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Three solutions to the two-body problem



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Abstract

The two-body problem consists of determining the motion of two gravitationally interacting bodies with given masses and initial velocities. The problem was first solved by Isaac Newton in 1687 using geometric arguments. In this thesis, we present selected parts of Newton's solution together with an alternative geometric solution by Richard Feynman and a modern solution using differential calculus. All three solutions rely on the three laws of Newton and treat the two bodies as point masses; they differ in their approach to the three laws of Kepler and to the inverse-square force law. Whereas the geometric solutions aim to prove some of these laws, the modern solution provides a method for calculating the positions and velocities given their initial values.

It is notable that Newton in his most famous work *Principia*, where the general law of gravity and the solution to the two-body problem are presented, used mathematics that is not widely studied today. One might ask if today's low emphasis on classical geometry and conic sections affects our understanding of classical mechanics and calculus.

1 Introduction

The two-body problem consists of determining the paths of two gravitational interacting bodies of known masses and initial velocities. The bodies are moving in three dimensional space and are affected by no other forces than the gravitational forces between them. The first solution of the two-body problem was published by Isaac Newton in 1687 in his epoch-making work *Principia*¹. In the seventeenth century Johannes Kepler had shown the planetary orbits around the sun are elliptic. This was astonishing to his contemporaries who regarded the circle to be the divine shape that governed the heavenly bodies. Kepler put a lot of effort into his calculations, using data from observations provided by Tycho Brahe, but despite his discovery he never explained why the planets move as they do. Some believed that they were pushed forward by angels. Another popular idea, proposed by René Descartes, explained the motion by vortices of particles that filled all of space, Densmore [2003].

Newton revolutionized the world by giving it the universal law of gravity, a law that could explain motion in the constant gravitational field here on earth as well as the motions of planets. The theory was laughed at by some, arguing such a invisible force to be supernatural and occult.

Now, a few hundred years later, when Newtonian mechanics is one of the main pillars of modern physics, we find theories involving angels or vortices in outer space amusing. Knowing what we know today it is hard to imagine the challenges Newton encountered as a scientist and from what mindset he had to break lose.

One obstacle scientists faced in the seventeenth century was the lack of data that could verify new theories. When Newton first developed the idea of gravity during a lengthy stay with his mother in Woolsthorpe, he wanted to test it by calculating the orbit of the moon

¹Throughout this thesis a translation into English, made by Andrew Motte and published in 1729, will be used.

around the earth. The result was unsatisfying because of inaccurate measurements of the distance to the moon and supposedly a disappointment to Newton. Without supporting calculations Newton left the matter and did not pursue it until years later, Chandrasekhar [1995].

Another difficulty scientists faced was the lack of knowledge of what makes up the universe. Newton's law is called the general law of gravity, applying to all objects with mass. But which phenomena observed in the sky have mass? When in 1681, fifteen years after Newton discovered gravity, a comet was seen wandering across the sky, and then a few months later another comet was seen going the other way, only one observer declared this to be one and the same object - and it was not Newton. Confronted with the idea Newton ridiculed it and it would take years until he published descriptions of the paths of comets, together with the solution of the two-body problem, in *Principia*.² When Newton, late in life, was asked how he had discovered the law of universal gravitation he replied "By thinking on it continually", Westfall [1999].

Even to Newton the knowledge of the celestial powers started off like a small plant, in need of care and nourishment. The plant was cultivated and pruned and in time it gave fruit in the form of new concepts, such as the universal law of gravity. Sometimes we forget that the knowledge did not just fall down on his head.

In addition to his tremendous contributions to physics, Newton is known for the invention of differential calculus. Today this is the branch of mathematics normally used for solving the two-body problem. It is somewhat surprising that Newton himself does not use this powerful tool in his solutions, the reason for this scholars do not agree on, but his publications concerning calculus date much later than *Principia*, Chandrasekhar [1995].

2 Different approaches to the two-body problem

In this theses three solutions to the problem is presented:

- a) a modern solution,
- b) Newton's solution,
- c) Feynman's solution.

The solutions share some assumptions of which the notable are the laws of Newton and the assumption that the two bodies can be treated as point masses. They vary in method and use of the three laws of Kepler.

Newton's laws can be stated as follows:

1. Every body preserves in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.
2. The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.
3. To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts. Newton [1729] p 19, 20.

Kepler's three laws:

1. The planets move in ellipses with the Sun at one focus.
2. A radius vector from the Sun to a planet sweeps out equal areas in equal times.

²The person was John Flamsteed, royal astronomer in London. To Newton's defense Flamsteed argued that the comet made its turn in front of the sun, not around it.

3. The square of the orbital period of a planet is proportional to the cube of its semi-major axis, Murray and Dermott [1999] p. 3.

2.1 Modern solution

We first present a modern solution which employs ideas from modern physics such as Newton's law of gravity, here referred to as the inverse-square law. A modern solution can be calculated in different ways, here the inverse-square law is used to prove the first and second law of Kepler. In appendix A calculations are included where the opposite is done; Kepler's first law of elliptic orbits is used to show the inverse-square law.

The problem is solved using differential equations and the masses and initial positions and velocities of the bodies. In the main part of the thesis we focus on how a solution is found, in appendix B and C calculations are performed given certain initial values. The modern approach, using differential equations has the advantage that it is applicable to any set of initial data, in this thesis we will focus on the case of an elliptic orbit³.

2.2 Newton's solution

Newton's contribution to the understanding of motion of bodies is extensive and the presentation of his solution of the two-body problem is here limited to a few propositions in *Principia*. We will see how Newton proves Kepler's second law and how he uses this law together with the assumption of elliptic orbits, Kepler's first law, to prove the inverse-square law. This proof is commonly regarded as the gem of *Principia*, Chandrasekhar [1995].

In the second and third editions of *Principia*, printed in 1713 and 1726, Newton added to his proof a *Idem aliter*, the same otherwise, proving elliptic orbit using the inverse-square law. Densmore writes in her book *Newton's Principia The Central Argument* that the original proof is presented the way Newton thought of it and if he wanted to exchange it for another one he had the choice to do so in later editions and not only write a short addition to the proof, Densmore [2003]. A study of this proof would be interesting but since the added text in *Principia* is brief and not believed central to Newton it is not included here.

The solution is performed using geometry, widely studied among scholars of math in the seventeenth century but the text is hard to digest for a modern reader. Essential parts of the proofs involve the concept of limit, lacking the rigor we are used to today. Some believe Newton originally used calculus to calculate proofs in *Principia* and then translated it into something more comprehensible, others believe this unlikely, Chandrasekhar [1995]. Presumably the idea of translated proofs was cherished among the Englishmen who took Newton's side against Leibniz in the priority dispute regarding the invention of calculus.

2.3 Feynman's solution

Feynman presents a geometrical solution to the simplified problem where one body is held still. Like Newton, Feynman proves Kepler's second law but to continue he uses Kepler's third law to prove the inverse-square law and then shows how this implies elliptical orbits.

2.4 Table of used assumptions and of what is proved

To summarize the relations to Kepler's three laws (K1, K2, K3) and the inverse-square law (ISQL), the solutions are presented in a table, see table 1, which also includes the alternative modern solution in appendix.

³As Newton showed, the other possible cases are the parabola and the hyperbola.

Solution	Uses	Proves
Modern	ISQL	K1, K2
Modern in appendix	K1	K2, ISQL
Newton's	K1	K2, ISQL
Feynman's	K3	K1, K2, ISQL

Table 1: Table of used assumptions and of what is proved by the different solutions.

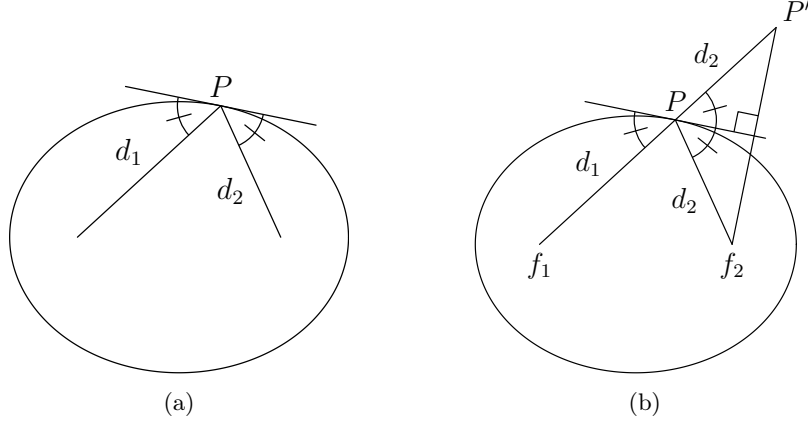


Figure 1

3 Properties of the ellipse

While the circle can be drawn using a thumbtack and a stretched string, the ellipse is drawn using two tacks, see figure 1a. The total length of the string stays the same as the pencil goes around the curve, making the positions of the tacks the ellipse's two foci. In the figure the sum of the distances d_1 and d_2 is constant. Another property is that a tangent, at any point P on the ellipse, gives the same angle to the lines drawn to any of the two foci.

Using an isosceles triangle these properties can be shown to be interconnected, see figure 1b. Given the foci f_1 and f_2 and the distance $d_1 + d_2$, an ellipse can be drawn. The line f_1P can be extended to a point P' making f_1PP' equal to f_1Pf_2 . The line bisecting f_2P' will be tangent to the ellipse and from construction it will make the same angle to both lines drawn from P to each focus. As a consequence the point f_1 can be viewed as the center of a circle and for every point f_2 , within the circle, an ellipse can be drawn, as in figure 2, Goodstein and Goodstein [1997]. The closer the foci are to one another the more resemblance in shape between the ellipse and the circle. For a more thorough exposition of the connection between the properties see Goodstein and Goodstein [1997].

As a measurement of the deviation from a circle we use eccentricity. The eccentricity for an ellipse is a number between zero and one and is defined as

$$e = \sqrt{1 - \frac{b^2}{a^2}} \quad (3.1)$$

where a is half the major axis and b half the minor axis. For a circle, the major and minor axis have the same length and the eccentricity is zero. A large deviation from the circle will give a value of e close to one. In such an ellipse the foci will be far apart and the eccentricity can also be expressed using c where c denotes the distance between the center of the ellipse and one of its focus points,

$$e = \frac{c}{a}. \quad (3.2)$$

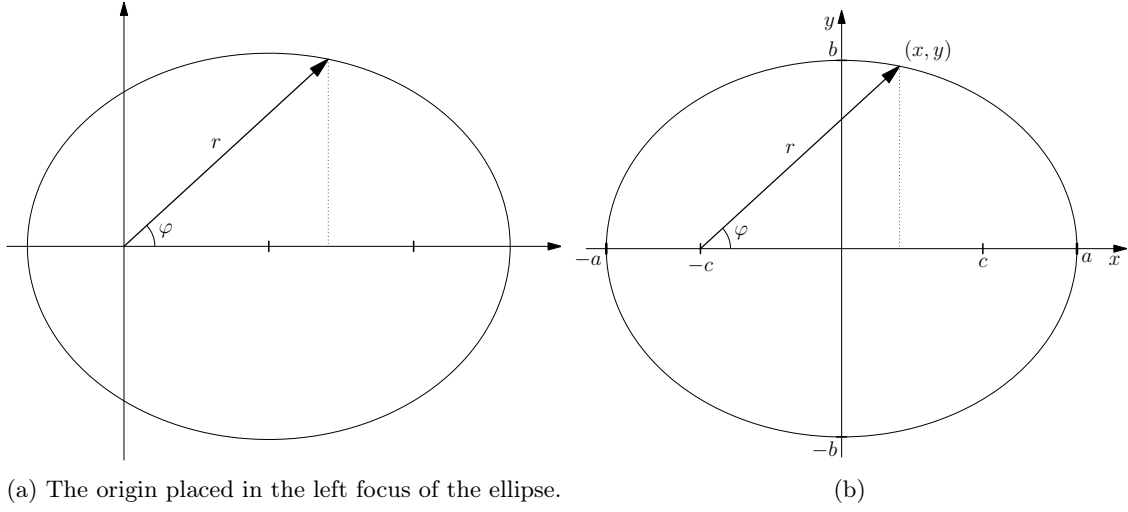


Figure 4

This equation follows from (3.3), expressing x and y using r and φ , see figure 4b,

$$x = r \cos \varphi - c \quad (3.9)$$

$$y = r \sin \varphi \quad (3.10)$$

$$\frac{(r \cos \varphi - c)^2}{a^2} + \frac{(r \sin \varphi)^2}{b^2} = 1. \quad (3.11)$$

By substituting b and c with expressions of a and e , see (3.1) and (3.2), equation (3.8) now follows.

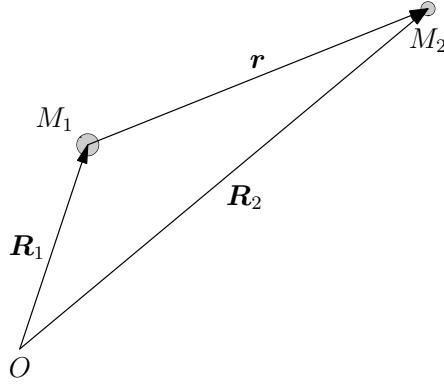


Figure 5: The two position vectors \mathbf{R}_1 and \mathbf{R}_2 give the locations of the bodies M_1 and M_2 . In the simplified problem, holding M_1 stationary, \mathbf{r} gives the position of M_2 .

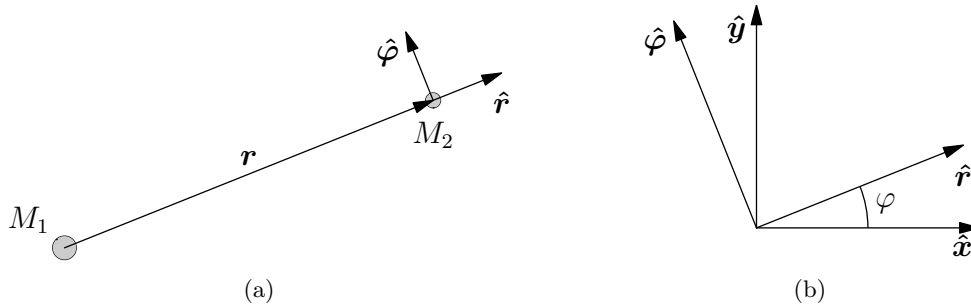


Figure 6

4 A modern solution

In a modern solution to the problem differential equations are used to find the paths for the bodies. Let M_1 and M_2 be two spherical masses moving in space and \mathbf{R}_1 and \mathbf{R}_2 be their position vectors, originating from an arbitrary point in space.

To simplify the problem we divide it into two parts, one part where M_1 is held still and M_2 is moving and a second part where both bodies are allowed to move.

4.1 One moving body

With M_1 being stationary so is \mathbf{R}_1 , this gives the option of expressing \mathbf{R}_2 as the sum of \mathbf{R}_1 and a vector \mathbf{r} going between the masses, see figure 5. The motion of \mathbf{r} will give the motion of M_2 and in this, the first part of the problem, the position of M_1 is treated as the origin.

