PHYS 325 - Lecture 15 21 Nov 24

$L^2(\mathbb{R}^2)$ $\int_{\mathbb{R}^n} d^n x \, |f(x)|^2 < \infty$ $x = (x_1, x_2, \ldots, x_n)$

$$\langle f | g \rangle = \int_{\mathbb{R}^n} d^n x \, f^*(x) g(x)$$

Unitarily equivalent

$w : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ $L^2_w(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} d^n x \, \underbrace{w(x)}_{\text{weight function}} |f(x)|^2$$

$$\int_{\mathbb{R}^n} d^n x \, w(x) f^*(x) g(x) = \langle f | g \rangle_w$$

$$U : L^2_w(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

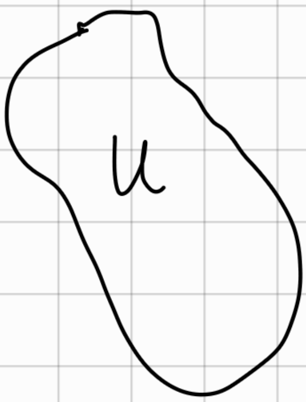
$$f \longmapsto \sqrt{w} \, f$$

$$\langle Uf | Ug \rangle = \int d^n x \, (\sqrt{w} f)^* (\sqrt{w} g) = \int d^n x \, w f^* g$$

$$= \langle f | g \rangle_w$$

• Sometimes you have only one region, Then we specify the region.

→ $L^2(U)$ ($U$ is a subset of $\mathbb{R}^n$)




$$\int_U d^n x \, |f(x)|^2 < \infty$$

→ $w : U \to \mathbb{R}^+$

$L^2_w(U)$

## Orthonormal Polynomial

$$P_n(x) = \frac{1}{w(x)} \frac{d^n}{dx^n}\left[ w(x)\, s^n(x) \right] \qquad \int_U dx \, w(x)\, P_n^*(x) P_m(x) = \delta_{nm}$$

order of polynomial

sol. of harmonic oscillator

# ⊛ – ORTHONORMAL POLYNOMIALS – ⊛

| | $\mathcal{U}$ | $w(x)$ | $s(x)$ | |
|---|---|---|---|---|
| Hermite Polynomials | $\mathbb{R}$ | $e^{-x^2}$ | $1$ | $H_n(x)$ |
| Lagurre | $[0,\infty)$ | $x^{\nu}e^{-x}(\nu>-1)$ | $x$ | $L_n^{\nu}(x)$ |
| Legendre | $[-1,1]$ | $1$ | $1-x^2$ | $P_n(x)$ |
| Chebichev 1 | $[-1,1]$ | $(1-x^2)^{-1/2}$ | $1-x^2$ | $T_n(x)$ |
| Chebichev 2 | $[-1,1]$ | $(1-x^2)^{1/2}$ | $1-x^2$ | $U_n(x)$ |

("For each value of $\nu$, I have an orthonormality relation")

Rest is in DK. 207-208 (Jacobian etc..)

⊛ They form an orthonormal basis for $L^2_w(\mathcal{U})$

Ex: Any $f \in L^2(\mathbb{R})$ can be expanded as:

$$f(x) = \sum_{n=0}^{\infty} c_n H_n(x) \quad (c_n \in \mathbb{C})$$

Any $f \in L^2([-1,1])$ can be expanded as:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \quad (c_n \in \mathbb{C})$$

Ex: Suppose we have an orthonormal basis for an inner product space V.

$$\{ |e_1\rangle, |e_2\rangle, |e_3\rangle, \ldots \}$$

let $|v\rangle \in V$, $\quad |v\rangle = \sum_{n=0}^{\infty} c_n |e_n\rangle \quad (c_n \in \mathbb{C})$

$$\langle e_m | v \rangle = \sum_{n=0}^{\infty} \langle e_m | c_n e_n \rangle = \sum_{n=0}^{\infty} c_n \overbrace{\langle e_m | e_n \rangle}^{\delta_{mn}} = c_m$$

$$c_m = \langle e_m | v \rangle$$

So, I can write:

$$|v\rangle = \sum_{n=0}^{\infty} c_n |e_n\rangle = \sum_{n=0}^{\infty} (\langle e_m | v \rangle) | e_n \rangle$$

$$= \sum_{n=0}^{\infty} |e_n\rangle \langle e_m | c \rangle$$

Completeness Relation (Resolution of Identity) $\quad \sum_{n=0}^{\infty} |e_n\rangle\langle e_n| = \mathbb{1}$

