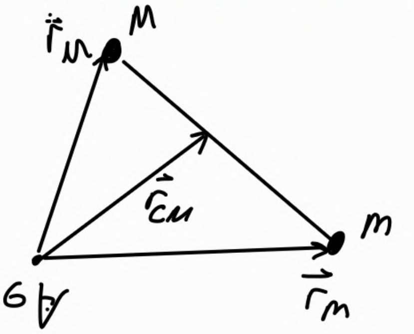
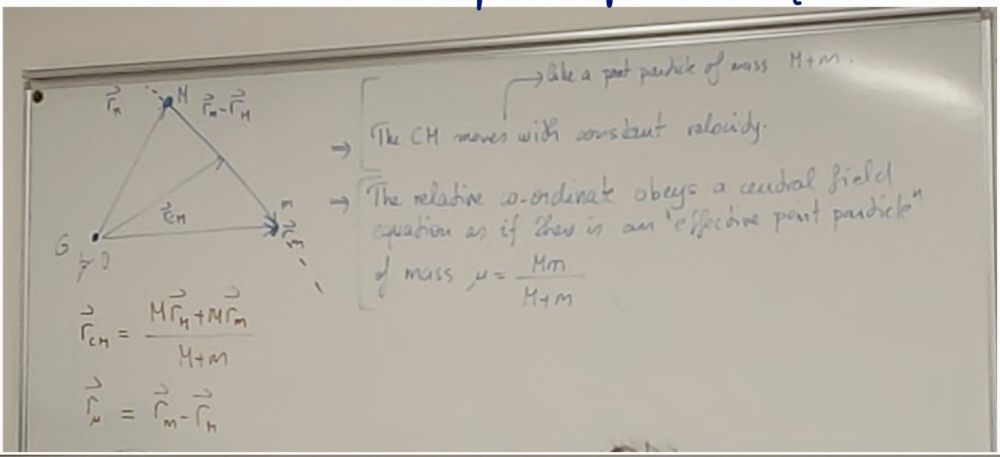
PHYS 301 Lecture 12 06.11.24

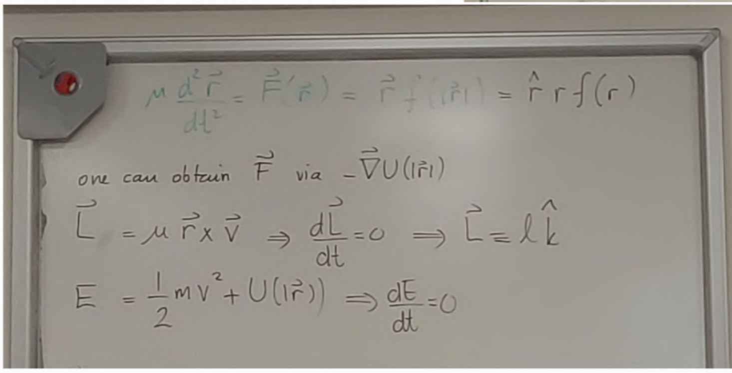


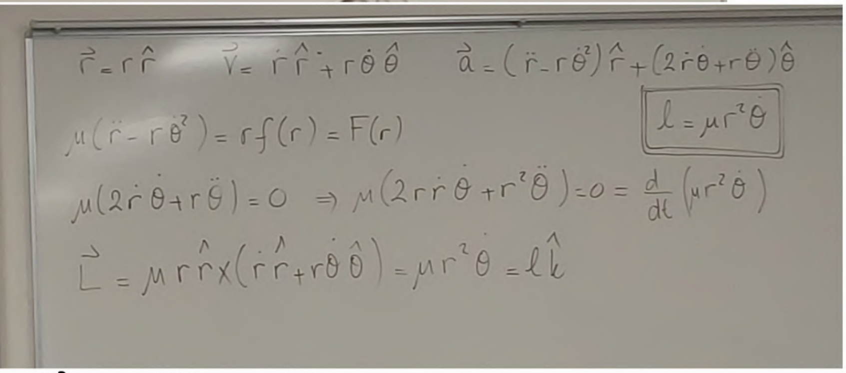
$$\vec{r}_{CM} = \frac{M\vec{r}_M + m\vec{r}_m}{M+m}$$

$$\vec{r}_\mu = \vec{r}_m - \vec{r}_M$$

$\Rightarrow$ The CM moves with constant velocity.
(Take a point particle of mass $M+m$)







$$\mu\left(\ddot{r} - r\frac{\ell^2}{\mu^2 r^4}\right) = F(r) = -\frac{dU}{dr} \qquad \mu\ddot{r} = \underbrace{\frac{\ell^2}{\mu r^3}}_{-\frac{d}{dr}\left(\frac{\ell^2}{2\mu r^2}\right)} - \frac{dU}{dr} = -\frac{dU_{eff}}{dr}$$

"what does on $U_{eff} = \infty$ do?"

