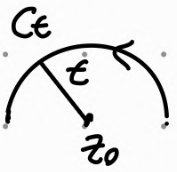
PHYS 325 Lecture 11 05.11.24

General Form of Principle Value Integral

$$P\int_A^B dx\, f(x) = \lim_{x \to x_0} \left[\int_A^{x_0-\epsilon} dx\, f(x) + \int_{x_0+\epsilon}^B dx\, f(x) \right]$$

Integration along small semi-circles



$$\int_{C_\epsilon} dz \frac{1}{z-z_0} = \int d\theta\, i\epsilon e^{i\theta} \frac{1}{\epsilon e^{i\theta}} = \underbrace{i\pi}_{\text{because it's a semi circle.}}$$

on C_ϵ : $z - z_0 = \epsilon e^{i\theta}$

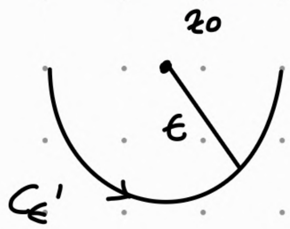
$dz = \epsilon\, i\, d\theta\, e^{i\theta}$

$n = 0, 1, 2, \ldots$

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} dz\, (z-z_0)^n = \lim_{\epsilon \to 0} \int_0^\pi d\theta\, i\epsilon e^{i\theta} \epsilon^n e^{in\theta}$$

$$= \lim_{\epsilon \to 0} \left[i\epsilon^{n+1} \int_0^\pi d\theta\, e^{i(n+1)\theta} \right] = 0$$

(This int. can be different than 0 but it's finite.)



$$\lim_{\epsilon \to 0} \int_{C_\epsilon'} dz \frac{1}{z - z_0} = i\pi$$

$n = 0, 1, 2, \ldots$

$$\lim_{\epsilon \to 0} \int_{C_\epsilon'} dz\, (z - z_0)^n = 0$$

EX:

?

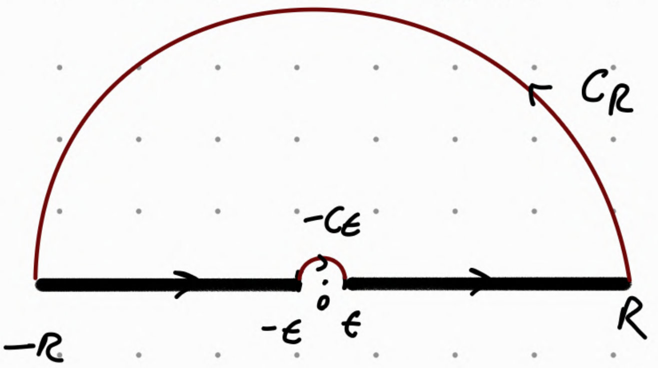
(odd & even function, so we can expand:)

$$I = \int_0^\infty dx\, \frac{\sin x}{x} = \frac{1}{2}\int_{-\infty}^{\infty} dx\, \frac{\sin x}{x} = \frac{1}{2} P \int_{-\infty}^{\infty} dx\, \frac{\sin x}{x} = \frac{1}{2} P\, \mathrm{Im} \int_{-\infty}^{\infty} dx\, \frac{e^{ix}}{x}$$

$$= \frac{1}{2}\, \mathrm{Im}\, P \int_{-\infty}^{\infty} dx\, \frac{e^{ix}}{x}$$

EXAMPLE:

In order to apply calculus of residues, I must close this contour.



$$C_T = (-R, -\epsilon) \cup (-C_\epsilon) \cup (\epsilon, R) \cup C_R$$

$$\int_{C_T} dz\, \frac{e^{iz}}{z} = 0$$ (Because pole of this integral is 0, and 0 is outside)

$$\rightarrow \int_{C_T} dz\, \frac{e^{iz}}{z} = 0 = \underbrace{\int_{-R}^{-\epsilon} dx\, \frac{e^{ix}}{x} + \int_{\epsilon}^{R} dx\, \frac{e^{ix}}{x}}_{?} + \int_{-C_\epsilon} dz\, \frac{e^{iz}}{z} + \int_{C_R} dz\, \frac{e^{iz}}{z}$$

$$\bullet \int_{-C_\epsilon} dz\, \frac{e^{iz}}{z} = \int_{-C_\epsilon} dz\, \frac{1}{z}\left[1 + iz + \frac{1}{2!}(iz)^2 + \ldots\right] = -\int_{C_\epsilon} dz\, \frac{1}{z} - \int_{C_\epsilon} dz\, \frac{iz}{z} - \int_{C_\epsilon} dz\, \frac{1}{2}\frac{(iz)^2}{z}$$

