PHYS 325 Lecture 1 11.02.2025

- V: Finite dimensional vector space reals

dual space V^*: set of all linear functions from V to $\mathbb{R}$.

- V^* is a vector space $\dim V^* = \dim V$

- $\alpha \in V^*$ α: co-vector. $v_1, v_2 \in V$ $c_1, c_2 \in \mathbb{R}$

$$\alpha(c_1 v_1 + c_2 v_2) = c_1 \alpha(v_1) + c_2 \alpha(v_2)$$

→ "How do they form a vector space?":

$$(c\alpha)(v) = c\alpha(v) \quad \text{(Multiplication)}$$

$$(\alpha+\beta)(v) = \alpha(v) + \beta(v) \quad \text{(Addition)}$$

Let's work with bases: Notice: subscript for basis el. of V, superscript for " " " V^*

$\{e_i\}$ basis for V.

A basis $\{\tilde{e}^i\}$ for V^* is called dual to $\{e_i\}$ if:

$$\tilde{e}^i(e_j) = \delta^i_j$$

Since placement of the script tells which space we're in, we'll drop the tilde from now on.

☆: $(V^*)^* = V$ for finite dimensional vector spaces.

Def: $V(\alpha) := \alpha(V)$ Linear:

- $V(c\alpha) = (c\alpha)(V) = c\alpha(v) = cV(\alpha)$

Also:

- $V(\alpha+\beta) = (\alpha+\beta)(V) = \alpha(V) + \beta(V) = V(\alpha) + V(\beta)$

"They are called Dual because they are linear

$\rightarrow \alpha, \beta \in V^*$ $\qquad$ $\alpha : V \to \mathbb{R}$ linear

$\qquad\qquad\qquad\qquad$ $\beta : V \to \mathbb{R}$ linear

$$\alpha \otimes \beta : V \times V \to \mathbb{R}$$

$$(u, v) \longmapsto \alpha(u)\,\beta(v)$$

In other words:

$$(\alpha \otimes \beta)(u, v) = \alpha(u)\,\beta(v)$$

! $\alpha \otimes \beta$ is bilinear (linear for both first and second entry.

Let's check:

$$(\alpha \otimes \beta)(c_1 u_1 + c_2 u_2, v) = \alpha(c_1 u_1 + c_2 u_2)\beta(v)$$

$$= c_1\, \alpha(u_1)\beta(v) + c_2\, \alpha(u_2)\beta(v)$$

$$= c_1 (\alpha \otimes \beta)(u_1, v) + c_2 (\alpha \otimes \beta)(u_2, v)$$

Similarly, (exercise):

$$(\alpha \otimes \beta)(u, c_1 v_1 + c_2 v_2) = c_1 (\alpha \otimes \beta)(u, v_1) + c_2 (\alpha \otimes \beta)(u, v_2)$$

Now, Let's Define:

$\rightarrow V^* \otimes V^*$: The space of all bilinear functions from $V \times V \to \mathbb{R}$

$\rightarrow \omega^{(2)}(u, v)$: linear in both entries.

if $\alpha, \beta \in V^*$ then $\alpha \otimes \beta \in V^* \otimes V^*$

What is $V^* \otimes V^* \otimes V^*$? → elements of this vec space are tensors of rank 3 and [illegible]

The space of all trilinear functions from $V \times V \times V$ to $\mathbb{R}$.

→ $\overbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}^{n\text{-factors}}$: the space of all n-linear functions from $\underbrace{V \times V \cdots \times V}_{n\text{-factors}}$ to $\mathbb{R}$.

→ Exercise: Check these form vector spaces.

Mixed Tensors

→ $V^* \otimes V$? The space of all bilinear functions from $V \times V^*$ to $\mathbb{R}$

n: covariant rank

m: contravariant rank

$T, S \in \overbrace{V^* \otimes \cdots V^*}^{n\text{-factors}} \otimes \overbrace{V \otimes \cdots \otimes V}^{m\text{-factors}}$: the set of all $(n+m)$-linear functions from

$$\underbrace{V \times \cdots V}_{n\text{-factors}} \times \underbrace{V^* \times \cdots V^*}_{m\text{-factors}} \text{ to } \mathbb{R}$$

Adding Two Tensors:

- $(cT)(v_1, \dots, v_n, \alpha_1, \dots \alpha_m) = c\,T(v_1, \dots, v_n, \alpha_1, \dots \alpha_m)$

- $(T+S)(v_1, \dots v_n, \alpha_1 \cdots \alpha_m) = T(v_1, \dots v_n, \alpha_1 \dots \alpha_m)$
$+ S(v_1, \dots v_n, \alpha_1, \dots \alpha_m)$

Note

→ If no n → contravariant ~~contravariant~~ tensor,

no m → covariant tensor.

$\omega \in V^* \otimes V^*$ $\qquad \{e_i\}$ basis for V.

$$u = u^i e_i = \sum_{i=1}^{\dim V} u^i e_i \qquad (u^i \in \mathbb{R})$$ (einstein summation convention)

$$v = v^j e_j$$

- $$\omega(u, v) = \omega(u^i e_i, v^j e_j) = u^i \omega(e_i, v^j e_j) = u^i v^j \underbrace{\omega(e_i, e_j)}_{\omega_{ij}} = \omega_{ij} u^i v^j = \omega_{ij} (e^i \otimes e^j)(u,v)$$

- Let $\{e^k\}$ be a basis in V^* dual to $\{e_i\}$

$$e^k(e_i) = \delta_i^{\ k}$$

$$e^k(u) = e^k(u^i e_i) = u^i e^k(e_i) = u^i \delta^k_i = u^k$$

$$\left.\begin{aligned} u^i &= e^i(u) \\ v^j &= e^j(v) \end{aligned}\right\} \quad u^i v^j = e^i(u)\, e^j(v) = (e^i \otimes e^j)(u, v)$$

(Tensor product of two covectors)

Thus...

$$\boxed{\omega = \omega_{ij}\, e^i \otimes e^j}$$

__Exercise:__ Show that $\{e^i \otimes e^j\}$ are linearly independent.

__Hint:__ Show ~~...~~ $\sigma_{ij} e^i \otimes e^j = 0$

How? → 4

$(\sigma_{ij}\, e^i \otimes e^j)(e_1, e_1) = \sigma_{ij}\, e^i(e_1)\, e^j(e_1)$

$= \sigma_{ij}\, \delta_1^i\, \delta_1^j = \sigma_{11} \Rightarrow \sigma_{11} = 0$

Therefore $\{e^i \otimes e^j\}$ is a basis for $V^* \otimes V^*$

You can generalize this

$\underbrace{\{e^i \otimes e^j \otimes e^k\}}_{\text{Basis for } V^* \otimes V^* \otimes V^*}$ $(e^i \otimes e^j \otimes e^k)(u, v, w) := e^i(u)\, e^j(v)\, e^k(w)$

The most general case:

$\{e^{i_1} \otimes \dots e^{i_n} \otimes e_{j_1} \otimes \dots \otimes e_{j_m}\}$ is a basis for

$\underbrace{V^* \otimes \dots \otimes V^*}_{n\text{-factors}} \otimes \underbrace{V \otimes \dots \otimes V}_{m\text{-facters.}}$

Let's figure out dimensions:

$\dim V = \dim V^* = N$, therefore $\dim (V^* \otimes V^*) = N^2$

- Similarly, $\dim V^* \otimes V^* \otimes V^* = N^3$

and N^{n+m} for the most general case