The vector \mathbf{r} is divided into length and direction:

$$\mathbf{r} = r(t) \hat{\mathbf{r}}(t). \quad (4.1)$$

With only two bodies in the system and one of them held still, the initial velocity of the moving body decides the plane in which all motion will take place. Accordingly, only polar coordinates are needed to describe the orbit. The angular movement is expressed by using $\hat{\boldsymbol{\varphi}}$, see figure 6a.

Imagine our sun and a comet traveling in space, the large difference in their masses will make the movement of the sun, due to the comet, insignificant. At a given time the comet has a position and a velocity. The greater the velocity is away from the sun, in the direction of $\hat{\mathbf{r}}$, the further away from the sun the comet will reach. Like Newton, we assume the force affecting the comet to be directed towards the sun and the change of

velocity to be proportional to the force. Unlike Newton we can use the inverse-square law that tells us how the force depends on the distance.

The gravitational force only acts in the direction of $\hat{\mathbf{r}}$ and not in the direction of $\hat{\boldsymbol{\varphi}}$.

4.1.1 Calculating the acceleration of \mathbf{r}

From (4.1) we get the first and second derivative of \mathbf{r} ;

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \quad (4.2)$$

$$\ddot{\mathbf{r}} = \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\frac{d\hat{\mathbf{r}}}{dt} + r\frac{d^2\hat{\mathbf{r}}}{dt^2}. \quad (4.3)$$

By expressing $\hat{\mathbf{r}}$ using $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, this time all three originated from the same point, see figure 6b, the equation for $\ddot{\mathbf{r}}$ can be simplified. We let:

$$\hat{\mathbf{r}} = \cos \varphi \cdot \hat{\mathbf{x}} + \sin \varphi \cdot \hat{\mathbf{y}} \quad (4.4)$$

$$\hat{\boldsymbol{\varphi}} = -\sin \varphi \cdot \hat{\mathbf{x}} + \cos \varphi \cdot \hat{\mathbf{y}} \quad (4.5)$$

and calculate the derivatives:

$$\frac{d\hat{\mathbf{r}}}{dt} = -\sin \varphi \cdot \dot{\varphi}\hat{\mathbf{x}} + \cos \varphi \cdot \dot{\varphi}\hat{\mathbf{y}} \quad (4.6)$$

$$\frac{d\hat{\boldsymbol{\varphi}}}{dt} = -\cos \varphi \cdot \dot{\varphi}\hat{\mathbf{x}} - \sin \varphi \cdot \dot{\varphi}\hat{\mathbf{y}}. \quad (4.7)$$

Put together, these four equations allow us to write:

$$\frac{d\hat{\mathbf{r}}}{dt} = \hat{\boldsymbol{\varphi}}\dot{\varphi} \quad (4.8)$$

$$\frac{d\hat{\boldsymbol{\varphi}}}{dt} = -\hat{\mathbf{r}}\dot{\varphi}. \quad (4.9)$$

Using (4.8), the second derivative of $\hat{\mathbf{r}}$ can be calculated:

$$\frac{d^2\hat{\mathbf{r}}}{dt^2} = \frac{d\hat{\boldsymbol{\varphi}}}{dt}\dot{\varphi} + \hat{\boldsymbol{\varphi}}\ddot{\varphi} \quad (4.10)$$

and simplified, using (4.9):

$$\frac{d^2\hat{\mathbf{r}}}{dt^2} = -\hat{\mathbf{r}}\dot{\varphi}^2 + \hat{\boldsymbol{\varphi}}\ddot{\varphi}. \quad (4.11)$$

Using (4.8) and (4.11) the acceleration given in (4.3) can be calculated:

$$\begin{aligned} \ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\hat{\boldsymbol{\varphi}}\dot{\varphi} + r(t)(-\hat{\mathbf{r}}\dot{\varphi}^2 + \hat{\boldsymbol{\varphi}}\ddot{\varphi}) \\ &= \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\dot{\varphi}\hat{\boldsymbol{\varphi}} - r(t)\dot{\varphi}^2\hat{\mathbf{r}} + r(t)\ddot{\varphi}\hat{\boldsymbol{\varphi}} \\ &= (\ddot{r} - r(t)\dot{\varphi}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\varphi} + r(t)\ddot{\varphi})\hat{\boldsymbol{\varphi}}. \end{aligned} \quad (4.12)$$

This acceleration is compounded of two parts, one in the direction of $\hat{\mathbf{r}}$, that is between the bodies, and one in the direction of $\hat{\boldsymbol{\varphi}}$, perpendicular to $\hat{\mathbf{r}}$. The acceleration is in the direction of the force which is in the direction of $\hat{\mathbf{r}}$. Hence the magnitude of the acceleration is equal to the coefficient of $\hat{\mathbf{r}}$ while the other part of the equation must be zero

$$\ddot{\mathbf{r}} = \underbrace{(\ddot{r} - r(t)\dot{\varphi}^2)}_{|\ddot{\mathbf{r}}|}\hat{\mathbf{r}} + \underbrace{(2\dot{r}\dot{\varphi} + r(t)\ddot{\varphi})}_0\hat{\boldsymbol{\varphi}}. \quad (4.13)$$

This gives us two differential equations to solve.

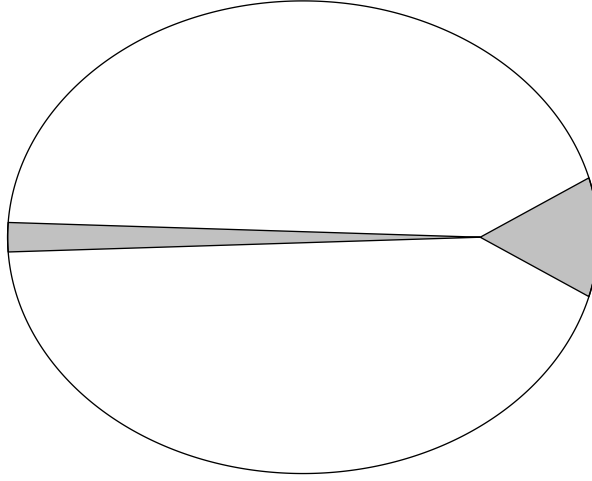


Figure 7: Illustration of Kepler's second law. When a planet is close to the sun it moves faster than when it is far from the sun, but for a certain time interval it sweeps out equal areas.

4.1.2 The acceleration along $\hat{\varphi}$

From (4.13) and the assumption that the force is directed to M_1 we get

$$2\dot{r}\dot{\varphi} + r(t)\ddot{\varphi} = 0. \quad (4.14)$$

The equation contains \dot{r} , which is the change of distance between M_1 and M_2 , and $\dot{\varphi}$ being the angular velocity of M_2 . Denoting the angular velocity ω separation of variables yields:

$$2\frac{dr}{dt}\omega + r\frac{d\omega}{dt} = 0 \quad (4.15)$$

$$r\frac{d\omega}{dt} = -2\frac{dr}{dt}\omega \quad (4.16)$$

$$\frac{1}{\omega} d\omega = -2\frac{1}{r} dr. \quad (4.17)$$

This gives:

$$\dot{\varphi} = \omega = \frac{h}{r^2}. \quad (4.18)$$

where h is a constant. This equation shows that the angular velocity is inversely proportional to the square of the distance, which is another way of stating Keplers second law; A radius vector from the Sun to a planet sweeps out equal areas in equal times, Murray and Dermott [1999] p. 3. The law is illustrated in figure 7.

It is worth noting that the law is true even when there is no gravitational force and the motion is along a straight line.

4.1.3 The acceleration along \hat{r}

So far we have used the same presumptions as Newton did but where he chooses to use the knowledge of elliptical orbits we choose the result of Newton; the gravitational force is inversely proportional to the square of the distance, and directed along the line between M_1 and M_2 . With r starting in M_1 and ending in M_2 the force, affecting M_2 , will be in the opposite direction. By F_r we denote the scalar value of the force, λ is the proportional constant and G is the gravitational constant

$$\mathbf{F}(\mathbf{r}) = -\frac{\lambda}{r^2}\hat{\mathbf{r}} = F_r(r)\hat{\mathbf{r}}, \quad \lambda = GM_1M_2. \quad (4.19)$$

In addition to the inverse-square law we will also use Newton's second law

$$\mathbf{F} = m\ddot{\mathbf{r}} \quad (4.20)$$

where m , in this case, is the moving mass M_2 . In view of (4.13) we can write

$$\mathbf{F}(\mathbf{r}) = m(\ddot{r} - r\dot{\varphi}^2)\hat{\mathbf{r}} \quad (4.21)$$

$$F_r(r) = m(\ddot{r} - r\dot{\varphi}^2). \quad (4.22)$$

By using (4.18) and (4.19), $\dot{\varphi}$ and the force can be replaced. The result is a nonlinear differential equation

$$-\frac{\lambda}{r^2} = m\ddot{r} - m\frac{h^2}{r^3}. \quad (4.23)$$

4.1.4 Expressing r as a function of φ

Instead of solving (4.23) and express the distance as a function of time, r can be written as a function of φ , which is sufficient for numerical calculations and data simulations.

Equation (4.23) can be solved for r , as a function of φ , using two substitutions. In the first substitution the independent variable is changed from t to φ . The way the variables r , φ and t are interdependent allows us to write $r(t(\varphi))$ or just $r(\varphi)$ instead of $r(t)$ but the acceleration \ddot{r} has to be calculated. Using the chain rule and (4.18) we can write

$$\frac{d}{dt}(r(\varphi(t))) = \frac{dr}{d\varphi} \frac{d\varphi}{dt} = \frac{h}{r^2} \frac{dr}{d\varphi} \quad (4.24)$$

This gives the acceleration

$$\ddot{r} = \frac{h}{r^2} \frac{d}{d\varphi} \underbrace{\left(\frac{h}{r^2} \frac{dr}{d\varphi} \right)}_{\dot{r}}. \quad (4.25)$$

We use this to rewrite equation (4.23) and simplify the expression

$$-\lambda = mh \frac{d}{d\varphi} \left(\frac{h}{r^2} \frac{dr}{d\varphi} \right) - m \frac{h^2}{r}. \quad (4.26)$$

In the second substitution we will substitute r for the inverse of a new variable u :

$$u = \frac{1}{r} \Leftrightarrow r = \frac{1}{u} \Rightarrow \frac{dr}{du} = -r^2. \quad (4.27)$$

This gives

$$\frac{dr}{d\varphi} = \frac{dr}{du} \frac{du}{d\varphi} = -r^2 \frac{du}{d\varphi}. \quad (4.28)$$

We rewrite (4.26)

$$-\lambda = mh \frac{d}{d\varphi} \left(-h \frac{du}{d\varphi} \right) - mh^2 u \quad (4.29)$$

and simplify the expression

$$\frac{\lambda}{mh^2} = \frac{d^2 u}{d\varphi^2} + u. \quad (4.30)$$

With the positive constant denoted K the equation and the solution can be written

$$u'' + u = K \quad (4.31)$$

$$u = A \cos \varphi + B \sin \varphi + K \quad (4.32)$$

where A and B depends on the initial velocity of the moving body, M_2 . Note that u has to be positive, being the inverse of the distance r . According to (4.27) we can substitute r for u :

$$r = \frac{1}{A \cos \varphi + B \sin \varphi + K}. \quad (4.33)$$

If we know how to decide the angle φ we now have a solution to the first part of our problem. Newton showed a body in orbit to describe an elliptic path, by comparing the solution with the equation for an ellipse, (3.8) the angle φ can be better understood. The equation of the ellipse where the angle θ is counted counter clockwise from the major axis, assuming r reaches its maximum value for $\theta = 0$:

$$r(\theta) = \frac{a(1 - e^2)}{1 - e \cos \theta}. \quad (4.34)$$

To rewrite the solution (4.33) to resemble the equation of the ellipse we use the trigonometric formula:

$$\cos \alpha \cos \beta + \sin \alpha \sin \beta = \cos(\alpha - \beta).$$

The formula allows us to replace A and B with the constants φ_0 and C :

$$\underbrace{-C \cos \varphi_0}_{A} \cos \varphi + \underbrace{-C \sin \varphi_0}_{B} \sin \varphi = -C \cos(\varphi_0 - \varphi) \quad (4.35)$$

$$A = -C \cos \varphi_0 \quad (4.36)$$

$$B = -C \sin \varphi_0 \quad (4.37)$$

$$A \cos \varphi + B \sin \varphi = -C \cos(\varphi_0 - \varphi). \quad (4.38)$$

By quadrating A and B an expression for C can be found

$$A^2 = C^2 \cos^2 \varphi_0 \quad (4.39)$$

$$B^2 = C^2 \sin^2 \varphi_0 \quad (4.40)$$

$$A^2 + B^2 = C^2(\cos^2 \varphi_0 + \sin^2 \varphi_0) \Rightarrow C = \sqrt{A^2 + B^2}. \quad (4.41)$$

C is chosen to be the positive root. We can now write:

$$r(\varphi) = \frac{1}{K - C \cos(\varphi - \varphi_0)}. \quad (4.42)$$

For a body in orbit, $\cos(\varphi - \varphi_0)$ varies between 1 and -1 and thus K needs to be greater than C for the distance, r , to be positive. If (4.42) is divided by K and φ_0 is set to be zero the solution is the equation for an ellipse

$$r(\varphi) = \frac{\frac{1}{K}}{1 - \frac{C}{K} \cos(\varphi)}. \quad (4.43)$$

With a value of K greater than C which makes the ratio between them suitable for the eccentricity, which in an ellipse is a number between zero and one⁴. (How the constants C , K and φ are calculated given certain initial values is shown in appendix B).

⁴In the cases of a hyperbolic path the eccentricity is more than one, suggesting C being larger than K which limits the range of the angle φ for which the solution is valid. For values of φ going to the limit of the values for which it is defined, r is going to infinity.

4.2 Two moving bodies

So far, one of the bodies has been held still in spite of the gravitational force from the other body. In many cases this is close enough to the real conditions, for example calculating the orbit of a satellite around the earth - the effect on the earth is negligible. If the bodies are more equal in mass the motion of both the bodies needs to be considered. We will use Newton's second law, commonly phrased: The force is equal to the mass times the acceleration. When Newton states the law in *Principia* he writes of change in motion: *The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed*, Newton [1729] p 19. That is; the force is equal to the rate of change of the mass times the velocity. Mass times velocity is also called linear momentum, denoted by \mathbf{P} ,

$$\mathbf{F} = m \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}. \quad (4.44)$$

If the linear momentum is constant the sum of the forces that affect the object is zero. For our system the total linear momentum is constant because there are no outside forces. Newton understood that the motion of a system of many bodies, for example a bag of marbles, could be divided into different parts and calculated separately. The motion of the marbles inside the bag would be one part of the problem, the motion of the whole bag, which could be regarded as a point mass, would be another part. If the bag was thrown here on earth the trajectory would be affected by the gravity of the earth, if it was thrown in empty space no outer forces would affect it and the center of mass of the bag would for ever continue with the same velocity as it initially had. This division of the problem into parts will help us to describe the motion of the two bodies.

As before, we use vectors to represent the locations of the bodies. In the first part of the problem the origin was placed in one of the bodies, now a good location of the origin is not as easily found. Let us start with the placing it somewhere outside the two bodies and describe the motion with vectors \mathbf{R}_1 and \mathbf{R}_2 .