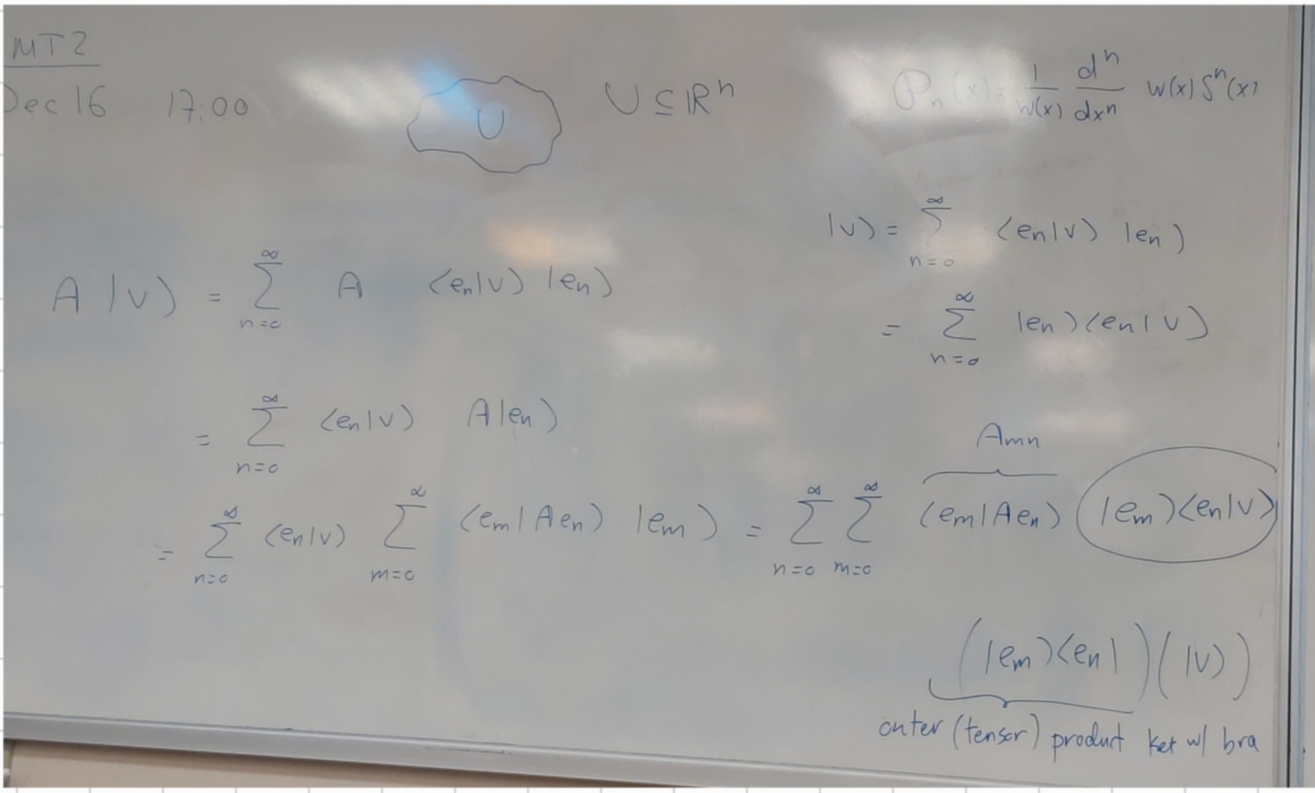
Tensor product of a ket with a bra: (dyad)

$$\left( \sum_{n=0}^{\infty} |e_n\rangle\langle e_n| \right) |v\rangle$$

Bra's are linear operators on kets (linear functional)

$$\sum_{n=0}^{\infty} (|e_n\rangle\langle e_n|) |v\rangle = \sum_{n=0}^{\infty} |e_n\rangle (\langle e_n | v \rangle) = |v\rangle$$

Example:




$$(*) \quad A = \sum_{n,m} \langle e_m|A|e_n\rangle \quad |e_m\rangle\langle e_n|$$

$$= \sum_{n,m} |e_m\rangle\langle e_m|A|e_n\rangle\langle e_n| = A$$

These must form an ortho-normal basis for the vectorspace

Remember the completeness relation: $\sum_m |e_m\rangle\langle e_m| = \mathbb{1}$

$$\sum_n |e_n\rangle\langle e_n| = \mathbb{1}$$

$$(*) \; A = \mathbb{1} A \mathbb{1} = \sum_m |e_m\rangle\langle e_m| A \sum_n |e_n\rangle\langle e_n|$$

$$= \sum_{m,n} |e_m\rangle\langle e_m|A|e_n\rangle\langle e_n|$$

$$= \sum_{m,n} \langle e_m|A|e_n\rangle \; |e_m\rangle\langle e_n|$$

HINT: If you can't remember (*) just start with (*) to derive it.

