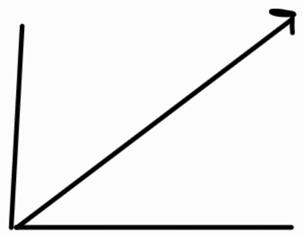
Systems of Particles (Continued)

30.09.24



$\vec{r}_i \quad i = 1, \ldots, N$

$$\vec{r}_{CM} = \frac{1}{m_T} \sum_{i=1}^{N} \vec{r}_i m_i$$

CM behaves as a point particle of mass m_T under the influence of external forces.

$$\vec{F}_i = \vec{F}_i^{(ext)} + \vec{F}_i^{(int)} \vdots \quad \Downarrow \quad \vec{F}_{NET}^{(ext)} = m_T \vec{a}_{cm}$$

$$\vec{p}_i = m_i \vec{v}_i \qquad \vec{P}_{Total} = \sum_{i=1}^{N} \vec{p}_i \qquad \Bigg| \qquad \frac{d\vec{p}_{Tot}}{dt} = \sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} \vec{F}_i^{(EXT)} = \vec{F}_{NET}^{(EXT)}$$

Just another conservation law.

Total momentum of a closed system is conserved.

↳ Better definition for the 3rd law. Action - Reaction is just one of the solutions.

Define $K_i = \frac{1}{2} m_i u_i^2 = \frac{1}{2} m_i \vec{u}_i \cdot \vec{u}_i$

$$\frac{dK_i}{dt} = m_i \vec{u}_i \cdot a_i = \boxed{\begin{array}{l}\text{(Power)} \\ \vec{u}_i \cdot \vec{F}_i \overset{?}{=} -\dfrac{dV_i}{dt}\end{array}}$$

→ a scalar quantity (potential)

$$\Rightarrow \frac{d}{dt}(K_i + V_i) = 0$$

- V_i : a function of particle's position.

Remember the chain rule from calculus:

- $$\frac{d}{dt} f(x(t)) = \frac{dx}{dt}\frac{df}{dx}$$

- $$\frac{d}{dt} f(x(t), y(t), z(t)) = \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{dz}{dt}\frac{\partial f}{\partial z}$$

Lets Continue..

- V_i : a function of particle's position.

$$u_i \cdot \vec{F}_i = \frac{dx_i}{dt} F_i^{(x)} + \frac{dy_i}{dt} F_i^{(y)} + \frac{dz_i}{dt} F_i^{(z)}$$

$$\text{iff: } \left.\begin{aligned} F_i^{(x)} &= \frac{\partial V_i}{\partial x} \\ F_i^{(y)} &= \frac{\partial V_i}{\partial y} \\ F_i^{(z)} &= \frac{\partial V_i}{\partial z} \end{aligned}\right\} \vec{F}_i \text{ is a conservative force}$$

Every force changes K_i. But some change it so that the force gives you a conserved quantity.

→ Friction for example is not conserved.

Conservative Forces

$$\vec{\nabla}_i = \hat{\imath}\frac{\partial}{\partial x_i} + \hat{\jmath}\frac{\partial}{\partial y_i} + \hat{k}\frac{\partial}{\partial z_i}$$

if $\boxed{\vec{F}_i = -\vec{\nabla} V_i}$ with V_i a function of x, y, z.

$$V\Big(\underbrace{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}_{s = \text{distance}}\Big) \equiv V_{21}$$

$$\frac{\partial V}{\partial x_2} = (x_2 - x_1)\frac{dV}{ds} = F_{21}^{(x)}$$

$$\frac{\partial V}{\partial x_1} = -(x_2 - x_1)\frac{dV}{ds} = F_{12}^{(x)} = F_{21}^{(x)}$$

Central forces: Forces who are aligned with and only depend on distance. They are conservative.