Newton's second law helps us to express the forces in the system,

$$M_1 \ddot{\mathbf{R}}_1 = \mathbf{F}_{12}, \quad (4.45)$$

$$M_2 \ddot{\mathbf{R}}_2 = \mathbf{F}_{21} \quad (4.46)$$

where \mathbf{F}_{12} is the force acting on M_1 and \mathbf{F}_{21} the force acting on M_2 , see figure 8a. According to Newton's third law, *To every action there is always opposed an equal reaction*, Newton [1729] p 20, the sum of these forces is zero

$$\mathbf{F}_{12} + \mathbf{F}_{21} = 0, \quad (4.47)$$

$$M_1 \ddot{\mathbf{R}}_1 + M_2 \ddot{\mathbf{R}}_2 = 0. \quad (4.48)$$

This tells us there is no change of the total linear momentum. As with the bag of marbles, our system has a center of mass traveling through space with constant velocity. Let us call this center \mathbf{R} and use equation (4.48) to find an expression for this vector, \mathbf{P} refers to the total linear momentum,

$$\frac{d\mathbf{P}}{dt} = M_1 \ddot{\mathbf{R}}_1 + M_2 \ddot{\mathbf{R}}_2 = 0, \quad (4.49)$$

$$\mathbf{P} = M_1 \dot{\mathbf{R}}_1 + M_2 \dot{\mathbf{R}}_2. \quad (4.50)$$

The vector \mathbf{R} can be defined in order to let the total linear momentum be the product of

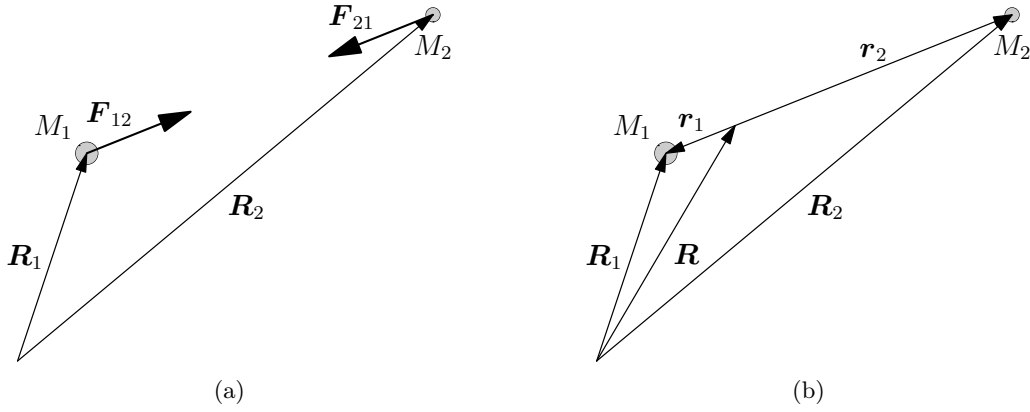


Figure 8

the total mass of the system and $\dot{\mathbf{R}}$

$$(M_1 + M_2)\dot{\mathbf{R}} = M_1\dot{\mathbf{R}}_1 + M_2\dot{\mathbf{R}}_2 \quad (4.51)$$

$$\dot{\mathbf{R}} = \frac{M_1\dot{\mathbf{R}}_1 + M_2\dot{\mathbf{R}}_2}{M_1 + M_2} \quad (4.52)$$

$$\mathbf{R} = \frac{M_1\mathbf{R}_1 + M_2\mathbf{R}_2}{M_1 + M_2} + C_1. \quad (4.53)$$

We choose \mathbf{R} to make C_1 zero.

With the center of mass traveling with constant velocity the location of M_1 and M_2 can be given compared to this center using vectors \mathbf{r}_1 and \mathbf{r}_2 , see figure 8b,

$$\mathbf{r}_1 = \mathbf{R}_1 - \mathbf{R} \quad (4.54)$$

$$\mathbf{r}_2 = \mathbf{R}_2 - \mathbf{R}. \quad (4.55)$$

As implied in figure 8b and by the term *center of mass*, \mathbf{R} is positioned somewhere on the line between M_1 and M_2 with \mathbf{r}_1 and \mathbf{r}_2 pointing in opposite directions. To show this we use (4.53) and the definitions of \mathbf{r}_1 and \mathbf{r}_2

$$(M_1 + M_2)\mathbf{R} = M_1\mathbf{R}_1 + M_2\mathbf{R}_2 \quad (4.56)$$

$$(M_1 + M_2)\mathbf{R} = M_1(\mathbf{R} + \mathbf{r}_1) + M_2(\mathbf{R} + \mathbf{r}_2) \quad (4.57)$$

$$(M_1 + M_2)\mathbf{R} = (M_1 + M_2)\mathbf{R} + M_1\mathbf{r}_1 + M_2\mathbf{r}_2 \quad (4.58)$$

$$0 = M_1\mathbf{r}_1 + M_2\mathbf{r}_2. \quad (4.59)$$

Both M_1 and M_2 are positive which means \mathbf{r}_1 and \mathbf{r}_2 point in opposite direction, consequently \mathbf{R} is positioned on the line between M_1 and M_2 .

4.2.1 The motion around the center of mass is planar

We know that \mathbf{R} is traveling with constant velocity and that it is positioned on the line between M_1 and M_2 . As it turns out the motion of the bodies around \mathbf{R} can be described in a plane as in the case of one moving body. This might seem natural for the solar system, presented to us like a flat disc even though we know it is moving compared to other stars. It is harder to imagine the planar motion in the case of two bodies passing each other, never going into orbit. The reason for the planar motion is the gravitational forces, always attracting the bodies towards one another and towards the center of mass. One could argue that the whole system might spin around the center of mass like a bag of marbles put into spin by an initial force. To show that no such spin could occur we use the concept of angular momentum.

The angular momentum is the cross product between the position vector and the linear momentum. If all motion takes place within a plane that always has the same normal vector, we expect the angular momentum for each body, as well as the total angular momentum, \mathbf{L} , to be constant

$$\mathbf{L} = \mathbf{R}_1 \times M_1 \dot{\mathbf{R}}_1 + \mathbf{R}_2 \times M_2 \dot{\mathbf{R}}_2. \quad (4.60)$$

Looking at the right hand side, we do not know this to be constant, but if it is constant its time derivative should be zero

$$\frac{d\mathbf{L}}{dt} = \dot{\mathbf{R}}_1 \times M_1 \dot{\mathbf{R}}_1 + \mathbf{R}_1 \times M_1 \ddot{\mathbf{R}}_1 + \dot{\mathbf{R}}_2 \times M_2 \dot{\mathbf{R}}_2 + \mathbf{R}_2 \times M_2 \ddot{\mathbf{R}}_2. \quad (4.61)$$

Disregarding cross products between identical vectors and making exchanges according to (4.45) och (4.46) we get

$$\frac{d\mathbf{L}}{dt} = \mathbf{R}_1 \times \mathbf{F}_{12} + \mathbf{R}_2 \times \mathbf{F}_{21}. \quad (4.62)$$

To cancel some of the cross products we write \mathbf{R}_2 as $\mathbf{R}_1 + (\mathbf{R}_2 - \mathbf{R}_1)$

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \mathbf{R}_1 \times \mathbf{F}_{12} + (\mathbf{R}_1 + (\mathbf{R}_2 - \mathbf{R}_1)) \times \mathbf{F}_{21} \\ &= \mathbf{R}_1 \times \mathbf{F}_{12} + \mathbf{R}_1 \times \mathbf{F}_{21} + (\mathbf{R}_2 - \mathbf{R}_1) \times \mathbf{F}_{21}. \end{aligned} \quad (4.63)$$

Since \mathbf{F}_{21} is positioned along the same line as $\mathbf{R}_2 - \mathbf{R}_1$ the crossproduct between them is zero. By exchanging \mathbf{F}_{21} to $-\mathbf{F}_{12}$ according to (4.47) we see that the derivative of \mathbf{L} is zero which makes \mathbf{L} constant:

$$\frac{d\mathbf{L}}{dt} = \mathbf{R}_1 \times \mathbf{F}_{12} + \mathbf{R}_1 \times (-\mathbf{F}_{12}) = 0. \quad (4.64)$$

We still do not know, however, if the angular momentum for each body separately is constant. Instead of viewing \mathbf{L} as composed of the angular momentum for the two bodies it can be divided to express the motion of \mathbf{R} and the motion of \mathbf{r}_1 and \mathbf{r}_2 . To rewrite (4.60) \mathbf{R}_1 and \mathbf{R}_2 are replaced by \mathbf{r}_1 , \mathbf{r}_2 and \mathbf{R} according to the definitions (4.54) and (4.55),

$$\begin{aligned} \mathbf{L} &= (\mathbf{R} + \mathbf{r}_1) \times M_1(\dot{\mathbf{R}} + \dot{\mathbf{r}}_1) + (\mathbf{R} + \mathbf{r}_2) \times M_2(\dot{\mathbf{R}} + \dot{\mathbf{r}}_2) \\ &= \mathbf{R} \times M_1(\dot{\mathbf{R}} + \dot{\mathbf{r}}_1) + \mathbf{r}_1 \times M_1(\dot{\mathbf{R}} + \dot{\mathbf{r}}_1) \\ &\quad + \mathbf{R} \times M_2(\dot{\mathbf{R}} + \dot{\mathbf{r}}_2) + \mathbf{r}_2 \times M_2(\dot{\mathbf{R}} + \dot{\mathbf{r}}_2) \\ &= \mathbf{R} \times M_1 \dot{\mathbf{R}} + \mathbf{R} \times M_1 \dot{\mathbf{r}}_1 + \mathbf{r}_1 \times M_1 \dot{\mathbf{R}} + \mathbf{r}_1 \times M_1 \dot{\mathbf{r}}_1 \\ &\quad + \mathbf{R} \times M_2 \dot{\mathbf{R}} + \mathbf{R} \times M_2 \dot{\mathbf{r}}_2 + \mathbf{r}_2 \times M_2 \dot{\mathbf{R}} + \mathbf{r}_2 \times M_2 \dot{\mathbf{r}}_2. \end{aligned} \quad (4.65)$$

In this expression we find the angular momentum for \mathbf{R} with respect to the origin, \mathbf{L}_R , and the angular momentum of \mathbf{r}_1 and \mathbf{r}_2 with respect of the center of mass, \mathbf{L}_C together with the sums $M_1 \dot{\mathbf{r}}_1 + M_2 \dot{\mathbf{r}}_2$ and $M_1 \mathbf{r}_1 + M_2 \mathbf{r}_2$ which we, according to (4.59), know to be zero

$$\begin{aligned} \mathbf{L} &= \underbrace{\mathbf{R} \times (M_1 \dot{\mathbf{R}} + M_2 \dot{\mathbf{R}})}_{\mathbf{L}_R} + \underbrace{\mathbf{r}_1 \times M_1 \dot{\mathbf{r}}_1 + \mathbf{r}_2 \times M_2 \dot{\mathbf{r}}_2}_{\mathbf{L}_C} + \\ &\quad \mathbf{R} \times \underbrace{(M_1 \dot{\mathbf{r}}_1 + M_2 \dot{\mathbf{r}}_2)}_0 + \underbrace{(M_1 \mathbf{r}_1 + M_2 \mathbf{r}_2)}_0 \times \dot{\mathbf{R}}. \end{aligned} \quad (4.66)$$

This gives

$$\mathbf{L} = \mathbf{L}_R + \mathbf{L}_C \quad (4.67)$$

$$\mathbf{L}_R = \mathbf{R} \times (M_1 \dot{\mathbf{R}} + M_2 \dot{\mathbf{R}}) \quad (4.68)$$

$$\mathbf{L}_C = \mathbf{r}_1 \times M_1 \dot{\mathbf{r}}_1 + \mathbf{r}_2 \times M_2 \dot{\mathbf{r}}_2. \quad (4.69)$$

The sum of the angular momentum is constant, therefore if either \mathbf{L}_R and \mathbf{L}_C is shown to be constant this is true for both terms. First we develop \mathbf{L}_R . Rewriting (4.68):

$$\mathbf{L}_R = \mathbf{R} \times (M_1 + M_2)\dot{\mathbf{R}}. \quad (4.70)$$

If \mathbf{R}_0 is the initial vector to the center of mass, \mathbf{R} can be expressed as $\mathbf{R}_0 + \dot{\mathbf{R}}t$

$$\begin{aligned} \mathbf{L}_R &= (\mathbf{R}_0 + \dot{\mathbf{R}}t) \times (M_1 + M_2)\dot{\mathbf{R}} \\ &= \mathbf{R}_0 \times (M_1 + M_2)\dot{\mathbf{R}}. \end{aligned} \quad (4.71)$$

The cross product of $\dot{\mathbf{R}}$ with itself is zero. The vectors \mathbf{R}_0 and $\dot{\mathbf{R}}$ are constant and consequently their cross product is constant. With \mathbf{L} and \mathbf{L}_R constant, the same is true for \mathbf{L}_C . The angular momentum for the two bodies compared to the center of mass is composed of two crossproducts and because \mathbf{r}_1 and \mathbf{r}_2 points along the same line they are both constant. We show this by expressing \mathbf{r}_1 and \mathbf{r}_2 using $\hat{\mathbf{r}}$, pointing in the direction between the bodies

$$\mathbf{r}_1 = r_1(-\hat{\mathbf{r}}) \Rightarrow \dot{\mathbf{r}}_1 = -r_1 \frac{d\hat{\mathbf{r}}}{dt} - \dot{r}_1 \hat{\mathbf{r}}, \quad (4.72)$$

$$\mathbf{r}_2 = r_2 \hat{\mathbf{r}} \Rightarrow \dot{\mathbf{r}}_2 = r_2 \frac{d\hat{\mathbf{r}}}{dt} + \dot{r}_2 \hat{\mathbf{r}}. \quad (4.73)$$

We use this to rewrite and simplify (4.69)

$$\begin{aligned} \mathbf{L}_C &= -r_1 \hat{\mathbf{r}} \times M_1 \left(-r_1 \frac{d\hat{\mathbf{r}}}{dt} - \dot{r}_1 \hat{\mathbf{r}} \right) + r_2 \hat{\mathbf{r}} \times M_2 \left(r_2 \frac{d\hat{\mathbf{r}}}{dt} + \dot{r}_2 \hat{\mathbf{r}} \right) \\ &= r_1^2 M_1 \left(\hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{dt} \right) + r_1 \dot{r}_1 M_1 \underbrace{(\hat{\mathbf{r}} \times \hat{\mathbf{r}})}_0 + r_2^2 M_2 \left(\hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{dt} \right) + r_2 \dot{r}_2 M_2 \underbrace{(\hat{\mathbf{r}} \times \hat{\mathbf{r}})}_0 \\ &= (r_1^2 M_1 + r_2^2 M_2) \left(\hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{dt} \right). \end{aligned} \quad (4.74)$$

This shows that if \mathbf{L}_C is constant the motion of M_1 and M_2 will always be in the plane spanned by $\hat{\mathbf{r}}$ and its derivative. Consequently the motions of the bodies M_1 and M_2 can be divided into the motion of their center of mass, which is constant, and their motions within a plane.

