PHYS 326 Lecture 7 (March 4)

#phys326

Midterm 1: March 24 (17:00)
Midterm 2: May 5 (17:00)

Algebraic operations

$$f_1(x) \sim a_o + a_1 x + a_2 x^2 + \dots \text{ as } x \to 0$$
$$f_2(x) \sim b_o + b_1 x + b_2 x^2 + \dots \text{ as } x \to 0$$

As $x \to 0$

Basic Algebraic Properties of Asymptotic Expansion

1. $f_1(x) + f_2(x) \sim (a_0 + b_o) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots$
2. $f_1(x) f_2(x) \sim \sum_{l=0}^{\infty} [\sum_{k=0}^{l} a_k b_{l-k}] x^l$
3. if $a_0 \neq 0$:

$$\frac{1}{f_1(x)} \sim \frac{1}{a_0 + a_1 x + a_2 x^2 + \dots}$$

4. $f_1(f_2(x)) = S_1(S_2(x))$

Gaussian Integrals

Let's say:

$$I(\alpha) = \int_0^{\infty} dx e^{-\alpha x^2} \; (\alpha > 0)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dx e^{-\alpha x^2} \; (\sqrt{\alpha} x = 0)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{du}{\sqrt{\alpha}} e^{-u^2} = \frac{1}{2} \frac{1}{\sqrt{\alpha}} \sqrt{\pi} = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}$$

$$= \int_{-\infty}^{\infty} dx e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

Let's say:

$$I(\alpha) = \int_{-\infty}^{\infty} dx e^{(-\alpha x^2 + \beta x)} \; (\alpha > 0)$$

$$= \int_{-\infty}^{\infty} dx e^{-\alpha(x^2 - 2\frac{\beta}{2\alpha}x + (\frac{\beta}{2\alpha})^2 - (\frac{\beta}{2\alpha})^2)} \; (\alpha > 0)$$

$$= e^{\alpha(\frac{\beta}{2\alpha})^2} \int_{-\infty}^{\infty} dx e^{-\alpha(x - \frac{\beta}{2\alpha})^2}$$

Say that $u = x - \frac{\beta}{2\alpha}$ and thus $du = dx$

Then,

$$I(\alpha) = e^{\frac{\beta^2}{4\alpha}} \int_{-\infty}^{\infty} du e^{-\alpha u^2} = \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{4\alpha}}$$

$$I_n(\alpha) = \int_{-\infty}^{\infty} dx x^n e^{\alpha x^2}$$

If n is odd, $I_n(\alpha) = 0$

What happens when n is even?
$n = 2k$

$$I_{2k}(\alpha) = \int_{-\infty}^{\infty} dx x^{2k} e^{-\alpha x^2}$$

$$\int_{-\infty}^{\infty} dx x e^{-alphax^2} = \frac{\partial}{\partial \beta} \int_{-\infty}^{\infty} dx e^{-\alpha x^2 + \beta x}|_{\beta=0}$$

$$= \int_{-\infty}^{\infty} dx \frac{\partial^2}{\partial \beta^2} e^{-\alpha x^2 + \beta x}|_{\beta=0} = \int_{-\infty}^{\infty} dx \; x^2 e^{-\alpha x^2 + \beta x}|_{\beta=0}$$

Gaussian Moments: $I_n(x) = \int_{-\infty}^{\infty} dx x^n e^{-\alpha x^2 + \beta x}$

$$I_n(x) = \int_{-\infty}^{\infty} dx x^n e^{-\alpha x^2 + \beta x} = \frac{\partial^n}{\partial \beta} \int_{-\infty}^{\infty} dx e^{-\alpha^2 x^2 + \beta x}|_{\beta=0} = \frac{\partial^n}{\partial \beta^n} \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{2\alpha}}|_{\beta=0}$$

$$I_1(x) = \frac{\partial}{\partial \beta} \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{2\alpha}}|_{\beta=0} = \sqrt{\frac{\pi}{\alpha}} \frac{\beta}{2\alpha} e^{\frac{\beta^2}{4\alpha}}|_{\beta=0} = 0$$

$$I_2(x) = \frac{\partial^2}{\partial\beta^2}\sqrt{\frac{\pi}{\alpha}}e^{\frac{\beta^2}{2\alpha}}|_{\beta=0} = \frac{\partial}{\partial\beta}[\sqrt{\frac{\pi}{\alpha}}\frac{\beta}{2\alpha}e^{\frac{\beta^2}{4\alpha}}]$$

$$I_2(x) = \sqrt{\frac{\pi}{\alpha}}\frac{1}{2\alpha}[e^{\frac{\beta^2}{4\alpha}} + \beta\frac{\beta}{2\alpha}e^{\frac{\beta^2}{4\alpha}}]_{\beta=0}$$

$$= \sqrt{\frac{\pi}{\alpha}}\frac{1}{2\alpha} = O(\frac{1}{\alpha\sqrt{\alpha}}) \text{ as } (\alpha \to \infty)$$

