot \frac{\lfloor V/2 \rfloor (\lfloor V/2 \rfloor + 1)}{2}}{\frac{V(V-1)}{2}}$$ Simplifying:

$$= \frac{\lfloor V/2 \rfloor (\lfloor V/2 \rfloor + 1)}{V-1}$$

4. Consider the Impact of k:

For a k-ring, the maximum distance between connected vertices should not exceed k, so you only need to sum up to k rather than $\lfloor V/2 \rfloor$ if $k < \lfloor V/2 \rfloor$.

Thus, the formula is:

$$\text{Average Path Length} = \frac{\min(k, \lfloor V/2 \rfloor)(\min(k, \lfloor V/2 \rfloor) + 1)}{V-1}$$

This accounts for the effect of k when k is small compared to V.