$\longrightarrow -i\pi + 0 + 0 + \dots = \boxed{-i\pi}$

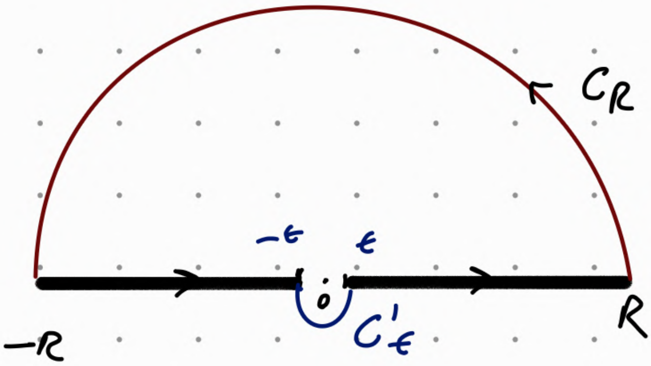
Exercise: Apply Jordan's Lemma to show that:

$$\lim_{R\to\infty} \int_{C_R} dz \, \frac{e^{iz}}{z} = 0$$

$$P\int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x} = i\pi \qquad \left(\text{Because } \int_{C_T} dz \, \frac{e^{iz}}{z} = 0\right)$$

$$\int_0^{\infty} dx \, \frac{\sin x}{x} = I = \frac{1}{2} \, \mathrm{Im} \, P\int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x} = \frac{\pi}{2}$$

- What if I closed the contour with a lower semi circle?
  - Result shouldn't depend on it.



$$a_{1,0} = \lim_{z\to 0} z \, \frac{e^{iz}}{z} = 1$$

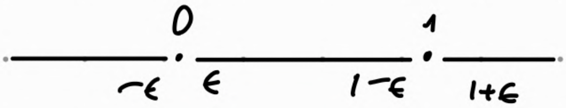
$$\rightarrow \int_{C_T} dz \, \frac{e^{iz}}{z} = 2\pi i = \int_{-R}^{-\epsilon} dx \, \frac{e^{ix}}{x} + \int_{\epsilon}^{R} dx \, \frac{e^{ix}}{x} + \underbrace{\int_{-C'_\epsilon} dz \, \frac{e^{iz}}{z}}_{i\pi \text{ (as } \epsilon\to 0)} + \int_{C_R} dz \, \frac{e^{iz}}{z}$$

EXERCISE:

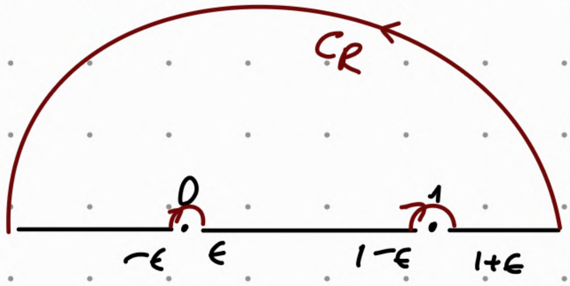
- $I = \int_{-\infty}^{\infty} dx \, \frac{\cos x}{x(x-1)}$! Blows up ! we must write it as:

(See: 2 poles on the real axis) 0 & 1

$$I = P\int_{-\infty}^{\infty} dx \, \frac{\cos x}{x(x-1)}$$



$$= \lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon} dx \ldots + \int_{\epsilon}^{1-\epsilon} dx \ldots + \int_{1+\epsilon}^{\infty} dx \ldots\right]$$



