PHYS326 Lecture 3

#phys326 #phys326/lecturenotes #physics

Last time: Derivative operators and differentials.

- $T_p\mathbb{R}^n$: Tangent Space to $\mathbb{R}^n$ at $p \in \mathbb{R}^n$
- $T_p^*\mathbb{R}^n$: Cotangent Space to $\mathbb{R}^n$ at $p \in \mathbb{R}^n$. This is the dual space of $T_p\mathbb{R}^n$

Basis for $T_p\mathbb{R}^n$ (notation):

$$\frac{\partial}{\partial x^i}|_p = \partial_i|_p = \partial_{ip}$$

Dual Basis for $T_p^*\mathbb{R}^n$:

$$dx_p^j$$

The usual rule is:

$$dx_p^j(\partial_{ip}) = \delta_i^j$$

Example: $\mathbb{R}^2$. Let's look at cartesian coordinates

$$x^1 = x, x^2 = y$$

At a generic point we have:

$$\frac{\partial}{\partial x} = \partial_x, \frac{\partial}{\partial y} = \partial_y$$

Dual basis:

$$dx, dy$$

Then,

$$dx(\partial_x) = 1, dx(\partial_y) = 0$$

$$d_y(\partial_x) = 0, dy(\partial_y) = 1$$

This is the definition for dual basis. We can do the same for polar coordinates.

Polar Coordinates

$$x'^1 = r, x'^2 = \theta$$
$$\partial_r = \frac{\partial}{\partial r}, \partial_\theta = \frac{\partial}{\partial \theta}$$

Dual basis:

$dr\ d\theta$

Then,

$$dr(\partial_r) = 1, dr(\partial_\theta) = 0$$

$$d_\theta(\partial_r) = 0, d_\theta(\partial_\theta) = 1$$

Relationship Between The Two Coordinate Systems

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
r &= \sqrt{x^2+y^2} \\
\theta &= \tan^{-1}\frac{y}{x}
\end{aligned}
$$

Let's start with basis vectors. (Remember the chain rule from last time)

$$\partial_x = \frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r} + \frac{-y}{x^2}\frac{1}{1+\frac{y}{x}^2}\frac{\partial}{\partial \theta}$$

See that:

$$\frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2}$$

Then everything is equal to,

$$= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta$$

Similarly (**Exercise 1**):

$$\partial_y = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$dx = dr\cos\theta + r(-\sin\theta)d\theta = \cos\theta dr - r\sin\theta d\theta$$
$$dy = \sin\theta dr + r\cos\theta d\theta$$

Remember we had the conditions:

$$dr(\partial_r) = 1, dr(\partial_\theta) = 0$$

$$d_\theta(\partial_r) = 0, d_\theta(\partial_\theta) = 1$$

$$dx(\partial_x) = 1, dx(\partial_y) = 0$$

$$d_y(\partial_x) = 0, dy(\partial_y) = 1$$

Let's check $dx(\partial_x)$:

$$dx(\partial_x) = dx(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta) =$$

$$= \cos\theta dr(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta) - r\sin\theta d\theta(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)$$

Then

$$= \cos^2\theta = r\sin\theta(-\frac{\sin\theta}{r}) = 1$$

Exercise 2: Check the other ones.

The General Case

I can have a tensor like this:

$$T^{i_1,\dots i_n}_{j_1\dots j_m}\partial_{i_1}\otimes\dots\partial_{i_n}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_m}$$

This is in a certain coordinate system such that:

$$\partial_{i_1}=\frac{\partial}{\partial x^{i_1}}$$

Let's go to the primed coordinate system:

$$\partial_{i_1}=\frac{\partial}{\partial x^{i_1}}=\frac{\partial x'^{k_1}}{\partial x^{i_1}}\frac{\partial}{\partial x'^{k_1}}=\frac{\partial x'^{k_1}}{\partial x^{i_1}}\partial'_{k_1}$$

$$-dx^{j_1}=\frac{\partial x^{j_1}}{\partial x'^{l_1}}dx'^{l_1}$$

- Jacobian Factor

Then,

$$T=[\frac{\partial x'^{k_1}}{\partial x^{i_1}}\dots\frac{\partial x'^{k_n}}{\partial x^{i_n}}\frac{\partial x^{j_1}}{\partial x'^{l_1}}\dots\frac{\partial x^{j_m}}{\partial x^{l_m}}T^{i_1,\dots i_n}_{j_1\dots j_m}]\partial'_{k_1}\otimes\dots\partial'_{k_n}\otimes dx'^{l_1}\otimes\dots\otimes dx'^{l_m}$$

Tensor Field

You start moving around your manifold

$$T_p=T^{i_1,\dots i_n}_{j_1\dots j_m}(p)\partial_{i,p}\otimes\dots dx^{j_m}_p$$

Metric Tensor Field g

> **Important**

We won't say field all the time. When we say tensor, we will mean field.

Note

$$g = g_{ij} dx^i \otimes dx^j$$

Conditions

1. Covariant rank 2 tensor
2. Symmetric:
   $g(v, u) = g(u, v) \forall u, v \in T\mathbb{R}^n$
   in particular,
   $$g(\partial_i, \partial_j) = g(\partial_j, \partial_i)$$
   $$g_{ij} = g_{ji}$$
   I can represent g_ij as a matrix, then it'd be a Symmetric Matrix
3. Non-degenerate:
   Think of g_{ij} as a square matrix. Non-degenerate means this matrix is an Invertible Matrix. Thus, $\det g_{ij} \neq 0$

Hint

$$g'_{kl} = \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x' l} g_{ij}$$

They are also invertible. (Product of invertible matrices are invertible)

Such properties are called Geometric Properties, meaning they are independent of the coordinate system.

What do we do with this metric?

