PHYS 325 - Lecture 4

03.10.2024

$z = x + iy \in \mathbb{C} \qquad x, y \in \mathbb{R}$

$z^* = x - iy$

$$x = \frac{z + z^*}{2} \qquad y = \frac{z - z^*}{2i}$$

Ex: $z = e^{i\theta} = \cos\theta + i\sin\theta \qquad (z^* = e^{-i\theta})$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{i\theta}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\underbrace{\sin z = \frac{e^{iz} - e^{iz}}{2i}}$$

Can get power series expansion for these

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \rightarrow \boxed{\cos z = \cosh iz}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \rightarrow \boxed{\sin z = \frac{1}{i}\sinh iz}$$

* See how convinient and useful the relation ship between complex n. and cosh, sinh ...

Returning to functions.. : A General function

$$f(z) = u(x,y) + i\,v(x,y)$$

EX $f(z) = z^2 = (x+iy)^2 = x^2 + 2ixy + (iy)^2$

$$= x^2 - y^2 + i(2xy)$$

$$u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$$

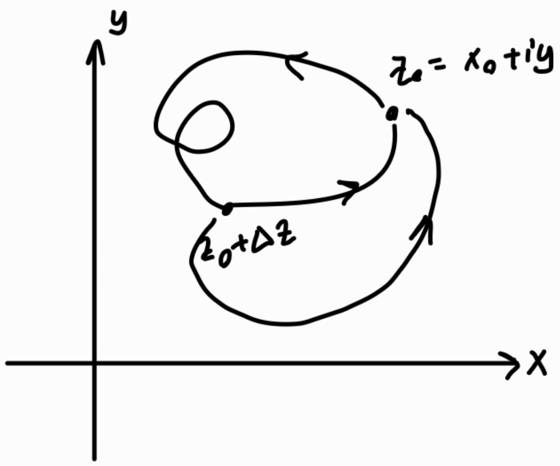
Derivative:

★ : Let $z_0 \in \mathbb{C}$

If $\displaystyle\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ exists, then f is called analytic at z_0. The limit is called the derivative of f at z_0 and it is $f'(z_0)$

⊛ lim exist means that the result is independent of the path. path independence is important!

Analytic Functions: Taylor expandible functions are called analytic functions



Improved definition:

★ : Let $z_0 \in \mathbb{C}$

If $\displaystyle\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ is path independent and finite ...

then f is called analytic at z_0. The limit is called the derivative of f at z_0 and it is $f'(z_0)$

EX 1 $f(z)=z$ $\quad f(z_0+\Delta z)=z_0+\Delta z$

$$\lim_{\Delta z\to 0}\frac{z_0+\Delta z-z_0}{\Delta z}=\lim_{\Delta z\to 0}1=1$$

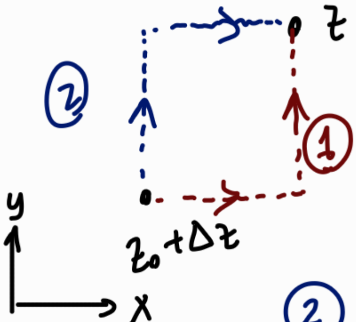
f is analytic everywhere on $\mathbb{C}$

EX 2: $f(z)=\bar{z}$

$$f(z_0+\Delta z)=\bar{z}_0+\Delta\bar{z}$$

$$\lim_{\Delta z\to 0}\frac{\bar{z}_0+\Delta\bar{z}-\bar{z}_0}{\Delta z}=\lim_{\Delta z\to 0}\frac{\Delta\bar{z}}{\Delta z}=\lim_{\Delta z\to 0}\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}$$

Writing the paths:



$$① \lim_{\Delta z\to 0}\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}=\lim_{\Delta y\to 0}\lim_{\Delta x\to 0}\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}$$

$$=\lim_{\Delta y\to 0}(-1)=-1$$

$$② \lim_{\Delta z\to 0}\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}=\lim_{\Delta x\to 0}\lim_{\Delta y\to 0}\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}$$

$$=\lim_{\Delta x\to 0}1=1$$

$\neq$

The limits are not equal. So f is not analytic.

(*) Taking limits of paths like this is an easy way of checking if f is analytic.

