PHYS 325 Lecture 10 31.10.24

Example

$$I = \int_{-\infty}^{\infty} dx \frac{\cos x}{x^2+a^2} \quad (a>0)$$

- $e^{ix} = \cos x + i \sin x$
- $\cos x = \mathrm{Re}\, e^{ix}$

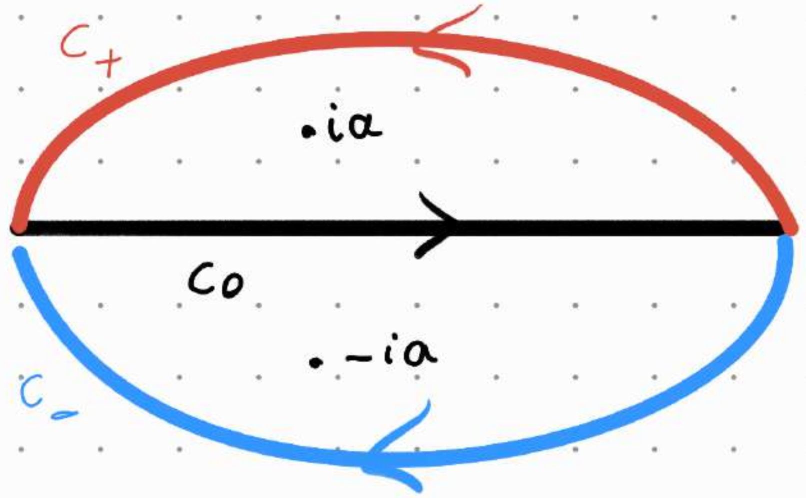
$$\int_{-\infty}^{\infty} dx \frac{e^{ix}}{x^2+a^2} = \int_{-\infty}^{\infty} dx \frac{\cos x}{x^2+a^2} + i \int_{-\infty}^{\infty} dx \frac{\sin x}{x^2+a^2}$$

0 (sin is an odd function)

$$\Rightarrow \mathrm{Re} \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x^2+a^2} = \int_{-\infty}^{\infty} dx \frac{\cos x}{x^2+a^2}.$$

- $\displaystyle\int_{-\infty}^{\infty} dx \frac{e^{ix}}{x^2+a^2}$

$$f(z) = \frac{e^{iz}}{z^2+a^2} = \quad e^{iz}$$



Analytic except at the zeros.

(ia, $-ia$ are simple poles)

• Which way should I close it?
C_+ or C_- (Contour trick)

C_- : $z = x + iy = Re^{i\theta}$ $\quad -\pi \leq \theta \leq 0$

$$e^z = e^{x+iy} = e^{iR(\cos\theta + i\sin\theta)} = e^{iR\cos\theta}\, e^{-R\sin\theta}$$

(+) in this interval. The integral diverges

• So C_- is not the correct choice. @ $\lim R \to \infty$

• C_+ is the correct choice

$C_T = C_0 \cup C_+$ encloses the simple pole at ia.

Calculating C_T : $$\underbrace{\int_{C_T} dz\, f(z)}_{\substack{2\pi i\, a_{-1,ia} \\ \text{(residue)}}} = \int_{C_0} dz\, f(z) + \int_{C_+} dz\, f(z)$$

$$= \frac{\pi e^{-a}}{a}$$

$$a_{-1,\,ia} = \lim_{z \to ia} (z-ia) f(z) = \lim_{z \to ia} \frac{(z-ia)\,e^{iz}}{(z+ia)(z-ia)} = \frac{e^{-a}}{2ia}$$

$\underline{C_+}$

$$\int_{C_+} dz\, f(z) = \int R i\, d\theta\, e^{i\theta}\, \frac{e^{iR\cos\theta}\, e^{-R\sin\theta}}{R^2 e^{2i\theta} + a^2}$$

we want to show:

PS

$$\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\lim_{R \to \infty} \int_{C_+} dz\, f(z) = 0$$

$$\left| \int_{C_+} dz\, f(z) \right| = \left| \int_0^{\pi} R i\, d\theta\, e^{i\theta}\, \frac{e^{iR\cos\theta}\, e^{-R\sin\theta}}{R^2 e^{2i\theta} + a^2} \right|$$

$$\leq \int_0^{\pi} d\theta\, R \underbrace{|i\, e^{i\theta}|}_{1}\, \frac{\overbrace{|e^{iR\cos\theta}|}^{1}\, e^{-R\sin\theta}}{|R^2 e^{2i\theta} + a^2|}$$