$$K_{Tot} = \sum_{i=1}^{N} \frac{1}{2} m_i u_i^2$$

$$\frac{dK_{Tot}}{dt} = \sum_{i=1}^{N} m_i \vec{u}_i \cdot \vec{F}_i = \sum_{i=1}^{N}\left(\vec{u}_i \cdot \vec{F}_i^{(cons)} + \vec{u}_i \cdot \vec{F}_i^{(n\ cons)}\right)$$

$$\frac{dK_{Tot}}{dt} = \sum_{i=1}^{N} \vec{u}_i \cdot \underbrace{\sum_{\substack{j=1 \\ j\neq i}}^{N} \vec{F}_{ij}^{(cons)}}_{-\vec{\nabla}_i V_{ij}} + \text{n. cons.}$$

$$E_{Total} = K_{Total} + V_{Total}$$

$$V_{Total} = \sum_{i=1}^{N} \sum_{\substack{j=i \\ i\neq j}}^{N} V_{ij}$$

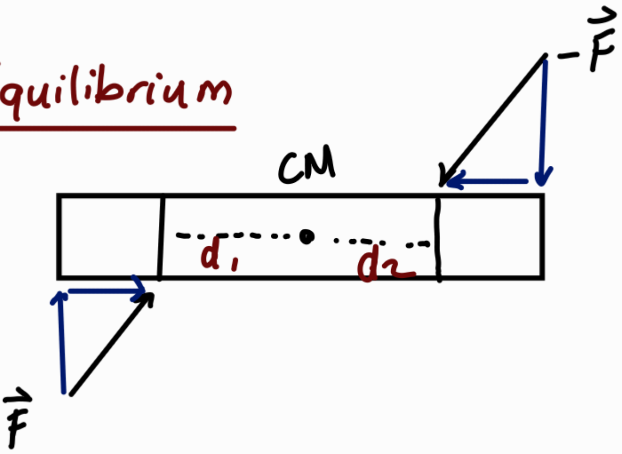
This is done because we don't want to double count.

$$\frac{dE_{Tot}}{dt} = 0 \quad \text{iff}$$

$$\sum_{i=1}^{N} \vec{u}_i \cdot \vec{F}_i^{(non\ cons)} = 0$$

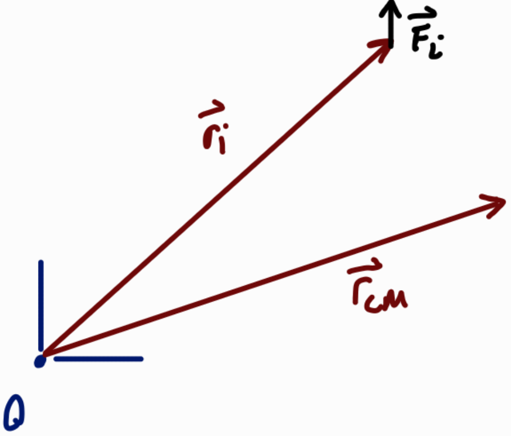
"Closed system which All Particle Interacts with Conservative Forces"

Equilibrium



- CM is stationary, but one can not say there is no motion. Actually K increases.

- I need an object that gives me the orientation of the rod:
  - torque and angular momentum.



- $\vec{\tau}_i^{(0)} \equiv \vec{r}_i \times \vec{F}_i$

if $\sum_{i=1}^{N} \vec{\tau}_i^{(0)} = 0$ the system is in mechanical equilibrium.

But we also said:

$$\sum_{i=1}^{N} \vec{F}_i = 0$$

Let us remember:

$$\vec{F}_i = \frac{d\vec{p}_i}{dt} = m_i \frac{d\vec{u}_i}{dt} = m_i \vec{a}_i$$

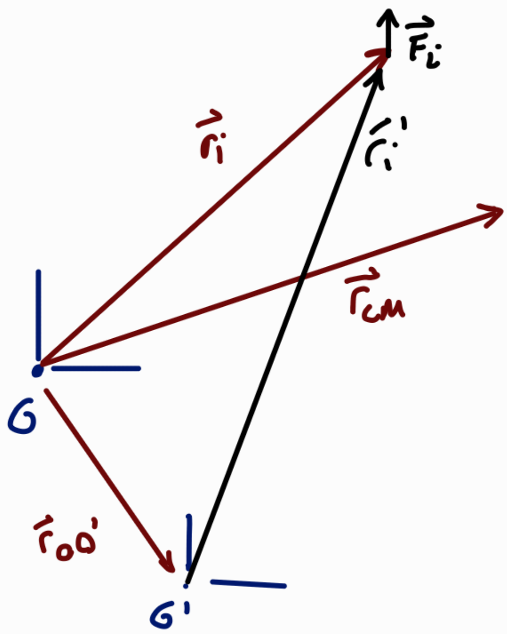
- $\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$

$$\vec{\tau}_i^{(0)} = \vec{r}_i \times \frac{d\vec{p}_i}{dt} = m_i \vec{r}_i \times \frac{du_i}{dt}$$

$$= \frac{d}{dt}\Big(\underbrace{m_i \vec{r}_i \times \vec{u}_i}_{\equiv \vec{L}_i^{(0)}}\Big)$$

