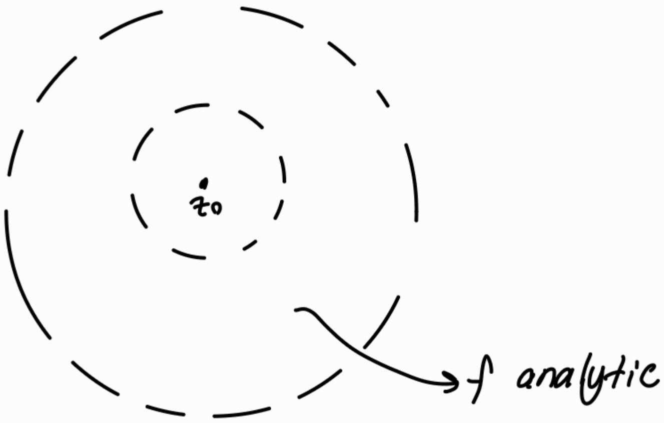
PHYS 325 Lecture 8 22.10.2024

Remember

Laurent Expansion



$$\boxed{f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n}$$

- If $f(z)=\sum_{n=-N}^{\infty} a_n (z-z_0)^n$ $a_{-N} \neq 0$ "large negative"

  then z_0 is called a pole order of N.

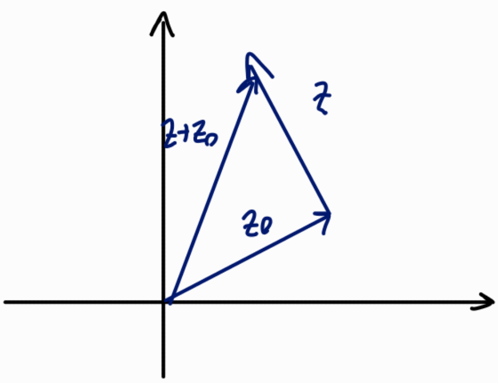
- If there is no such N then z_0 is called an essential singularity

so there are arbitrarily large negative powers in the expansion.

Examples with Complex Functions

(i) Let z_0 be a fixed complex number.

$$f(z) = z+z_0 \quad \text{(translation)}$$
$$= (x+iy)+(x_0+iy_0)$$
$$= (x+x_0)+i(y+y_0)$$



$f^{-1}(z)=z-z_0$. Both f & f^{-1} are analytic.

1 - 1 map (map = function)

This is why it's called a translation.

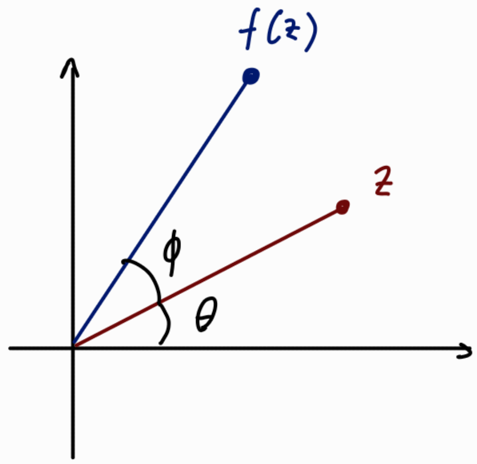
ii) Let $z_0 \neq 0$ and $|z_0| = 1$ be a fixed complex numbers.

$$f(z) = z_0 z \quad \text{(rotation)}$$

$$f^{-1}(z) = \frac{z}{z_0} \quad \text{1-1 map.}$$

Write it in polar coordinates to see it's a rotation:

$$z = re^{i\theta}$$
$$z_0 = e^{i\phi}$$
$$f(z) = z_0 z = e^{i\phi} re^{i\theta} = r e^{i(\theta+\phi)}$$



