PHYS311 Lecture 3 — 30.09.24

Laws of Motion in Special Relativity.

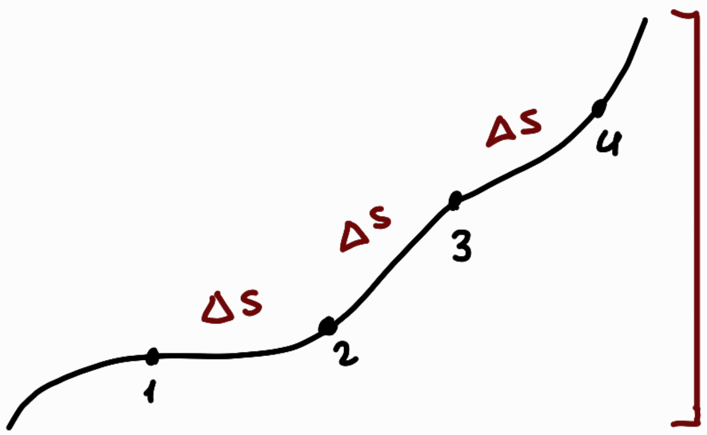
Invariant Interval

$$\Delta s^2 \equiv c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$
$$\equiv c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2$$

$\Delta s^2 > 0$: timelike intervals

time order of such supported events is the same for all inertial observers. (Contains causality)

Motion: Any causally connected set of events.



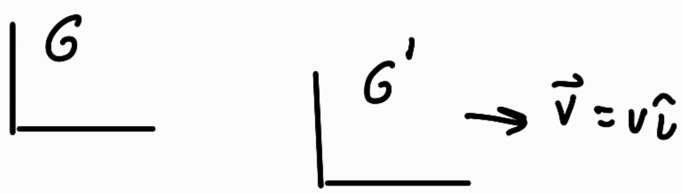
affine parameter

Time can be made the affine parameter (Same for all observers)

But it can't be the affine parameter for special relativity. Invariant interval can be.

Remember the transformation laws:

$$ct' = \gamma(ct - \beta x)$$
$$x' = \gamma(x - \beta ct)$$
$$y' = y$$
$$z' = z$$



Define a new object: Four Vector (A^0, A^1, A^2, A^3)

$A^{0'} = \gamma(A^0 - \beta A^1)$
$A^{1'} = \gamma(A^1 - \beta A^0)$
$A^{2'} = A^2$
$A^{3'} = A^3$

- if A^0 is like a scalar under ordinary 3d rotations. (acts like time)
- A^1, A^2, A^3 are vectors under ordinary rotations.

Let $A \circ A \equiv (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$

$$= A' \circ A'$$

*inner product of this 4 vectors is the same for all observers.

Let $ds^2 = \lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \Delta s^2 \Rightarrow ds$ is an invariant infinitesimal

s.t. $\Delta s^2 \geq 0$

$$\int_{\text{event start}}^{\text{event end}} ds = s_{\text{start-end}}$$

$A(s)$ a four-vector related to the motion.

$\Rightarrow \frac{dA}{ds}$ is also a four vector!

$= B$

$X(s)$: (x^0, x^1, x^2, x^3)
$\quad\;\; ct \quad x \quad y \quad z$

$$U(s) \equiv \frac{dX(s)}{ds}$$

For a given particular frame one can say:

$$ds = \sqrt{c^2 dt^2 - dx^2 - dy^2 \ldots}$$
$$= c\,dt \sqrt{1 - \frac{u^2}{c^2}} \qquad u^2 : \vec{u}\cdot\vec{u} \qquad \vec{u} = \frac{d\vec{r}}{dt}$$

$$X^0 = ct, \quad X^1 = x, \ldots$$

$$U^0 = \frac{dx^0}{d\tau} = \frac{c\,dt}{dt\sqrt{1-u^2/c^2}} = \boxed{\frac{c}{\sqrt{1-u^2/c^2}}}$$

4-velocity

$$U^1 = \frac{dx}{dt\sqrt{1-u^2/c^2}} = \frac{u^1}{\sqrt{1-u^2/c^2}} \Longrightarrow \boxed{\vec{U} = \frac{\vec{u}}{\sqrt{1-u^2/c^2}}}$$

$$U \cdot U = (U^0)^2 - (U^1)^2 - (U^2)^2 - (U^3)^2$$
$$= \frac{c^2}{1-u^2/c^2} - \frac{u^2}{1-u^2/c^2} = c^2$$

$$\boxed{U \cdot U = c^2}$$

$$P \equiv \overset{\text{inertial mass}}{m}\, U$$: Four-Momentum

$$P \cdot P = \underbrace{m^2 c^2}_{\text{constant}}$$

$$\mathcal{F} \equiv \frac{dP}{d\tau}$$: Four-Force (Minkowski)

