PHYS 325 - Lecture 3 01.10.24

## Properties of Uniform Convergence

(i) Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ (uniformly)

for all $x \in S \subseteq \mathbb{R}$

Then for $x_0 \in S$,

$$\lim_{x\to x_0} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x)$$

$$\Rightarrow \lim_{x\to x_0} f(x) = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x)$$

- Now assume $f_n$'s are continuous on $S$.

Then
$$\lim_{x\to x_0} f(x) = \lim_{n\to\infty} f_n(x) = f(x_0)$$

So $f(x)$ is continuous on $S$.

ii) Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ (uniformly) on $S \subseteq \mathbb{R}$

Let $(a,b) \subseteq S$ ⊛ (Provided $\int_a^b dx\, f_n(x)$ exists)

$$\lim_{n\to\infty} \int_a^b dx\, f_n(x) = \int_a^b dx \lim_{n\to\infty} f_n(x)$$

$$= \int_a^b dx\, f(x) \quad \text{(By definition)}$$

iii) Suppose $\lim_{n \to \infty} f_n(x) = f(x)$ (pointwise)

$f_n'(x)$ exists & continuous. And:

$$\lim_{n \to \infty} f_n'(x) = g(x) \quad \text{(uniformly)}$$

Then $g(x) = f'(x)$

## Series of Functions

$$\sum_{n=0}^{\infty} f_n(x)$$

is called uniformly convergent on $S \subseteq \mathbb{R}$ if the sequence of partial sums $\{S_N(x)\}_{N=0}^{\infty}$ is uniformly convergent on $S$ $\left[ S_N(x) = \sum_{n=0}^{N} f_n(x) \right]$

## Weierstrass M-Test (Proof in Arfken)

The series $\sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent on $S \subseteq \mathbb{R}$ provided there exists a sequence $\{M_n\}_{n=0}^{\infty}$ such that:

(i) $|f_n(x)| < M_n \quad \forall n \in \mathbb{Z}_{\geq 0}$ and $\forall x \in S$

(ii) $\sum_{n=0}^{\infty} M_n$ is convergent

**Ex:** $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $e^x$ Let $R>0$ and $\mathcal{D}=(-R,R)$ For $x\in\mathcal{D}$

$$\left|\frac{x^n}{n!}\right| = \frac{|x|^n}{n!} < \frac{R^n}{n!} \qquad \underline{\text{Claim:}} \quad M_n = \frac{R^n}{n!}$$

- check (ii) $\sum_{n=0}^{\infty} \frac{R^n}{n!}$ is convergent.

- Apply ratio test:

$$\lim_{n\to\infty}\left|\frac{M_{n+1}}{M_n}\right| = \lim_{n\to\infty} \frac{R^{n+1}}{(n+1)!}\,\frac{n!}{R^n} = \lim_{n\to\infty} \frac{R}{n} = 0$$

$\therefore \sum_{n=0}^{\infty} M_n$ $\underline{\text{is}}$ convergent.

So, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is uniformly convergent.

**Properties of Uniformly Convergent Series** $F(x)=\sum_{n=0}^{\infty} f_n(x)$

(i) $\lim_{x\to x_0} \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \lim_{x\to x_0} f_n(x)$

$x\in\mathcal{D}$

$$\lim_{x\to x_0} \underbrace{\lim_{N\to\infty} S_N(x)}_{} = \lim_{N\to\infty} \underbrace{\lim_{x\to x_0} S_N(x)}_{}$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} f_n(x) \qquad\qquad = \lim_{x\to x_0} \sum_{n=0}^{N} f_n(x)$$

$$= \sum_{n=0}^{\infty} f_n(x) \qquad\qquad = \sum_{n=0}^{N} \lim_{x\to x_0} f_n(x)$$

partial sum of the series

P.S) Uniform convergence implies pointwise convergence.

- Assume $f_n(x)$ is continuous $\forall n \in \mathbb{Z}_{\geq 0}$, $\forall x \in S$

$$\lim_{x \to x_0} F(x) = \sum_{n=0}^{\infty} f_n(x_0)$$
$$\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad F(x_0)$$

(ii) $(a,b) \subseteq S$

$$\int_a^b dx \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \int_a^b dx\, f_n(x)$$

provided $\int_a^b dx\, f_n(x)$ exists.

→ There are simpler tests for this making use of Lebeck integral.

