PHYS 325 Lecture 14

19.11.24

$A : V \to W$ linear for all $v \in V, w \in V$ $\exists A^\dagger : W \to V$

$$\langle w | Av \rangle_W = \langle A^\dagger w | v \rangle_V$$

Unitary operator: $\mathcal{U} : V \longrightarrow W$

if $\mathcal{U}^\dagger \mathcal{U} = \mathcal{U}\mathcal{U}^\dagger = 1$ then $\mathcal{U}$ is called a unitary operator.

$A : V \to V,$ $A^\dagger = A$ $Av = \lambda v$

eigenvalue

eigenvector

$\det(A - \lambda I) = 0$ (Secular Equation)

EX: $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y$

$$\det(\sigma_y - \lambda I) = \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0 \to \lambda_{1,2} = \pm 1$$

$\hat{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\hat{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\boxed{\langle v | u \rangle = \sum_i v_i^* u_i}$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 1 \begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} -ib \\ ia \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$b = ia$ $a = 1$ $b = i \longrightarrow v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\Rightarrow \quad \hat{V}_1 = \frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ (normalized eigenvector)

$$\|v\| = \sqrt{\langle v|v\rangle} = \sqrt{1+(-1)(1)} = \sqrt{2}$$

Properties of eigenvalue problem for Hermitian Operators

(i) All eigenvalues of A are real $\quad \langle v|u\rangle^* = \langle u|v\rangle$

$$Av = \lambda v \longrightarrow \langle v|Av\rangle = \lambda\langle v|v\rangle = \langle v|Av\rangle^* = \lambda^*\langle v|v\rangle^*$$

$$\langle Av|v\rangle = \lambda^*\langle v|v\rangle$$

$$\langle Av|v\rangle = \lambda\langle v|v\rangle$$

$$0 = (\lambda - \lambda^*)\langle v|v\rangle \rightarrow \lambda = \lambda^* \rightarrow \lambda \in \mathbb{R}$$

(ii) Let λ & μ $(\lambda \neq \mu)$ be eigenvalues of A, then the corresponding eigenvectors are orthogonal to each other.

$$Av = \lambda v \longrightarrow \langle u|Av\rangle = \lambda\langle u|v\rangle \rightarrow \langle Au|v\rangle = \lambda\langle u|v\rangle$$

$$Au = \mu u$$

$$\rightarrow \langle \mu u|v\rangle = \lambda\langle u|v\rangle$$

$$\rightarrow \mu^*\langle u|v\rangle = \lambda\langle u|v\rangle \rightarrow \mu\langle u|v\rangle = \lambda\langle u|v\rangle$$

$$\rightarrow (\mu - \lambda)\langle u|v\rangle = 0 \longrightarrow \langle u|v\rangle = 0$$

(iii) Spectral Theorem : There exists an orthonormal basis of eigenvectors of A for V

Diagonalization

Let's pick an orthonormal basis of eigenvectors of A $\{v_1, v_2, \dots, v_n\}$

$$\langle v_i | v_j \rangle = \delta_{ij} \qquad V = \operatorname{span}_{\mathbb{C}}\{v_1, \dots, v_n\}$$

$$U = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad U^\dagger = \begin{pmatrix} — & v_1^* & — \\ — & v_2^* & — \\ & \vdots & \\ — & v_n^* & — \end{pmatrix}$$

$$U^\dagger U = \begin{pmatrix} — & v_1^* & — \\ — & v_2^* & — \\ & \vdots & \\ — & v_n^* & — \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} \sum_i v_{1i}^* v_{1i} & \sum_i v_{1i}^* v_{2i} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

This is just the inner product.

$$= \begin{pmatrix} \langle v_1 | v_1 \rangle & \langle v_1 | v_2 \rangle & \cdots & \langle v_1 | v_n \rangle \\ \langle v_2 | v_1 \rangle & \vdots & \cdots & \langle v_2 | v_n \rangle \\ \vdots & \vdots & & \\ \langle v_n | v_1 \rangle & \langle v_n | v_2 \rangle & \cdots & \langle v_n | v_n \rangle \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

Turns out to be an Identity matrix.

