PHYS 326 Lecture 8 (March 11)

#phys326 #phys326/lecturenotes

MT 1: March 25 13:00

Continuing with asymptotic analysis

Laplace Method

(...)

$$I(lpha) \sim e^lpha \int_{-\infty}^\infty d heta e^{-lpha rac{ heta^2}{2}} e^{-lpha (rac{ heta^4}{4!} - rac{ heta^6}{6!}) + \dots}$$

(...)

Stirling Approximation

We have the Gamma Function:

$$\Gamma(x+1) = \int_0^\infty dt t^x e^{-t} \sim ? ext{ as } x o \infty$$

$$=\int_0^\infty dt e^{\ln t^lpha} e^{-t} = \int_0^\infty dt e^{lpha \ln t - t}$$

Let $t = \alpha u$ so $dt = \alpha du$ then,

$$=\int_0^\infty du \alpha e^{\alpha \ln \alpha u - \alpha u}$$

$$=\int_0^\infty du lpha e^{lpha \ln lpha} e^{lpha (\ln u - u)} = lpha e^{lpha \ln lpha} \int_0^\infty du e^{lpha (\ln u - u)}$$

Exercise 1:

Show that $\int_{1+\epsilon}^{\infty} du e^{\alpha(\ln u + u)}$ is exponentially small.

Show that $\int_0^{1-\epsilon} du e^{lpha(\ln u + u)}$ is exponentially small.

Continuing...

$$\Gamma(lpha+1) \sim e^{lpha \ln lpha} \int_{1-\epsilon}^{1+\epsilon} du e^{lpha [\ln u - u]}$$

$$f(u) = \ln u - u = -1 - rac{1}{2!}(u-1)^2 + O((u-1)^3)$$

$$f(1) = -1$$

$$f'(u) = \frac{1}{u} - 1, f'(1) = 0$$

 $f''(u) = \frac{1}{u^2}, f''(1) = -1$

$$egin{aligned} \Gamma(lpha+1) &\sim e^{lpha \ln lpha} \int_{1-\epsilon}^{1+\epsilon} du e^{-lpha rac{(u-1)^2}{2}-lpha} \left[1+O((u-1)^3)
ight] \ &\sim lpha e^{lpha \ln lpha} \int_0^\infty du e^{-lpha rac{(u-1)^2}{2}} \end{aligned}$$

Let v = u - 1

$$\sim lpha e^{lpha \ln lpha} \int_0^\infty dv e^{-lpha rac{v^2}{2}} \ \int_0^\infty dv e^{-lpha rac{v^2}{2}} = \int_{-\infty}^{-1} dv e^{-lpha rac{v^2}{2}} + \int_{-1}^\infty dv e^{-lpha rac{v^2}{2}}$$

Exercise: Show that this is exponentially small

Then,

$$egin{split} &\sim lpha e^{lpha \ln lpha - lpha} \int_{-1}^{\infty} dv e^{-lpha rac{v^2}{2}} &\sim lpha e^{lpha \ln lpha - lpha} \int_{-\infty}^{\infty} dv e^{-lpha rac{v^2}{2}} \ &\sim lpha e^{lpha \ln lpha - lpha} \sqrt{rac{2\pi}{lpha}} = \sqrt{2\pi lpha} e^{lpha \ln lpha - lpha} \end{split}$$

This is the stirling approximation.

Then,

$$\Gamma(lpha+1) \sim \sqrt{2\pilpha} e^{lpha \lnlpha-lpha} \ \sim e^{lpha \lnlpha-lpha+rac{1}{2}\ln(2\pilpha)} \ \sim e^{lpha \lnlpha-lpha+rac{1}{2}\ln(lpha)+rac{1}{2}\ln(2\pi)}$$

Consider this integral:

$$I(lpha)=\int_0^\infty dx e^{-lpha x^p}x^q=?$$

$$p > 0, q \ge 0$$

We will make use of the gamma function.

$$u=lpha x^p \ x=\left(rac{u}{lpha}
ight)^{1/p}$$

$$dx=rac{rac{1}{p}u^{rac{u}{p}-1}du}{lpha^{1/p}}$$

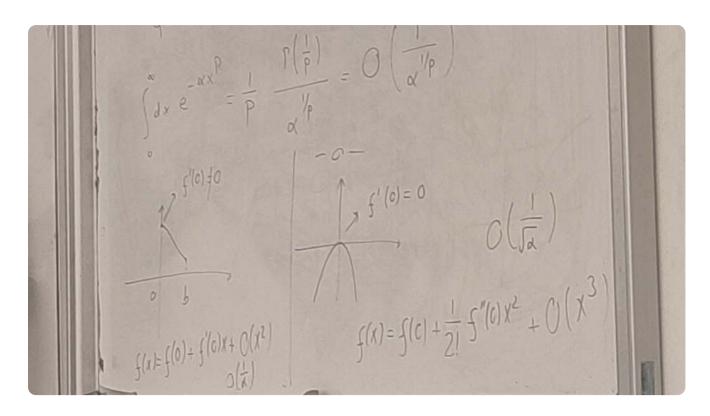
$$I(lpha)=\int_0^\infty du rac{rac{1}{p}u^{rac{u}{p}-1}du}{lpha^{1/p}}e^{-u}rac{u^{q/p}}{lpha^{rac{q}{p}}}$$

$$=rac{1}{p}rac{1}{lpha^{q+1/p}}\int_0^\infty du u^{(q+1/p)-1}e^{-u}=rac{1}{p}rac{\Gamma\left(rac{q+1}{p}
ight)}{lpha^{(q+1)/p}}$$

Special case:

$$q = 0$$

$$\int_0^\infty dx e^{-lpha x^p} = rac{1}{p} rac{p\left(rac{1}{p}
ight)}{lpha^{1/p}} = O\left(rac{1}{lpha^{1/p}}
ight)$$



Leading contribution will come from local maximum (critical point \rightarrow derivative is zero)