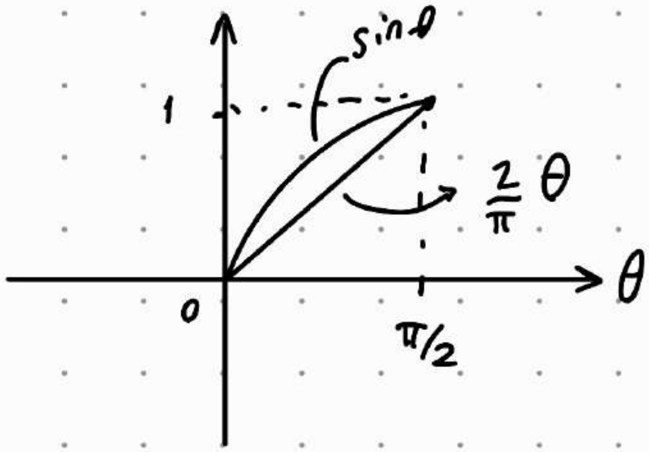
$$\leq R \int_0^{\pi} d\theta\, \frac{e^{-R\sin\theta}}{|R^2 e^{2i\theta} + a^2|} \leq R \int_0^{\pi} d\theta\, \frac{e^{-R\sin\theta}}{|R^2 - a^2|}$$

$$= \frac{R}{R^2 - a^2} \int_0^{\pi} d\theta\, e^{-R\sin\theta}$$

EXERCISE: $\int_0^{\pi} d\theta\, e^{-R\sin\theta} = 2\int_0^{\pi/2} d\theta\, e^{-R\sin\theta} \leq \frac{\pi}{2}(1-e^{-R})$

(Jordan's Lemma)

$$\int_0^{\pi} d\theta\, e^{-R\sin\theta}$$



$$\sin\theta > \frac{2}{\pi}\theta$$

$$e^{-R\sin\theta} \leq e^{-R\frac{2}{\pi}\theta}$$

Therefore:

$$2\int_0^{\pi/2} d\theta\, e^{-R\sin\theta} \leq 2\int_0^{\pi/2} d\theta\, e^{-R\frac{2\theta}{\pi}} = -\frac{\pi}{R}(e^{-R}-1)$$

$\frac{\pi e^{-a}}{a}$:

$$\left|\int_{C_+} dz\, f(z)\right| \leq \frac{R}{R^2-a}\,\frac{\pi}{R}(1-e^{-R}) \xrightarrow[R\to\infty]{} 0$$

- You can apply Jordan's lemma for fourier integrals.

$$\int_{-\infty}^{\infty} dx \, \frac{\cos x}{x^2+a^2} = \frac{\pi e^{-a}}{a} \qquad \int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x^2+a^2} = \frac{\pi e^{-a}}{a}$$

Fourier Type integrals Let's say I have:

- $$\int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x^2+a^2} = \frac{\pi e^{-a}}{a}$$

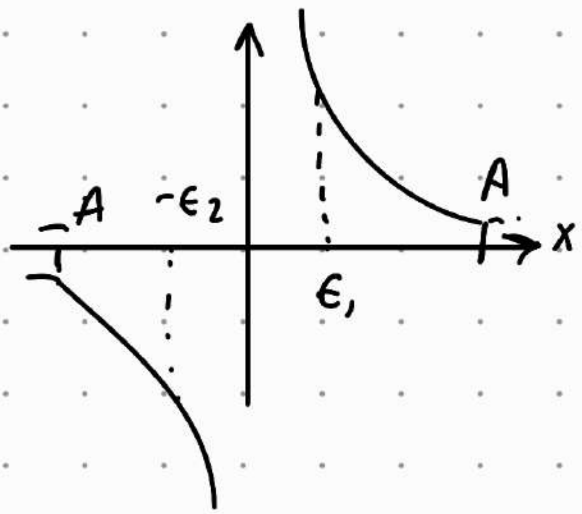
$$e^{-iz} = e^{-i(x+iy)} = e^{-ix} e^{y}$$

- $$\left[\int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x^2+a^2} \right]^* = \left(\frac{\pi e^{-a}}{a} \right)^*$$

EXAM IS UP TO THIS POINT

improper integral definition

- $$\int_{-A}^{A} dx \, \frac{1}{x}$$



$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Let's say we cut the integral at ϵ_1 and ϵ_2.