iii) $f: \mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$

$$f(z) = \frac{1}{z} \quad \text{(inversion)}$$

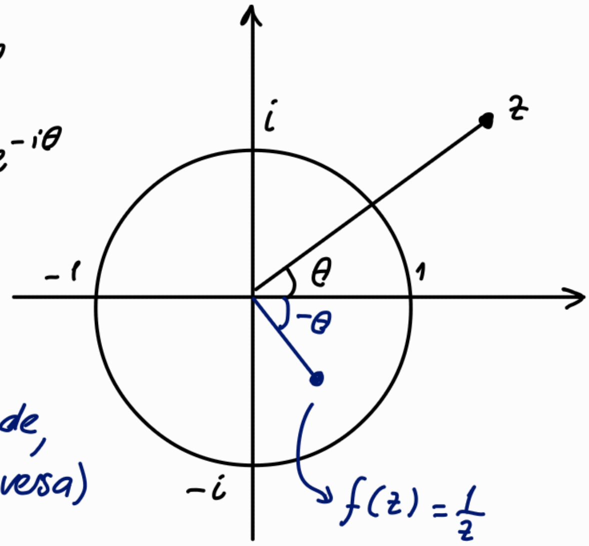
$$f^{-1}(z) \quad \frac{1}{z}$$

$z = re^{i\theta}$

$\Rightarrow f(z) = \frac{1}{r} e^{-i\theta}$

1-1 map

inversion with respect to the unit circle (if z is outside, it will go inside the circle vice versa)



iv) $f(z) = z^2$

$$z = re^{i\theta} \Rightarrow f(z) = r^2 e^{2i\theta}$$

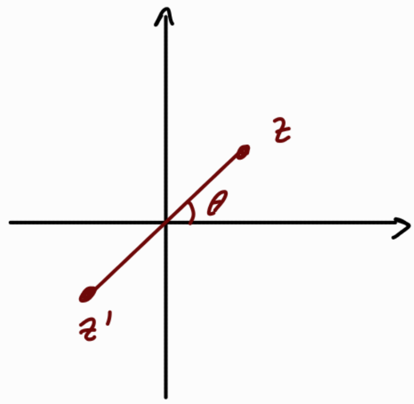
Let's look at another point z'

$$z' = re^{i(\theta+\pi)} \Rightarrow f(z') = r^2 e^{2i(\theta+\pi)}$$

$$f(z') = r^2 e^{2i\theta} e^{2i\pi}$$

you can read it off the unit circle graph (there's one in Ex (iii))

$$\boxed{f(z') = r^2 e^{i2\theta} \cdot 1 = f(z)}$$



2-1 map (No inverse! (Must be 1-1 & onto))

- For $\mathbb{R}$ too...

v) Multivalued Functions

$f(z) = z^{1/2}$

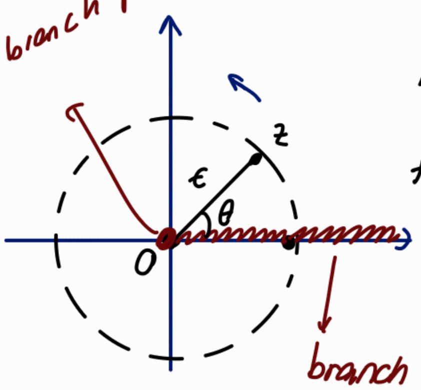
Write this in polars ... $\quad z = re^{i\theta} = re^{i(\theta+2\pi)}$

$f(z) = z^{1/2} = \boxed{\sqrt{r}\, e^{i\frac{\theta}{2}}}$

$f(z) = z^{1/2} = \sqrt{z}\, e^{i(\frac{\theta}{2}+\pi)} = \boxed{-\sqrt{r}\, e^{i\frac{\theta}{2}}}$ $\quad (e^{i\pi} = -1)$

multivalued!

P.S:



Suppose we are moving z around the circle.

initially: $\theta = 0$ $\qquad f_{initial} = \sqrt{r}\, e^{i0} = \sqrt{r}$

finally: $\theta = 2\pi$ $\qquad f_{final} = \sqrt{r}\, e^{i2\pi} = \sqrt{r}\, e^{i\pi} = -\sqrt{r}$

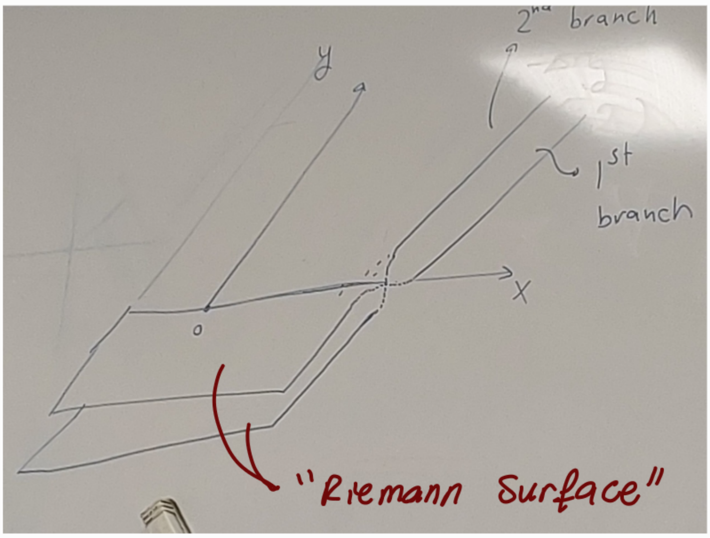
"Once I pass, I reset the angles and put a $\ominus$ sign"