$\equiv \vec{L}_i^{(0)}$ angular momentum of particle i.

$$\boxed{\vec{\tau}_i^{(0)} = \frac{d\vec{L}_i^{(0)}}{dt}}$$



Remember

We proved that forces are the same for both observers.

$$\boxed{\vec{r}_i = \vec{r}_{OO'} + \vec{r}_i'}$$

- Anything depending on velocity & position depends on the reference frame.

→ See them as a tool.

$$\underbrace{\sum_{i=1}^{N} \vec{\tau}_i^{\,(0)}}_{\vec{\tau}_{Net}^{\,(0)}} = \sum_{i=1}^{N} \frac{d\vec{L}_i^{\,(0)}}{dt} = \frac{d}{dt} \underbrace{\sum_{i=1}^{N} \vec{L}_i^{\,(0)}}_{\vec{L}_{Tot}^{\,(0)}}$$

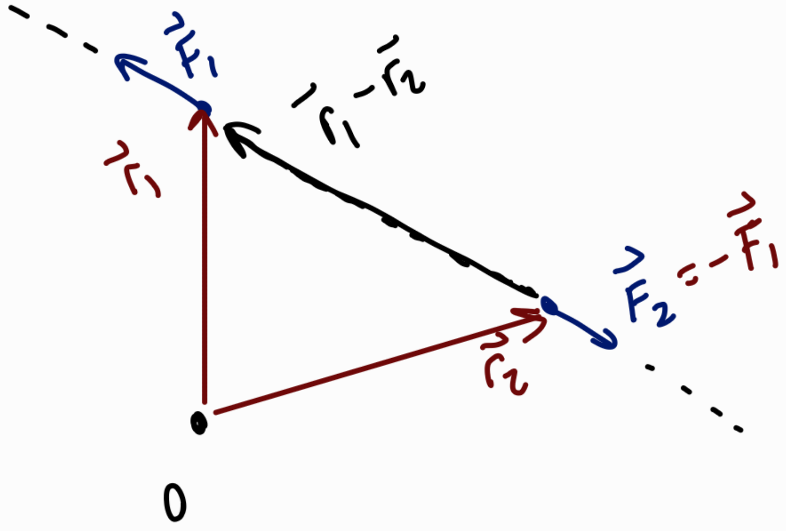
A stronger form of the 3rd law.

- Consider a closed system of particles:

All interactions between particles are along the "line" the torque vanishes. (Central Forces)

$\Downarrow$

Conservative



- $\vec{r}_1 \times \vec{F}_1 + r_2 \times \vec{F}_2 = (r_1 - r_2) \times \vec{F}_{12}$

  $\parallel (\vec{r}_1 - \vec{r}_2)$

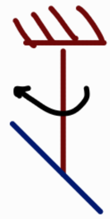
- $V((x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2)$

Question for the next time.

→ Does angular momentum "keep track" of the orientation of the object?

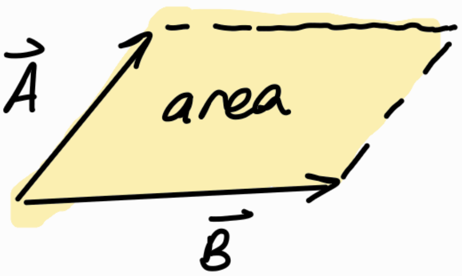
→ Is $\omega \parallel L$? Just like $p \parallel v$.

They can be, iff axis of rotation stays the same remember this is arbitrary choice however. Here we tilted the rod. $\vec{L}$ is not $\parallel \omega$. $\Leftarrow$



Remember:

We can associate an area to a cross product.



$\vec{A} \times \vec{B} \propto$ area