- $$\int_{-A}^{A} dx \, \frac{1}{x} = \lim_{\epsilon_1 \to 0^+} \int_{\epsilon_1}^{A} dx \, \frac{1}{x} + \lim_{\epsilon_2 \to 0^+} \int_{-A}^{-\epsilon_2} dx \, \frac{1}{x}$$

$$\rightarrow \int_{\epsilon_1}^{A} dx \, \frac{1}{x} = \ln|x| \Big|_{\epsilon_1}^{A}$$

$$\ln|A| - \ln|\epsilon_1| = +\infty$$

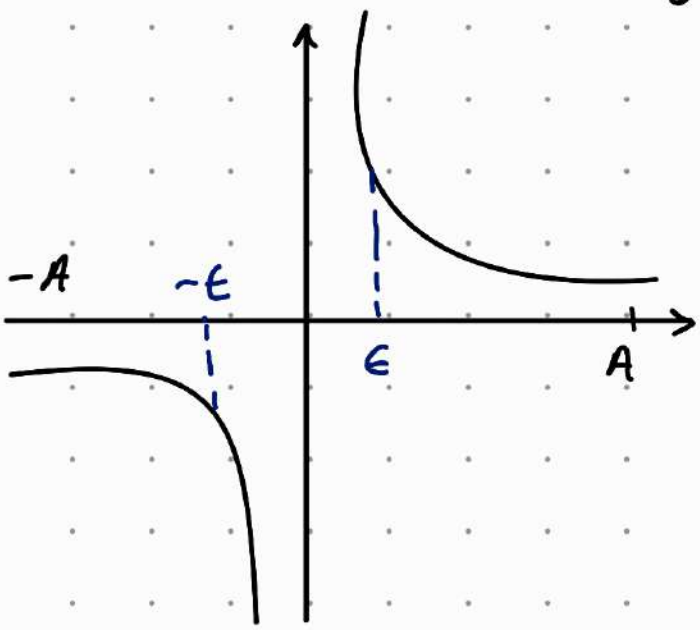
Cauchy's Principle Value Integral

P: Cauchy's principle value integral

$$P\int_{-A}^{A} dx\,\frac{1}{x} = \lim_{\epsilon\to 0^+}\left[\int_{-A}^{-\epsilon} dx\,\frac{1}{x} + \int_{\epsilon}^{A} dx\,\frac{1}{x}\right]$$

$$= \lim_{\epsilon\to 0^+}\left[\ln\frac{|\epsilon|}{|A|} + \ln\frac{|A|}{|\epsilon|}\right]$$

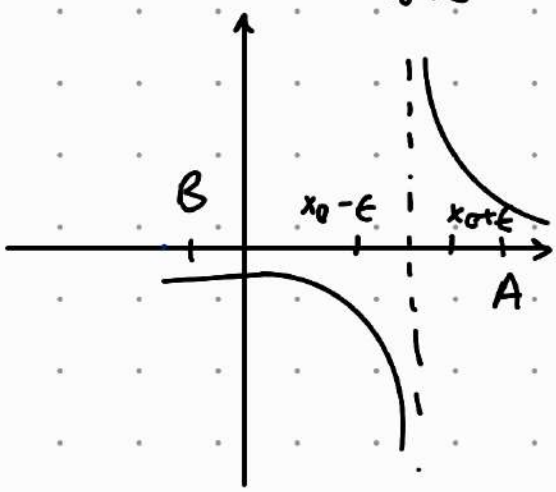
$$= \lim_{\epsilon\to 0^+} \ln 1 = 0$$



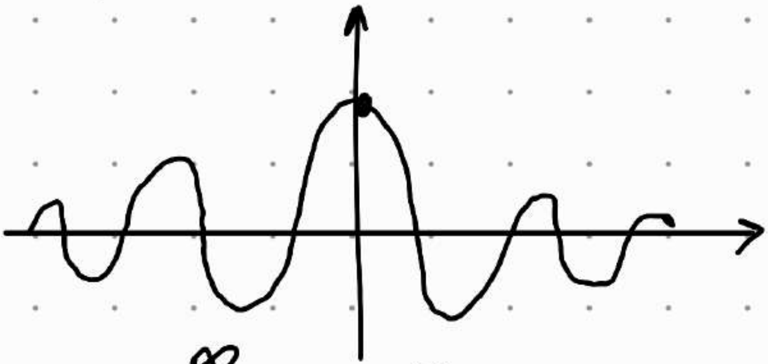
Why is this useful?

$$P\int_{B}^{A} dx\,\frac{1}{x-x_0} = \lim_{\epsilon\to 0^+}\left[\int_{B}^{x_0-\epsilon} dx\,\frac{1}{x-x_0} + \int_{x_0+\epsilon}^{A} dx\,\frac{1}{x-x_0}\right]$$

