PHYS 325 Lecture 19

05.12.24

2nd order diff. operator

$$L_x = p_2(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_0(x) \qquad a \leqslant x \leqslant b$$

$$L_x^{+} = \frac{1}{w(x)}\left[\frac{d^2}{dx^2} w(x)\, p_2^*(x) - \frac{d}{dx} w(x)\, p_1^*(x) + w(x)\, p_0^*(x)\right] \quad \text{adjoint of } L_x$$

$$\langle v | L_x u\rangle_w - \langle L_x^{+} v | u\rangle_w = \underbrace{\int_a^b dx \frac{d}{dx} Q[v^*(x), u(x)]}$$

$$S = \left[w(x) v^*(x) p_2(x) u'(x) - (w(x) v^*(x) p_2(x))' u(x) + w(x) v^*(x) p_1(x) u(x)\right]_a^b$$

L_x is called **self-adjoint** if $L_x^{+} = L_x$

Boundary Conditions

$(\alpha_1, \beta_1, \gamma_1, \delta_1) \neq C(\alpha_2, \beta_2, \gamma_2, \delta_2)$ otherwise B.C. would be the same.

$$B_j(u) = \alpha_i u(a) + \beta_i u'(a) + \gamma_i u(b) + \delta_i u'(b)$$

$(i=1,2)$ $\alpha, \beta, \gamma, \delta \in \mathbb{C}$

$$\left.\begin{array}{l} B_1(u)=0 \\ B_2(u)=0 \end{array}\right\} \text{linearly independent}$$

Fact: Given the boundary conditions $B_1(u) \neq 0$ and $B_2(u) \neq 0$ there exists unique boundary conditions on V.

$$\underbrace{B_3(v)=0, \quad B_4(v)=0 \quad \text{so that} \quad S=0}$$

These are called adjoints of $B_1(u) = 0 = B_2(u)$

$$L_x = p_2(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_0(x) \qquad a \leq x \leq b$$

$$L_x^+ = \frac{1}{w(x)}\left[\frac{d^2}{dx^2} w(x) p_2^*(x) - \frac{d}{dx} w(x) p_1^*(x) + w(x) p_0^*(x)\right]$$ adjoint of L_x

with b.c.s:

$$B_1(u) = 0 = B_2(v)$$
$$B_3(u) = 0 = B_4(v) \longrightarrow S' = 0$$

L_x is called Hermitian if $L_x^+ = L_x$ AND $B_1 = B_3$, $B_2 = B_4$

$$\alpha_1 = \alpha_3 \qquad \alpha_2 = \alpha_4$$
$$\beta_1 = \beta_3 \qquad \beta_2 = \beta_4$$
$$\vdots \qquad \vdots$$

$$\langle v | L_x u\rangle_w = \langle L_x^+ u | v\rangle_w$$

Example: $L_x = -\frac{d^2}{dx^2}$ $\quad a \leq x \leq b \quad w = 1 \quad (L_x^+ = L_x)$

$$B(u) = \alpha u(a) + \beta u'(a) + \gamma u(b) + \delta u'(b)$$

$$\langle v | L_x u\rangle - \langle L_x^+ v | u\rangle = v^{*\prime}(b) u(b) - v^*(b) u'(b) - v^{*\prime}(a) u(b) + v^*(a) u'(a)$$

$B_1(u) = u(a)$ $\quad (\alpha_1 = 1, \beta_1 = \gamma_1 = \delta_1 = 0)$

$B_2(u) = u(b)$ $\quad (\gamma_2 = 1, \alpha_2 = \beta_2 = \delta_2 = 0)$

Remember: Conditions on v do not depend on u.

$$\left.\begin{aligned} u(a) &= 0 \\ u(b) &= 0 \end{aligned}\right\}$$ Dirichlet boundary conditions

$$S = -v^*(b) u'(b) + v^*(a) u'(a) = 0 \qquad \overbrace{v(a) = 0 \quad v(b) = 0}$$

"Being Hermitian depends on boundary condition!"

