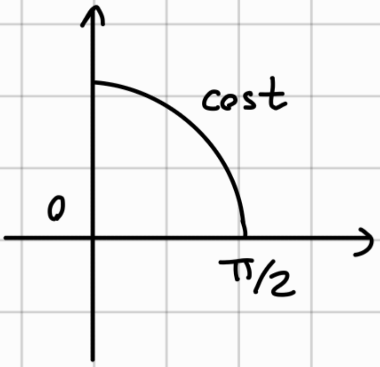
PHYS 326 — 18 March 2025

Ex: $I(\alpha) = \int_0^{\pi/2} dt \; e^{i\alpha \cos t} \sim ?$ as $\alpha \to \infty$

$\frac{d}{dt} \cos t = -\sin t \longrightarrow$ at $t_0 = 0$ we have a max.



$$I(\alpha) = \int_0^{\epsilon} dt \; e^{i\alpha \cos t} + \underbrace{\int_{\epsilon}^{\pi/2} dt \; e^{i\alpha \cos t}}_{O(\frac{1}{\alpha})}$$

$h(t) = 1$

$f(t_0) = \cos 0 = 1 \qquad f''(t_0) = 1$

$\cos t = 1 - \frac{1}{2} t^2 + O(t^4)$

$$I(\alpha) \sim \int_0^{\infty} dt \; e^{i\alpha(1 - \frac{1}{2}t^2)} = \frac{1}{2} \int_{-\infty}^{\infty} dt \, e^{i\alpha(1-\frac{1}{2}t^2)}$$

$$\sim \frac{1}{2} e^{i\alpha} e^{-i\frac{\pi}{4}} \sqrt{\frac{2\pi}{\alpha}}$$

$$I(\alpha) = \int_{C_0} dz \;\; h(z) \; e^{\alpha f(z)} \sim ? \quad \text{as} \quad \alpha \to \infty$$

Saddle point method

Let $f(z)$ be in a region $R \subseteq \mathbb{C}$. $z_0 \in R$ is called **critical point** of $f(z)$ if $f'(z_0) = 0$.

A critical point z_0 of $f(z)$ is called a **regular critical point** if $f''(z_0) \neq 0$.

$z = x + iy$ $\qquad x = \dfrac{z + z^*}{2} \qquad y = \dfrac{z - z^*}{2i}$

$z^* = x - iy$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

Exercise: Show that $\dfrac{\partial}{\partial z^*} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$

$f = u + iv$

$$f'(z) = \frac{\partial f(z)}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv)$$

$$= \frac{1}{2}(u_x + iv_x - iu_y + v_y) = \frac{1}{2}\left[(u_x + v_y) + i(v_x - u_y)\right]$$

Exercise: Use $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$ and CR equations to get this result

$f''(z) = u_{xx} - iu_{xy} = v_{xy} + iv_{xx}$

Recall Cauchy-Riemann equations.

$u_x = v_y \qquad u_y = -v_x$

$$\begin{array}{rl} & u_{xx} = v_{xy} \\ + & u_{yy} = -v_{xy} \\ \hline & u_{xx}+u_{yy} = v_{xy} - v_{xy} = 0 \end{array}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\nabla^2 u = 0$ (Laplace equation)

Exercise: Show that $\nabla^2 v = 0$

Note: Solutions to Laplace equation are called harmonic functions.

Let z_0 be a regular critical point of $f(z)$

$f'(z_0) = 0$, $f''(z_0) \neq 0$ $\quad (z_0 = x_0 + iy_0)$

$u_{xx}(x_0, y_0)$ and $u_{xy}(x_0, y_0)$ can't both vanish

In particular, $u_{xx}^2(x_0, y_0) + u_{xy}^2(x_0, y_0) \neq 0$

$u_{xx} + u_{yy} = 0$

$$\begin{vmatrix} u_{xx}(x_0, y_0) & u_{xy}(x_0, y_0) \\ u_{yx}(x_0, y_0) & u_{yy}(x_0, y_0) \end{vmatrix} = u_{xx}(x_0; y_0)\underbrace{u_{yy}(x_0, y_0)}_{-u_{xx}(x_0, y_0)} - u_{xy}^2(x_0, y_0)$$

$$= -\left[u_{xx}^2(x_0,y_0) + u_{xy}^2(x_0,y_0)\right] < 0$$

$$f'(z) = \frac{1}{2}\left[(u_x + v_y) + i\,(v_x - u_y)\right]$$

$$= \frac{1}{2}\left[2u_x - 2\,i\,u_y\right]$$

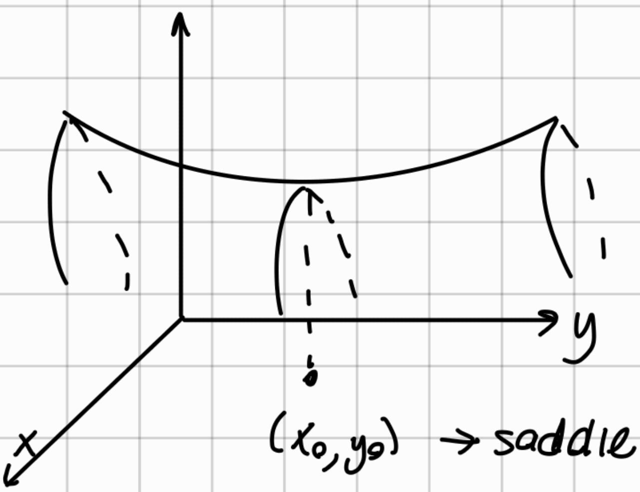
CR

$u_x = v_y$

$u_y = -v_x$

$u_x(x_0, y_0) = 0 \qquad u_y(x_0, y_0) = 0$

So, (x_0, y_0) is a saddle point.