1st branch: $0 \le \theta < 2\pi$ $\qquad f_1(z) = z^{1/2} = \sqrt{r}\, e^{i\frac{\theta}{2}}$

2nd branch: $0 \le \theta < 2\pi$ $\qquad f_2(z) = -\sqrt{r}\, e^{i\frac{\theta}{2}}$

f_1 and f_2 are the branches of the function.

Picture of The Branches:



"Two copies of the plane"

- All multivalued functions can be defined in bigger surfaces where they are single valued (Riemann Surface)
  ↳ manifold

- I go over the origin in the circle and detect the multivaluedness, (Branch Point)

Definition.

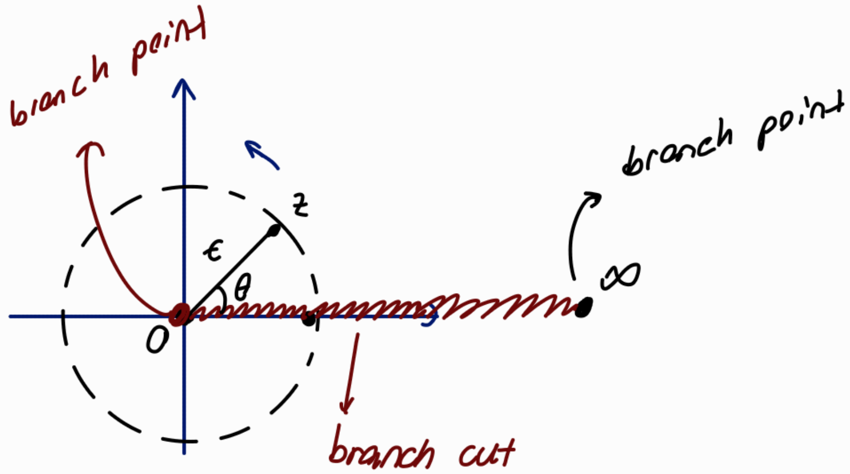
- A branch Point z_0 is a point with the following property:
    - if we go around a small enough circle centered around at z_0, then at the end of the cycle we get a different value of the function.

$f(z) = z^{1/2}$ $\qquad z = \frac{1}{z'}$

$\tilde{f}(z') = f\left(\frac{1}{z}\right) = \frac{1}{z'^{1/2}} = (z')^{-1/2}$

- We have a branch at $z' = 0$ which corresponds to $z \to \infty$

Usually a branch cut is a curve connecting two branch points.



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vi) $f(z) = e^z = e^x e^{iy}$ (Here, cartesian coordinates are more convinient)

- Let's calculate : $f(z + i2n\pi) = f(x + i(y + 2n\pi)) = e^x e^{i(y+2n\pi)}$

$$= e^x e^{iy} e^{i2n\pi}$$

$$= e^x e^{iy} = f(z)$$

$(n \in \mathbb{Z})$ $\quad \infty : 1$ map

vii) $f(z) = \ln(z)$ ("inverse" of e^z) ("" because ∞ values)

$$z = re^{i\theta} = re^{i(\theta+2n\pi)}$$

$$f(z) = \ln z = \ln(re^{i\theta}) = \ln(r) + \ln(e^{i\theta}) = \ln(r) + i\theta$$

$$= \ln r + i\theta + i2n\pi$$

(∞'ly valued function)

∞ branches

∞ planes!

($n=0$ is called the principle branch)

- Let's look at branch cuts here

initially: $\theta = 0$ $\quad f_{initial} = \ln r$

finally: $\theta = 2\pi$ $\quad f_{final} = \ln r + i2\pi$

$\rightarrow$ $z=0$ is a branch point.