(ii) $u(a) = u(b)$ $\quad$ $u(a) - u(b) = 0$ $\quad$ Periodic Boundary Conditions

$u'(a) = u'(b)$ $\quad$ $u'(a) - u'(b) = 0$

$v(a) = v(b)$ $\quad$ $v'(a) = v'(b)$

$$S = [v^{*\prime}(b) - v^{*\prime}(a)]\, u(a) + [v^*(a) - v^*(b)]\, u'(a) = 0$$

(iii) $u'(a) = 0$ $\quad$ Neumann Boundary Conditions

$u'(b) = 0$

- Why do we use weights (w)?

- Assume that the coefficient functions of the diff. eqn. are real valued. (p_2, p_1, p_0)

$$p(x) := p_2(x)\, w(x) \quad \text{or} \quad p_2(x) = \frac{p(x)}{w(x)}$$

Assume: $$p_1(x) = \frac{p'(x)}{w(x)} = \frac{p_2(x)\, w'(x) + p_2'(x)\, w(x)}{w(x)}$$

$$L_x = \frac{p(x)}{w(x)} \frac{d^2}{dx^2} + \frac{p'(x)}{w(x)} \frac{d}{dx} + p_0(x)$$

$$L_x^+ = \frac{1}{w(x)} \left[\frac{d^2}{dx^2} w(x)\, p_2(x) - \frac{d}{dx} w(x)\, p_1(x) + w(x)\, p_0(x) \right]$$

$$L_x^+ u = \frac{1}{w(x)} \left[\frac{d^2}{dx^2} (p(x)\, u(x)) - \frac{d}{dx} \left(p'(x)\, u(x) \right) + w(x)\, p_0(x)\, u(x) \right]$$

Which simplifies to:

$$L_x^{+}u = \frac{1}{w}\left[p\frac{d^2u}{dx^2} + p'\frac{du}{dx} + wp_0u\right] = L_x u$$

I can turn any operator to a self adjoint operator with appropriate choice of w.

That's why we use weight!

Exercise: $p_2(x)\,w'(x) + (p_2'(x) - p_1(x))\,w(x)$

solution= $w(x) = w(x_0)\dfrac{p_2(x)}{p_2(x_0)}\, e^{\int_{x_0}^{x} dx' \frac{p_1(x')}{p_2(x')}}$

Check Surface Term S: I once again choose w so the conditions hold

$$S = \left[v^*(x)p(x)u'(x) + p(x)v^{*\prime}(x)u(x)\right]_a^b \quad (*)$$

(exercise) (should simplify a lot)

Let's look at some boundary conditions.

$(*) = v^*(b)p(b)u'(b) + p(b)v^{*\prime}(b)u(b)$

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$(v^*(a)p(a)u'(a) + p(a)v^{*\prime}(a)u(a))$

Main Examples of B.C.s:

(i) $u(a) = 0$ $\quad v(a) = 0$ (Dirichlet B.C)

$u(b) = 0$ $\quad v(b) = 0$ L_x hermitian

(ii) $u(a) = u(b)$ $\quad v(a) = v(b)$ (Periodic)

$u'(a) = u'(a)$ $\quad v'(a) = v'(b)$

(iii) $u'(a) = 0$ $\quad v'(a) = 0$ (Neumann)

$u'(b) = 0$ $\quad v'(b) = 0$ L_x hermitian

You might also have: $u(a) = 0$ $\quad v(a) = 0$
(in string th.) $u'(b) = 0$ $\quad v'(b) = 0$

Hermitian

- Apply results from Hilbert space here ...

$$\langle v | L_x u \rangle_w = \langle L_x v | u \rangle_w$$

- Eigenvectors satisfy ...
completeness and orthonormality

(We will come back to this)

Green's Functions of Differential Operators

Suppose I want to solve:

$$L_x \underbrace{u(x)}_{\text{unknown}} = \underbrace{f(x)}_{\text{given}} \qquad a \leq x \leq b \qquad (w(x) > 0)$$

solve for $u(x)$ subject to boundary conditions:

$$B_1(u) = 0 \qquad B_2(u) = 0$$

Consider the following eqn:

$$L_x \underbrace{G(x,x')}_{\text{Green's function}} = \frac{\delta(x-x')}{w(x)}$$