Another notation used in literature

$$\langle x^{2k} \rangle := \frac{I_{2k}(x)}{I_0(x)} = O(\frac{1}{\alpha^k})$$

$$I_{2k}(x) = \int_{-\infty}^{\infty} dx x^{2k} e^{-\alpha x^2} (u = \sqrt{\alpha}x)$$
$$= \int_{-\infty}^{\infty} \frac{du}{\sqrt{\alpha}} \frac{u^{2k}}{(\alpha)^{2k}} e^{-u} = \frac{1}{\alpha^k\sqrt{\alpha}} \int_{-\infty}^{\infty} du u^{2k} e^{-u^2}$$
$$= O(\frac{1}{\alpha^k\sqrt{\alpha}}) \text{ as } \alpha \to \infty$$

Example:

$$F_n(\alpha) = \int_0^{\infty} dx x^n e^{-\alpha x^2}$$

$v = \alpha x^2$, $dv = 2\alpha x dx = 2\alpha\sqrt{\frac{v}{\alpha}}dx$

$$dx = \frac{dv}{2\alpha\sqrt{\frac{v}{\alpha}}} = \frac{dv}{2\sqrt{\alpha v}}$$

Remember Gamma Function:

$$\Gamma(s) = \int_0^{\infty} dv v^{s-1} e^{-v}$$

Anyway,

$$F_n(\alpha) = \int_0^{\infty} \frac{dv}{2\sqrt{\alpha v}} \left(\frac{v}{\alpha}\right)^{n/2} e^{-v}$$

$$= \frac{1}{2}\frac{1}{\alpha^{(n+1)/2}}\int_0^\infty dv v^{\frac{n+1}{2}-1}e^{-v}$$

$$= \frac{1}{2}\frac{1}{\alpha^{n+1/2}}\Gamma\left(\frac{n+1}{2}\right) = O\left(\frac{1}{\sqrt{\alpha}\alpha^{\frac{n}{2}}}\right)$$

See also: Error Functions
Now we are going to generalize these for multi variable functions.

Multivariable Gaussian Integrals

Warning: NOT POWERS (tensor notation)

$$I(A) = \int_{\mathbb{R}^n} d^n x e^{-\sum_{ij} x^i A_{ij} x^j}$$

$$A_{ij} = A_{ji}$$

A is a positive definite matrix (all its eigenvalues are positive or, equivalently $\sum_{i,j} x^i A_{ij} x^i > 0$ for all $x \neq 0$)

$D = O^T A O$ where $O^T O = 1$

$D_{kl} = (O^T)_{ki} A_{ij} O_{jl}$ (summation over $i\&j$)

$$= O_{ik} A_{ij} O_{jl}$$

Now, let's make the change of variable:

$$x_j = O_{jl} u_l$$

Remember Jacobian:

$$J = \left|\det \frac{\partial x_j}{\partial u_l}\right| = |\det O_{jl}| = 1$$

$$\begin{aligned} I(A) &= \int_{\mathbb{R}^n} d^n u e^{-O_{ik} u_k A_{ij} O_{jlul}} \\ &= \int_{\mathbb{R}^n} d^n u e^{-u_k (O_{ik} A_{ij} O_{jl}) u_l} \end{aligned}$$

$$\int_{\mathbb{R}^n} d^n u e^{-(\lambda_1 u_1^2 + \lambda_2 u_2^2 + \cdots + \lambda u_n^2)}$$

$$\int_{-\infty}^\infty du_1 e^{-\lambda_1 u_1^2} \int_{-\infty}^\infty du_2 e^{-\lambda_2 u_{2^2}} \ldots$$

Then,

$$I(A) = \int_{\mathbb{R}} d^n x e^{-\sum_{ij} x_i A_{ij} x_j}$$

$$= \sqrt{\frac{\pi}{\lambda_1}} \sqrt{\frac{\pi}{\lambda_2}} \cdots \sqrt{\frac{\pi}{\lambda_n}}$$

