PHYS 325 Lecture 12 12.11.2024

Some Theorems

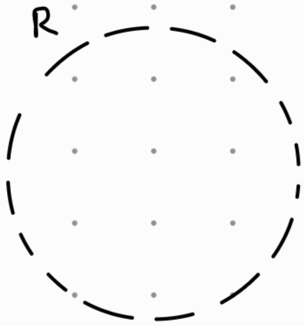
(i) Fundamental Theorem of Algebra

Counting with multiplicities, n^{th} order polynomial equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0 \quad (a_n \neq 0)$$

has n solutions (Proof on DK pg 86)

(ii) Derivatives of a series of analytic functions



Let $\{f_n(z)\}_{n=0}^{\infty}$ be a sequence of analytic functions on R. Assume at every $z \in R$ the series $\sum_{n=0}^{\infty} f_n(z)$ is convergent:

$$F(z) = \sum_{n=0}^{\infty} f_n(z)$$

Then $F(z)$ is analytic and $F'(z) = \sum_{n=0}^{\infty} f_n'(z)$

(DK pg 45)

! Branch points, essential singularities, removable singularities...

(iii) Mittag-Leffler Expansions

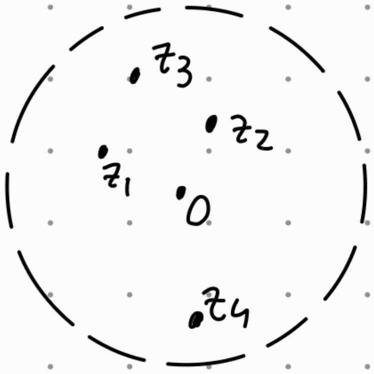
? Example of singularities that aren't poles?

Let $f(z)$ be a meromorphic function with poles at $z = z_j$ $(j = 1, 2, 3, ..)$ so that $0 < |z_1| \leq |z_2| \leq \cdots$

Assume all poles are simple.

Def:
Meromorphic Function:
$f(z)$ is meromorphic if only singularities of f are poles.

Let C_n be a circle centered at the origin and enclosing n poles.



Then inside C_n:

$$f(z) = f(0) + \sum_{j=1}^{n} a_{1,z_j}\left(\frac{1}{z-z_j} + \frac{1}{z_j}\right)$$

(DK, pg 84)

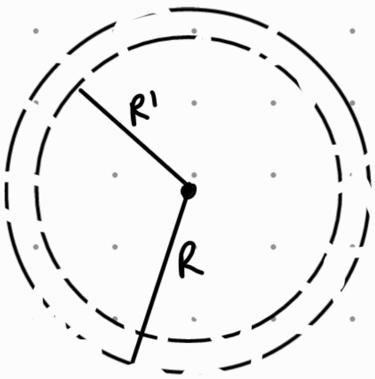
(Useful for more advanced topics)

(iv) Analytic Continuation

Example: $F(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ (Sum of geometric series)

by the ratio test the series is convergent for $|z| < 1$.
Divergent for $z = 1$.

Using item (ii) $F(z)$ is analytic in any $R' \subset R$

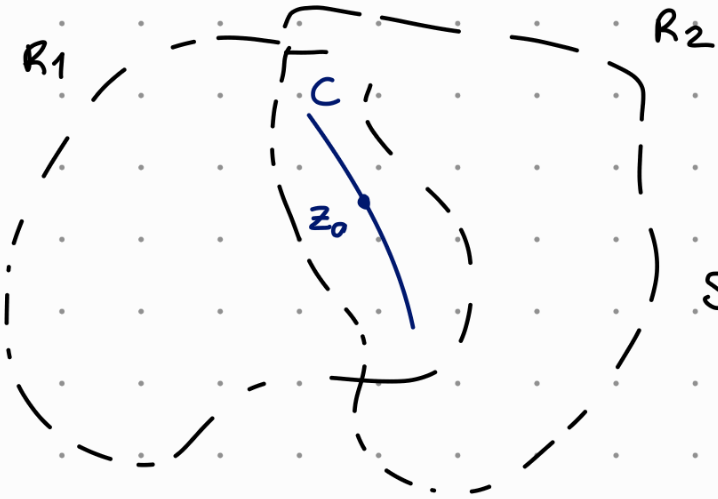


Now let $G(z) = \frac{1}{1-z} \rightarrow G|_{R'} = F$

We say that G is an analytic continuation of $F(z)$ to $\mathbb{C} - 1$

(Dener and Krizwincky pg 76)

The important result:



Let F_1 be analytic on R_1,
Let F_2 be analytic on R_2.