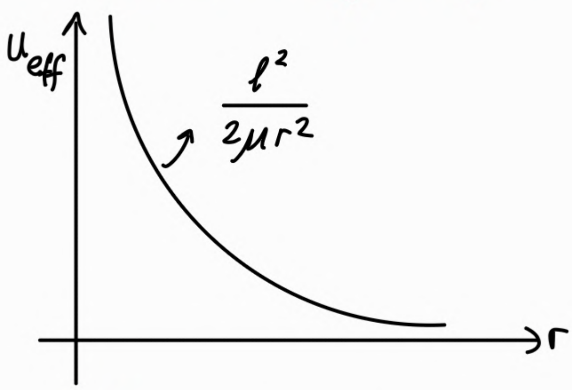
When particle has $L \neq 0$ it can never reach e. (origin)

(Even the minutest amount of $\dot{\theta}$)

$$U_{eff} = \frac{\ell^2}{2\mu r^2} + U(r)$$

PS: Of course planet's collide, but if they were point particles they wouldn't, they would scatter.

Bounded Trajectory



A bounded trajectory for finite energy $r \in [r_{min}, r_{max}]$ one must have:

1. $U(r) \neq 0$

2. One needs at least one minima (one point where the force is 0)

$\frac{dU}{dr} = 0$ at some points, & $\frac{d^2U}{dr^2} > 0$ at these points.

Constant Radius Solutions

$\ddot{r} = 0 \longrightarrow r = R$

$-\frac{\ell^2}{\mu R^3} = F(R)$ this must have a solution.

Remember

$\vec{L} = \ell \hat{k}$, $\ell = \mu r^2 \theta$

$\ell = \mu R^2 \dot{\theta}_{(\omega(R))}$

$$\Rightarrow \boxed{\mu R^3 \omega^2 = -F(R)}$$

Uniform circular motion

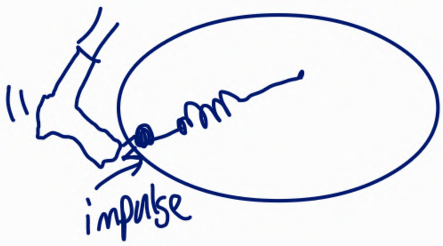
$F_{Newton} = -\frac{GMm}{r^2}$ $F_{Hook} = -kR$

(doesn't depend on R)

"inward kick can't change $\vec{L}$"

↳ it is radial.

"Bertland's Theorem"



(ℓ is constant)

$$\left.\begin{aligned} &\to r(t) = R + \delta r(t) \\ &\dot{r} = \delta \dot{r} \\ &\ddot{r} = \delta \ddot{r} \end{aligned}\right\} \mu \delta \ddot{r} - \frac{\ell^2}{\mu (R + \delta r)^3} = F(R + \delta r)$$

$$\to \mu \delta \ddot{r} - \underbrace{\frac{\ell^2}{\mu R^3}\left(1 + \frac{\delta r}{R}\right)^{-3}}_{\text{Taylor expand this}} = F(R + \delta r)$$

$$\mu \ddot{\delta r} - \frac{\ell^2}{\mu R^3} + \frac{\ell^2}{\mu R^3} \cdot 3\frac{\delta r}{R} = F(R) + \delta r \left.\frac{dF}{dr}\right|_{r=R}$$

$$\mu \ddot{\delta r} = \left(-\frac{3\ell^2}{\mu R^4} + \left.\frac{dF}{dr}\right|_{r=R} \right) \delta r$$

Equation that the small change has to obey!