EX3 $f(z)=z^2$

$$f(z_0+\Delta z) = (z_0+\Delta z)^2 = z_0^2 + 2z_0\Delta z + (\Delta z)^2$$

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{\cancel{z_0^2} + 2z_0\Delta z + (\Delta z)^2 - \cancel{z_0^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2z_0\Delta z + (\Delta z)^2}{\Delta z} = \lim_{\Delta z \to 0} [2z_0 + \Delta z]$$

$$= 2z_0 + \underbrace{\lim_{\Delta z \to 0} \Delta z}_{0} = 2z_0 .$$

Therefore, $f'(z) = 2z$. f is analytic everywhere

(EXERCISE)

Ex4: $f(z) = z^n$ $n \in \mathbb{Z}_{\geq 0}$ show that:

analytic everywhere and $f'(z) = nz^{n-1}$

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{(z_0+\Delta z)^n - z_0^n}{\Delta z}$$

- Binomial Expansion : $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $(\ldots)$

Derivatives of Polynomials

EX5: $f(z)=\sum_{n=0}^{N} c_n z^n \quad (c_n \in \mathbb{C})$ analytic everywhere

$$f'(z)=\sum_{n=1}^{N} c_n n z^{n-1}$$

→ Let's start with a function, that we know is analytic.

Let f be analytic at $z_0 \in \mathbb{C}$ $\quad z_0 = x_0 + i y_0$

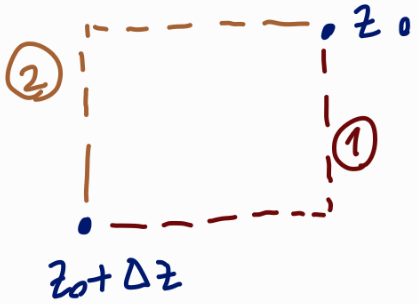
$$f(z) = u(x,y) + i v(x,y)$$

$$f(z_0+\Delta z) = u(x_0+\Delta x,\, y_0+\Delta y) + i v(x_0+\Delta x,\, y_0+\Delta y)$$

$$f(z_0+\Delta z) - f(z_0) = \overbrace{\left[u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)\right]}^{\Delta u} + i \overbrace{\left[v(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)\right]}^{\Delta v}$$

$$= \Delta u + \Delta v$$

Let's take the limit from different paths.



We already know f is path independent!

$$\lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

$$\lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{[u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)] + i[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)]}{\Delta x + i\Delta y}$$

$$\lim_{\Delta y \to 0} \frac{[u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)] + i[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)]}{i\Delta y}$$

$$= \frac{1}{i}\left[\frac{\partial u}{\partial y}\bigg|_{(x_0,y_0)} + i\,\frac{\partial v}{\partial y}\bigg|_{(x_0,y_0)}\right] = \frac{\partial v}{\partial y}\bigg|_{(x_0,y_0)} - i\,\frac{\partial u}{\partial y}\bigg|_{(x_0,y_0)}$$

$$\lim_{\Delta y \to 0}\lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = \frac{\partial v}{\partial y}\bigg|_{(x_0,y_0)} - i\,\frac{\partial u}{\partial y}\bigg|_{(x_0,y_0)}$$

$$\lim_{\Delta x \to 0}\lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = \frac{\partial u}{\partial x}\bigg|_{(x_0,y_0)} + i\,\frac{\partial v}{\partial x}\bigg|_{(x_0,y_0)}$$

So...

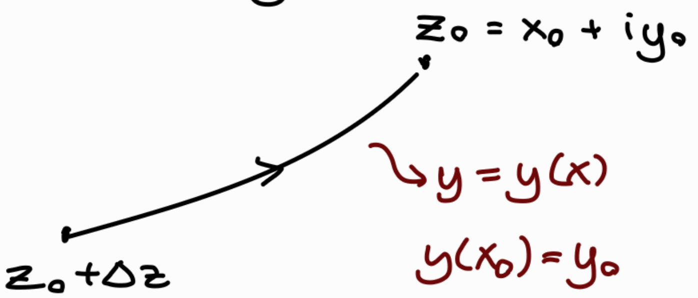
Theorem If f is analytic at $z_0 = x_0 + iy_0$ then at z_0 we have:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(Cauchy - Riemann Equations) (CR)

→ This is a useful negative test!

Conversely, assume CR are satisfied at $z_0 = x_0 + iy_0$



$u = u(x, y(x))$
$v = v(x, y(x))$
along the curve.