Suppose there is a curve C in $R_1 \cap R_2$ so that:

$$F_1\Big|_C = F_2\Big|_C$$

Taylor expanding F_1 & F_2 at z_0 one can show that:

$$F_1\Big|_{R_1 \cap R_2} = F_2\Big|_{R_1 \cap R_2}$$

$$\left[\begin{aligned} F_1(z) &= F_1(z_0) + F_1'(z_0)(z-z_0) + F_1''(z_0)(z-z_0)^2\tfrac{1}{2} + \cdots \\ &= F_2(z_0) + F_2'(z_0)(z-z_0) + F_2''(z_0)(z-z_0)^2\tfrac{1}{2} + \cdots \end{aligned}\right]$$

$$F(z) = \begin{cases} F_1(z) & \text{if } z \in R_1 \\ F_2(z) & \text{if } z \in R_2 \end{cases}$$

which is defined on $R_1 \cup R_2$ $\therefore$ F is analytic continuation of both F_1 & F_2 to $R_1 \cup R_2$

"sequence of points with an accumilation point"

Complex Linear Algebra

$$\mathbb{C}^n = \{(z_1, z_2, \dots, z_n) : z_j \in \mathbb{C} \text{ for } j = 1, 2, 3, \dots, n\}$$

$$(z_1, z_2, \dots, z_n) + (w_1, w_2, \dots, w_n) = (z_1 + w_1, z_2 + w_2, \dots, z_n + w_n)$$

$$\lambda \in \mathbb{C} \quad \lambda(z_1, z_2, z_3, \dots, z_n) = (\lambda z_1, \lambda z_2, \dots, \lambda z_n)$$

$\mathbb{C}^n$ is a vector space over $\mathbb{C}$.

Defining The Inner Product

Recall $\mathbb{R}^n$:

$$\vec{x} = (x_1, x_2, \dots, x_n) \qquad \vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

$$\vec{y} = (y_1, y_2, \dots, y_n) \qquad \vec{x} \cdot \vec{x} = x_1^2 + \cdots + x_n^2 \geq 0$$

Inner Product on $\mathbb{C}^n$: (Standard inner Product)

$$u = (u_1, u_2, \dots u_n) \in \mathbb{C}^n$$

$$v = (v_1, v_2, \dots v_n) \in \mathbb{C}^n$$

$$\langle u | v \rangle := u_1^* v_1 + u_2^* v_2 + \cdots + u_n^* v_n = \sum_{i=1}^{n} u_i^* v_i$$

$$\langle u | u \rangle = \sum_{i=1}^{n} u_i^* u_i = \sum_{i=1}^{n} |u_i|^2 \geq 0$$

$$\|u\| := \sqrt{\langle u | u \rangle}$$ (Length or magnitude of u)

Properties of The inner Product

(i) Linearity: $\langle u | \alpha v_1 + \beta v_2 \rangle \quad \alpha, \beta \in \mathbb{C} \quad u_1, v_1, v_2 \in \mathbb{C}^n$

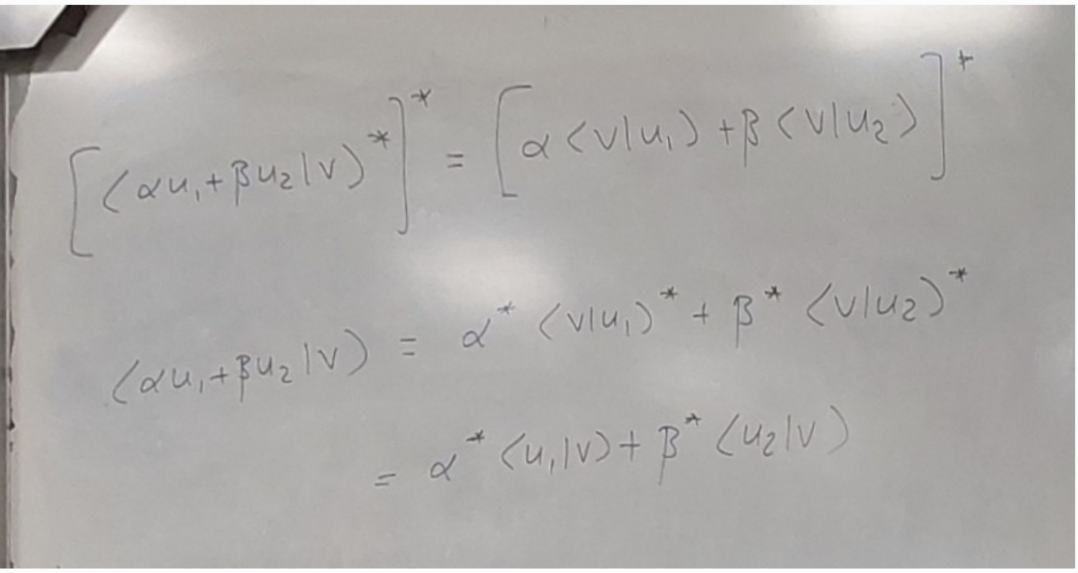
$$\langle u | \alpha v_1 + \beta v_2 \rangle = \alpha \langle u | v_1 \rangle + \beta \langle u | v_2 \rangle$$ (Exercise)