- There are many ways to approach this. We are trying to unpeel the orange.

$$P \circ P = m^2c^2 \rightarrow \text{constant. So:}$$

! You take the idea from Newtonian physics that the momentum changes in the presence of a force

$$\frac{d}{d\tau}(P \circ P) = 0 = \mathcal{F} \circ P$$

$$\vec{\mathcal{F}} = \frac{d\vec{P}}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}}\frac{d\vec{P}}{dt}$$

$$\underbrace{\vec{P} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}}_{\text{relativistic momentum}}$$

$$\vec{\mathcal{F}} \equiv \frac{1}{\sqrt{1-u^2/c^2}}\vec{F} \rightarrow \vec{F} = \frac{d\vec{p}}{dt}$$

$$\mathcal{F}^0 P^0 - \vec{\mathcal{F}} \cdot \vec{P} = 0 \qquad \mathcal{F}^0 = \frac{dP^0}{d\tau}, \quad P^0 = mU^0 = \frac{mc}{\sqrt{1-u^2/c^2}}$$

$$\mathcal{F}^0 = \frac{\vec{\mathcal{F}} \cdot \vec{P}}{P^0}$$

$$\frac{dP^0}{dt}\frac{1}{\cancel{\sqrt{1-u^2/c^2}}} = \frac{\cancel{m}\vec{F} \cdot \vec{u}}{\cancel{\sqrt{1-u^2/c^2}}} \cdot \frac{1}{\sqrt{1-\cancel{u^2/c^2}}} \cdot \frac{\cancel{\sqrt{1-u^2/c^2}}}{\cancel{m}c}$$

$$= \boxed{\frac{dP_c^0}{dt} = \vec{F} \cdot \vec{u}}$$ (concept of power)

$$P^0 \equiv \frac{E}{c} \qquad \boxed{E = \frac{mc^2}{\sqrt{1-u^2/c^2}}}$$ Relativistic Kinetic Energy

Let's open this in Taylor Series:

$$(1-x)^b = 1 - bx + \cdots$$

$$b = -1/2$$

$$\frac{1}{\sqrt{1-u^2/c^2}} \simeq 1 - \left(-\frac{1}{2}\right)\frac{u^2}{c^2} + \cdots$$

$$\frac{mc^2}{\sqrt{1-u^2/c^2}} \simeq mc^2 + \underbrace{\frac{1}{2}mu^2}_{\text{Newtonian}}$$

A particle at rest, must have energy.

↳ inertial mass is a form of energy.

$$\rightarrow \sum_{i=1}^{N} E_i = \sum_{i=1}^{N} \frac{m_i c^2}{\sqrt{1-u_i^2/c^2}}$$

$E_{mass\,i}$ K_i

$$\approx c^2 \sum_{i=1}^{N} m_i + \sum_{i=1}^{N} \frac{1}{2} m_i u_i^2$$

In Einsteinian relativity, there is no mass conservation.

$$= c^2 \sum_{i=1}^{N'} \hat{m}_i + \sum_{i=1}^{N'} \frac{1}{2} \hat{m}_i \hat{u}_i^2$$

$E_{mass\,f}$ K_f

$m \neq 0$

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} \qquad E = \frac{mc^2}{\sqrt{1-u^2/c^2}} \qquad \frac{d\vec{p}}{dt} = \vec{F} \qquad \frac{dE}{dt} = \vec{F}\cdot\vec{u}$$

$$\boxed{\begin{gathered} \mathcal{F}^0 = \frac{\vec{\mathcal{F}}\cdot\vec{P}}{P_0} = \frac{1}{\sqrt{1-u^2/c^2}}\,\vec{F}\cdot\frac{\vec{u}}{c} \\ \vec{\mathcal{F}} = \frac{1}{\sqrt{1-u^2/c^2}}\,\vec{F} \end{gathered}}$$

$$\mathcal{F}^{0\prime} = \frac{1}{\sqrt{1-u'^2/c^2}}\,\vec{F}'\cdot\frac{\vec{u}'}{c}$$

$$\vec{\mathcal{F}}' = \frac{1}{\sqrt{1-u'^2/c^2}}\,\vec{F}'$$

$$\boxed{\vec{F} = 0 \iff \vec{F}' = 0}$$

- Being "free" is the same for each inertial observer.

But forces are not (Unless they are all on the same line / parallel)

(?)

- See that $\vec{F}$ is NOT parallel to acceleration.
  So it has a component // to v.

* There will be a finite time where particle will reach the speed of time.

This can't be! v has to change.

$$\frac{\frac{mc^2}{c^2}\,\vec{u}\cdot\frac{d\vec{u}}{dt}}{(1-u^2/c^2)^{3/2}}$$