$B < x_0 < a$



Example: Consider: $\displaystyle\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x}$



$$P\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x} = \lim_{\epsilon \to 0^+} \left[\int_{-\infty}^{-\epsilon} dx \, \frac{\sin x}{x} + \int_{\epsilon}^{\infty} dx \, \frac{\sin x}{x} \right]$$

$$\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x}$$

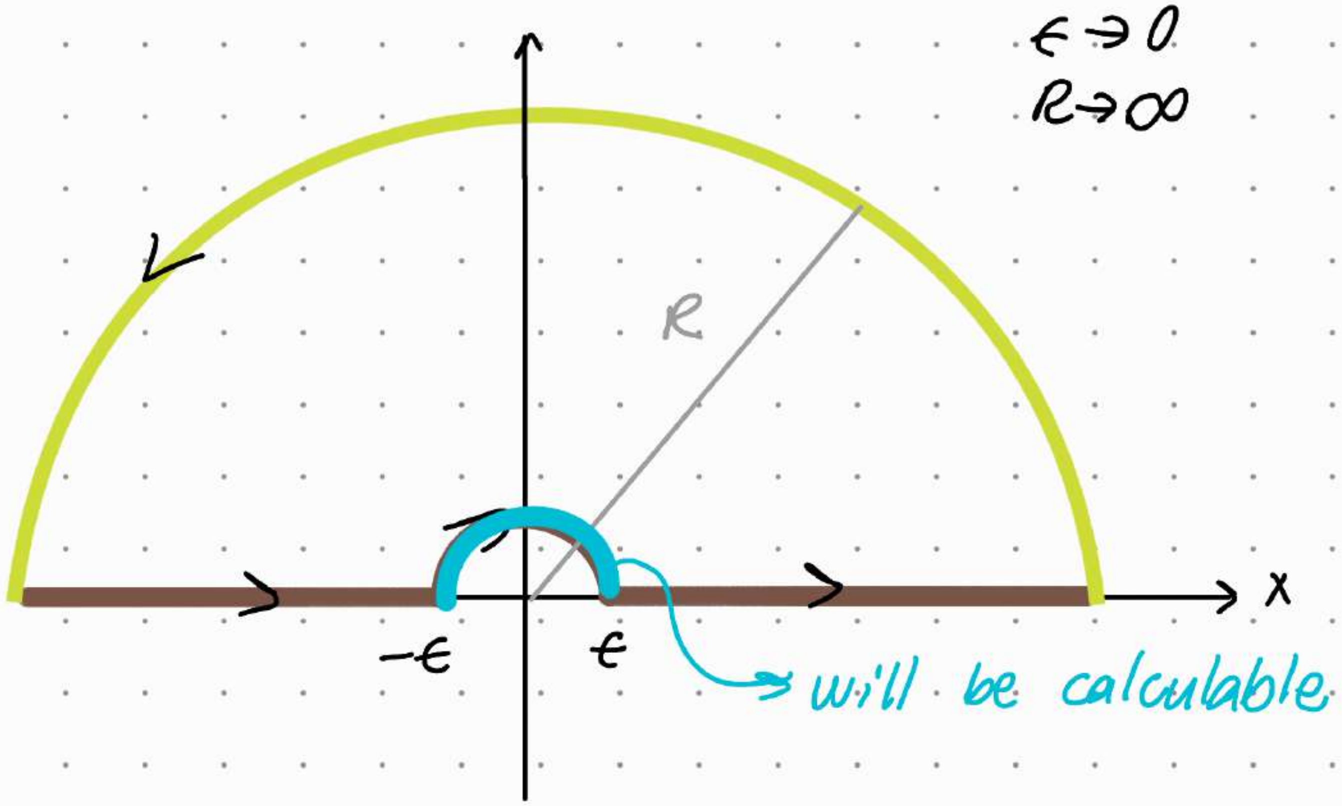
Example

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} = P\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x}$$

$\sin x = \operatorname{Im} e^{ix}$

$$= \operatorname{Im} P\int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x}$$

$$= \operatorname{Im} \lim_{\epsilon \to 0^+} \left[\int_{-\infty}^{-\epsilon} dx \, \frac{e^{ix}}{x} + \int_{\epsilon}^{\infty} dx \, \frac{e^{ix}}{x} \right]$$



$$C_T = (-\infty, -\epsilon) \cup C_\varepsilon \cup (\epsilon, \infty) \cup C_R$$