PHYS 326 Lecture 7 (March 4)

#phys326

Midterm 1: March 24 (17:00) Midterm 2: May 5 (17:00)

Algebraic operations

$$f_1(x) \sim a_o + a_1x + a_2x^2 + \ldots ext{ as } x o 0 \ f_2(x) \sim b_o + b_1x + b_2x^2 + \ldots ext{ as } x o 0$$

As x o 0

Basic Algebraic Properties of Asymptotic Expansion

- 1. $f_1(x) + f_2(x) \sim (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots$
- 2. $f_1(x)f_2(x) \sim \sum_{l=0}^{\infty} [\sum_{k=0}^{l} a_k b_{l-k}] x^l$
- 3. if $a_0 \neq 0$:

$$rac{1}{f_1(x)} \sim rac{1}{a_0 + a_1 x + a_2 x^2 + \dots}$$

4. $f_1(f_2(x)) = S_1(S_2(x))$

Gaussian Integrals

Let's say:

$$egin{align} I(lpha) &= \int_0^\infty dx e^{-lpha x^2} \ (lpha > 0) \ &= rac{1}{2} \int_{-\infty}^\infty dx e^{-lpha x^2} \ (\sqrt{lpha} x = 0) \ &= rac{1}{2} \int_{-\infty}^\infty rac{du}{\sqrt{lpha}} e^{-u^2} = rac{1}{2} rac{1}{\sqrt{lpha}} \sqrt{\pi} = rac{1}{2} \sqrt{rac{\pi}{lpha}} \ &= \int_{-\infty}^\infty dx e^{-lpha x^2} = \sqrt{rac{\pi}{lpha}} \ &= \int_{-\infty}^\infty dx e^{-lpha x^2} = \sqrt{rac{\pi}{lpha}} \ &= 0$$

Let's say:

$$egin{align} I(lpha) &= \int_{-\infty}^{\infty} dx e^{(-lpha x^2 + eta x)} \ (lpha > 0) \ &= \int_{-\infty}^{\infty} dx e^{-lpha (x^2 - 2rac{eta}{2lpha} x + (rac{eta}{2lpha})^2 - (rac{eta}{2lpha})^2)} \ (lpha > 0) \ &= e^{lpha (rac{eta}{2lpha})^2} \int_{-\infty}^{\infty} dx e^{-lpha (x - rac{eta}{2lpha})^2} \ &= e^{lpha (rac{eta}{2lpha})^2} \int_{-\infty}^{\infty} dx e^{-lpha (x - rac{eta}{2lpha})^2} \ &= e^{-lpha (rac{eta}{2lpha})^2} \int_{-\infty}^{\infty} dx e^{-lpha (x - rac{eta}{2lpha})^2} \ &= e^{-lpha (rac{eta}{2lpha})^2} \ &= e^{-lpha (x - rac{eta}{2lpha})^2} \ &= e^{-lpha (x - lpha)^2} \ &= e^{-lpha (x - lpha)^2} \ &= e^{-$$

Say that $u=x-rac{\beta}{2lpha}$ and thus du=dx

Then,

$$I(lpha)=e^{rac{eta^2}{4lpha}}\int_{-\infty}^{\infty}du e^{-lpha u^2}=\sqrt{rac{\pi}{lpha}}e^{rac{eta^2}{4lpha}}$$

$$I_n(lpha) = \int_{-\infty}^{\infty} dx x^n e^{lpha x^2}$$

If n is odd, $I_n(\alpha) = 0$

What happens when n is even?

n=2k

$$egin{align*} I_{2k}(lpha) &= \int_{-\infty}^{\infty} dx x^{2k} e^{-lpha x^2} \ &\int_{-\infty}^{\infty} dx x e^{-alpha x^2} = rac{\partial}{\partial eta} \int_{-\infty}^{\infty} dx e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, rac{\partial^2}{\partial eta^2} e^{-lpha x^2 + eta x} |_{eta = 0} = \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \ &= \int_{-\infty}^{\infty} dx \, x^2 e^{-lpha x^2 + eta x} |_{eta = 0} \$$

Gaussian Moments: $I_n(x) = \int_{-\infty}^{\infty} dx x^n e^{-\alpha x^2 + eta x}$

$$I_n(x) = \int_{-\infty}^{\infty} dx x^n e^{-lpha x^2 + eta x} = rac{\partial^n}{\partial eta} \int_{-\infty}^{\infty} dx e^{-lpha^2 x^2 + eta x} |_{eta = 0} = rac{\partial^n}{\partial eta^n} \sqrt{rac{\pi}{lpha}} e^{rac{eta^2}{2lpha}} |_{eta = 0}$$

$$I_1(x)=rac{\partial}{\partialeta}\sqrt{rac{\pi}{lpha}}e^{rac{eta^2}{2lpha}}|_{eta=0}=\sqrt{rac{\pi}{lpha}}rac{eta}{2lpha}e^{rac{eta^2}{4lpha}}|_{eta=0}=0$$

$$egin{align} I_2(x) &= rac{\partial^2}{\partialeta^2}\sqrt{rac{\pi}{lpha}}e^{rac{eta^2}{2lpha}}|_{eta=0} &= rac{\partial}{\partialeta}[\sqrt{rac{\pi}{lpha}}rac{eta}{2lpha}e^{rac{eta^2}{4lpha}}] \ I_2(x) &= \sqrt{rac{\pi}{lpha}}rac{1}{2lpha}[e^{rac{eta^2}{4lpha}}+etarac{eta}{2lpha}e^{rac{eta^2}{4lpha}}]_{eta=0} \ &= \sqrt{rac{\pi}{lpha}}rac{1}{2lpha} = O(rac{1}{lpha\sqrt{lpha}}) ext{ as } (lpha o\infty) \ \end{aligned}$$