4.2.2 The method of reduced mass

We now know the velocity of the center of mass is constant and that if it is subtracted from the velocity of the bodies their motion is two dimensional. This simplification of the problem enables us to use the solution to the problem of one moving body found in 4.1. To do this we will express the motion of M_2 relatively M_1 , using \mathbf{r} , and set up an equation using Newton's second law

$$\ddot{\mathbf{R}}_2 - \ddot{\mathbf{R}}_1 = \frac{\mathbf{F}_{21}}{M_2} - \frac{\mathbf{F}_{12}}{M_1} \quad (4.75)$$

$$\ddot{\mathbf{r}} = \left(\frac{1}{M_2} + \frac{1}{M_1} \right) \mathbf{F}_{21} \quad (4.76)$$

$$\frac{1}{\frac{1}{M_2} + \frac{1}{M_1}} \ddot{\mathbf{r}} = \mathbf{F}_{21}. \quad (4.77)$$

This equation has the same structure as the one solved in 4.1 and we know how to solve it for r as a function of φ . We know $\ddot{\mathbf{r}}$ to be the acceleration of M_2 relatively M_1 and \mathbf{F}_{21} to

be the gravitational force acting on M_2 and interpret the ratio containing M_1 and M_2 as a mass denoted μ

$$\mu \ddot{\mathbf{r}} = \mathbf{F}_{21}. \quad (4.78)$$

For any set of M_1 and M_2 the smallest body will be larger than μ , which is why μ is called the reduced mass. This reduced mass can be understood as the mass that will take the path $\mathbf{r}(\varphi)$ if attracted by the force \mathbf{F}_{21} towards the origin of \mathbf{r} . Before reducing the mass we had two masses attracted by two forces making two different paths, by using this method we have reduced this part of the problem to contain only one path.

4.2.3 Finding equations for the locations of the bodies

Assuming we know \mathbf{r} as a function of φ and the motion of the center of mass, we want to find expressions for the locations of the bodies, that is the position vectors \mathbf{R}_1 and \mathbf{R}_2 . To do this we use the division of \mathbf{r} into the vectors \mathbf{r}_1 and \mathbf{r}_2 , both emanating from the center of mass, see figure 8b. To express \mathbf{r}_1 and \mathbf{r}_2 we use (4.59):

$$M_1 \mathbf{r}_1 + M_2 \mathbf{r}_2 = 0. \quad (4.79)$$

Substituting \mathbf{r}_2 for $\mathbf{r} + \mathbf{r}_1$

$$M_1 \mathbf{r}_1 + M_2 (\mathbf{r} + \mathbf{r}_1) = 0 \quad (4.80)$$

$$(M_1 + M_2) \mathbf{r}_1 = -M_2 \mathbf{r} \quad (4.81)$$

$$\mathbf{r}_1 = -\frac{M_2}{M_1 + M_2} \mathbf{r}. \quad (4.82)$$

Similarly an expression can be found for \mathbf{r}_2

$$\mathbf{r}_2 = \frac{M_1}{M_1 + M_2} \mathbf{r}. \quad (4.83)$$

Now we know the motions of the two bodies to be

$$\mathbf{R}_1 = \mathbf{R}(0) + \dot{\mathbf{R}}t + \mathbf{r}_1 = \mathbf{R}(0) + \dot{\mathbf{R}}t - \frac{M_2}{M_1 + M_2} r(\varphi(t)) \hat{\mathbf{r}}(\varphi(t)) \quad (4.84)$$

$$\mathbf{R}_2 = \mathbf{R}(0) + \dot{\mathbf{R}}t + \mathbf{r}_2 = \mathbf{R}(0) + \dot{\mathbf{R}}t + \frac{M_1}{M_1 + M_2} r(\varphi(t)) \hat{\mathbf{r}}(\varphi(t)). \quad (4.85)$$

where $r(\varphi(t))$ can be chosen to be (4.33) or in the case of an orbit (4.43). Having no explicit function $r(t)$ the values of \mathbf{R}_1 and \mathbf{R}_2 are to be found numerically. To see how the problem with two moving bodies is solved given a certain set of initial data, see appendix C.

5 Newton's solution

Newton's masterpiece the *Principia* revolutionized the scientific world. It was first published in 1687, more than twenty years after Newton began his research in celestial mechanics. It was most likely the result of a visit by the astronomer and mathematician Edmond Halley in 1684. Halley told Newton about a conversation he had participated in at the Royal Society in London, where Robert Hooke had claimed to be able to demonstrate all the laws of celestial motion by assuming a power inversely as the square of the distance between the celestial bodies. A prize had been offered but no one had presented a solution. Halley asked Newton whether he knew what sort of orbit an inverse-square law would produce, Densmore [2003]. To Halley's surprise Newton gave him a rapid answer, declaring the path to be elliptic and that he already had the proof worked out. Unable to find his calculations he promised to send a proof to Halley, a promise he kept later that year, Chandrasekhar [1995]. Newton was a hard working and very productive scientist but had been unwilling to share most of his discoveries. Halley was eager to have the document published but Newton wanted to expand it and spent three years writing *Principia*, Densmore [2003].

Principia is divided into three volumes and contains laws and solutions to numerous problems, the two-body problem is found in Proposition I, VI, XI and LVII to LXIII. In the first three propositions Newton treats systems with one stationary body, he proves Kepler's second law and use Kepler's first law of elliptic orbits to prove the inverse-square law. The last seven propositions deal with two moving bodies.

5.1 One moving body

To decide under what law a heavenly body finds an elliptic orbit Newton pictured a small planet orbiting the sun. He lets the sun stand still, which is close to the truth if the planet is small.

5.1.1 Proposition I

Newton starts with proving Kepler's second law: a radius vector from the Sun to a planet sweeps out equal areas in equal times. To do this he uses a discrete model, see figure 9. According to Newton's first law, a planet that starts in A and travels to B during one unit of time, would continue to c if not affected by the gravitational force of the sun, S . According to Newton's second law, the gravitational force of the sun will alter the path of the planet in the direction of the Sun. In Newton's discrete model the force is applied momentarily at position B , and in the next unit of time the planet will travel to C . In the same manner it will continue to D and E . In figure 9, SB and Cc are parallel. The distance Cc depends on the change of velocity, that is the acceleration, and thus proportional to the force causing the change. To prove Kepler's second law the two triangles SAB and SBC need to have the same area. According to simple geometry the triangles SAB and SBC have the same area since the distance AB is the same as Bc and if these are the bases in each triangle, both have the same height to S . In the same manner SBC and SBC have the same area, if AB is the base the triangles share the same height since Cc is parallel with SB . Newton completes his proof by arguing that if the number of those triangles is augmented and their breadth diminished *in infinitum*, their perimeter will be a curved line and therefore the centripetal force will act continually.

5.1.2 Proposition VI

Newton here shows how the gravitational force is proportional to the ratio of the change in position due to the gravitational force and the square of the area swept out by the radius. When Newton reaches Proposition VI he has changed type of the figure, see figure 10. Now

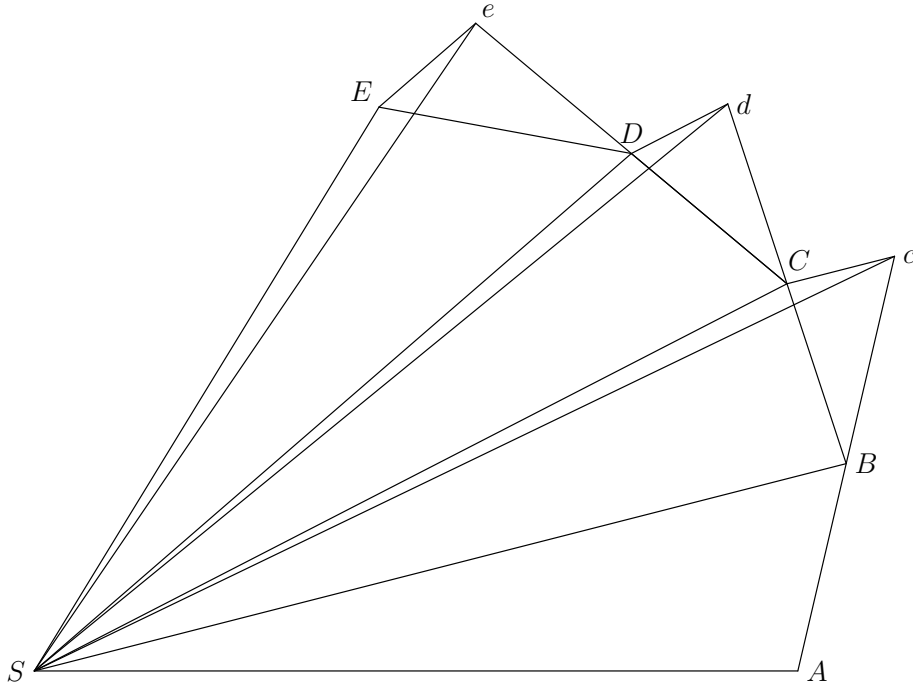


Figure 9: Modified version of Newton's illustration to Prop. I. The planet starts at A and travels counter clockwise around the sun, S .

the orbit is smooth and the planet moves from P to Q where the distance between these positions is thought to diminish until they *coincide*. The central force, C.F., is proportional to the acceleration and the acceleration is, according to Galileo, the distance divided by the square of time

$$C.F. \propto \text{acceleration} = \frac{\text{distance}}{(\text{time})^2}. \quad (5.1)$$

According to Kepler's second law, time is proportional to the area swept out by the radius. When Q is close to P this area could be approximated with the triangle SPQ . We get

$$\frac{\text{distance}}{(\text{time})^2} \propto \frac{QR}{(\text{area of } SPQ)^2} = \frac{QR}{(\frac{1}{2}SP \cdot QT)^2} \quad (5.2)$$

and can write

$$C.F. \propto \frac{QR}{SP^2 QT^2}. \quad (5.3)$$

To show the force to be proportional to the inverse square of the distance, Newton needs to show the ratio between QR and QT^2 to be a constant.

5.1.3 Proposition XI

In Proposition XI Newton proves the inverse-square law. As mentioned earlier there is disagreement over what methods Newton initially used to found solutions to his problems. Chandrasekhar writes in his thorough exposition of *Principia* that he believes "Newtons geometrical insights were so penetrating that the proofs emerged whole in his mind" and "the geometrical construction, that must have left its readers in helpless wonder, came quite naturally", Chandrasekhar [1995] p. 273.

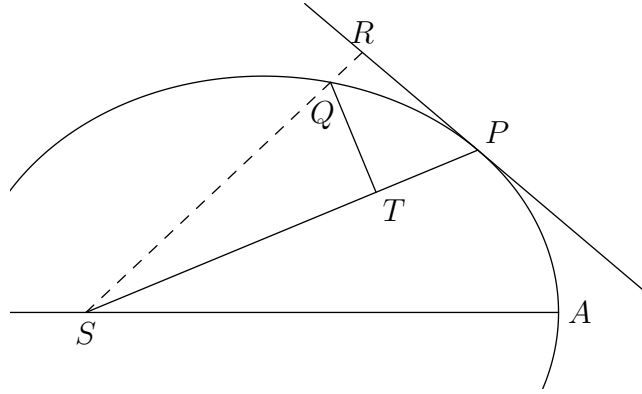


Figure 10: Modified version of Newton's illustration to Prop. VI

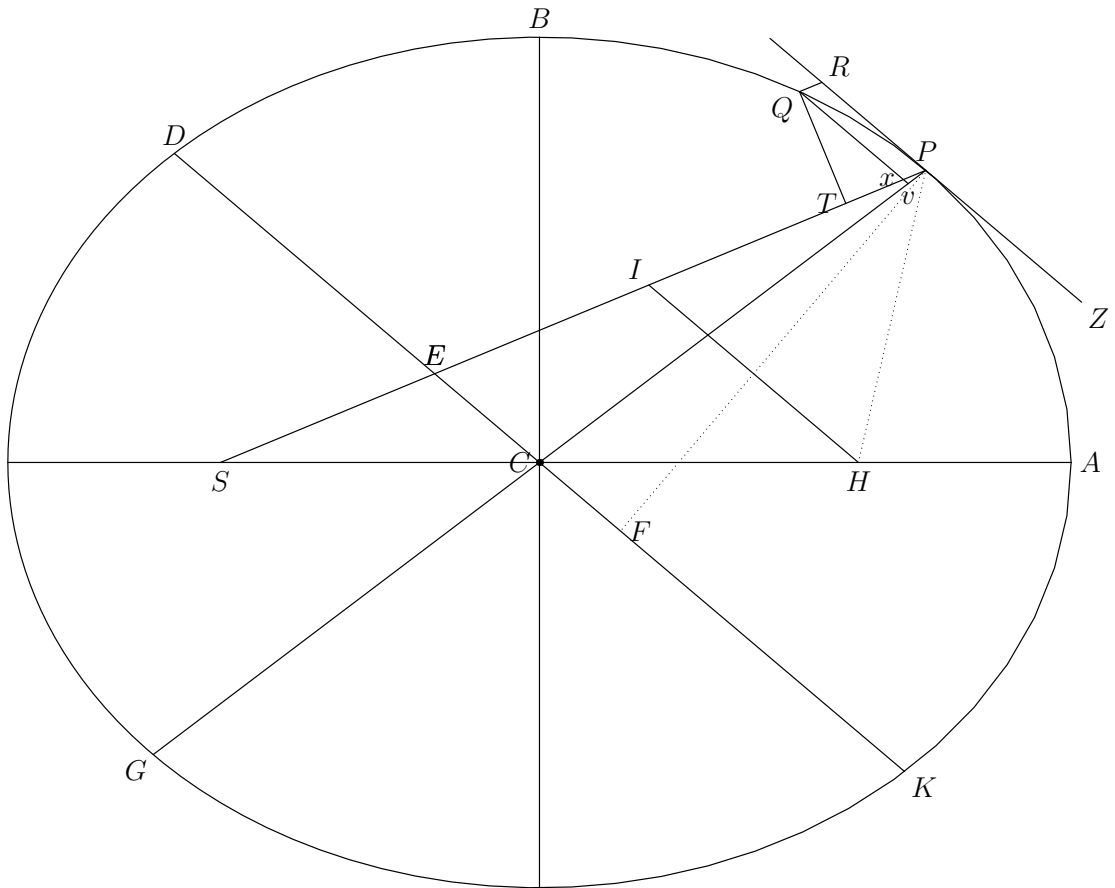


Figure 11: Newton's illustration to Prop. XI. The illustrations to the propositions alter between the editions. This illustration is found in the third edition of *Principia*, Chandrasekhar [1995].