(if a function is continuous, it has a Reimann & Lebeck integral and they are equal to each other)

## Complex Numbers

imaginary number : $i$ : $i^2 = -1$

complex number : $z = x + iy$

$x = \mathrm{Re}\, z$

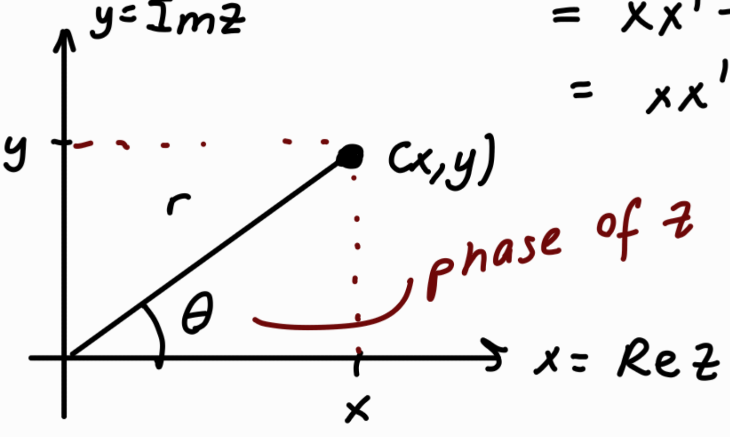
$y \in \mathrm{Im}\, z$

$x, y \in \mathbb{R}$

**Properties:** $(x+iy) + (x'+iy') = (x+x') + i(y+y')$

let $c \in \mathbb{R}$ $\quad c(x+iy) = cx + icy$

$$(x+iy)(x'+iy') = xx' + xiy' + iyx' + iyiy'$$
$$= xx' + ixy' + iyx' + i^2 yy'$$
$$= xx' - yy' + (xy' + yx')i$$




Argan Plane $\approx \mathbb{R}^2$

$(x,y)$ Cartesian Coordinates

$(r,\theta)$ Polar Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$\tan\theta = y/x$

modulus (magnitude) / length of z $\leftarrow r = \sqrt{x^2+y^2}$

z from Cartesion to polar:

$$z = x+iy = r\cos\theta + r\sin\theta\, i$$
$$= r(\cos\theta + \sin\theta\, i)$$

- $z = x+iy = r(\cos\theta + \sin\theta\, i)$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Taylor series: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{i\theta} = \sum^{\infty} \frac{(i\theta)^n}{n!}$$

Separate this sum into even terms and odd terms.

- $$e^{i\theta} = \sum^{\infty} \frac{(i\theta)^n}{n!} = \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!}$$

$$\boxed{\begin{array}{l} \bullet\ i^{2k+1} = i\, i^{2k} = i\,(-1)^k \\ \bullet\ i^{2k} = (-1)^k \end{array}}$$

$= \cos\theta + i\sin\theta$ (You must've recognized the series)

(There is a suggested problems for a similar problem)

The tests we used so far has close analogues for complex numbers.

EX:

- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ Convergent for all $z \in \mathbb{C}$

Ratio Test.

$$\lim_{n\to\infty} \frac{|z|^{n+1}}{(n+1)!} \frac{n!}{|z|^n} = \lim_{n\to\infty} \frac{|z|}{n+1} = 0$$

EX:

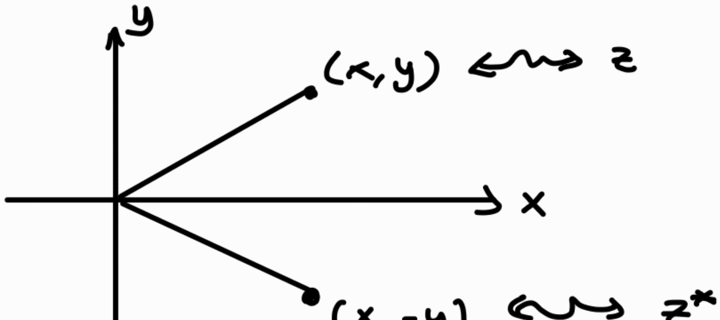
$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$

$$\cosh z = \frac{e^z + e^{-z}}{2} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}$$

## Complex Conjugate

$z = x + iy$

$z^* = \bar{z} = x - iy$




$z\,z^* = x^2 + y^2 = \underbrace{|z|^2}$ modulus $z^2$

**Ex:** $e^{i\theta} = \cos\theta + i\sin\theta$

cos θ : even function

sin θ : odd function

$$(e^{i\theta})^* = \cos\theta - i\sin\theta$$

$$= \cos(-\theta) - i\left[-\sin(-\theta)\right]$$

$$= \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$$

So, $e^{i\theta}(e^{i\theta})^*$ is:

$$e^{i\theta}(e^{i\theta})^* = e^{-i\theta}e^{i\theta} = e^0 = 1$$

## Division of Complex Numbers

$z = x + iy$

$w = a + ib$

$$\frac{z}{w} = \frac{x+iy}{a+ib} = \frac{(x+iy)(a-ib)}{a^2+b^2}$$

$$\frac{z}{w} = \frac{(xa+yb) + i(ay - bx)}{a^2+b^2}$$