PHYS 325 Lecture 16 — 26. 11. 2024

$A = A^\dagger$

$A|e_n\rangle = \lambda_n |e_n\rangle \qquad \langle e_m|e_n\rangle = \delta_{mn}$

$$f(A) = \sum_n f(\lambda_n)\,|e_n\rangle\langle e_n|$$

$\{v_k\}$ an arbitrary orthonormal basis

$$\sum_m \delta_{mn} A_m = A_n \qquad \sum_n \Big[\underbrace{\sum_m \delta_{mn}\langle v_k|e_m\rangle}_{\langle v_k|e_n\rangle}\Big] f(\lambda_n)\langle e_n|v_\ell\rangle$$

$$\langle v_k|f(A)|v_\ell\rangle = \sum_n f(\lambda_n)\langle v_k|e_n\rangle\langle e_n|e_\ell\rangle$$

$$= \sum_{m,n} \underbrace{\big[f(\lambda_n)\,\delta_{mn}\big]}_{D_{m,n}} \underbrace{\langle v_k|e_m\rangle}_{U_{km}} \underbrace{\langle e_n|v_\ell\rangle}_{(U^\dagger)_{n\ell}} = \sum_{m,n} U_{km} D_{mn} (U^\dagger)_{n\ell} = (UDU^\dagger)_{k\ell}$$

$$D_{mn} \rightsquigarrow \begin{pmatrix} f(\lambda_1) & & 0 \\ & f(\lambda_2) & \\ 0 & & \ddots \end{pmatrix} \qquad U_{km} := \langle v_k|e_m\rangle$$

$$(U^\dagger)_{n\ell} = (U_{\ell n})^* = \langle v_\ell|e_n\rangle^* = \langle e_n|v_\ell\rangle$$

$$(U^\dagger U)_{nm} = \sum_\ell (U^\dagger)_{n\ell}(U)_{\ell m} = \sum_\ell \langle e_n|v_\ell\rangle\langle v_\ell|e_m\rangle$$

$$= \langle e_n|\underbrace{\sum_\ell |v_\ell\rangle\langle v_\ell|}_{I}|e_m\rangle = \langle e_n|e_m\rangle = \delta_{nm}$$

U is a unitary matrix. Exercise: $(UU^\dagger)_{k\ell} = \delta_{k\ell}$

Let's look at a special case:

$$A = \sum_n \lambda_n |e_n\rangle\langle e_n|$$

$$A_{kl} = \langle v_k | A | v_l \rangle = \left[U \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} U^+ \right]_{kl}$$

Hermitian $\rightarrow$ can be diagonalized unitary matrix

Example: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad A^T = A \quad \sqrt{A} = ?$

$$(\sqrt{A}) = \left[U \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} U^+ \right]_{kl}$$

U is the matrix that diagonalizes A.

$D = U^+ A U \; (\Leftrightarrow A = U D U^+)$

$U = O$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left[O \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} O^T \right]_{kl}$$

$D = O^T A O \qquad O^T O = O O^T = 1$

$(\cdots)$

Some Properties $\qquad [A, B] = AB - BA$ commutator

(i) Baker-Campbell-Hausdorff Formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B] + \frac{1}{12}([A,[A,B]] - [B,[A,B]]) + \cdots}$$

Special case: $[A, B] = c\mathbb{1}$

$$e^A e^B = e^{A+B+\frac{c}{2}\mathbb{1}} = e^{\frac{c}{2}\mathbb{1}} e^{A+B}$$

Special Case 2: if $[A,B]=0$, $e^A e^B = e^{A+B}$

(ii) $e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots$

Exercise: Check the special cases:

$[A,B] = 0$

$[A,B] = cI$

(iii) Similarity Transformation

Let B be an invertable operator.

$B^{-1} f(A) B = f(B^{-1}AB)$

$B f(A) B^{-1} = f(BAB^{-1})$

Ex: $f(A) = A^2$

$$B f(A) B = BA^2B^{-1} = BA\overset{B^{-1}B}{\vee}A^{-1}B^{-1}$$

$$= BAB^{-1}BAB^{-1} = (BAB^{-1})^2$$

$$= f(BAB^{-1})$$

(Distribution)

Generalized Function: A special type of linear functionals.

