PHYS 325 Lecture 5 08.10.24

Some properties of analytic functions

(i) $\overset{=u+iv}{f}$ is analytic at $z_0 = x_0 + iy_0$ if and only if at $z_0 = x_0 + iy_0$ we have:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Ex: $f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + i2xy$

$$u = (x^2 - y^2) \qquad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = 2x \; \checkmark \qquad \frac{\partial u}{\partial y} = -2y \quad -\frac{\partial v}{\partial x} = -2y \; \checkmark$$

$f(z)$ is analitic. everywhere

EX: $f(z) = \bar{z} = x - iy$

$$u = x \qquad v = -y$$

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$$

not analitic everywhere.

(ii) $$f'(z) = \frac{d f(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

(iii) If f & g are analitic at $z = z_0$ then for any two complex numbers c_1 and c_2 the linear combination: $c_1 f(z) + c_2\, g(z)$ is also analitic at z_0.

In fact,

$$\frac{d}{dz}\Big[c_1 f(z) + c_2\, g(z)\Big]_{z_0} = c_1 \left.\frac{df}{dz}\right|_{z_0} + c_2 \left.\frac{dg}{dz}\right|_{z_0}$$

$\frac{d}{dz}$ is a linear operator on set analytic functions.

(This tells us analytic function at a point z_0 form a linear space.)

(important in quantum mechanics)

- For one thing, superposition principle holds.
- Hilbert spaces (important in wave phenomena)

(iv) If f & g are analytic at $z = z_0$ then $f \cdot g$ is analytic at z_0. In fact we have:

$$\left.\frac{d}{dz} f(z) g(z)\right|_{z_0} = \left.\frac{df(z)}{dz}\right|_{z_0} g(z_0) + f(z_0) \left.\frac{dg(z)}{dz}\right|_{z_0}$$

(v) If f & g are analytic at $z = z_0$ and $g(z_0) \neq 0$ then $\frac{f(z_0)}{g(z_0)}$ is analytic at z_0. Moreover:

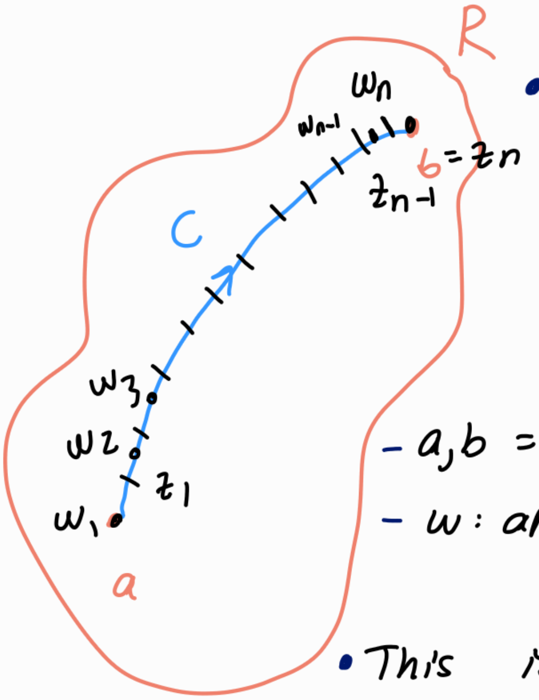
$$\left.\frac{d}{dz} \frac{f(z_0)}{g(z_0)}\right|_{z_0} = \frac{f'(z_0) g(z_0) - f(z_0) g'(z_0)}{g^2(z_0)}$$

(vi) If g is analitic at $z = z_0$ and f is analitic at $g(z_0)$ then:

$$\frac{d}{dz} f(g(z))\Big|_{z_0} = f'(g(z_0))\, g'(z_0)$$

Contour Integrals

Contour: A path where parametrization isn't important.



C: finite

- Let f be analytic on R.

$$\int_C f(z)\, dz = \lim_{\substack{z_k - z_{k-1} \to 0 \\ n \to \infty}} \sum_{k=1}^{n} f(w_k)(z_k - z_{k-1})$$

- a, b = complex numbers. C is divided into n pieces.
- w: arbitrary complex numbers picked from n portions

- This is an imitation of line integrals.

$f = u + iv \qquad z = x + iy \qquad dz = dx + i\,dy$

$$\int_C (u+iv)(dx + i\,dy) = \int_C (u\,dx - v\,dy) + i \int_C v\,dx + u\,dy$$

→ You can think of a counter int. as a couple of line int.s.