Such a basis is called a complete set / basis.

in infinite dimensional spaces
(in infinite dimensional bases, there's a small difference)

## Gram - Schmidt orthogonalization

Start with an arbitrary basis $\{ v_1, v_2, \dots \}$

$$e_1 : \frac{v_1}{\| v_1 \|} \longrightarrow \| e_1 \| = 1$$


Substract the projection to get an orthogonal vector (to $e_1$)
(And normalize)

$\langle e_1 | v_2 \rangle e_1$ → projection of $v_2$ onto $e_1$

- Can be written as: $\big( |e_1\rangle\langle e_1| \big) |v_2\rangle$

projection operator

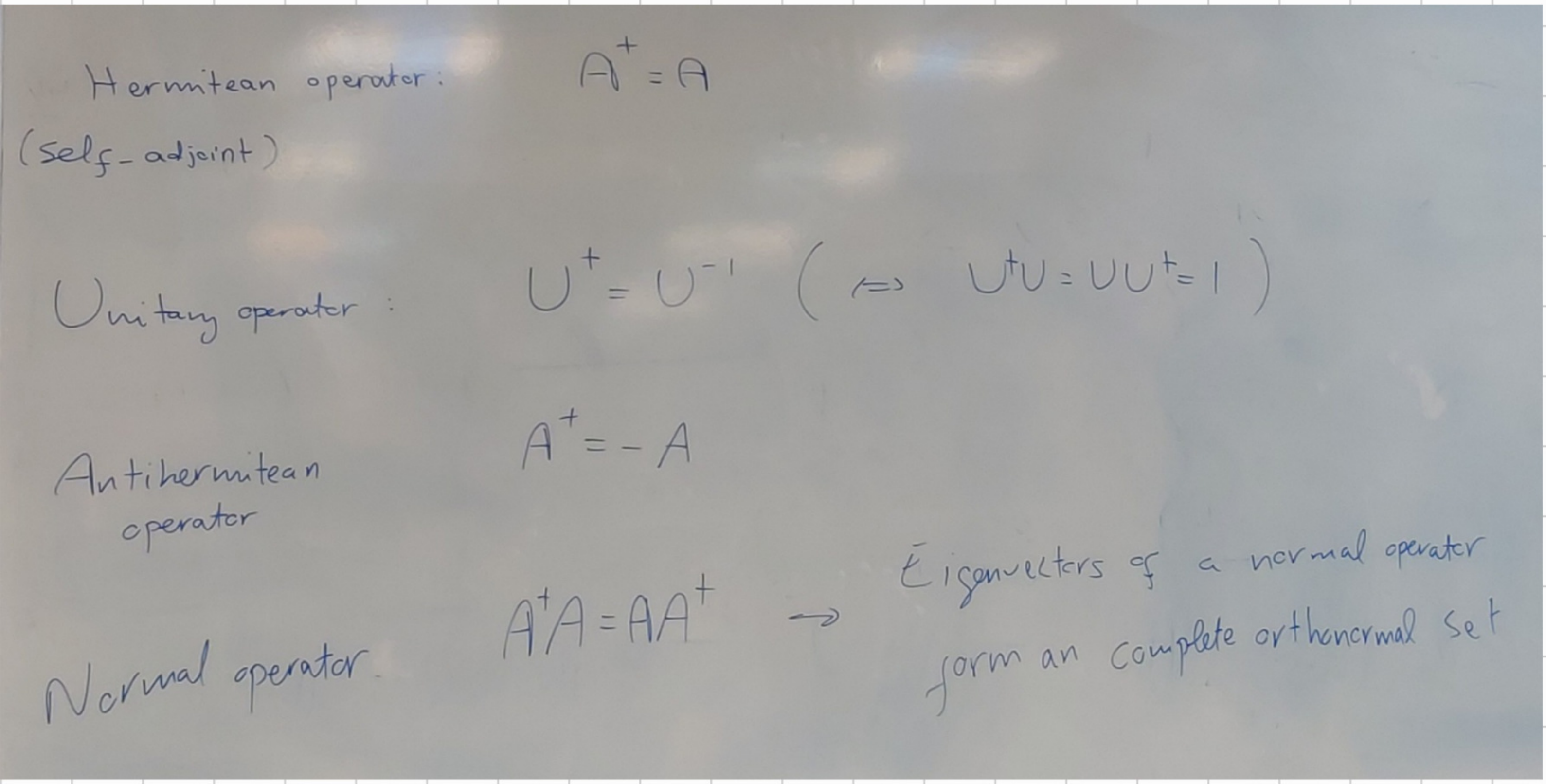
$$e_2 : \frac{v_2 - \text{projection}}{\| v_2 - \text{projection} \|} = \frac{v_2 - \langle e_1 | v_2 \rangle e_1}{\| v_2 - \langle e_1 | v_2 \rangle e_1 \|} \qquad \| e_2 \| = 1$$