Let $\{h_n(x)\}_{n=1}^{\infty}$ be a sequence of functions $\mathbb{R} \to \mathbb{R}$

given $g(x)$ we define the following linear functional.

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} dx\, h_n(x) g(x)$$

(Assume everything is well defined)

"Schwartz test functions"

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}dx\, h_n(x)[\alpha g(x)+\beta s(x)]$$

$$\lim_{n\to\infty}\left[\alpha\int_{-\infty}^{\infty}dx\, h_n(x)\, g(x)+\beta\int_{-\infty}^{\infty}dx\, h_n(x)\, s(x)\right]$$

$$= \alpha \lim_{n\to\infty}\int_{-\infty}^{\infty}dx\, h_n(x) g(x) + \beta\int_{-\infty}^{\infty}dx\, h_n(x) s(x)$$

$$\int_{-\infty}^{\infty}dx\, \chi(x)\, g(x) := \lim_{n\to\infty}\int_{-\infty}^{\infty}dx\, h_n(x)\, g(x)$$

$$\chi(x) = \lim_{n\to\infty} h_n(x) \quad \text{(distributional sense)}$$

NOT POINTWISE!

eg: $\chi(1) \neq \lim_{n\to\infty} h_n(1)$

Dirac Delta Function

$$\lim_{n\to\infty}\sqrt{\frac{n}{\pi}}\, e^{-n(x-x_0)^2} = \delta(x-x_0)$$

$$\delta_0(g) = g(0)$$

$$\delta_a(g) = g(a)$$

$$D_n(x) = \sqrt{\frac{n}{\pi}}\, e^{-nx^2} \text{ (Gaussian)}$$

$$\int_{-\infty}^{\infty}dx\, \delta(x)\, g(x) = \lim_{n\to\infty}\int_{-\infty}^{\infty}dx\, \sqrt{\frac{n}{\pi}}\, e^{-nx^2} g(x) = g(0)$$

$$\int_{-\infty}^{\infty} dx\, \delta(x-x_0)\, g(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} dx \sqrt{\frac{n}{\pi}}\, e^{-n(x-x_0)^2} g(x) = g(x_0)$$

$$S_n(x) = \begin{cases} n/2 & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

$$\lim_{n\to\infty} S_n(x) = \delta_n$$

(Lorentzian)

$$L_n(x) = \frac{1}{\pi n} \frac{n^2}{n^2x^2+1} = \frac{n}{\pi(n^2x^2+1)}$$

$$\lim_{n\to\infty} L_n(x) = \delta(x)$$

$$P_n(x) = \frac{\sin nx}{\pi x} \qquad \lim_{n\to\infty} P_n(x) = \delta(x)$$

"The point is that they are not uniformly convergent"

Properties of δ.

(i) $\delta(x-x_0)\, g(x) = \delta(x-x_0)\, g(x_0)$

(ii) $\delta(x-x_0) = \delta(x_0-x)$

(iii) $\delta(ax) = \dfrac{\delta(x)}{|a|} \qquad (a \in \mathbb{R})$

$$\frac{\partial x}{\partial y} = \frac{1}{a} \qquad ax = y$$

$$\int_{-\infty}^{\infty} dx\, \delta(ax) g(x) = \int_{-\infty}^{\infty} dy\, \frac{1}{|a|}\, \delta(y)\, g\left(\frac{y}{a}\right)$$

$$= \frac{1}{|a|}\, g(0) = \int_{-\infty}^{\infty} dx \left[\frac{1}{|a|}\, \delta(x)\right] g(x)$$

(iv) $x = (x_1, \dots, x_n)$

$$\delta(x) = \delta(x_1)\, \delta(x_2) \cdots \delta(x_n)$$

(v) $\delta(f(x)) = \sum_i \dfrac{\delta(x-x_i)}{|f'(x_i)|}$

x_i's are the zeroes of $f(x)$. $f(x_i) = 0$

$$\int_{-\infty}^{\infty} dx'\, \delta(x-x')\, g(x') = g(x)$$

$$\sum_{j} \delta_{ij} g_j = g_i$$

You can think of Dirac Delta as generaliz-ation of Kronecker delta