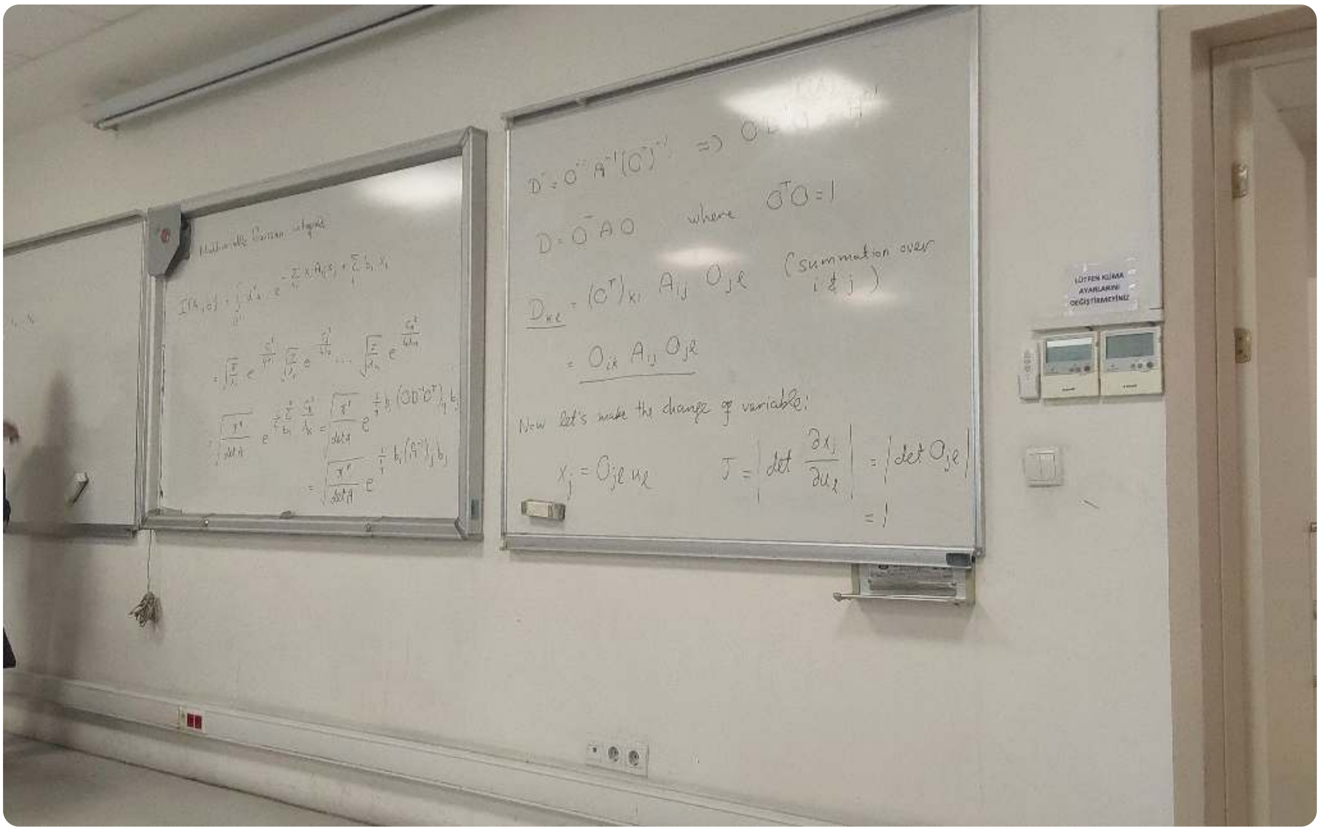
$$= \sqrt{\frac{\pi^n}{\det A}}$$

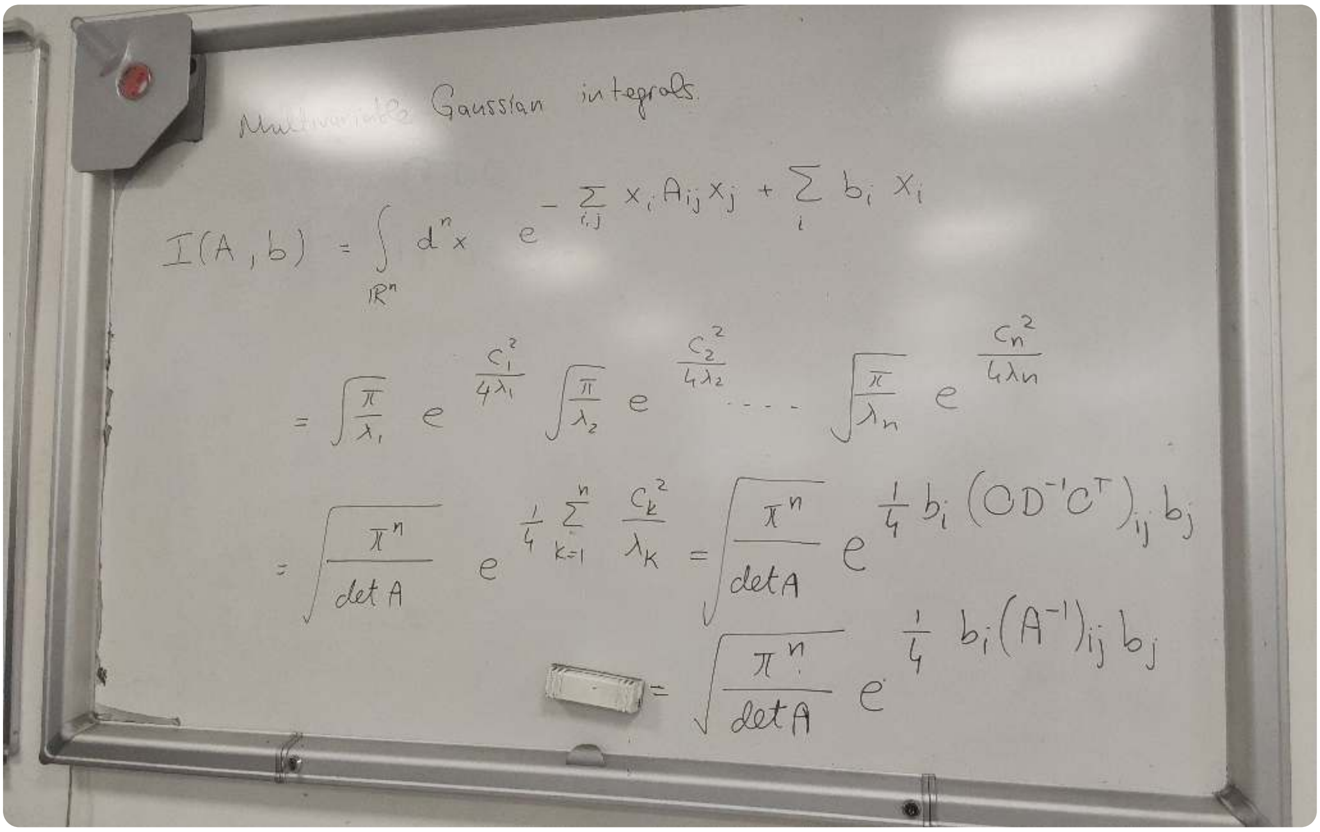
Let's make the problem even more complicated:

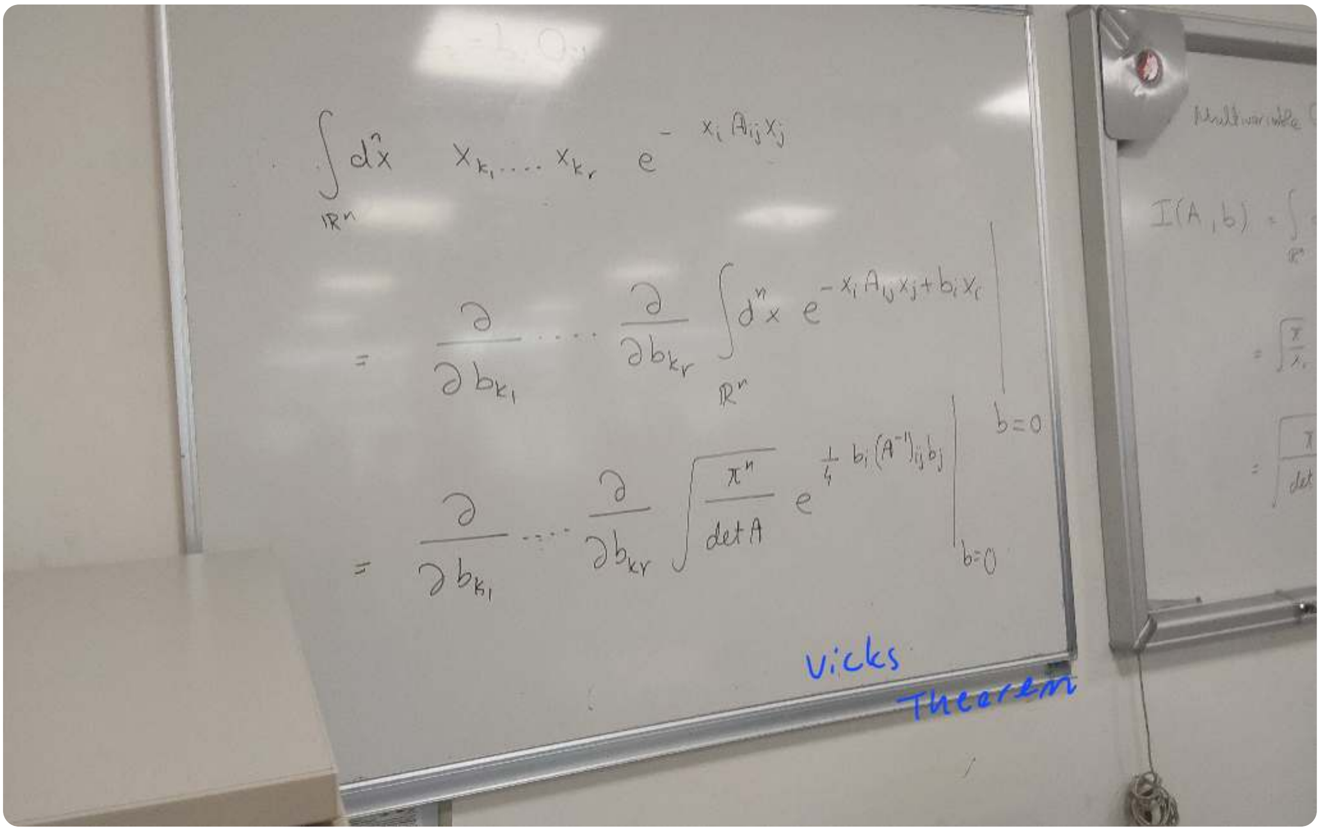
$$I(A, b) = \int_{\mathbb{R}} d^n x e^{-\sum_{ij} x_i A_{ij} x_j + \sum_i b_i x_i}$$