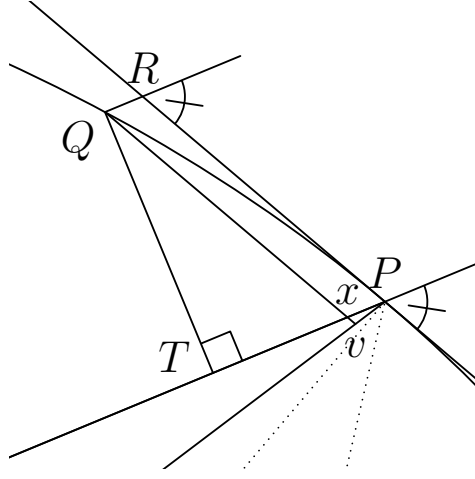


Figure 12: Enlargement of figure 11

To make his proof Newton uses geometry not commonly known today but probably more so to his contemporary readers. Here follows the proposition in Mottes translation from 1729 as a taste of the works of Newton. The text is accompanied by a figure and an enlargement, see figures 11 and 12. (Note that Newton writes GvP when we today would write $Gv \cdot Pv$, Densmore [2003]).

Proposition XI.

If a body revolves in an ellipsis: it is required to find the law of the centripetal force tending to the focus of the ellipsis.

Let S be the focus of the ellipsis. Draw SP cutting the diameter DK of the ellipsis in E , and the ordinate Qv in x ; and complete the parallelogram $QxPR$. It is evident that EP is equal to the greater semi-axis AC : for drawing HI from the other focus H of the ellipsis parallel to EC , because CS , CH are equal ES , EI will be also equal, so that EP is the half sum of PS , PI , that is, (because of the parallels HI , PR , and the equal angles IPR , HPZ) of PS , PH , which taken together are equal to the whole axis $2AC$. Draw QT perpendicular to SP , and putting L for the principal latus rectum of the ellipsis (or for $\frac{2BC^2}{AC}$) we shall have $L \times QR$ to $L \times Pv$ as QR to Pv , that is, as PE or AC to PC ; and $L \times Pv$ to GvP as L to Gv ; and GvP to Qv^2 as PC^2 to CD^2 ; and (by corol.2 lem.7) the points Q and P coinciding, Qv^2 is to Qx^2 in the ration of equality; and Qx^2 or Qv^2 is to QT^2 as EP^2 to PF^2 , that is, as CA^2 to PF^2 or (by lem. 12) as CD^2 to CB^2 . And compounding all those ratios together, we shall have $L \times QR$ to QT^2 as $AC \times L \times PC^2 \times CD^2$ or $2BC^2 \times PC^2 \times CD^2$ to $PC \times Gv \times CD^2 \times CB^2$, or as $2PC$ to Gv . But the points Q and P coinciding, $2PC$ and Gv are equal. And therefore the quantities $L \times QR$ and QT^2 , proportional to these, will be also equal. Let those equals be drawn into $\frac{SP^2}{QR}$, and $L \times SP^2$ will become equal to $\frac{SP^2 \times QT^2}{QR}$. And therefore (by corol. 1 and 5 prop 6) the centripetal force is reciprocally as $L \times SP^2$, that is, reciprocally in the duplicate ration of the distance SP . Q.E.I. Newton [1729] p 79, 80

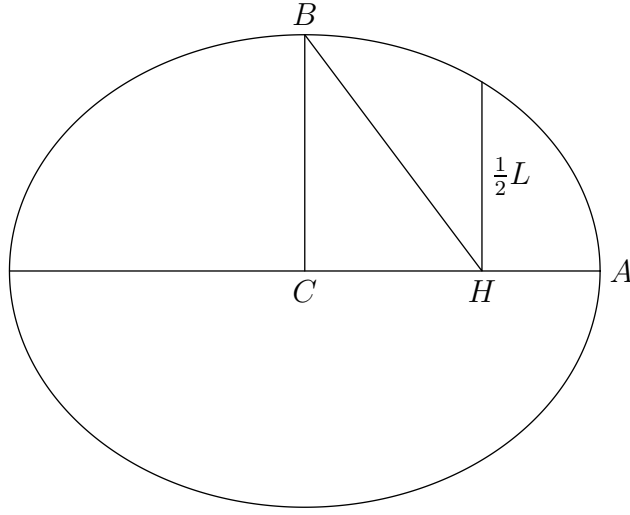


Figure 13: Modified version of figure 11 with half *latus rectum* drawn to the focus H .

We will now go through the proposition.

The first property of the ellipse Newton addresses is

$$EP = AC \quad (5.4)$$

for which he gives a thorough explanation. The second is

$$L = \frac{2BC^2}{AC} \quad (5.5)$$

and is not further explained. To prove it we can use the equation of the ellipse in cartesian coordinates, (3.3). The distance L is the *principal latus rectum*, the line perpendicular to the great axis going through one of the foci. See figure 13 where half this distance is marked.

If we write the equation using the coordinates of the point where L intersect the ellipse we get

$$\frac{CH^2}{AC^2} + \frac{(\frac{1}{2}L)^2}{BC^2} = 1. \quad (5.6)$$

According to (3.5) the distance from B to one focus is the same as half the great axis, or $BH = AC$. BH is the hypotenuse in right-angled triangle BCH which gives us the relation

$$CH^2 + BC^2 = BH^2 = AC^2. \quad (5.7)$$

If CH in (5.6) is substituted using (5.7) we get

$$\frac{AC^2 - BC^2}{AC^2} + \frac{(\frac{1}{2}L)^2}{BC^2} = 1 \quad (5.8)$$

and the property follows if we solve for L .

Newton continues with four equations.

1.

$$\frac{L \cdot QR}{L \cdot Pv} = \frac{QR}{Pv} = \frac{PE}{PC} = \frac{AC}{PC} \quad (5.9)$$

In this equation Newton uses the fact that $PxQR$ is a parallelogram, which makes $QR = Px$, and the similar triangles PCE and Pvx . We can write

$$\frac{QR}{Pv} = \frac{Px}{Pv} = \frac{PE}{PC} \quad (5.10)$$

and as shown EP is equal to AC .

2.

$$\frac{L \cdot Pv}{Gv \cdot Pv} = \frac{L}{Gv} \quad (5.11)$$

3.

$$\frac{Gv \cdot Pv}{Qv^2} = \frac{PC^2}{CD^2} \quad (5.12)$$

A modern way of treating this equation is to use CP as an x -axis and CD as a y -axis in a non orthogonal system, Hollingdale [1989] p 213. If the point Q is (x, y) the distances Gv , Pv and Qv can be expressed using x and y .

$$\frac{Gv \cdot Pv}{Qv^2} = \frac{(PC + x)(PC - x)}{y^2} = \frac{PC^2 - x^2}{y^2}. \quad (5.13)$$

By using the equation of the ellipse, an expression for y^2 can be found

$$\frac{x^2}{PC^2} + \frac{y^2}{CD^2} = 1 \quad \Rightarrow \quad y^2 = \frac{CD^2(PC^2 - x^2)}{PC^2}. \quad (5.14)$$

By substituting for y^2 in (5.13) using (5.14) and by simplify the expression, the right hand side of Newtons third equation follows.

4.

$$\frac{Qv^2}{QT^2} = \frac{Qx^2}{QT^2} = \frac{EP^2}{PF^2} = \frac{CA^2}{PF^2} = \frac{CD^2}{CB^2} \quad (5.15)$$

When posing his forth equation Newton argues that when Q coincides with P , Qv will coincides with Qx . To take the second step of the equation Newton uses the similarity between the triangles QxT and PEF and exchanges EP to CA according to (5.4)⁵. Newton then refers to Lemma 12 where he states that all bounding parallelograms inscribed in an ellipse has the same area, given that its diagonals are two conjugate diameters of the ellipse. Newton never presents a proof but refers to "the writers on the conic sections". According to the lemma we know the area of the parallelogram having the great and minor axis as its conjugate diameters is $2CA \cdot CB$ and the area of the parallelogram $PDGK$ is $2CD \cdot PF$. We can write

$$2CA \cdot CB = 2CD \cdot PF \quad \Rightarrow \quad \frac{CA}{PF} = \frac{CD}{CB} \quad (5.16)$$

which enables the last step in the equation.

After posing these four equations; (5.9), (5.11), (5.12) and (5.15), Newton uses the pattern made of the ratios on the left hand sides; the denominator in one equation turns up in the numerator in the next. The product of the left hand sides is

$$\frac{L \cdot QR \cdot L \cdot Pv \cdot Gv \cdot Pv \cdot Qv^2}{L \cdot Pv \cdot Gv \cdot Pv \cdot Qv^2 \cdot QT^2} = \frac{L \cdot QR}{QT^2}. \quad (5.17)$$

⁵Newton uses both the notations EP and PE .

When simplified the left hand side includes the ratio QR and QT^2 , sought to do so by Newton since the same ratio is found in Proposition VI, (5.3). The product of the right hand sides of the four equations is

$$\frac{AC \cdot L \cdot PC^2 \cdot CD^2}{PC \cdot Gv \cdot CD^2 \cdot CB^2} = \frac{2PC}{Gv}. \quad (5.18)$$

here simplified using (5.5). Together this gives

$$\frac{L \cdot QR}{QT^2} = \frac{2PC}{Gv}. \quad (5.19)$$

Continuing his reasoning on limits Newton argues that when the points P and Q coincide, $2PC$ is equal to Gv and since L is a constant, so is the ratio between QR and QT^2 . Together with Proposition VI and (5.3) Newton now has shown

$$C.F. \propto \frac{1}{SP^2} \quad (5.20)$$

where SP is the distance between the planet and the sun. This is the sought relation between the distance and the force; the force is proportional to the inverse of the square of the distance. Hence, Newton has proved that the assumption of Kepler's second law of elliptic planetary orbits results in the inverse-square law.

5.2 Two moving bodies

The solution to the two-body problem containing two bodies in motion is found in Proposition LVII to LXIII in *Principia*. When proving the inverse-square law Newton used attractions of bodies toward an immovable centre even though he thought no such thing existent in nature. Newton writes

For attractions are made towards bodies; and the actions of the bodies attracted and attracting, are always reciprocal and equal by law 3. so that if there are two bodies, neither the attracted nor the attracting body is truly at rest, but both (by cor. 4. of the laws of motion) being as it were usually attracted, revolve about a common centre of gravity. Newton [1729] p. 218

Newton refers in the quotation to law 3, which we today call Newton's third law. The law says that to each force there is a counter force, equal in size but opposite in direction. He also refers to a corollary in which he writes that if there are no outside forces a common center of gravity does not alter its state of motion or rest by the actions of the bodies among themselves.

Newton assumes this center of gravity to be positioned on the line between the bodies due to the masses M_1 and M_2 , as described in 4.2. As a consequence, if M_2 describes an elliptic orbit around M_1 with M_1 in one of the foci, the same is true for the orbit M_2 makes around the center of gravity. The same is true for all other kinds of paths, which Newton proves in *Principia* to be either parabolas or hyperbolas.

Newton uses the elliptic orbits around the center of mass to treat this center as a point mass to which the bodies are attracted. In 4.2 we used the concept of reduced mass, holding one body still and letting the other body orbit around it. Newton adds a stationary supposed mass in the center, keeps the orbits for both bodies and treats each body separately. If the supposed mass is known, the paths of the bodies can be calculated one at the time. Newton does not give mathematical expressions showing how this is done but provides a lengthy explanation in words. To exemplify this, Proposition LXI is included in appendix E.

If we want to know what path M_1 takes around the center of gravity, instead of around M_2 , we can calculate the supposed mass M'_2 . According to Newton's second law the force is equal to the change of the linear momentum, or, if the mass is constant, more commonly phrased, the force is equal to the mass times the acceleration. Hence the force is proportional to the mass. As Newton showed the force is also proportional to the square of the distance. Using the notations from 4.1 we can write

$$|\mathbf{F}_{12}| \propto \frac{M_2}{|\mathbf{r}|^2}. \quad (5.21)$$

If we want the force to stay the same and the distance to be changed (from \mathbf{r} to \mathbf{r}_1), the mass needs to be adjusted for the ratio to be held constant

$$\frac{M_2}{|\mathbf{r}|^2} = \frac{M'_2}{|\mathbf{r}_1|^2}. \quad (5.22)$$

If we use the relationship between \mathbf{r} and \mathbf{r}_1 according to (4.82) we can solve for M'_2

$$\mathbf{r}_1 = -\frac{M_2}{M_1 + M_2}\mathbf{r} \quad \Rightarrow \quad M'_2 = M_2 \left(\frac{M_2}{M_1 + M_2} \right)^2. \quad (5.23)$$

Using the concept of placing a new mass in the center of mass of the system Newton has in this way provided a model for calculating the orbits of planetary objects⁶.

⁶In Proposition XVII Newton shows how to decide which conic section a body will describe given a certain initial velocity.

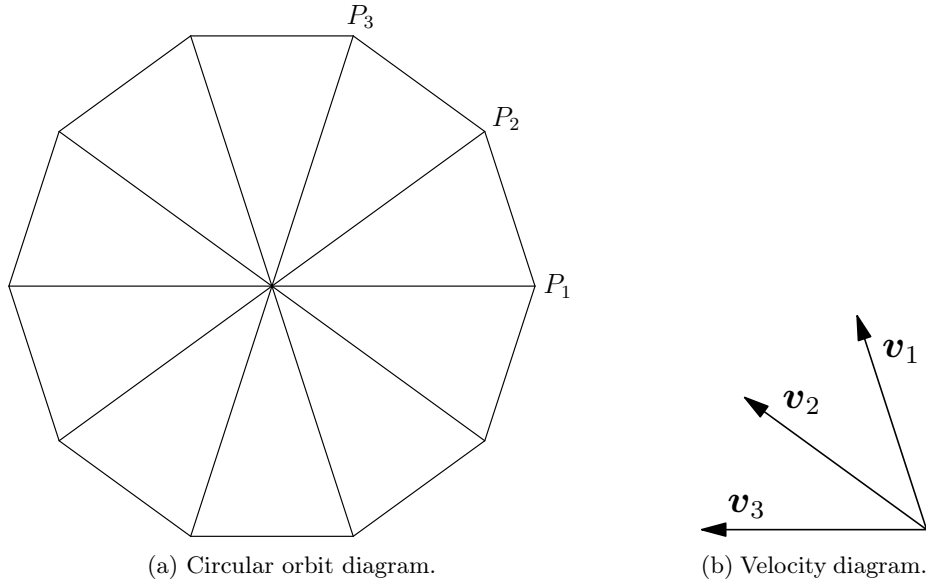


Figure 14: The object moves from P_1 to P_2 with the velocity \mathbf{v}_1 , from P_2 to P_3 with the velocity \mathbf{v}_2 and so on. The size of the velocity diagram is not related to the size of the orbit diagram.

6 Feynman's solution

Richard Feynman (1918-1988) was an american theoretical physicist, awarded with the nobel price in physics for the invention of quantum electrodynamics and also known for the textbook *The Feynman Lectures on Physics* based on lectures held by Feynman to undergraduate students at Caltech. One of the lectures that never made it to the book is about planetary motion around the sun and corresponds to our first part of the problem where one of the bodies is held still. After Feynman's death the lecture has been recapitulated and published, Goodstein and Goodstein [1997].

In his lecture Feynman wanted to give a geometric proof for the first part of the two-body problem and went to Newton and *Principia*. He essentially follows Proposition I, proving Kepler's second law. According to Goodstien and Goodstein, Feynman initially had in mind to present the proof of Newton but after Proposition I "finds himself unable to follow Newton's line of argument any further, and so sets out to invent one of his own" p. 111. In his deviation from Newton's proof Feynman succeeds in proving both the inverse-square law and elliptical orbits, he does this by using Kepler's third law. The law states that the time it takes for a planet to complete an orbit is proportional to $\sqrt{a^3}$ where a , as before, is half the great axis of the ellipse.