Döner macht schöner

- Behaviour at ∞.

$$z = \frac{1}{z'} \qquad \tilde{f}(z') = f\left(\frac{1}{z'}\right) = \ln \frac{1}{z'}$$

$$= \ln \frac{1}{r'e^{i\theta'}}$$

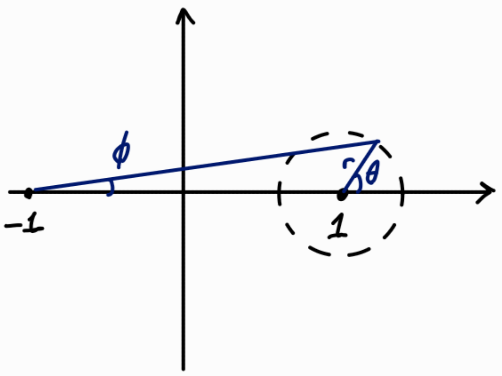
$$= \ln \frac{1}{r'} - i\theta' = -\ln r' - i\theta'$$

$z'=0$ is a branch point of $\tilde{f}$ $\qquad$ $z=\infty$ is a branch point of f.

- Branch points are intrinsic to function.
- Branch cuts are up to your convinience. Chose whatever you like.

- When dealing with multivalued function, you must specify the branch you are working on.

(viii) $f(z) = (z^2-1)^{1/2} = (z-1)^{1/2}(z+1)^{1/2}$



$(z-1) = re^{i\theta}$

$(z+1) = \rho e^{i\phi}$

Initial: $\theta = 0$, $\phi = 0$

$(z-1)^{1/2} = \sqrt{r}$

$(z+1)^{1/2} = \sqrt{\rho}$

$f_{initial} = \sqrt{r}\sqrt{\rho}$

Final: $\theta = 2\pi$, $\phi = 0$

$(z-1)^{1/2} = -\sqrt{r}$

$(z+1)^{1/2} = \sqrt{\rho}$

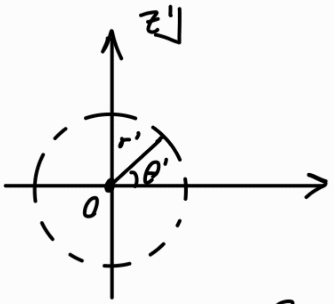
$f_{final} = -\sqrt{r}\sqrt{\rho}$

→ $z=1$ is a branch point.

- Similarly $z=-1$ is also a branch point.

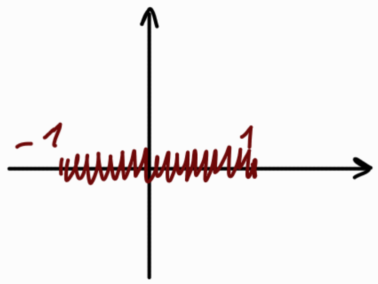
Behaviour in ∞:

$$\tilde{f}(z') = f\left(\frac{1}{z'}\right) = \frac{(1-z'^2)^{1/2}}{z'}$$

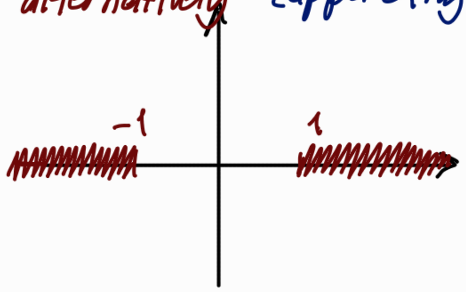


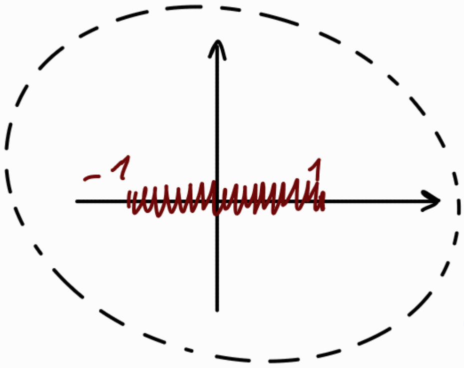
- initially: $\theta' = 0$ $\quad f_{initial} = \dfrac{(1-r'^2)^{1/2}}{r'}$
- finally: $\theta' = 2\pi$ $\quad f_{final} = \dfrac{(1-r'^2)^{1/2}}{r'}$
- $z = \infty$ not a branch point of f.

Choosing a branch cut:



alternatively (apparently useful in some calculations)





Behaviour in a big circle:

Doesn't change because "I don't cross the branch cut."

- Riemann S. you construct with either branch cut is analytically equivalent.
  → They differ by a change of variable.

Exercise: Follow the big circle

next time: calculus of residues