PHYS 326 Lecture 4

#phys326 #physics

$\mathbb{R}^2$

$x = r\cos\theta$

$y = r\sin\theta$

$dx^1 = dx$

$dx^2 = dy$

$$\begin{aligned} dx &= dr\cos\theta - r\sin\theta d\theta \\ dy &= dr\sin\theta + r\cos\theta d\theta \end{aligned}$$

$$\begin{gathered} g = dx \otimes dx + dy \otimes dy = (dx)^2 + (dy)^2 = \delta_{ij} dx^i \otimes dx^j \\ = (dr\cos\theta - r\sin\theta d\theta) \otimes (dr\cos\theta - r\sin\theta d\theta) \\ +(dr\sin\theta + r\cos\theta d\theta) \otimes (dr\sin\theta + r\cos\theta d\theta) \\ = cos^2\theta dr \otimes dr - r\sin\theta\cos\theta(dr \otimes d\theta + d\theta \otimes dr) \\ +r^2 sin^2 d\theta \otimes d\theta \\ +\sin^2\theta dr \otimes dr + r\sin\theta\cos\theta(dr \otimes d\theta + d\theta \otimes dr) \\ +r^2\cos^2\theta d\theta \otimes d\theta \end{gathered}$$

$$g = dr \otimes dr + r^2 d\theta \otimes d\theta = (dr)^2 + r^2(d\theta)^2$$

$$g'_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

Riemannian Value Element

$\mathbb{R}^2$

$$\begin{gathered} dV = \sqrt{|\det g|} dx^1 dx^2 \dots dx^n \\ = \sqrt{|\det g'_{ij}|} dx'^2 dx'^2 \dots dx'^n \end{gathered}$$

$$\int dV_f = \int dx^1 dx^2 \dots dx^n \sqrt{|\det g|} f$$

Cartesian Coordinates

$dV = \sqrt{|\det \delta_{ij}|} dxdy = dxdy$

Polar Coordinates

$\det g'_{ij} = r^2 \ (r \geq 0)$

$dV = r dr d\theta = dxdy$

Note

Whenever you have a metric, you have a volume form.

See also:

- Lie Derivative
- Covariant Derivative

Laplace Operator (Laplacian)

Definition

$g^{ik} g_{kj} = \delta^i_k$

$$\nabla^2 := \frac{1}{\sqrt{|\det g|}} \partial_i \sqrt{|\det g|} g^{ij} \partial_j$$

Cartesian Coordinates

$$\sqrt{|\det \delta_{ij}|} = 1$$

$$g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \nabla^2 &= \partial_1 g^{11} \partial_1 + \partial_1 g^{12} \partial_2 + \partial_2 g^{21} \partial_1 + \partial_2 g^{22} \partial_2 \\ &= \partial_1 \partial_1 + \partial_2 \partial_2 \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{aligned}$$

Polar Coordinates

$$\sqrt{|\det g'|} = r$$

$$g'_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

$$g'^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1/r \end{bmatrix}$$

$$\begin{aligned} \nabla^2 &= \frac{1}{r}[\partial_r + r\partial_r^2 + \frac{1}{r}\partial_\theta^2] \\ &= \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{aligned}$$

Note

- You can show this is independent of coordinate system
- You can use chain rule but it's more difficult to calculate that way.

Asymptotic Analysis

O-symbol

- $f(x) = O(g(x))$ as $x \to x_0$

$$\begin{gathered} \exists C, \epsilon > 0 \text{ s.t} \\ |f(x)| < C|g(x)| \text{ for } x \in (x_0 - \epsilon, x_0 + \epsilon) \end{gathered}$$

- $f(x) = O(g(x))$ as $x \to \infty$

$$\begin{gathered} \exists C, N > 0 \text{ s.t} \\ |f(x)| < C|g(x)| \text{ for } x > N \end{gathered}$$

Examples

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Exercise 1: $\alpha > \beta \geq 0$ show that $\frac{1}{x^\alpha} = 0(\frac{1}{x^\beta})$ as $x \to \infty$

Hint: First take the limit and use the definition of limit.

o-symbol

$f(x) = o(g(x))$ as $x \to x_0$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

Given $\epsilon > 0 \; \exists \delta(\epsilon) > 0$ s.t.

$|\frac{f(x)}{g(x)}| < \epsilon \text{ for } x \in (-\delta(\epsilon), \delta(\epsilon))$

$|f(x)| < \epsilon |g(x)|$

> **Note**
>
> if $f(x) = o(g(x))$ then, $f(x) = O(g(g))$

But the converse in general isn't true. There are counter examples.

In some modern textbooks, this notation is used:

$$f(x) << g(x), x \to x_0$$

Examples

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> **Remark**
>
> You can't apply L'hospital to any ratio. You must have $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminacy.

~ symbol (Asymptotic Equivalence)

$f(x) \sim g(x)$ as $x \to x_0 \; g \neq 0$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

Examples

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Exercise 2: $\sinh \sim -\frac{1}{2}e^{-x}$ as $x \to \infty$

A Useful Result

Assume $f(x) = O(g(x))$ as $x \to x_0$ and $g(x) = o(h(x))$ as $x \to x_0$ then $f(x) = o(h(x))$ as $x \to x_0$

$$\begin{gathered}\exists C, \epsilon > 0 \text{ s.t.} \\ |f(x)| \leq C|g(x)| \forall x \in (x_0 - \epsilon, x_0 + \epsilon)\end{gathered}$$

$$0 \leq \lim_{x \to x_0} \frac{|f(x)|}{|h(x)|} \leq \lim_{x \to x_0} \frac{|g(x)|}{|h(x)|} = 0$$

(Taking limit preserves inequalities)

$$\lim_{x \to x_0} \frac{|g(x)|}{|h(x)|} = 0$$

Asymptotic Expansion of a Function as $x \to x_0$

Let $\phi_0(x), \phi_1(x), \phi_2(x), \ldots$ be a sequence of functions s.t.

$$\phi_{k+1}(x) = o(\phi_k(x)) \text{ as } x \to x_0$$

$$\lim_{x \to x_0} \frac{\phi_{k+1}(x)}{\phi_k(x)} = 0$$

Examples

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A function $f(x)$ is said to have the asymptotic expansion:

$$f(x) \sim c_0\phi_0(x) + c_1\phi_1(x) + \dots$$

as $x \to x_0$

if

$$\begin{gathered} f(x) = O(\phi_0(x)) \\ f(x) - c_0\phi_0(x) = O(\phi_1(x)) \\ f(x) - c_0\phi_0(x) - c_1\phi_1(x) = O(\phi_2(x)) \end{gathered}$$

For any $N \in \mathbb{Z}_{\geq 0}$

$$f(x) - \sum_{k=0}^{N} c_k\phi_k(x) = O(\phi_{N+1}(x))$$

as $x \to x_0$

Examples

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