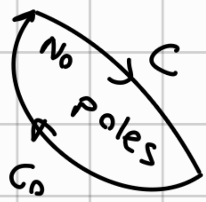
$(x_0, y_0) \rightarrow$ saddle point

Exercise: Show that the same result also holds for $v(x,y)$

Saddle Point Approximation

$$I(\alpha) = \int_{C_0} dz\, h(z)\, e^{\alpha f(z)} \sim ? \quad \text{as} \quad \alpha \to \infty$$

Let C be another contour so that



(i) $$\int_{C_0} dz\, h(z)\, e^{\alpha f(z)} = \int_{C} dz\, h(z)\, e^{\alpha f(z)}$$

(ii) On C, $\operatorname{Im} f = \beta =$ constant real number

(iii) The restriction of $\operatorname{Re} f$ to C has at least one max on C.

On C $\quad f(z) = u + i\beta$

$$\int_C dz \; h(z) e^{\alpha u} e^{i\alpha\beta} = e^{i\alpha\beta} \int dz \; h(z) e^{\alpha u}$$

$$= e^{i\alpha\beta} \int_a^b ds \; w'(s) \, h(w(s)) \, e^{\alpha u(w_r(s), w_i(s))}$$

$$f(z) = u(x,y) + i v(x,y)$$

- we reduced complex integration to a real integration. And then we apply Laplace method.

$$I(\alpha) \sim e^{i\alpha\beta} \; w'(s_0) \; h(w(s_0))$$

$$\int_C dz \; h(z) e^{\alpha u} e^{i\alpha\beta} = e^{i\alpha\beta} \int dz \; h(z) e^{\alpha u}$$

$$= e^{i\alpha\beta} \int_a^b ds \; w'(s) \, h(w(s)) \, e^{\underbrace{\alpha u(w_r(s), w_i(s))}_{g(s)}}$$

$$\left. \frac{d u(w_r(s), w_i(s))}{ds} \right|_{s=s_0} = 0 \; , \quad \left. \frac{d^2 u(w_r(s), w_i(s))}{ds} \right|_{s=s_0} < 0$$

assuming $s_0 \neq a$, $s_0 \neq b$

$$I(\alpha) \sim e^{i\alpha\beta} \, w'(s_0) \, \overbrace{h(w(s_0))}^{h(z_0)} \, e^{\alpha g(s_0)}$$

Note that $w(s_0)$ is a saddle point of f

$$\frac{2\pi}{\sqrt{\alpha |g''(s_0)|}}$$

$$\frac{d g(s)}{ds} = \frac{d}{ds} \operatorname{Re} f(w(s)) = \operatorname{Re} \frac{d}{ds} f(w(s))$$

$$= \operatorname{Re}\left(f'(w(s))\, w'(s)\right)$$

$$\left.\frac{d^2 g(s)}{ds^2}\right|_{s=s_0} = \operatorname{Re}\left[f''(w(s))\left[w'(s)\right]^2 + \underbrace{\cancel{f'(w(s))}}_{0} w''(s)\right]_{s=s_0}$$

$$f(z) = u(x,y) + i\, v(x,y) \qquad f'(\underbrace{w(s_0)}_{z_0}) = 0$$

$$f''(z_0) = |f''(z_0)|\, e^{i\gamma_0}$$

$$w'(s_0) = |w'(s_0)|\, e^{i\phi_0}$$

$$\left.\frac{d^2 g(s)}{ds^2}\right|_{s=s_0} = \operatorname{Re}\Big(\underbrace{f''(w(s_0))\,(w'(s_0))^2}_{|f''(z_0)|\,|w'(s_0)|^2 \underbrace{e^{2i\phi_0}\, e^{i\gamma_0}}_{e^{i(2\phi_0+\gamma_0)}}}\Big)$$

$$= |f''(z_0)|\,|w'(s_0)|^2 \cos(2\phi_0 + \gamma_0)$$

$$I(\alpha) \sim e^{i\alpha f(z_0)} h(z_0)\, e^{i\phi_e} \sqrt{\frac{2\pi}{\alpha |f''(z_0)||\cos(2\phi_0+\gamma_0)|}}$$

On C $\quad \mathrm{Im}\, f(w(s)) = \beta =$ const. real number

$$\mathrm{Im}\left[f''(z_0)\, w'(s_0)\right] = |f''(z_0)||w'(s_0)| \sin(2\phi_0+\gamma_0) = 0$$

$$\rightarrow \sin(2\phi_0+\gamma_0) = 0$$

represent the sone line

$\sin(2\phi_e, \gamma_0) = 0$

$\phi_0 = \frac{\gamma}{2}, \quad -\frac{\gamma}{2}+\frac{\pi}{2}, \quad -\frac{\gamma}{2}+\pi, \quad -\frac{\gamma}{2}+\frac{\pi}{2}+\pi$

$\cos(2\phi_0+\gamma_0)\ \pm 1$

represent the sone line