EXAMPLE: Integral involving a multivalued function

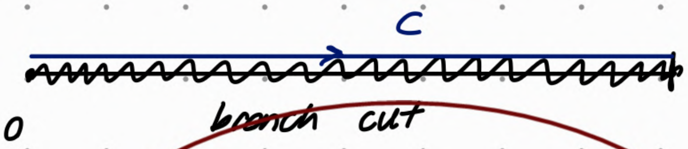
$-1 < a < 1$

$$I = \int_0^\infty dx \, \frac{x^a}{x^2+1}$$

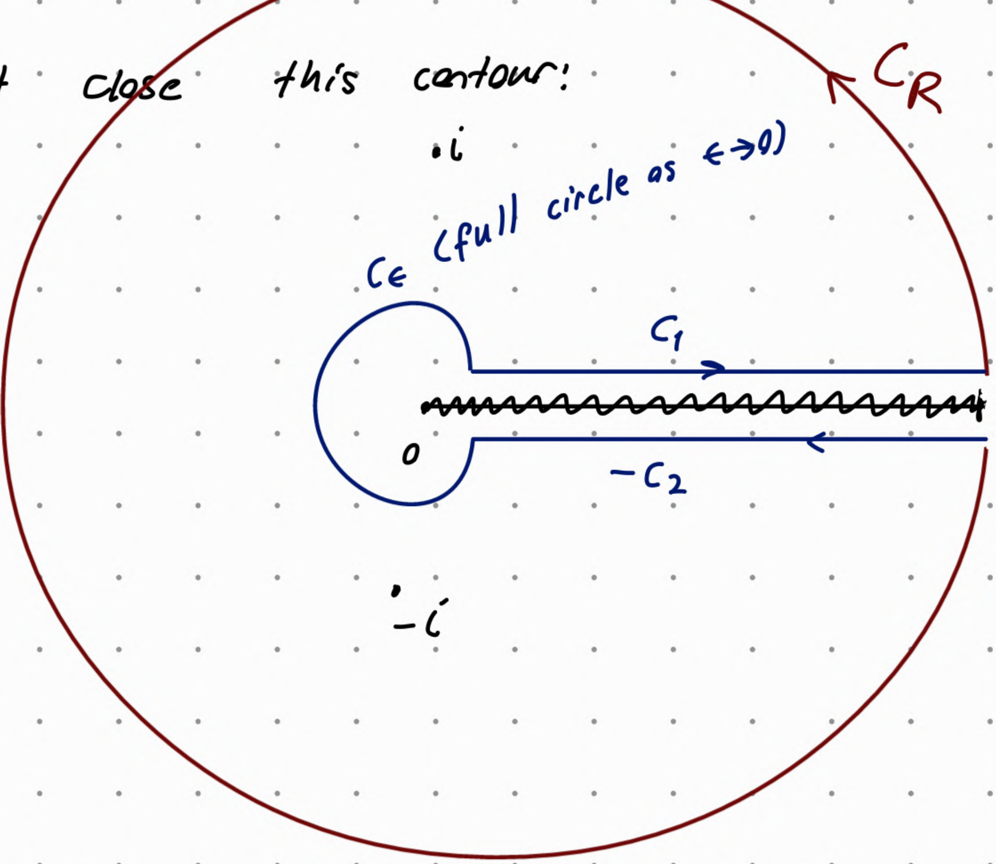
x^a — multivalued function.

x^2+1 — polynomial

Has a branch point at 0 and ∞:



I must close this contour:



Remember:

Pole: Singularity of finite order.

Laurent exp: $\sum_{n=-N}^{\infty} a_n (z-z_0)^n$

Remember: Interested only in the poles inside.

