(I was late to class) 14.11.24

PHYS 325 Lecture 13

Let $\{e_1, e_2, \dots, e_n\}$ be a basis for $\mathbb{C}^n$ if

$\langle e_i | e_j \rangle = \delta_{ij}$ then the basis is called orthonormal if $i \neq j$ $\langle e_i | e_j \rangle = 0 \rightarrow$ we say e_i & e_j are orthogonal to each other

$$\|e_i\| = \sqrt{\langle e_i | e_j \rangle} = 1$$

e_i is a normalized (unit) vector.

Magnitude 1.

- $F : \mathbb{C}^n \rightarrow \mathbb{C}$ linear

F is called a linear functional (co-vector)

Let F & G be a linear functional.

$$(F+G)(v) = F(v) + G(v)$$

$c \in \mathbb{C}$ $\quad (cF)(v) = cF(v)$

$$F(c_1 v + c_2 u) = F(c_1 v) + F(c_2 u)$$

The set of all linear functionals (co-vectors) is a vector space. It is denoted as V^* and is called the dual space of V.

Example: Let's pick $v \in V$. To v we associate

$$F_v : V \longrightarrow \mathbb{C} \qquad c_1, c_2 \in \mathbb{C} \quad u_1, u_2 \in V$$
$$u \longmapsto \langle v | u \rangle$$

$$F_v(c_1 u_1 + c_2 u_2) = \langle v | c_1 u_1 + c_2 u_2 \rangle$$

$$= c_1 \langle v | u_1 \rangle + c_2 \langle v | u_2 \rangle$$

$$= c_1 F_v(u_1) + c_2 F_v(u_2).$$

Example: (Continued)

Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of V.

$$F_{e_i}(e_j) = \langle e_i | e_j \rangle = \delta_{ij}$$

Let's now pick an arbitrary linear functional F

$$F(u) = F(u_1 e_1 + \dots + u_n e_n) = u_1 F(e_1) + \dots u_n F(e_n)$$

$$= \sum_{i=1}^{n} u_i F(e_i)$$

$$F_{e_j}(u) = \sum_{i=1}^{n} u_i F_{e_j}(e_i) = \sum u_i \langle e_j | e_i \rangle = \sum_{i=1}^{n} u_i \delta_{ij} = u_j$$

$$F = \sum_{i=1}^{n} \underbrace{F(e_i)}_{\mathbb{C}} F_{e_i}$$

Exercise: Show that they are linearly independent.

$$F(u) = \sum_{i=1}^{n} F(e_i) F_{e_i}(u)$$

- This basis is called the dual basis to $\{e_1, e_2, \ldots, e_n\}$

Dirac Notation (Bra-Ket Notation)

The elements of V will be denoted by $|v\rangle$ (ket)
" " " V^* " " " " $\langle F|$ (bra)

$$\langle e_i| \neq |e_i\rangle$$

$|e_i\rangle$ → just the vector

$\langle e_i|$ → covector, linear functional

$$\langle e_i| = \langle F_{e_i}|$$

$$F_{e_i}(e_j) = \langle e_i | e_j\rangle = \langle e_i| \, |e_j\rangle$$

Bra & Ket can be connected.

Finite vs infinite vec spaces

↳ Get results by using finite dimentionality of the space so proofs don't apply.

(Liner functional analysis)

Linear Operator

- Let $(V, \langle | \rangle_V)$ and $(W, \langle | \rangle_W)$ be two inner product spaces and consider a linear function.

$$A: V \longrightarrow W \text{ linear}$$

$$A(c_1 v_1 + c_2 v_2) = c_1 A(v_1) + c_2 A(v_2) \quad (c_1, c_2 \in \mathbb{C})$$

A is called a linear operator.

Example: Let's pick orthonormal basis

$$\{e_1, \cdots, e_n\} \text{ for } V \qquad \langle e_i | e_j \rangle_V = \delta_{ij}$$

$$\{f_1, \cdots, f_m\} \text{ for } W \qquad \langle f_a | f_b \rangle_W = \delta_{ab}$$

$A|e_j\rangle$ (A acting on a vector in V will be an element of W)

$$A e_j = \sum_{a=1}^{m} A_{aj} f_a$$

A_{aj} → matrix elements of A relative to the basis given above

$$\langle f_b | A e_j \rangle_W = \langle f_b | \sum_{a=1}^{m} A_{aj} f_a \rangle_W$$

$$= \sum_{a=1}^{m} A_{aj} \langle f_b | f_a \rangle = \sum_{a=1}^{m} A_{aj} \delta_{ab} = A_{bj}$$

$$\longrightarrow A_{bj} = \langle f | A e_j \rangle_W$$

- Let's take $V = W$ and use the orthonormal basis $\{e_1, \dots e_n\}$

$$A_{ij} = \langle e_i | A e_j \rangle$$

- Let A be a linear operator on V ($A: V \to V$) There exists an operator $A^\dagger$ on V (in fact it is unique) such that:

$$\langle u | A v \rangle = \langle A^\dagger u | v \rangle$$

$A^\dagger$ is called the Hermitian Conjugate (adjoint) of A. (Proof on DK)

$$A_{ij} = \langle e_i | A e_j \rangle \Rightarrow (A_{ij})^* = \langle e_i | A e_j \rangle^* = \langle A e_j | e_i \rangle$$

$$(A_{ji})^* = \langle A e_i | e_j \rangle = \langle (A^\dagger)^\dagger e_i | e_j \rangle$$
$$= \langle e_i | A^\dagger e_j \rangle = (A^\dagger)_{ij}$$

Exercise: $(A^\dagger)^\dagger = A$

Like transposing but you also take the complex conjugate.

$$(A^\dagger)_{ij} = (A_{ji})^*$$

- Let A be a linear operator on V $(A: V \to V)$

$$A = \begin{pmatrix} i & 1 \\ 0 & i+1 \end{pmatrix} \qquad A^{+} = \begin{pmatrix} -i & 0 \\ 1 & 1-i \end{pmatrix}$$

If $A = A^{+}$ then A is called Hermitian (self-adjoint)

i.e. $\langle u | Av \rangle = \langle A^{+} u | v \rangle = \langle Au | v \rangle$

Dirac Notation for Hermition Operators:

$$\langle u | A | v \rangle$$

- Let's take $V = W$ and use the orthonormal basis $\{e_1, \dots e_n\}$

A linear operator U so that:

$$U^{+} U = U U^{+} = 1$$

is called a unitary operator $(U^{-1} = U^{+})$

$$\langle Uu | Uv \rangle = \langle u | v \rangle$$

$$\langle U^{+} U u | v \rangle = \langle 1 u | v \rangle = \langle u | v \rangle$$

- isometry (Preserves the inner product)

Exercise: $(AB)^{+} = B^{+} A^{+}$