Similar to Newton, Feynman does not present his proof with the rigor expected of a modern proof.

6.1 One moving body

In Proposition I Newton uses a discrete model where the time dependent steps are thought to be infinitely small. Feynman uses the same model but when proving the inverse-square law he makes a temporary assumption of a circular orbit, see the position diagram in figure 14a. When traveling counter clockwise from point P_1 to P_2 the planet has the velocity \mathbf{v}_1 , when traveling from point P_2 to P_3 it has the velocity \mathbf{v}_2 and so on as shown in the velocity diagram, see figure 14b. The directions of the velocity vectors comes from the circular orbit diagram but in the velocity diagram the vectors have the same origin. (Note that the size of one diagrams does not affect the size of the other).

As the planet orbits around the sun, it is affected by a gravitational force proportional

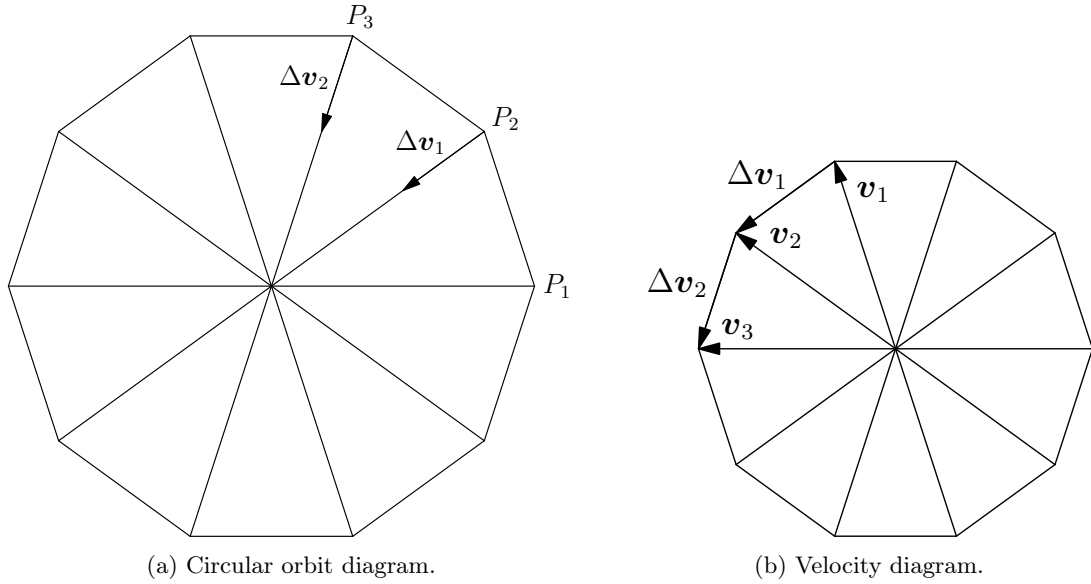


Figure 15: As a result of the gravitational force acting towards the center of the orbit the velocity of the orbit is changed. In Feynman's discrete model these changes occur at the points P_i . The orbit is circular, the gravitational attraction is constant and the vectors Δv_i are equal in size.

to the change in velocity, that is the difference between \mathbf{v}_1 and \mathbf{v}_2 , see figure 15. Assuming the gravitational force is not dependent on the angle to the sun, the magnitude of the velocity is constant due to the circular orbit and accordingly all the vectors \mathbf{v}_i are equal in size. If φ is the angle counting from P_1 the angles $\Delta\varphi$, at the center of both diagrams, are of the same size, $\Delta\varphi$ is also the angle observed between any two following vectors Δv_i and Δv_{i+1} . With a circular orbit like this one, the velocity diagram will be circular.

Assuming the relationship between the force and the distance not to be affected by the shape of the orbit Feynman uses the circular orbit to prove the inverse-square law. If T is the time it takes for the planet to make one orbit and r_c the radius in the circular position diagram, the magnitude of the velocity is

$$|\mathbf{v}| = \frac{2\pi r_c}{T}. \quad (6.1)$$

The magnitude of the velocity is the radius of the velocity diagram, figure 15b, and the change of velocity, summed up for one orbit, is the same as the perimeter of this diagram: $2\pi|\mathbf{v}|$. If we divide by the time it takes for the planet to make one orbit we get the acceleration, which in this case is constant in magnitude. Using (6.1) the magnitude of the acceleration can be written as

$$\frac{2\pi|\mathbf{v}|}{T} = \frac{4\pi^2 r_c}{T^2}. \quad (6.2)$$

The magnitude of the force is proportional to the magnitude of the acceleration

$$|\mathbf{F}| \propto \frac{4\pi^2 r_c}{T^2} \propto \frac{r_c}{T^2}. \quad (6.3)$$

Making use of Kepler's third law, stating that T^2 is proportional to r_c^3 , Feynman rewrites (6.3)

$$|\mathbf{F}| \propto \frac{1}{r_c^2} \quad \Rightarrow \quad |\mathbf{F}| \propto \frac{1}{r^2}. \quad (6.4)$$

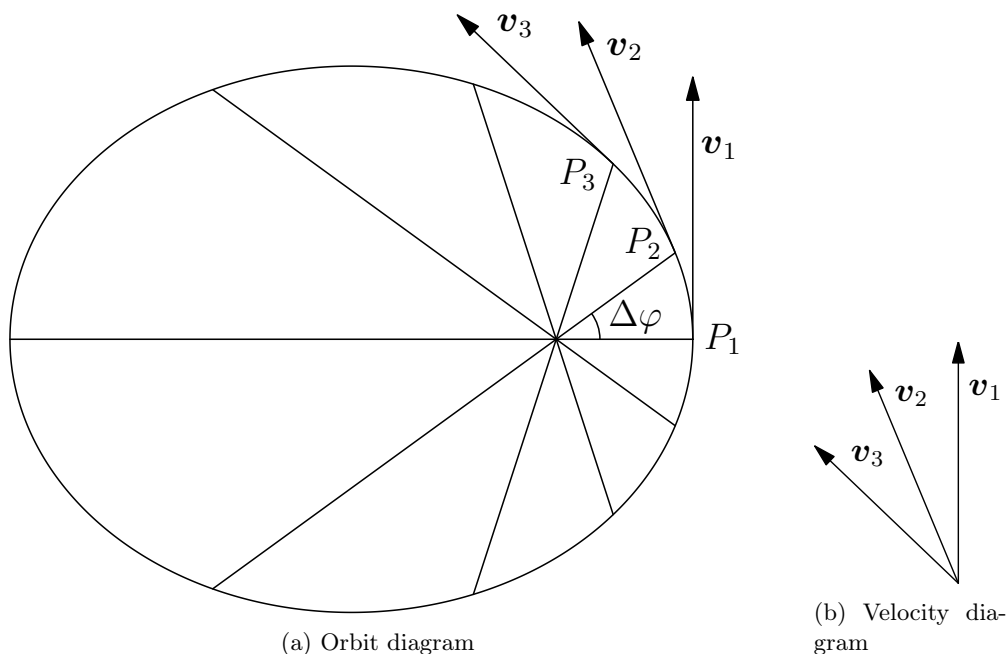


Figure 16: Diagrams of a planet in orbit and its velocity vectors.

Feynman now intends to show that the inverse-square law produces elliptical orbits and to do so he leaves the temporary assumption of a circular orbit. Before moving on to the diagrams showing a non circular orbit let us look back at figure 15. In the circular orbit diagram the planet moves from P_1 to P_2 and so on, each step taking the same amount of time. Another way of dividing the planet's orbit would be by the angle it travels compared to the sun in the center, $\Delta\varphi$. Since the orbit is circular this would not alter the drawing of the diagrams. Observe how this angle decides the direction of any vector $\Delta\mathbf{v}_i$ which is directed towards the sun. The shape of the orbit decides the origin of each such vector but for a given $\Delta\varphi$ the angles between two following vectors will be the same. (In figure 15 $\Delta\varphi$ is 36° and thus the angle between any vectors $\Delta\mathbf{v}_i$ and $\Delta\mathbf{v}_{i+1}$ is also 36°).

Now let us look at the orbit diagram in figure 16a. Here the step length depends not on time but on the angle $\Delta\varphi$ to the sun. If the steps become infinitely small the orbit will have an oval shape as the one in the diagram.

Velocity vectors can be drawn for every step in the orbit and a velocity diagram be made, see figures 16b and 17. As indicated in figure 17 the diagram takes on the outer shape of a circle. This velocity diagram is essential to Feynmans proof and we will show why it is circular.

If we look at figure 16a we know, from Kepler's second law, the velocity at point P_2 to be lower than the velocity at P_1 and even lower at P_3 . The change of magnitude of the velocity makes the time it takes to move the angle $\Delta\varphi$ change as well. According to Kepler's second law this time is proportional to the area swept out by a line between the sun and the planet. In figure 16a the lines from the sun to each point P_i divide the area within the orbit in pieces. When $\Delta\varphi$ approaches zero the number of pieces will increase and the pieces will look more and more like triangles. For small values of $\Delta\varphi$ the area of such a triangle can be approximated to be

$$\frac{r \cdot r\Delta\varphi}{2}, \quad (6.5)$$

where r is the distance between the sun and the planet and $r\Delta\varphi$ is an approximation of the other side of the triangle. Using the second law of Kepler we can write

$$\Delta t \propto \frac{r^2\Delta\varphi}{2} \propto r^2, \quad (6.6)$$

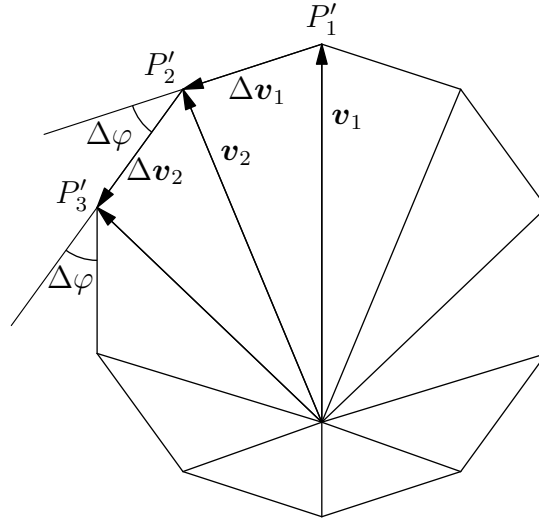


Figure 17: The velocity diagram will approach the shape of a circle when $\Delta\varphi$ approaches zero.

that is, the time it takes for the planet to travel between points P_i and P_{i+1} is proportional to the square of the distance between the sun and the planet. Like Newton, Feynman assumes the change of velocity to be proportional to the force:

$$\frac{|\Delta\mathbf{v}|}{\Delta t} \propto |\mathbf{F}| \quad \Leftrightarrow \quad |\Delta\mathbf{v}| \propto |\mathbf{F}| \Delta t. \quad (6.7)$$

The force is also proportional to the inverse of the square of the distance, see 6.4:

$$|\mathbf{F}| \propto \frac{1}{r^2}. \quad (6.8)$$

Combining these proportionals Feynman writes

$$|\Delta\mathbf{v}| \propto \frac{1}{r^2} r^2 = 1 \quad (6.9)$$

and concludes that the change in velocity, $|\Delta\mathbf{v}|$, is constant, given that $\Delta\varphi$ is constant. If we look at the velocity diagram in figure 17 this means that the vectors $\Delta\mathbf{v}_i$ all have the same length. From the choice of deciding the step length by using the angle $\Delta\varphi$, the angle between any two following vectors $\Delta\mathbf{v}_i$ and $\Delta\mathbf{v}_{i+1}$ will also be $\Delta\varphi$ ⁷. The velocity diagram has now been shown to be a circle.

Feynman's goal is now to show the orbit to be an ellipse. In his geometric proof he uses the properties of the ellipse presented in 3; the distance from a point on the ellipse to one focus added to the distance from the same point to the other focus is constant, and a tangent, at any point on the ellipse, gives the same angle to the lines drawn to any of the two foci. He builds the proof on a relation he discovers between the orbit and the velocity diagrams in figures 16a and 17. When drawing the diagrams, the information to draw the velocity diagram is thought to come from the orbit diagram. Feynman realized he could do the procedure backwards, using the velocity diagram to draw the orbit, and while doing so prove the orbit to be an ellipse. We will go through the procedure step by step.

In order to draw the orbit we will use the points P_1 , P_2 and so on, showing the positions of the planet in the orbit. We will try to find these points in the velocity diagram in figure

⁷While this might seem plausible it is not proven, nor by Goodstein and Goodstein or in the article by Hall and Higson [1998] explaining Feynman's lecture.

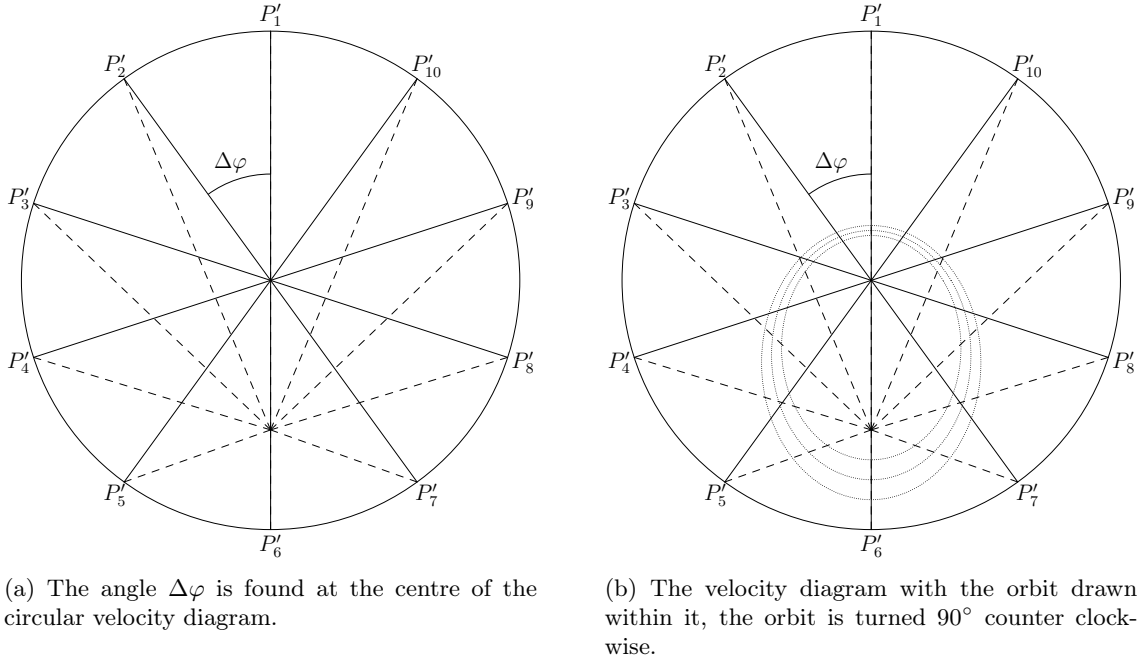


Figure 18

17. If $\Delta\varphi$ is small enough, the points will give us the orbit. We know from figure 16a that even though the distances between the sun and a point on the orbit varies, the angle between two following points, compared to the sun, is constant. We can find these angles in the velocity diagram.