$C_T = C_1 \cup C_R \cup (-C_2) \cup C_\epsilon$

Two simple poles: $x = \pm i$

$$\oint_{C_T} dz \frac{z^a}{z^2+1} = \oint_{C_T} dz \frac{e^{a\ln z}}{(z+i)(z-i)} = 2\pi i \left[a_{-1,i} + a_{-1,-i}\right]$$

$$= 2\pi i \left[\frac{e^{a i \frac{\pi}{2}}}{2i} + \frac{e^{a i \frac{3\pi}{2}}}{-2i}\right]$$

$$a_{-1,i} = \lim_{z\to i} \frac{e^{a\ln z}(z-i)}{(z+i)(z-i)}$$

$$= \boxed{\pi\left[e^{ia\frac{\pi}{2}} - e^{ia\frac{3\pi}{2}}\right]}$$

$i = e^{i\frac{\pi}{2}}$ $\qquad -i = e^{i\frac{3}{2}\pi}$

$\ln i = i\frac{\pi}{2}$ $\qquad \ln(-i) = -i\frac{3\pi}{2}$

C_2:

$$\int_{-C_2} dz \frac{z^a}{z^2+1}$$

On $(-C_2)$: $z = t e^{i2\pi}$

$dz = e^{i2\pi} dt$

$z^a = e^{a\ln z}$ $\quad \ln z = \ln t + i2\pi$

1st branch.

$$= \int_\infty^0 dt \, \underbrace{e^{i2\pi}}_{1} \frac{e^{(a\ln t + ia2\pi)}}{t^2 \underbrace{e^{i4\pi}}_{1} + 1} = -e^{ia2\pi}\int_0^\infty dt \frac{t^a}{t^2+1} = \boxed{-e^{ia2\pi} I}$$

→ Because I have a prefactor, C_1 and $-C_2$ don't cancel each other.

$$\overset{(C_1)\,(-C_2)}{I\left(1-e^{ia2\pi}\right)} + \int_{C_R} + \int_{C_\epsilon}$$

C_ϵ: $z^a = e^{a\ln z}$

$$\left|\int_{C_\epsilon} dz \frac{z^a}{z^2+1}\right| = \left|\int_0^{2\pi} d\theta \; i\epsilon e^{i\theta} \frac{\epsilon^a e^{ia\theta}}{\epsilon^2 e^{2i\theta}+1}\right| = \left|\int_0^{2\pi} d\theta \, i\epsilon^{a+1} \frac{e^{i(a+1)\theta}}{\epsilon^2 e^{2i\theta}+1}\right|$$

$$\leq \epsilon^{a+1} \int_0^{2\pi} d\theta \; \frac{|i\, e^{i(a+1)\theta}|}{|\epsilon^2 e^{2i\theta}+1|}$$

$$\leq \epsilon^{a+1} \int_0^{2\pi} d\theta \frac{1}{1-\epsilon^2} = \frac{\epsilon^{a+1}}{1-\epsilon^2} 2\pi \xrightarrow[\epsilon\to 0]{} 0$$

(Triangle inequality: $|\epsilon^2 e^{2i\theta}+1| \geq \left||\epsilon^2 e^{i2i\theta}|-1\right| = |\epsilon^2-1| = 1-\epsilon^2$)

C_n C_R: $\quad z = Re^{i\theta} \qquad dz = R\, i d\theta\, e^{i\theta}$

$$\left| \oint_{C_R} dz \frac{z^a}{z^2+1} \right| \leq \frac{R^{a+1}}{R^2-1} 2\pi \xrightarrow[R\to\infty]{} 0 \qquad (a+1<2)$$

$$\boxed{I = \pi \frac{e^{ia\frac{\pi}{2}} - e^{ia\frac{3\pi}{2}}}{1-e^{ia2\pi}}}$$

- As a check, we want ~~have~~ the answer the way, (Function is a real number)

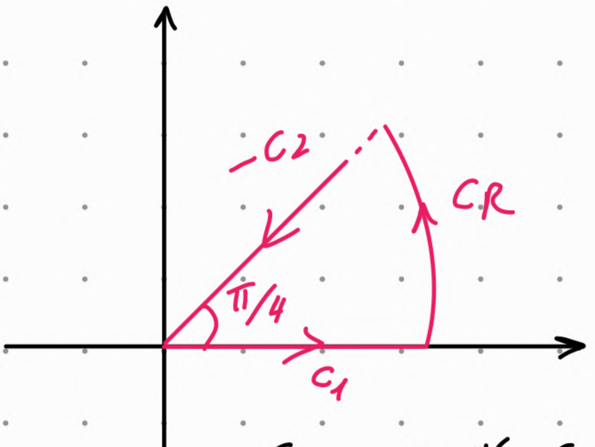
$$= \pi \; e^{ia\frac{\pi}{2}} \frac{(1-e^{ia\pi})}{(1-e^{2ia\pi})}$$

$$= \pi \; e^{ia\frac{\pi}{2}} \frac{1-e^{ia\pi}}{(1-e^{ia\pi})(1+e^{ia\pi})} = \frac{\pi}{e^{-\frac{ia\pi}{2}} + e^{\frac{ia\pi}{2}}} = \frac{\pi}{2\cos\frac{a\pi}{2}}$$

indeed the answer is real.