Think x' as a parameter in the same interval $a \leq x' \leq b$.

$$B_1(G) = 0 \qquad B_2(G) = 0$$

Suppose I formed this:

$$\int_a^b dx' \, w(x') \, G(x,x') \, f(x')$$

apply L_x on it:

$$L_x \int_a^b dx' \, w(x') \, G(x,x') \, f(x')$$

Bring L_x inside

$$\int_a^b dx'\, w(x') \underbrace{\left[L_x G(x,x')\right]}_{\frac{\delta(x-x')}{w(x)} = \frac{\delta(x-x')}{w(x')}} f(x')$$

property of δ

$$= \int_a^b dx'\, \delta(x-x') f(x')$$

$$= f(x)$$

Exercise : Check B.C.s

So: $\underbrace{L_x \int_a^b dx'\, w(x') G(x,x') f(x')}_{u(x)}$

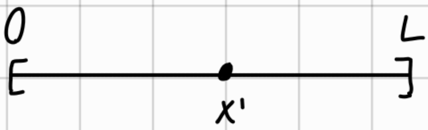
Thus,

$$L_x u(x) = f(x) \qquad B_1(u)=0 \quad B_2(u)=0$$

Note That: $u(x) = \int_a^b dx'\, w(x') G(x,x') f(x')$

$$\lim_{x\to x'} G(x,x') = G(x',x)$$

$$u(x) = \int_a^b dx' \, w(x') \, G(x,x') \, f(x')$$



Example: $-\dfrac{d^2}{dx^2} u(x) = f(x)$ $\quad u(0) = 0 \quad u(L) = 0 \quad 0 \leqslant x \leqslant L \quad w(x) = 1$

$$-\frac{d^2}{dx^2} G(x,x') = \delta(x-x')$$

$G(0,x') = 0$
$G(L,x') = 0$

$x < x'$

$$-\frac{d^2}{dx^2} G(x,x') = 0$$

$G(x,x') = a(x')x$
$G(0,x') = b(x') = 0$

$x > x'$

$$-\frac{d^2}{dx^2} G(x,x') = 0$$

(means it's linear)

$$G(x,x') = c(x')x + d(x')$$

$$G(L,x') = c(x')L + d(x')$$

$$\lim_{x \to x'^-} G(x,x') = \lim_{v \to x'^+} G(x,x')$$

$$a(x')x' = c(x')(x'-L)$$

$$c(x') = \frac{a(x')x'}{x'-L}$$

$$G(x,x') \begin{cases} a(x')x & x < x' \\ \dfrac{a(x')x'}{x'-L}(x-L) & x > x' \end{cases}$$

$$-\frac{d^2}{dx^2} G(x,x') = \delta(x\ x')$$

$$-\int_{x'-\epsilon}^{x'+\epsilon} dx \frac{d^2}{dx^2} G(x,x') = \int_{x'-\epsilon}^{x'+\epsilon} dx\, \delta(x-x')$$

$$-\left[\left.\frac{dG(x,x')}{dx}\right|_{x'+\epsilon} - \left.\frac{dG(x,x')}{dx}\right|_{x'-\epsilon} \right] = 1$$ (Jump discontinuity)

$$\lim_{\epsilon \to 0^+} \left[\left.\frac{dG(x,x')}{dx}\right|_{x'+\epsilon} - \left.\frac{dG(x,x')}{dx}\right|_{x'-\epsilon} \right] = 1$$

Derivative of the second one ... So ...

$$G(x,x') \begin{cases} \frac{L-x'}{L} x & x < x' \\ \frac{x'}{L}(L-x) & x > x' \end{cases}$$

Solution to Boundary value Problem:

$$u(x) = \int_a^b dx' \, w(x') G(x, x') f(x')$$

$$= \int_a^x dx' \frac{x'}{L} (L-x) f(x') + \int_x^b dx' \; \frac{L-x'}{L} \, x f(x)$$

$$= \frac{L-x}{L} \int_0^x dx' \, x' f(x') + \frac{x}{L} \int_x^L dx' (L-x') f(x')$$