Another notation used in literature

$$egin{aligned} \langle x^{2k}
angle &:= rac{I_{2k}(x)}{I_0(x)} = O(rac{1}{lpha^k}) \ &I_{2k}(x) = \int_{-\infty}^{\infty} dx x^{2k} e^{-lpha x^2} (u = \sqrt{lpha} x) \ &= \int_{-\infty}^{\infty} rac{du}{\sqrt{lpha}} rac{u^{2k}}{(lpha)^{2k}} e^{-u} = rac{1}{lpha^k \sqrt{lpha}} \int_{-\infty}^{\infty} du u^{2k} e^{-u^2} \ &= O(rac{1}{lpha^k \sqrt{lpha}}) ext{ as } lpha o \infty \end{aligned}$$

Example:

$$F_n(lpha) = \int_0^\infty dx x^n e^{-lpha x^2}$$

$$v=lpha x^2$$
 , $dv=2lpha x dx=2lpha \sqrt{rac{v}{lpha}}dx$

$$dx=rac{dv}{2lpha\sqrt{rac{v}{lpha}}}=rac{dv}{2\sqrt{lpha v}}$$

Remember Gamma Function:

$$\Gamma(s) = \int_0^\infty dv v^{s-1} e^{-v}$$

Anyway,

$$F_n(lpha) = \int_0^\infty rac{dv}{2\sqrt{lpha v}} \Big(rac{v}{lpha}\Big)^{n/2} e^{-v}$$

$$=rac{1}{2}rac{1}{lpha^{(n+1)/2}}\int_0^\infty dv v^{rac{n+1}{2}-1}e^{-v}$$

$$=\frac{1}{2}\frac{1}{\alpha^{n+1/2}}\Gamma\left(\frac{n+1}{2}\right)=O\left(\frac{1}{\sqrt{\alpha}\alpha^{\frac{n}{2}}}\right)$$

See also: Error Functions

Now we are going to generalize these for multi variable functions.

Multivariable Gaussian Integrals

Warning: NOT POWERS (tensor notation)

$$I(A) = \int_{\mathbb{R}^n} d^n x e^{-\sum_{ij} x^i A_{ij} x^j}$$

$$A_{ij} = A_{ji}$$

A is a positive definite matrix (all its eigenvalues are positive or, equivalently $\sum_{i,j} x^i A_{ij} x^i > 0$ for all $x \neq 0$)

 $D = O^T A O$ where $O^T O = 1$

 $D_{kl} = (O^T)_{ki} A_{ij} O_{jl}$ (summation over i & j)

$$=O_{ik}A_{ij}O_{jl}$$

Now, let's make the change of variable:

$$x_j = O_{jl}u_l$$

Remember Jacobian:

$$J = \left| \det rac{\partial x_j}{\partial u_l}
ight| = \left| \det O_{jl}
ight| = 1$$

$$egin{align} I(A) &= \int_{\mathbb{R}^n} d^n u e^{-O_{ik} u_k A_{ij} O_{jlul}} \ &= \int_{\mathbb{D}^n} d^n u e^{-u_k (O_{ik} A_{ij} O_{jl}) u_l} \ \end{aligned}$$

$$\int_{\mathbb{R}^n} d^n u e^{-(\lambda_1 u_1^2 + \lambda_2 u_2^2 + \cdots + \lambda u_n^2)}$$

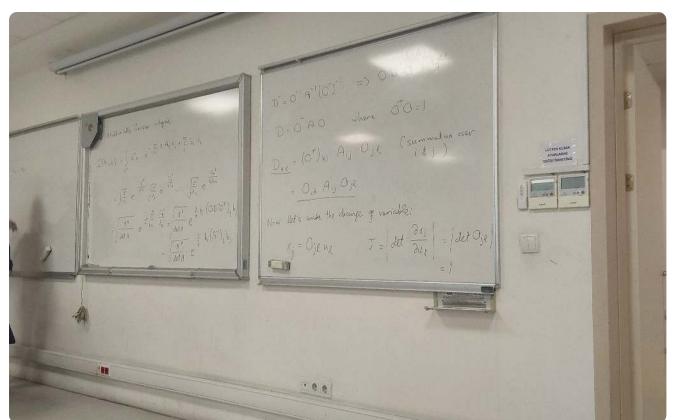
$$\int_{-\infty}^{\infty}du_1e^{-\lambda_1u_1^2}\int_{-\infty}^{\infty}du_2e^{-\lambda_2u_{2^2}}\dots$$

Then,

$$egin{aligned} I(A) &= \int_{\mathbb{R}} d^n x e^{-\sum_{ij} x_i A_{ij} x_j} \ &= \sqrt{rac{\pi}{\lambda_1}} \sqrt{rac{\pi}{\lambda_2}} \cdots \sqrt{rac{\pi}{\lambda_n}} \ &= \sqrt{rac{\pi^n}{\det A}} \end{aligned}$$

Let's make the problem even more complicated:

$$egin{align} I(A,b) &= \int_{\mathbb{R}} d^n x e^{-\sum_{ij} x_i A_{ij} x_j + \sum_i b_i x_i} \ &= \sqrt{rac{\pi}{\lambda_1}} e^{rac{c_1^2}{4\lambda_1}} \cdots \ &= \sqrt{rac{\pi^n}{\det A}} e^{rac{1}{4} \sum_{k=1}^n rac{c_k^2}{\lambda_k}} \end{aligned}$$



Multiplier Gaussian integrals.

$$I(A,b) = \int d^{n}x e^{-\frac{z}{4}} \frac{1}{x^{2}} \int_{-\frac{z}{4}}^{\frac{z}{4}} \frac{1}{x^{2}} \int_{-\frac{z}{$$