> We use it to transform a vector to a covector and vice versa.

Take our metric g.

$$g(u,v) \in \mathbb{R} \forall u,v \in T\mathbb{R}^n$$

Suppose I fixed the first entry (u) but let the second free.

$$g(u,\cdot) : T\mathbb{R}^n \to \mathbb{R}$$

- It is linear.

Therefore, $\tilde{u} = g(u,\cdot)$ is a co-vector. i.e. an element of the Cotangent Space.

Definition

$\tilde{u} := g(u,v)$

$g(u,\cdot)$
$u = u^i \partial_i$ and $v = v^j \partial_j$

Then,

$$\begin{aligned}\tilde{u}(v) = g(u,v) = g(u^i\partial_i, v^j\partial_j) &= u^i v^j g(\partial_i, \partial_j)\\ &= u^i v^j g_{ij}\\ &= [g_{ij}u^i]v^j\end{aligned}$$

$$\tilde{u}(v) = \tilde{u}_k dx(v^j \partial_j) = \tilde{u}_k v^j$$

This holds for any v^j!

We get:

$$\tilde{u}_j = g_{ij}u^i = g_{ji}u^i$$

This is called the lowering of the indices.

$$\tilde{u}_j = g_{ji}u^i$$

Remember that metric tensor is non-degenarte. Its inverse is denoted as:

$$g^{kj}g_{ji} = \delta^k_i$$

(See the summation over j)

$$\begin{aligned} g^{kj}\tilde{u}_j &= g^{kj}g_{ji}u^i \\ &= \delta^k_i u^i \\ &= u^k \end{aligned}$$

$u^k = g^{kj}\tilde{u}_j$ raising the index.

→ So using g we arrive at a 1-1 correspondence between the tangent space and the cotangent space.

This all depends on your choice of the metric tensor.

You can get rid of the tilde, the placement of the index will tell you whether its a tangent or cotangent vector.

$g(u, v) = g(v, u)$. g is non-degenerate.

Let's look at $\tilde{u}_j = g_{ji}u^i$ Therefore, $g_{ji}u^i = 0$ means $u^i = 0$

Extra Restriction on the Metric Tensor g

4. I can put an extra restriction on the metric tensor: that g_{ij} is a positive matrix.

$$g(u, u) \geq 0 \; \forall u \in T\mathbb{R}^n$$

with equality holding for only $u = 0$ (Positive semi definite.)

With this extra condition I can use g to define an inner product on our tangent space $T\mathbb{R}^n$

Definition

$u \cdot v := g(u, v)$

Remember the definition of the inner product:

1. Linear in both entries
2. $u \cdot v = v \cdot u$ because $g(u, v) = g(v, u)$
3. $u \cdot u = g(u, u) \geq 0$ with equality holding only for $u = 0$

Exercise 3: Show that 4th property implies the 3rd property.

Important

Remember, inner product may be position dependent

Like this,

$$u_p \cdot v_p := g_p(u_p, v_p)$$

where

$$g = g_{ij}(p) dx^i_p \otimes dx^j_p$$

Sometimes 4th is not satisfied.

> **Note**
>
> If conditions $1, 2$ and 4 are satisfied, I have a Riemannian Metric Tensor

> **Note**
>
> If conditions $1, 2, 3$ are satisfied, I have a Pseudo-Riemannian Metric Tensor

Examples of Riemannian and Pseudo-Riemannian Metric Tensors

Example: $\mathbb{R}^2$ Riemannian-Metric Tensor

Consider:

$$g = \delta_{ij} dx^i \otimes dx^j = dx^1 \otimes dx^1 + dx^2 \otimes dx^2$$

$$\delta_{ij} = \delta ji$$

$$\delta_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$g(u,u) = g(u^i\partial_i, u^j\partial_j) = u^i u^i g(\partial_i, \partial_j) = u^i u^j \delta_{ij}$$

$$= (u^1)^2 + (u^2)^2 \geq 0$$

Example: $\mathbb{R}^2$ Riemannian-Metric Tensor

Consider:

$$g = \eta_{ij} dx^i \otimes dx^j$$

$$\eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\eta_{ij} = \eta_{ji}$$

$$\det \eta_{ij} = -1 \neq 0$$

See that 4th condition it not satisfied.

Exercise 4 ($g(u,u)$ may even be negative)

Ex $\mathbb{R}^2$

$g = \eta_{ij}\, dx^i \otimes dx^j$ (i) $\eta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= dx^1 \otimes dx^1 - dx^2 \otimes dx^2$

$\eta_{ij} = \eta_{ji}$ (ii) (iv) is not satisfied

$u = \partial_1 + \partial_2$

$\det \eta_{ij} = -1 \neq 0$ (iii) $g(u,u) = 0$

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Why is pseudo-riemannian metric important?

Notice that the eta in the example 2 is the minkowski metric!

Example*

$\mathbb{R}^2$

$g = \delta_{ij} dx^i \otimes dx^j = dx \otimes dx + dy \otimes dy$
$x^1 = x, x^2 = y$

$$dx = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \sin\theta dr + r\cos\theta d\theta$$

$$\begin{aligned} g &= \cos^2\theta dr \otimes dr - r\cos\theta\sin\theta(dr \otimes d\theta + d\theta \otimes dr) + r^2\sin^2\theta d\theta \otimes d\theta) \\ &= \sin^2 dt \otimes dr + r\cos\theta\sin\theta(////) + r^2\cos^2\theta d\theta \otimes d\theta \\ &= dr \otimes dr + r^2 d\theta \otimes d\theta \end{aligned}$$

Therefore,

$$g = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

Again, I have a diagonal matrix. In polar coordinates, there is position dependence.

> **Note**
>
> Generally, when you change coordinates coefficients of the metric tensor become position dependent.