" Conditions on a so x doesn't grow/remains small "

$\ddot{g} = ag$

$$\left.\begin{array}{l} \ddot{g}_1 = ag_1 \\ \ddot{g}_2 = ag_2 \end{array}\right\} \alpha g_1 + \beta g_2 \text{ is also a solution.}$$

In an ODE we can always:

$g = e^{\lambda t}$ $\quad \ddot{g} = \lambda e^{\lambda t} \rightarrow \lambda^2 = a \rightarrow \boxed{\lambda = \pm\sqrt{a}}$

- When will g never grow? :

$$\boxed{g = \alpha e^{\sqrt{a}t} + \beta e^{-\sqrt{a}t}}$$

$a > 0$ will grow g.

$a = 0$ is a degenerate case, pass.

$a < 0$, $\quad g = \alpha e^{i\sqrt{a}t} + \beta e^{-i\sqrt{a}t}$

→ euler's formula : $e^{i\pi} = -1$ $\quad (e^{i\theta} = \cos\theta + i\sin\theta)$

$$g = A\cos(\sqrt{|a|}t) + B\sin(\sqrt{|a|}t)$$

$$\mu \delta \ddot{r} = \boxed{\left(-\frac{3\ell^2}{\mu R^4} + \frac{dF}{dr}\Big|_{r=R} \right)} \delta r$$

if <0, δr remains small.

Let $F(r) = -Ar^n$ $\boxed{\ell = \mu R^2 \omega}$

$$\mu R \omega^2 = +AR^2 \Rightarrow AR^{n-1} = \mu \omega^2$$

$$\frac{-3\mu^2 R^4 \omega^2}{\mu R^4} = -nAR^{n-1} < 0$$

$$-3\mu\omega^2 - n\mu\omega^2 < 0$$

$\boxed{(-3-n) < 0}$ condition on n.

attractive forces can have $\oplus$ or $\ominus$ n.

constraint on $\ominus$ ns.

$n = -k$ $k>0$, $-3+k<0$

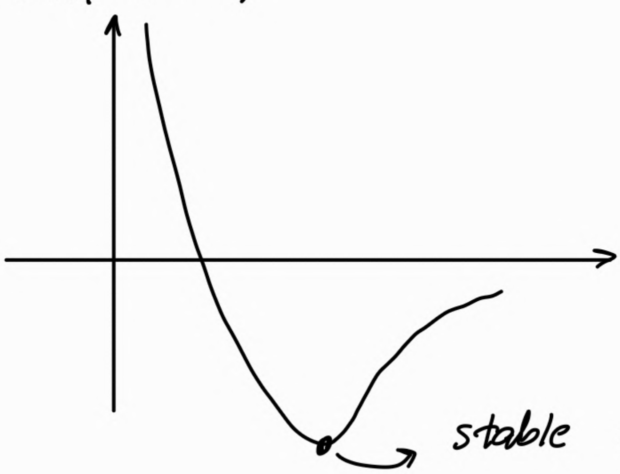
$k<3$

$k=1,2$

$$F = -Ar^n \rightarrow U = -\int F dr = \frac{Ar^{n+1}}{n+1}$$

$$U_{eff} = \frac{\ell^2}{2\mu r^2} + \frac{Ar^{n+1}}{n+1}$$

U_{eff} $(n=-2)$



stable circular orbit

$E \equiv \frac{1}{2}\mu v^2 + U(r)$

$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$ U_{eff}

$l = \mu r^2 \dot{\theta} \longrightarrow E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2\mu}\frac{l^2}{r^2} + U(r)$

$\frac{dE}{dt} = 0$ $\dot{r} = \sqrt{\frac{2}{\mu}(E - U_{eff}(r))}$

$$\frac{dr}{\sqrt{\frac{2}{\mu}(E - U_{eff})}} = dt \Rightarrow r(t)$$

Only interested in shape?

$\frac{d\theta}{dt} = \frac{l}{\mu r^2}$ $\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U_{eff}(r))}$

$$\Rightarrow \frac{dr}{d\theta} = \frac{\sqrt{\frac{2}{\mu}(E - U_{eff}(r))}}{l}\mu r^2$$

Will give you the eventual shape.

Write U_{eff}:

$$= \sqrt{2\mu r^4 E - l^2\mu r^2 - 2\mu r^4 U(r)}\ \frac{1}{2}$$

You can prove:
For newton's Law, solutions are conic sections.