$$= \sqrt{\frac{\pi}{\lambda_1}} e^{\frac{c_1^2}{4\lambda_1}} \cdots$$

$$= \sqrt{\frac{\pi^n}{\det A}} e^{\frac{1}{4} \sum_{k=1}^n \frac{c_k^2}{\lambda_k}}$$


$D^{-1} = O^{-1} A^{-1} (O^T)^{-1} \Rightarrow O$
$D = O^{-1} A O$ where $O^T O = 1$
$D_{kl} = (O^T)_{ki} A_{ij} O_{jl}$ (summation over i & j)
$= O_{ik} A_{ij} O_{jl}$
Now let's make the change of variable:
$x_j = O_{jl} u_l$
$J = \left| \det \frac{\partial x_i}{\partial u_l} \right| = \left| \det O_{jl} \right| = 1$
LÜTFEN KLİMA AYARLARINI DEĞİŞTİRMEYİNİZ


Multivariable Gaussian integrals.
$I(A,b) = \int_{\mathbb{R}^n} d^n x \; e^{-\sum_{i,j} x_i A_{ij} x_j + \sum_i b_i x_i}$
$= \sqrt{\frac{\pi}{\lambda_1}} e^{\frac{c_1^2}{4\lambda_1}} \sqrt{\frac{\pi}{\lambda_2}} e^{\frac{c_2^2}{4\lambda_2}} \cdots \sqrt{\frac{\pi}{\lambda_n}} e^{\frac{c_n^2}{4\lambda_n}}$
$= \sqrt{\frac{\pi^n}{\det A}} e^{\frac{1}{4}\sum_{k=1}^{n} \frac{c_k^2}{\lambda_k}} = \sqrt{\frac{\pi^n}{\det A}} e^{\frac{1}{4} b_i (OD^{-1}O^T)_{ij} b_j}$
$= \sqrt{\frac{\pi^n}{\det A}} e^{\frac{1}{4} b_i (A^{-1})_{ij} b_j}$



$\int_{\mathbb{R}^n} d^n x \; x_{k_1} \cdots x_{k_r} \; e^{-x_i A_{ij} x_j}$
$= \frac{\partial}{\partial b_{k_1}} \cdots \frac{\partial}{\partial b_{k_r}} \int_{\mathbb{R}^n} d^n x \; e^{-x_i A_{ij} x_j + b_i x_i} \Big|_{b=0}$
$= \frac{\partial}{\partial b_{k_1}} \cdots \frac{\partial}{\partial b_{k_r}} \sqrt{\frac{\pi^n}{\det A}} \; e^{\frac{1}{4} b_i (A^{-1})_{ij} b_j} \Big|_{b=0}$
Vicks Theorem
Multivariable
$I(A, b) =$