Let's now take the limit along the curve.

$$\lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}}{1 + i\frac{\Delta y}{\Delta x}}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \left.\frac{dy}{dx}\right|_{x_0}$$

$$\left.\frac{du}{dx}\right|_{x_0} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y(x_0 + \Delta x)) - u(x_0, y(x_0))}{\Delta x}$$

$$= \left.\frac{\partial u}{\partial x}\right|_{(x_0, y_0)} + \left.\frac{\partial u}{\partial y}\right|_{(x_0, y_0)} \left.\frac{dy}{dx}\right|_{(x_0)}$$

Remember chain rule:

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \left.\frac{\partial v}{\partial x}\right|_{(x_0, y_0)} + \left.\frac{\partial v}{\partial y}\right|_{(x_0, y_0)}$$

$$\left.\frac{dy}{dx}\right|_{x_0} = -\left.\frac{\partial u}{\partial y}\right|_{(x_0, y_0)} + \left.\frac{\partial v}{\partial x}\right|_{(x_0, y_0)} \left.\frac{dy}{dx}\right|_{(x_0)}$$

at (x_0, y_0):

$$\lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}}{1 + i\frac{\Delta y}{\Delta x}} = \frac{\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}\right) + i\left(-\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{dy}{dx}\right)}{1 + i\frac{dy}{dx}}$$

$$\frac{\frac{\partial u}{\partial x}\cancel{\left(1+i\frac{dy}{dx}\right)} - i\frac{\partial u}{\partial y}\cancel{\left(1+i\frac{dy}{dx}\right)}}{\cancel{1+i\frac{dy}{dx}}} = \underbrace{\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} - i\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}}$$

Result is independent of the curve: Analitic!

Why not just call it differentible?

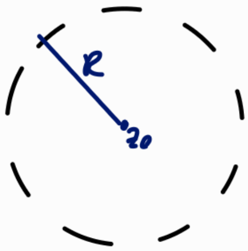
Because it also has to satisfy CR eqs.

it is more than being differentible

Some Ideas from Topology

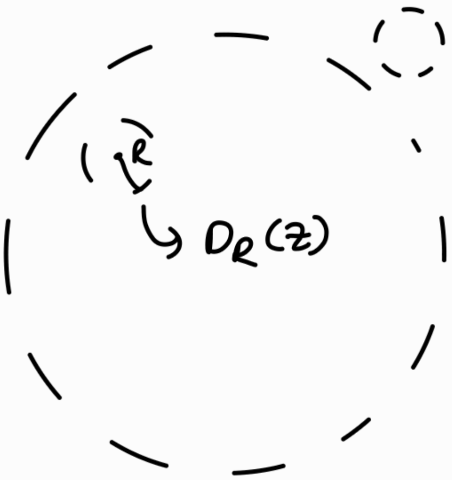
(Will be used when defining things rigorously)

Open Disk (Open meaning boundry not included)



$$D_R(z_0) = \{ z \in \mathbb{C} : |z - z_0| < R \}$$

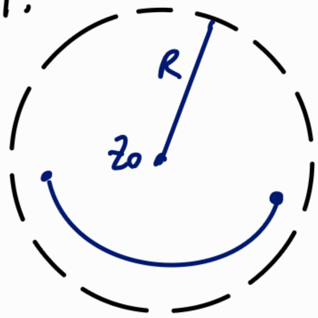
Open Set: O is called an open set if for any $z \in O$ $\exists\, D(z)$ s.t. $D_R(z) \subset O$



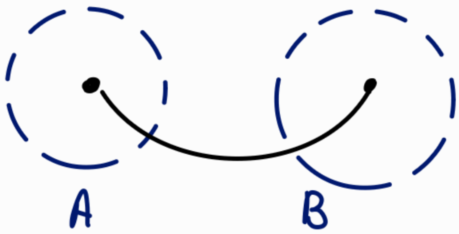
Closed Set: Is a set whose complement is open.

Path Connected Set: A set A is called path connected if any two points in it can be connected by a curve C lying inside A.

a → not by all curves.



Ex: Disconnected set:



$A \cup B$ is disconnected

Simply Connected: Assume a set A is simply connected.

Suppose i can deform one curve continuously, and end up in the other curve, it is called simply connected.

Ex: Not simply connected: Sets with holes in them are generally not connected.

Region: Is an open, connected set.