As shown above, the angles between the vectors $\Delta\mathbf{v}_i$ and $\Delta\mathbf{v}_{i+1}$ in figure 17 are always $\Delta\varphi$, as a consequence this angle can be found in the center of the circle, see figure 18a. Since the circumference of the velocity diagram is evenly divided by points P'_i , radii drawn from each such point will produce angles at the center, all of equal size. The number of points P_i on the orbit is the same as the number of points P'_i in the velocity diagram, which makes the angles between the radii equal to $\Delta\varphi$.

If we look at the orbit diagram 16a, the angles $\Delta\varphi$ is found around the sun, in the velocity diagram 18a the same angles is found around the center.

Feynman realized that the diagrams can be made to fit together if the sun in the orbit is placed at the center of the circular velocity diagram. For the purpose of the proof one diagram has to be turned 90° compared to the other. In figure 18b the orbit is turned 90° counter clockwise, the sun is placed in the center of the circle. As shown in the figure this alone does not decide the size of the orbit. Feynman found a way to draw the orbit in order to prove it to be an ellipse.

To draw the orbit Feynman use the velocity vectors connected to the orbit. In order to tell the velocity vectors apart, the ones connected to the orbit are denoted $\mathbf{v}'_1, \mathbf{v}'_2$ and so on. As a result of turning the orbit 90° all these velocity vectors are turned as well. Accordingly, vector \mathbf{v}'_1 will be perpendicular to \mathbf{v}_1 , see figure 19a, as with every pair of \mathbf{v}'_i and \mathbf{v}_i . (For the purpose of the proof, the lengths of the vectors \mathbf{v}'_i are not important).

If we want to draw the oval orbit within the circular diagram the starting point for \mathbf{v}'_1 lays somewhere on the line from the origin to P'_1 , see figure 19a. The starting point for \mathbf{v}'_2 lays somewhere along the radius drawn to P'_2 since the radius creates the angle $\Delta\varphi$ with \mathbf{v}_1 , see figure 19b and compare with the orbit in figure 16a. The starting points of the velocity vectors will all be points on the orbit.

In section 3, a method was shown of how to draw an ellipse within a circle, see figure 2. With this method in mind, the starting point of \mathbf{v}'_2 is found by intersecting a perpendicular bisector of \mathbf{v}_2 to the radius connected with P'_2 . From the way the point is found we know

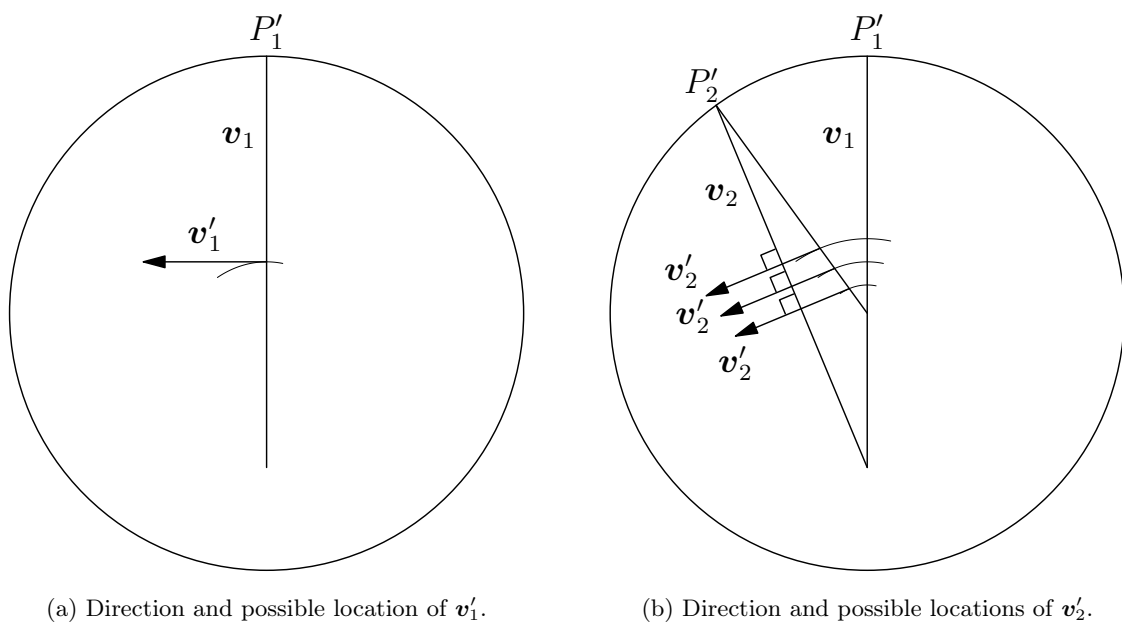


Figure 19

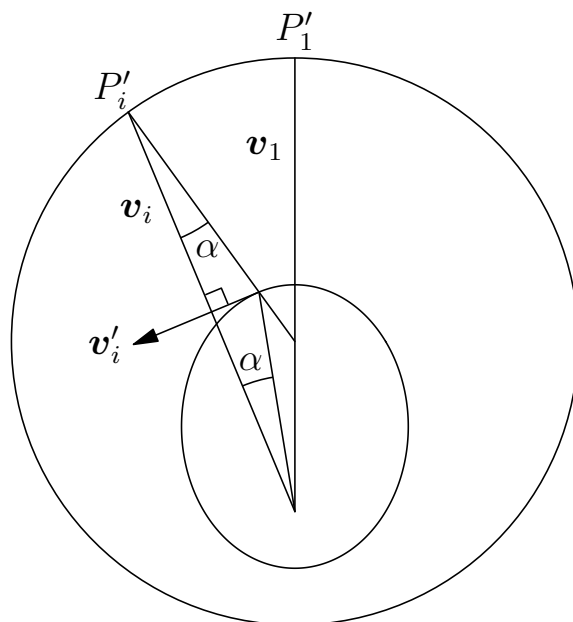


Figure 20: To a vector v_i in the velocity diagram a vector v'_i is found. The starting point for v'_i is a point on an ellipse.

this to be a point on an ellipse. The same thing can be done for every vector \mathbf{v}'_i see figure 20.

This method allows us to find any point P_i on the orbit. Since every such point is a point on an ellipse the orbit has to be an ellipse.

For more reading on Fenman’s lecture see Hall and Higson [1998].

7 Discussion

Throughout history mankind has marveled at the sky searching for connections between life on earth and the motions of heavenly bodies. The desire to explain and predict the movements of stars and planets has given fuel to the progress of science, a development that took a giant leap when Newton provided the general theory of gravity. The theory could explain the motions of objects on earth as well as in the sky and it provided a connection between us and the rest of the universe.

Considering the legacy of Newton, passed on to mathematicians and physicists, it is notable that in his most famous work *Principia*, where the general law of gravity and the solution to the two-body problem are presented, Newton used math that is not widely studied today. One might ask how today's low emphasis on classical geometry and conic sections affects our understanding of classical mechanics and calculus.

If we could ask Feynman he might answer that by neglecting the study of geometry we lose part of the beauty of math. In his tape recorded lecture, referred to above, Feynman says:

"It is not easy to use the geometrical method to discover things. It is very difficult, but the elegance of the demonstrations after the discoveries are made is really very great. The power of the analytic method is that it is much easier to discover things than to prove things. But not in any degree of elegance. It's a lot of dirty paper, with x 's and y 's and crossed out, cancellations and so on" Goodstein and Goodstein [1997] p. 164.

Another difference between our modern way of expressing mathematics and Newton's *Principia* is the mathematical text. Newtons lengthy reasoning stands in contrast to modern, more efficient, mathematical language. History shows how the use of many words has been abandoned in favor of short well defined concepts. Where Newton used ideas, such as two points *coinside* or distances diminishing *in infinitum*, we can now use the concept of limit. Clearly the mathematical language has evolved. The question could be raised, though, how the process of learning physics has been affected by this evolvment and the modern use of variables and mathematical signs. When stating his second law Newton wrote

The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed, Newton [1729] p 19.

The law was initially written as a proportionality, not as an equation. Nowhere in *Principia* does Newton say that a force is equal to any absolute quantity instead he compare ratios,⁸ Densmore [2003]. We can write Newton's second law as an equation:

$$\mathbf{F} = m\mathbf{a}$$

an equation that might conceal the depth of the concepts of the variables.

In a learning environment where constructed problems are solved it is possible that the capability of performing calculations exceeds the understanding of the concepts hidden behind the letters. Sometimes the algebraic structure of mathematical physics might provide a short cut that runs past deeper knowledge of concepts like force, mass and acceleration.

To anyone who want to grow in the knowledge of Newtonian mechanics, the study of Newton's own work is recommended.

⁸Our modern expression of the law was formulated by Euler in 1747, Densmore [2003].

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Appendices

A A modern approach of proving the inverse-square law using the equation of the ellipse

Like Newton, we could do the calculations the other way around; assuming elliptical orbits and calculate the relation between the force and the distance. Without using the inverse-square law we can get to the equation for the acceleration, (4.13), and conclude that the angular velocity is proportional to the inverse of the square of the distance, (4.18):

$$|\mathbf{F}| \propto \ddot{r} - r(t)\dot{\varphi}^2 \quad (\text{A.1})$$

$$\dot{\varphi} = \frac{h}{r^2} \quad (\text{A.2})$$

and combined

$$|\mathbf{F}| \propto \ddot{r} - r \left(\frac{h}{r^2} \right)^2. \quad (\text{A.3})$$

From the equation of the ellipse in polar coordinates we know $r(\varphi)$ to be

$$r(\varphi) = \frac{a(1 - e^2)}{1 - e \cos \varphi}. \quad (\text{A.4})$$

From (4.24) we can write

$$\dot{r} = \frac{dr}{d\varphi} \frac{h}{r^2}. \quad (\text{A.5})$$

By derivating $r(\varphi)$ in (A.4) and using $r(\varphi)$ from the same equation we can rewrite (A.5)

$$\dot{r} = -\frac{a(1 - e^2)e \sin \varphi}{(e \cos \varphi - 1)^2} \frac{h(1 - e \cos \varphi)^2}{(a(1 - e^2))^2} = -\frac{eh}{a(1 - e^2)} \sin \varphi. \quad (\text{A.6})$$

This allows us to calculate the second time derivative of r :

$$\ddot{r} = -\frac{eh}{a(1 - e^2)} \cos \varphi \dot{\varphi}. \quad (\text{A.7})$$

The expression can be simplified by solving (A.4) for $\cos \varphi$ and by exchanging $\dot{\varphi}$ for $\frac{h}{r^2}$:

$$\ddot{r} = -\frac{h^2}{a(1 - e^2)r^2} + \frac{h^2}{r^3}. \quad (\text{A.8})$$

This helps us to rewrite the right hand side of (A.3)

$$\ddot{r} - r \left(\frac{h}{r^2} \right)^2 = -\frac{h^2}{a(1 - e^2)r^2} + \frac{h^2}{r^3} - \frac{h^2}{r^3} \quad (\text{A.9})$$

and we get

$$|\mathbf{F}| \propto \frac{1}{r^2}. \quad (\text{A.10})$$

B Initial value problem, assuming one body in orbit

Let the initial distance between M_1 and M_2 be r_0 and the initial velocity of the moving body M_2 be \mathbf{v}_0 where \mathbf{v}_0 is expressed using two unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$

$$\mathbf{v}_0 = v_r(0) \hat{\mathbf{r}} + v_\varphi(0) \hat{\boldsymbol{\varphi}}. \quad (\text{B.1})$$

From (4.2) and (4.8) we know

$$\dot{\mathbf{r}} = \dot{r} \hat{\mathbf{r}} + r \dot{\varphi} \hat{\boldsymbol{\varphi}} \quad (\text{B.2})$$

which together with (B.1) gives us

$$v_r(0) = \dot{r}_0 \quad v_\varphi(0) = r_0 \dot{\varphi}(0). \quad (\text{B.3})$$

Our goal is to express r as a function of φ using the initial values r_0 , $v_r(0)$ and $v_\varphi(0)$. To do this we use the solution found in 4.1.4:

$$r(\varphi) = \frac{1}{K - C \cos(\varphi - \varphi_0)}. \quad (\text{B.4})$$

From (4.19) and (4.30) we know K to be

$$K = \frac{GM_1}{h^2} \quad (\text{B.5})$$

where G is the gravitational constant. From (4.18) we know the constant h to be equal to $r^2 \dot{\varphi}$ and together with (B.3) we can write

$$h = v_\varphi(0) r_0 \quad (\text{B.6})$$

thus

$$K = \frac{GM_1}{(v_\varphi(0) r_0)^2}. \quad (\text{B.7})$$

To proceed we need the constants A and B which appear in an equation where r has been substituted to be $1/u$ and u depends on φ . We will use the derivative of u to find the constants. From (4.32) we know

$$u = A \cos \varphi + B \sin \varphi + K, \quad (\text{B.8})$$

$$\frac{du}{d\varphi} = -A \sin \varphi + B \cos \varphi. \quad (\text{B.9})$$

At $t = 0$ we do not have a value of $(\varphi - \varphi_0)$ but we know φ to be zero which gives us

$$u(0) = A + K, \quad (\text{B.10})$$

$$\frac{du}{d\varphi}|_{\varphi=0} = B. \quad (\text{B.11})$$

The derivative of u can also be expressed using (4.27), (4.18) and (B.1)

$$\frac{du}{d\varphi} = \frac{du}{dr} \frac{dr}{d\varphi} = -\frac{1}{r^2} \frac{dr}{d\varphi} = -\frac{1}{r^2} \frac{dr}{dt} \frac{dt}{d\varphi} = -\frac{1}{r^2} \frac{dr}{dt} \frac{r^2}{h} = -\frac{1}{h} \frac{dr}{dt} \quad (\text{B.12})$$

$$\frac{du}{d\varphi}|_{\varphi=0} = -\frac{1}{h} v_r(0) = -\frac{v_r(0)}{v_\varphi(0) r_0}. \quad (\text{B.13})$$

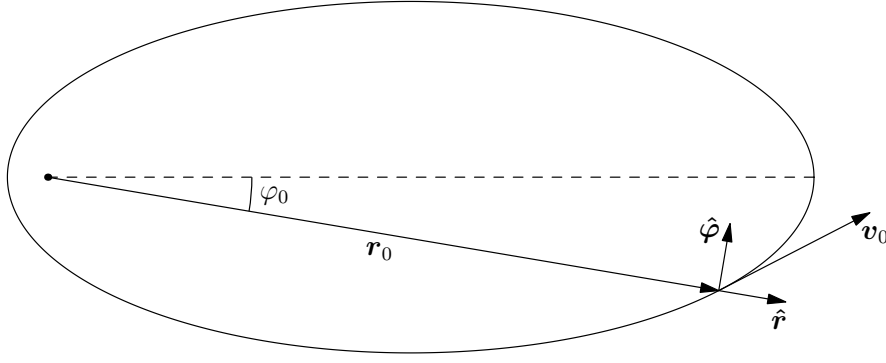


Figure 21: Together with the distance r the angle φ_0 gives the position of the object in the ellipse. This orbit is drawn using data for the trans-Neptunian object Sedna, reaching almost 1,000 AU away from the sun and never coming closer than 70 AU.