(ii) $\langle u | v \rangle^* = \langle v | u \rangle$ (Exercise)

(iii) (Follows from 1st and 2nd properties)

$$\langle \alpha u_1 + \beta u_2 | v_2 \rangle = \alpha^* \langle u_1 | v \rangle + \beta^* \langle u_2 | v \rangle$$

$$\langle \alpha u_1 + \beta u_2 | v \rangle^* = \langle v | \alpha u_1 + \beta u_2 \rangle$$
$$= \alpha \langle v | u_1 \rangle + \beta \langle v | u_2 \rangle$$



(iv) $\langle u | u \rangle \geq 0$ where equality holds if and only if:

$u = 0$.

General Definition of Inner Product.

In fact, any pairing
$$\mathbb{C}^n \times \mathbb{C}^n \longmapsto \mathbb{C}$$
$$(u,v) \longmapsto (u,v)$$

which satisfies the properties (i) – (iv) is called an inner product

→ There might be other inner products.

EX: A can be an $n \times n$ real symmetric, positive matrix.

Positive Matrix:
- all eigenvalues are positive.

$(A = A^T)$

$$\langle u | v \rangle_A = \sum_{i=1}^{n} \sum_{j=1}^{n} u_i^* A_{ij} v_j$$ (Should satisfy all conditions)

$$\delta_{ij} \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

We'll use the standard one.

Metric:

$$\| u \| = \sqrt{\langle u | u \rangle}$$

$$d(u,v) := \| u - v \| = \sqrt{\langle u - v | u - v \rangle}$$

↳ distance (metric)

$$\big| \|u\| - \|v\| \big| \leq \| u - v \| \leq \|u\| + \|v\|$$

Triangle Inequality

Cauchy - Schwartz Inequality

$u, v \in \mathbb{C}^n$

- $|\langle u|v\rangle| \leqslant \|u\| \cdot \|v\|$

Proof: Let x be an arbitrary real number.

$$\langle u+xv|u+xv\rangle = \langle u+xv|u\rangle + x\langle u+xv|v\rangle$$

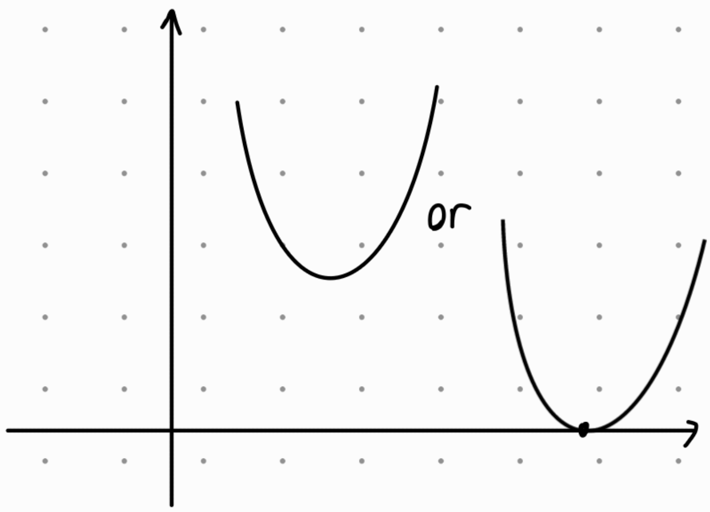
$$= \langle u|u\rangle + x^*\langle v|u\rangle + x\langle u|v\rangle + xx^*\langle v|v\rangle$$

$$= \|v\|^2 x^2 + x\left[\langle u|v\rangle + \underbrace{\langle v|u\rangle}_{\langle u|v\rangle^*}\right] + \|u\|^2$$

$$(*) = \|v\|^2 x^2 + 2x \operatorname{Re}\langle u|v\rangle + \|u\|^2 \geqslant 0$$

Should hold for $\forall x \in \mathbb{R}$

Call $(*)$ $f(x) \geqslant 0$:



$f(x) = 0$ must have at most one real root.

Look at the discriminant:

$$4(\operatorname{Re}\langle u|v\rangle)^2 - 4\|v\|^2\|u\|^2 \leq 0$$

$$\mathrm{Re}\langle u|v\rangle^2 \leq \|u\|^2\|v\|^2$$

Replace v by iv. ($\|v\| = \|iv\|$) (Check as an exercise)

$$\mathrm{Re}\underbrace{\langle u|iv\rangle}_{-i\langle u|v\rangle} \leq \|u\|^2\|v\|^2$$

(Continued next time)

$$(\mathrm{Im}\langle u|v\rangle)^2 \leq \|u\|^2\|v\|^2$$