$$I_{ij} = \delta_{ij}$$

Multiplication

$$AU = A \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_n \\ | & | & & | \end{pmatrix}$$

v's are eigenvectors. $Av_i = \lambda_i v_i$

$$= \begin{pmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{pmatrix}$$

Let's do one more thing:

$$U^\dagger A U = \begin{pmatrix} \text{---} & v_1^* & \text{---} \\ \text{---} & v_2^* & \text{---} \\ & \vdots & \\ \text{---} & v_n^* & \text{---} \end{pmatrix} \begin{pmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} \sum_i v_{1i}^* \lambda_i v_{1i} & \sum_i v_{1i}^* \lambda_2 v_{2i} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 \langle v_1 | v_1 \rangle & \lambda_2 \langle v_1 | v_2 \rangle & \cdots & \lambda_n \langle v_1 | v_n \rangle \\ \lambda_1 \langle v_2 | v_1 \rangle & \vdots & \cdots & \lambda_n \langle v_2 | v_n \rangle \\ \vdots & \vdots & & \\ \lambda_1 \langle v_n | v_1 \rangle & \lambda_2 \langle v_n | v_2 \rangle & \cdots & \lambda_n \langle v_n | v_n \rangle \end{pmatrix}$$

Now we see the diagonalization of a hermitian matrix:

$$D = U^{\dagger} A U = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$$

Ex: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = U^{\dagger} \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y} U$

$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$$

exercise: Check this explicitly

Hilbert Space ℓ^2 $L^2(\mathbb{R}^n)$

ℓ^2: set of all square summable complex sequences.

$(a_0, a_1, \dots)$ $\qquad \sum_{i=0}^{\infty} |a_i|^2 < \infty$

" ℓ^2 is a vectorspace "

Proof: $c(a_0, a_1, a_2, \dots) := (ca_0, ca_1, ca_2 \dots)$

$(a_0, a_1, a_2, \dots) + (b_0, b_1, b_2, \dots) := (a_0+b_0, a_1+b_1, a_2+b_2 \dots)$

"We must show that they are square summable"

Assume $\sum_{i=0}^{\infty} |a_i|^2 < \infty$:

$(ca_0, ca_1, \dots) \in \ell^2$ $\qquad \sum_{i=0}^{\infty} |ca_i|^2 = \sum_{i=0}^{\infty} |c|^2 |a_i|^2 = |c|^2 \sum_{i=0}^{\infty} |a_i|^2 < \infty$

$$\sum_{i=0}^{\infty} |a_i+b_i|^2 = \sum_{i=0}^{\infty} (a_i^*+b_i^*)(a_i+b_i) = \sum_{i=0}^{\infty} (|a_i|^2 + \underbrace{a_i^*b_i + b_i^*a_i}_{2\mathrm{Re}\, a_i^* b_i} + |b_i|^2)$$

$2\mathrm{Re}\, a_i^* b_i \left(\frac{z+z^*}{2} = \mathrm{Re}\, z \right)$

$$= \sum_{i=0}^{\infty} (|a_i|^2 + |b_i|^2 + 2\mathrm{Re}\, a_i^* b_i) \quad (*)$$

$$0 \leq |a_i - b_i|^2 = (a_i^* - b_i^*)(a_i - b_i) = |a_i|^2 - a_i^* b_i - b_i^* a_i + |b_i|^2$$

$$= |a_i|^2 + |b_i|^2 - 2\mathrm{Re}\, a_i^* b_i$$

$$2\mathrm{Re}\, a_i^* b_i \leq |a_i|^2 + |b_i|^2$$

$$(*) \quad \sum_{i=0}^{\infty} (|a_i|^2 + |b_i|^2 + 2\mathrm{Re}\, a_i^* b_i) \leq 2 \sum_{i=0}^{\infty} (|a_i|^2 + |b_i|^2) < \infty$$

The Hilbert Space of Square Integrable Function

- The set of all square integrible functions $\mathbb{R}^n \to \mathbb{C}$

$$\int_{\mathbb{R}^n} d^n x \, |f(x)|^2 < \infty$$

$c \in \mathbb{C} \qquad f, g \in L^2(\mathbb{R}^n)$

Exercise: Hint: imitate ℓ^2

$$(cf)(x) := c f(x)$$
$$f+g\,(x) := f(x) + g(x)$$

Also a vector space, and they are inner product spaces.

ℓ^2

$$\langle (a_0, a_1, \dots) | \underbrace{(b_0, b_1, \dots)} \rangle = \sum_{i=0}^{\infty} a_i^* b_i < \infty$$

- we can think of this sequence as a function $\mathbb{Z}_{\geq 0} \to \mathbb{C}$

(Domain is a discrete set)

$L^2(\mathbb{R}^n)$

$$\langle f | g \rangle = \int_{\mathbb{R}^n} d^n x \, f^*(x) g(x) < \infty$$

(Domain is a continuous set)