Now A and B can be calculated using (B.10) and (B.24)

$$A = \frac{1}{r_0} - \frac{GM_1}{(v_\varphi(0)r_0)^2}, \quad B = -\frac{v_r(0)}{v_\varphi(0)r_0}. \quad (\text{B.14})$$

To express C we use (4.41)

$$C = \sqrt{A^2 + B^2}. \quad (\text{B.15})$$

The last constant to be calculated is φ_0 . As concluded in 4.1.4, φ_0 is the angle between the initial location of M_2 to the great axis, see figure 21. As M_2 travels counter clockwise around the ellipse $\varphi - \varphi_0$ gives the deviation from the great axis. Its value is decided by (4.37) and (4.36)

$$-\frac{A}{C} = \cos \varphi_0, \quad (\text{B.16})$$

$$-\frac{B}{C} = \sin \varphi_0. \quad (\text{B.17})$$

Assuming we use the latter of the two equations to express φ_0 we need to take into account that arcsine only is defined for angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, which corresponds to $\cos \varphi_0$ being zero or positive. If $\cos \varphi_0$ is negative the angle received using arcsine has to be adjusted. The sign of $\cos \varphi_0$ depends on the sign of A , since C is chosen to be the positive root. We get

$$\text{if } A \leq 0 \quad \varphi_0 = \arcsin -\frac{B}{C}, \quad (\text{B.18})$$

$$\text{if } A > 0 \quad \varphi_0 = \pi - \arcsin -\frac{B}{C}. \quad (\text{B.19})$$

We can now express the three constants K , C and φ_0 using initial values and calculate r as a function of φ .

B.1 Path of Eris

In the calculations above initial values of r , v_r and v_φ are used. For most of us these are not the figures at hand. More likely the numbers found will give us the orbit period, its eccentricity and the distances to the sun when it is closest to the sun and furthest from the sun (perihelion and aphelion), but from these numbers the velocity of the object at different locations can be calculated. We will do some of these calculations for a dwarf planet.

The dwarf planet Eris was first spotted during a survey of the outer solar system in 2003. When the discovery was later confirmed, it triggered the debate leading to the International Astronomical Union's decision in 2006 to remove Pluto from the list of planets and to reclassify it as a dwarf planet, NASA [2013a].

At its homepage NASA provides us with some basic information about the orbit of Eris, NASA [2013b]. If we want to calculate the velocity of the dwarf planet we can use the following information:

Perihelion $5.766 \cdot 10^{12}\text{m}$,
Aphelion $1.459 \cdot 10^{13}\text{m}$,
 $e=0.4336$.

From these numbers and the mass of the sun we can find the velocity when the angle between Eris and the major axis is zero, that is when Eris is furthest away from the sun, in aphelion. At this location the only velocity is in the direction of v_φ , a velocity we know from (B.6) to be h divided by the aphelion. We use the equation of the orbit and the equation for the ellipse, φ_0 is zero since the choice of location of the dwarf planet:

$$r(\varphi) = \frac{\frac{1}{K}}{1 - \frac{C}{K} \cos \varphi}, \quad r(\varphi) = \frac{a(1 - e^2)}{1 - e \cos \varphi}. \quad (\text{B.20})$$

The variable a is half the semi major axis

$$a = \frac{\text{perihelion} + \text{aphelion}}{2}. \quad (\text{B.21})$$

The constant h is calculated using (B.5) and (B.20)

$$K = \frac{1}{a(1 - e^2)}, \quad (\text{B.22})$$

$$K = \frac{GM_1}{h^2} \Rightarrow h = \sqrt{GM_1 a(1 - e^2)}. \quad (\text{B.23})$$

Since v_r is zero in aphelion we want to calculate v_φ , to do this we use (B.6)

$$v_\varphi(0) = \frac{h}{r_0}. \quad (\text{B.24})$$

The numbers provided by NASA gives us the velocity 2271 m/s at aphelion.

The two cases aphelion and perihelion are easiest to treat, but if we know the constant h the velocity can be calculated for any angle. The angle gives us the distance which together with h gives us v_φ , see (B.24). From (B.3) we know v_r is equal to \dot{r} , calculated in (A.6)

$$\dot{r} = -\frac{eh}{a(1 - e^2)} \sin \varphi. \quad (\text{B.25})$$

The values of v_φ and v_r give us the total velocity

$$\mathbf{v} = \sqrt{v_r^2 + v_\varphi^2}. \quad (\text{B.26})$$

The angle 58.4° gives a velocity of Eris of $3.43 \cdot 10^3$ m/s which, according to NASA, is the average orbit velocity.

C Initial value problem, two bodies in orbit

To find \mathbf{R}_1 and \mathbf{R}_2 we need another set of initial values in three dimensional space. Now the velocity of the bodies are given with regard of an outside origin. The initial values

include the position of the bodies, $\mathbf{R}_1(0)$ and $\mathbf{R}_2(0)$, and their velocities, $\dot{\mathbf{R}}_1(0)$ and $\dot{\mathbf{R}}_2(0)$. The functions we want to express using initial values are

$$\mathbf{R}_1 = \mathbf{R}(0) + \dot{\mathbf{R}}t - \frac{M_2}{M_1 + M_2}r(\varphi(t))\hat{\mathbf{r}} \quad (\text{C.1})$$

$$\mathbf{R}_2 = \mathbf{R}(0) + \dot{\mathbf{R}}t + \frac{M_1}{M_1 + M_2}r(\varphi(t))\hat{\mathbf{r}}. \quad (\text{C.2})$$

The initial vector $\mathbf{R}(0)$ and the derivative $\dot{\mathbf{R}}$ are both constants. To calculate $\mathbf{R}(0)$ we use (4.53) and $\dot{\mathbf{R}}$ is the derivative of $\mathbf{R}(0)$:

$$\mathbf{R}(0) = \frac{M_1\mathbf{R}_1(0) + M_2\mathbf{R}_2(0)}{M_1 + M_2}, \quad (\text{C.3})$$

$$\dot{\mathbf{R}}(0) = \frac{M_1\dot{\mathbf{R}}_1(0) + M_2\dot{\mathbf{R}}_2(0)}{M_1 + M_2}. \quad (\text{C.4})$$

To express $\hat{\mathbf{r}}$ we need to find the plane that contains the velocities of the bodies after the velocity of the center of mass is subtracted and that has \mathbf{L}_C , the angular momentum of \mathbf{r}_1 and \mathbf{r}_2 with respect of the center of mass, as its normal. The plane is spanned by the vectors $\hat{\mathbf{r}}(0)$ and $\hat{\boldsymbol{\varphi}}(0)$. To find $\hat{\mathbf{r}}(0)$ we use $\mathbf{r}(0)$:

$$\mathbf{r}(0) = \mathbf{R}_2(0) - \mathbf{R}_1(0), \quad (\text{C.5})$$

$$\hat{\mathbf{r}}(0) = \frac{\mathbf{r}(0)}{|\mathbf{r}(0)|}. \quad (\text{C.6})$$

\mathbf{L}_C is constant and calculated from equation (4.69)

$$\mathbf{L}_C = \mathbf{r}_1 \times M_1\dot{\mathbf{r}}_1 + \mathbf{r}_2 \times M_2\dot{\mathbf{r}}_2. \quad (\text{C.7})$$

To get the angular momentum we need $\mathbf{r}_1(0)$ and $\mathbf{r}_2(0)$ and their derivatives which can be found using (4.54) and (4.55)

$$\mathbf{r}_1(0) = \mathbf{R}_1(0) - \mathbf{R}(0), \quad \dot{\mathbf{r}}_1(0) = \dot{\mathbf{R}}_1(0) - \dot{\mathbf{R}}(0), \quad (\text{C.8})$$

$$\mathbf{r}_2(0) = \mathbf{R}_2(0) - \mathbf{R}(0), \quad \dot{\mathbf{r}}_2(0) = \dot{\mathbf{R}}_2(0) - \dot{\mathbf{R}}(0). \quad (\text{C.9})$$

The vector $\hat{\boldsymbol{\varphi}}(0)$ is the cross product of the normalized vector \mathbf{L}_C and $\hat{\mathbf{r}}(0)$

$$\hat{\boldsymbol{\varphi}}(0) = \frac{\mathbf{L}_C}{|\mathbf{L}_C|} \times \hat{\mathbf{r}}(0). \quad (\text{C.10})$$

Now the vector $\hat{\mathbf{r}}$ can be calculated

$$\hat{\mathbf{r}} = \cos \varphi \hat{\mathbf{r}}(0) + \sin \varphi \hat{\boldsymbol{\varphi}}(0). \quad (\text{C.11})$$

The function $r(\varphi(t))$ is calculated in a similar manner as in section 4.1.4. Instead of the mass M_1 we use μ and the initial values $v_r(0)$ and $v_\varphi(0)$ are expressed using $\dot{\mathbf{r}}(0)$ calculated from (C.5)

$$\dot{\mathbf{r}}(0) = \dot{\mathbf{R}}_2(0) - \dot{\mathbf{R}}_1(0). \quad (\text{C.12})$$

We now have enough data calculate to \mathbf{R}_1 and \mathbf{R}_2 .

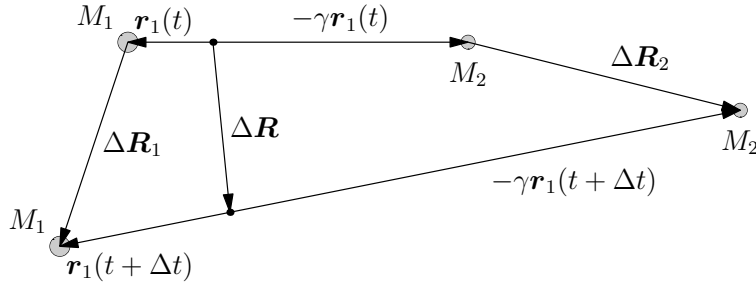


Figure 22

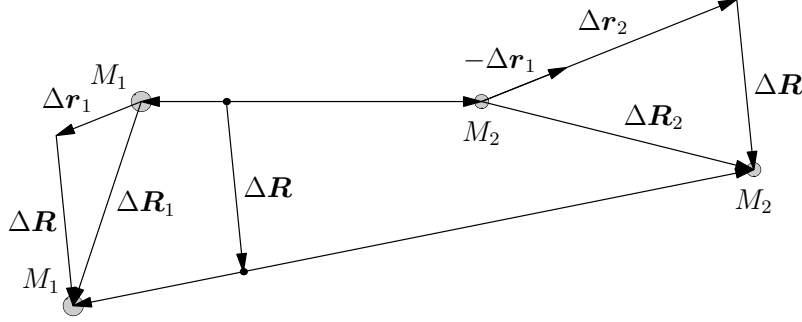


Figure 23

D The motion of \mathbf{r}_1 and \mathbf{r}_2 is parallel

Observe that no matter the initial velocity of the bodies, the motion of M_1 and M_2 is parallel if the motion of the center of mass is subtracted, as illustrated in figure 22 and 23. Using the same notations as before and comparing the system at time t and $t + \Delta t$, the two vectors $\Delta \mathbf{r}_1$ and $\Delta \mathbf{r}_2$ can be compared. From (4.82) and (4.83) it follows that the position vector \mathbf{r}_2 can be expressed as a scalar times \mathbf{r}_1

$$\gamma = \frac{M_1}{M_2}, \quad \mathbf{r}_2 = -\gamma \mathbf{r}_1. \quad (\text{D.1})$$

To find an expression of $\Delta \mathbf{r}_1$, $\Delta \mathbf{R}$ is subtracted from $\Delta \mathbf{R}_1$

$$\Delta \mathbf{r}_1 = \Delta \mathbf{R}_1 - \Delta \mathbf{R}. \quad (\text{D.2})$$

The vector $\Delta \mathbf{R}$ is substituted for $\mathbf{r}_1(t) + \Delta \mathbf{R}_1 - \mathbf{r}_1(t + \Delta t)$

$$\Delta \mathbf{r}_1 = \Delta \mathbf{R}_1 - \mathbf{r}_1(t) - \Delta \mathbf{R}_1 + \mathbf{r}_1(t + \Delta t) \quad (\text{D.3})$$

and the expression is simplified

$$\Delta \mathbf{r}_1 = -\mathbf{r}_1(t) + \mathbf{r}_1(t + \Delta t). \quad (\text{D.4})$$

To find an expression of $\Delta \mathbf{r}_2$, $\Delta \mathbf{R}$ is substituted for $-\gamma \mathbf{r}_1(t) + \Delta \mathbf{R}_2 + \gamma \mathbf{r}_1(t + \Delta t)$

$$\Delta \mathbf{r}_2 = \Delta \mathbf{R}_2 - \Delta \mathbf{R}, \quad (\text{D.5})$$

$$\Delta \mathbf{r}_2 = \Delta \mathbf{R}_2 + \gamma \mathbf{r}_1(t) - \Delta \mathbf{R}_2 - \gamma \mathbf{r}_1(t + \Delta t), \quad (\text{D.6})$$

$$\Delta \mathbf{r}_2 = \gamma(\mathbf{r}_1(t) - \mathbf{r}_1(t + \Delta t)). \quad (\text{D.7})$$

When comparing (D.4) and (D.7), $\Delta \mathbf{r}_1$ and $\Delta \mathbf{r}_2$ are shown to be parallel

$$\Delta \mathbf{r}_2 = -\gamma \Delta \mathbf{r}_1. \quad (\text{D.8})$$

E Proposition LXI

Proposition LXI.

If two bodies attracting each other with any kind of forces, and not otherwise agitated or obstructed, are moved in any manner whatsoever; those motions will be the same, as if they did not at all attract each other mutually, but were both attracted with the same forces by a third body placed in their common centre of gravity; and the law of the attracting forces will be the same in respect of the distance of the bodies from the common centre, as in respect of the distance between the two bodies.

For those forces with which the bodies attract each other mutually, by tending to the bodies tend also to the common centre of gravity lying directly between them; and therefore are the same as if they proceeded from an intermediate body. Q.E.D.

And because there is given the ratio of the distance of either body from that common centre to the distance between the two bodies, there is given of course the ratio of any power of one distance to the same power of the other distance and also the ratio of any quantity derived in any manner from one of the distances compounded any how with given quantities, to another quantity, derived in like manner from the other distance, and as many given quantities having that given ratio of the distances to the first. Therefore if the force with which one body is attracted by another be directly or inversely as the distance of the bodies from each other, or as any power of that distance; or lastly as any quantity derived after any manner from that distance compounded with given quantities; then will the same force with which the same body is attracted to the common centre of gravity, be in like manner directly or inversely as the distance of the attracted body from the common centre, or as any power of that distance, or lastly as a quantity derived in like sort from that distance compounded with analogous given quantities. That is, the law of attracting force will be the same with respect to both distances. Q.E.D. Newton [1729] p 225, 226.



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