The **clustering coefficient** in a graph is a measure of the degree to which nodes in the graph tend to cluster together. In a **k-ring graph**, which is a regular graph where each node is connected to its `k` nearest neighbors in a circular manner, the clustering coefficient quantifies how many of a node's neighbors are also neighbors with each other.

Global and Local Clustering Coefficients

- **Local Clustering Coefficient (LCC)** for a node measures the ratio of the number of edges between its neighbors to the number of possible edges between those neighbors.

  For a node `v` with `k` neighbors: $C(v) = \frac{\text{Number of edges between the neighbors of } v}{\frac{k(k-1)}{2}}$

  This gives the proportion of potential triangles (three nodes fully connected) that are actually formed.

- **Global Clustering Coefficient (GCC)** is the average of the local clustering coefficients for all nodes, or it can be computed as the ratio of the number of closed triplets (three nodes that form a triangle) to the total number of connected triplets (three nodes where at least two are connected).

Clustering Coefficient in a k-Ring Graph

In a **k-ring graph**:

1. Each node is connected to its `k` nearest neighbors (e.g., in a circular ring layout).
2. Neighbors of a node are also likely to be neighbors with each other when `k` is large enough to form triangles.

For example:

- In a **1-ring graph** (where each node is connected to its 1 nearest neighbor), the clustering coefficient is 0 because no triangles are formed (no neighbors are interconnected).
- In a **2-ring graph** (each node connected to 2 neighbors), again, there are no triangles, so the clustering coefficient is 0.
- In a **3-ring graph**, each node is connected to its 3 nearest neighbors. Depending on how the edges connect, triangles may start to form, and thus the clustering coefficient becomes greater than zero.

For large enough `k`, the clustering coefficient increases as more of a node's neighbors are connected to each other, forming triangles.

In summary, the clustering coefficient in a **k-ring graph** depends on `k`:

- Small `k` values (e.g., 1, 2) often lead to a clustering coefficient of 0, as neighbors are not interconnected.
- Larger `k` values can lead to a higher clustering coefficient as more neighbors of a node are also connected to each other.

The **clustering coefficient** in a k-ring graph can be computed using two main approaches: **local clustering coefficient** (for each node) and the **global clustering coefficient** (for the entire graph). Let's go over how each is computed.

1. Local Clustering Coefficient (LCC)

The **local clustering coefficient** for a node v measures how many of the possible connections between its neighbors actually exist. Here's how to compute it:

Formula:

For a node v with k neighbors, the local clustering coefficient $C(v)$ is given by:

$$C(v) = \frac{2 \times \text{Number of edges between neighbors of } v}{k(k-1)}$$

Steps to Compute LCC:

1. **Identify the neighbors** of node v. In a k-ring graph, each node is connected to its k nearest neighbors (in a circular fashion).

2. **Count the number of edges** between the neighbors of v. These are the edges that form triangles with v.

   For example, if node v has neighbors $N(v) = \{a, b, c\}$, check how many edges exist between a, b, and c.

3. **Calculate the possible number of edges** between the neighbors. If a node has k neighbors, there are $\frac{k(k-1)}{2}$ possible edges between them.

4. **Compute the ratio** of actual edges to possible edges. This gives the local clustering coefficient for that node.

Example:

Suppose in a 3-ring graph, node v has neighbors a, b, and c. If a, b, and c are all connected (fully forming a triangle), the local clustering coefficient is:

$$C(v) = \frac{2 \times 3}{3(3-1)} = 1$$

If only one edge exists between a and b, the local clustering coefficient would be:

$$C(v) = \frac{2 \times 1}{3(3-1)} = \frac{1}{3}$$

2. Global Clustering Coefficient (GCC)

The **global clustering coefficient** measures the overall tendency of nodes in the graph to cluster together. It is computed as the average of the local clustering coefficients for all nodes or as the ratio of the number of closed triplets (triangles) to the number of connected triplets (two nodes connected to a third node).

Formula (Triplet Definition):

The global clustering coefficient can be computed using the ratio of closed triplets to total triplets in the graph:

$$C_{global} = \frac{3 \times \text{Number of triangles in the graph}}{\text{Number of connected triplets of nodes}}$$

Steps to Compute GCC:

1. **Count the number of triangles** in the graph. A triangle is a closed triplet of nodes where each pair of nodes is connected by an edge.

2. **Count the number of connected triplets** of nodes. A connected triplet consists of three nodes where at least two edges are present.

3. **Compute the ratio** of the number of triangles to the number of connected triplets. Multiply the number of triangles by 3 because each triangle contributes to three triplets.

Example:

In a graph with 6 triangles and 12 triplets (connected sets of three nodes), the global clustering coefficient would be:

$$C_{global} = \frac{3 \times 6}{12} = 1.5$$

If there are 3 triangles and 10 triplets, then:

$$C_{global} = \frac{3 \times 3}{10} = 0.9$$

Special Case: k-Ring Graph

In a k**-ring graph**:

- When k is small (e.g., $k = 1$ or $k = 2$), the clustering coefficient is 0 because there are no triangles formed (no neighbors of a node are connected to each other).
- As k increases, the clustering coefficient increases because the neighbors of a node are more likely to be connected to each other, forming triangles.

For larger k-ring graphs, you can compute the clustering coefficients based on how many triangles are formed within the nearest neighbors.

In a k-ring graph, each node is connected to its k nearest neighbors on each side. This means each node has $2k$ connections.

Clustering Coefficient for a Node

The local clustering coefficient for a node i is defined as:

$$C_i = \frac{\text{Number of triangles through node } i}{\text{Number of possible triangles through node } i}$$

Counting the Number of Triangles

For a node i with $2k$ neighbors, the number of triangles that include node i is determined by the number of edges among its neighbors.

Each pair of neighbors can potentially form a triangle with node i.

The number of these pairs is given by the combination formula:

$$\binom{2k}{2} = \frac{2k(2k-1)}{2} = k(2k-1)$$

However, not all pairs will form a triangle. In a k-ring graph, each node is directly connected to only its k nearest neighbors on either side, which limits the potential pairs forming a triangle.

Number of Possible Triangles

For each node, the number of potential triangles is determined by the pairs of its neighbors it can connect to. As we established, each node has $2k$ neighbors, and any two neighbors can potentially form a triangle with the node itself. Thus, the number of potential triangles through a single node is:

$$\binom{k}{2} = \frac{k(k-1)}{2}$$

Step 5: Derivation of Clustering Coefficient

To calculate the overall clustering coefficient, we average the local clustering coefficients over all nodes. Given the k-ring structure, the number of actual triangles is approximately:

$\approx \frac{k(k-1)}{2}$ Therefore, the clustering coefficient averaged over all nodes is approximated by:

$$C = \frac{\text{Number of actual triangles}}{\text{Number of possible triangles}} = \frac{k(k-1)}{2k(2k-1)}$$