Complex Gaussian Integral (Important in Feynman Path Integral Method)

$$I = \int_0^\infty dx\, e^{iax^2} = \frac{1}{2}\sqrt{\frac{\pi}{-ia}} \quad (a>0)$$



$$C_T = C_1 \cup (-C_2) \cup C_R$$

$$\int_{C_1} dz\, e^{iaz^2}$$

No poles! Analytic everywhere. No residues!

$$\oint_{C_T} dz\, e^{iaz^2} = 0$$

(Cauchy-Goursat)

$$0 = \int_{C_1} dz\, e^{iaz^2} + \int_{-C_2} dz\, e^{iaz^2} + \int_{C_R} dz\, e^{iaz^2}$$

on $(-C_2)$: $\quad z = te^{i\frac{\pi}{4}} \qquad dz = e^{i\frac{\pi}{4}}dt$

$$\int_{-C_2} dz\, e^{iaz^2} = \int_\infty^0 dt\, e^{i\frac{\pi}{4}}\, e^{iat^2 \underbrace{e^{i\frac{\pi}{2}}}_{i}}$$

The reason we chose the angle as $\pi/4$.

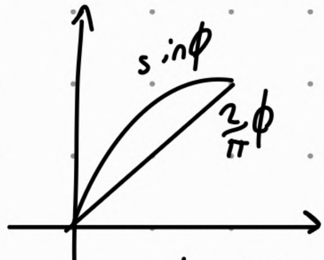
$$= -e^{i\frac{\pi}{4}} \int_0^\infty dt\, e^{-at^2} = -e^{i\frac{\pi}{4}}\, \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

(gaussian integral)

on C_R: $\quad z = Re^{i\theta} \qquad dz = Rid\theta\, e^{i\theta}$

$$\int_{C_R} dz\, e^{iaz^2} = \int_0^{\pi/4} d\theta\, iRe^{i\theta}\, e^{iaR^2(\cos 2\theta + i\sin 2\theta)}$$

Take absolute values:



$\sin\phi \geq \frac{2}{\pi}\phi$

$$\left| \int_{C_R} dz\, e^{iaz^2} \right| = \left| \int_0^{\pi/4} d\theta\; iRe^{i\theta}\, e^{iaR^2(\cos 2\theta + i\sin 2\theta)} \right|$$

(change of variable)

$$\leq \int_0^{\pi/4} d\theta\, R\, e^{aR^2 \sin 2\theta} = \int_0^{\pi/2} d\phi\; 2R\, e^{aR^2 \sin\phi}$$

Now: Jordan's Lemma

$$\leq \int_0^{\pi/2} d\phi\; 2R\, e^{-aR^2 \frac{2}{\pi}\phi}$$

$$= 2R\, \frac{e^{-aR^2 \frac{2}{\pi}\frac{\pi}{2}} - 1}{-aR^2\, \frac{2}{\pi}} \xrightarrow[R\to\infty]{} 0.$$

- $\int_{C_1} dz + \int_{C_2} dz = 0$

$$\int_{C_1} dz\, e^{iaz^2} = e^{i\frac{\pi}{4}} \frac{1}{2}\sqrt{\frac{\pi}{a}} = \frac{1}{2}\, \frac{\sqrt{\pi}}{e^{\frac{-i\pi}{4}}\sqrt{a}}$$

$(-i)^{1/2} = e^{-i\pi/4}$ $\qquad = \frac{1}{2}\sqrt{\frac{\pi}{-ia}}$

NO lecture on thursday.

- Principle value int. not included.