(vii) If f is analitic at $z = z_0$ then f is continuous at z_0.

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad |f(z) - f(z_0)| < \epsilon \quad \text{for} \quad |z - z_0| < \delta$$

Modulus: Absolute value of a complex function

Darboux Inequality

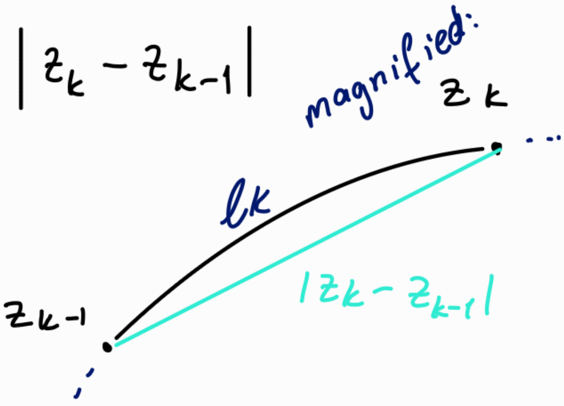
real valued function of z

Let's assume f is bounded on C. $|f| \leq f_{max}$ on C.

Let's look at the absolute value of the sum in definition:

remember $|a+b| \leq |a| + |b|$

$$\left|\sum_{k=1}^{n} f(w_k)(z_k - z_{k-1})\right| \leq \sum_{k=1}^{n} |f(w_k)|\,|z_k - z_{k-1}|$$



$$\leq \sum_{k=1}^{n} \underbrace{f_{max}}_{A}\, l_k = A \sum_{k=1}^{n} l_k = AL.$$

$|z_k - z_{k-1}| \leq l_k$

So the Darboux Inequality:

$$\boxed{\int_C f(z)\,dz \leq AL}$$

as a direct consequence of this we get:

$$\left|\sum_{k=1}^{N} z_k\right| \leq \sum_{k=1}^{N} |z_k|$$

Define the vector fields on $\mathbb{R}^2$

$$\vec{V}_1(x,y) = v(x,y)\,\hat{x} + u(x,y)\,\hat{y}$$

$$\vec{V}_2(x,y) = u(x,y)\,\hat{x} - v(x,y)\,\hat{y}$$

$f(z) = u(x,y) + iv(x,y)$

• Assume $f'(z)$ is continuous on $R \subseteq \mathbb{R}^2$

$$\Rightarrow \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \text{ continuous on } R.$$

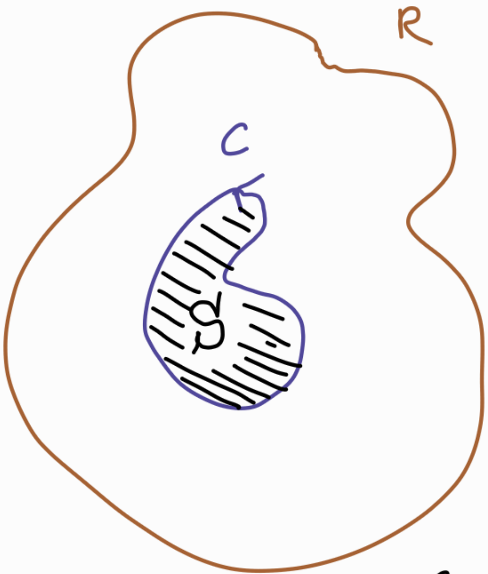
needed for Stoke's theorem.

$$\oint_C u\,dx - v\,dy = \oint_C \vec{V}_2 \cdot d\vec{\ell},$$

$$(*) \oint_C v\,dx + u\,dy = \oint_C \vec{V}_1 \cdot d\vec{\ell}$$

Let's apply Stoke's Theorem on $(*)$

Stoke's Theorem



Important: Integrate ccw.

$$(*) = \int_S ds \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) = 0$$

Because of cauchy rieman this is 0.

and similarly:

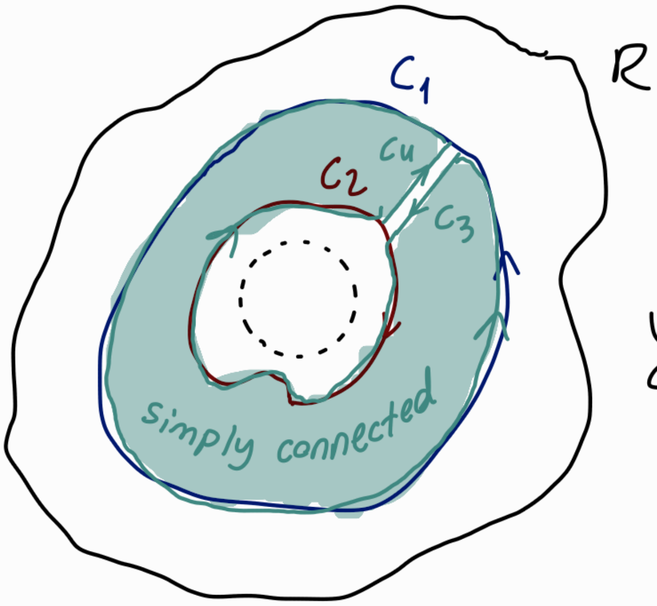
Stokes thm. is applied in such regions

If f is analitic on R (simply connected) and C is a closed curve on R, then:

$$\oint_C u\,dx - v\,dy = 0 \qquad \oint_C f(z)\,dz = 0$$

Cauchy – Gorsat Theorem

P.S: A non-simply connected domain is called **multiply connected domain.**



R: multiply connected

Suppose I'm following this path:

$$\int_{C_1} dz\, f(z) + \cancel{\int_{C_3} dz\, f(z)} + \cancel{\int_{C_4} dz\, f(z)} + \int_{C_2} dz\, f(z) = 0.$$

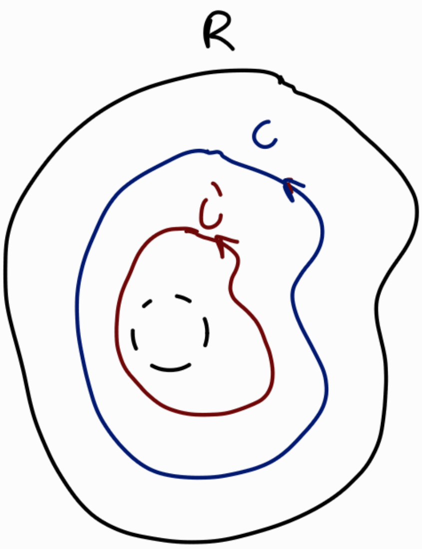
because the area is simply connected

"Calculus of residues"

$$\oint_{C_1} dz\, f(z) = \oint_{C_2} dz\, f(z)$$

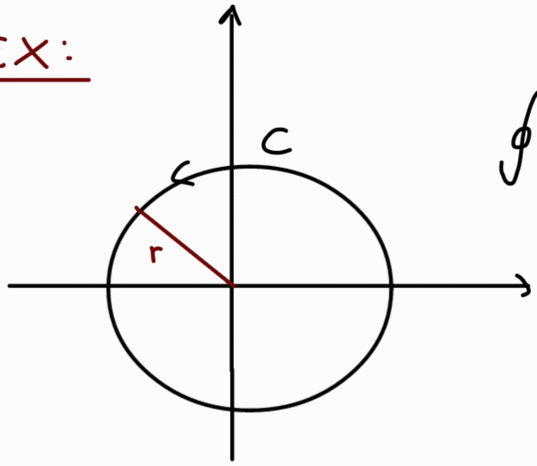
ccw ← C_1 $\qquad$ C_2 → clockwise

Can change the direction of integration.



$$\oint_{C} f(z)\, dz = \oint_{C'} f(z)\, dz$$

EX:



$$\oint z\,dz$$

$$z = re^{i\theta} \qquad dz = ri\,e^{i\theta}d\theta$$

$$\oint z\,dz = \oint_0^{2\pi} re^{i\theta}\,ri\,e^{i\theta}\,d\theta$$

$$= r^2 i \oint_0^{2\pi} e^{2i\theta}\,d\theta$$

Remember:

$$e^{4\pi i} = \cos 4\pi + i \sin 4\pi$$

$$= r^2 i \left.\frac{e^{2i\theta}}{2i}\right|_0^{2\pi} = r^2 i\,\frac{e^{4\pi i}}{2i} - r^2 i\,\frac{1}{2i}$$

$$0 = \frac{r^2 \cancel{i}}{2\cancel{i}}\left(e^{4\pi i} - 1\right)$$

$$((\cos 4\pi + i \sin 4\pi) - 1)\,\frac{r^2}{2} = 0$$

Exercise! $\oint z^n\,dz = ?$ $\qquad n > 1 \qquad n = 0 \qquad (n \rightarrow \text{integer})$