- Let's check orthogonality.

$$\langle e_1 | e_2 \rangle = \left\langle e_1 \middle| \frac{v_2 - \langle e_1 | v_2 \rangle e_1}{\| v_2 - \langle e_1 | v_2 \rangle e_1 \|} \right\rangle$$

$$= \frac{1}{\| v_2 - \langle e_1 | v_2 \rangle e_1 \|} \langle e_1 | v_2 - \langle e_1 | v_2 \rangle e_1 \rangle = \frac{\langle e_1 | v_2 \rangle - \langle e_1 | v_2 \rangle \overbrace{\langle e_1 | e_1 \rangle}^{1}}{\| v_2 - \langle e_1 | v_2 \rangle e_1 \|} = 0$$

$$e_3 := \frac{v_3 - \langle e_1 | v_3 \rangle e_1 - \langle e_2 | v_3 \rangle e_2}{\| v_3 - \langle e_1 | v_3 \rangle e_1 - \langle e_2 | v_3 \rangle e_2 \|}$$




$A = A^+$ Let $\{ |e_1\rangle, |e_2\rangle, \dots \}$ be a complete orthonormal set of eigenvalues of $A$.

$$A|e_n\rangle = \lambda_n |e_n\rangle \qquad A = \mathbb{1} A \mathbb{1}$$

$$A = \mathbb{1} A \mathbb{1} = \sum_{m,n} |e_m\rangle\langle e_m| A |e_n\rangle\langle e_n| = \sum_{m,n} \underbrace{\langle e_m | A | e_n \rangle}_{\langle e_m | \lambda_n e_n \rangle = \lambda_n \langle e_m | e_n \rangle = \lambda_n \delta_{mn}} |e_m\rangle\langle e_n|$$

$$= \sum_n \lambda_n |e_n\rangle\langle e_n|$$

✪ For hermitian op. eigenvalues were real, but this is not always the case for other normal operators.

✪ Suppose I have $f(x) = e^x$ how do I define $e^A$?

Taylor expansion!

(*) $f(A) = e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$

You can show each entry is convergent!

Another way of defining it (when no taylor expansion):

$$f(A) := \sum_n f(\lambda_n) |e_n\rangle\langle e_n|$$

(**) $$e^A := \sum_n e^{\lambda_n} |e_n\rangle\langle e_n|$$

(More general def. valid for normal operators)

- How can we relate (*) to (**)

$$A|e_n\rangle = \lambda_n |e_n\rangle \quad \rightarrow \quad A = \sum_n \lambda_n |e_n\rangle\langle e_n|$$

$$A^2 = \sum_n \lambda_n |e_n\rangle\langle e_n| \sum_m \lambda_m |e_m\rangle\langle e_m| = \sum_n \lambda_n^2 |e_n\rangle\langle e_n|$$

$$A^3 = \sum_n \lambda_n^3 |e_n\rangle\langle e_m|$$

$\vdots$

$$e^A = \sum_n |e_n\rangle\langle e_n| + \sum_n \lambda_n |e_n\rangle\langle e_n| + \frac{1}{2!}\sum_n \lambda_n^2 |e_n\rangle\langle e_n| + \cdots$$

$$= \sum_n \left(1 + \lambda_n + \frac{\lambda_n^2}{2!} + \cdots\right) |e_n\rangle\langle e_n| = \sum_n e^{\lambda_n} |e_n\rangle\langle e_n|$$

(**) is more general since not all functions are Taylor expandible.

Exercise: An operator is called positive if $\langle v|Av\rangle > 0$ for all $v \in V$.

Show that $A$ is a positive if and only if all its eigenvalues are positive.

EX: $A^{+} = A$, $A$ positive operator

$$\sqrt{A} := \sum \sqrt{\lambda_n}\, |e_n\rangle\langle e_n|$$