PHYS 325 Lecture 9 24.10.24

Complex Powers of Complex Numbers (You use logs!)

→ $f(z) = z^c$ $c \in \mathbb{C}$ $0 \leq \theta < 2\pi$

Riemann Surface is q sheeted

$$z^c = e^{c\ln z} = e^{c[\overbrace{\ln r + i\theta}^{\text{principle branch of } \ln} + i2n\pi]}$$

$z = re^{i\theta}$

(i) $c \in \mathbb{R}$ $c \in \mathbb{Q}$ $c = \frac{p}{q}$ p,q are prime integers

$ci2n\pi = i\frac{p}{q}n2\pi$ if $n = q$

$\Downarrow$

$= ip2\pi$

$\Rightarrow e^{\frac{p}{q}(\ln r + i\theta)} \overbrace{e^{ip2\pi}}^{1}$ (explains the periodicity)

no change in value

Riemann Surface is q sheeted.

(ii) $c \in \mathbb{R}$, $c \notin \mathbb{Q}$

$ci2n\pi \neq m2\pi$ ⊛ Riemann surface of z^c is the same as the Riemann surface of $\ln z$

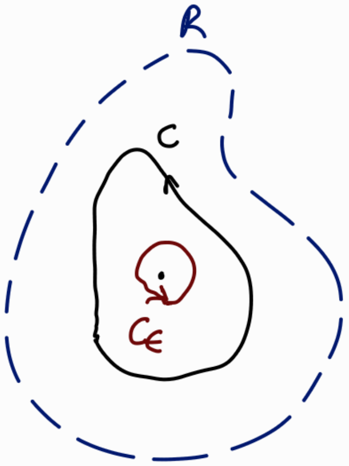
EX: Let's calculate the principle value of:

$$(i)^i = e^{i\ln i} = e^{i\ln(e^{i\frac{\pi}{2}})}$$

$$i = e^{i\frac{\pi}{2}} = e^{i[i\frac{\pi}{2}]} = e^{-\frac{\pi}{2}}$$

↳ it's real!

Calculus of Residues



$f(z)$ analytic on R except at z_0 where it has a pole.

Around z_0 we have the Laurent expansion:

$$f(z) = \sum_{n=-N}^{\infty} a_n (z-z_0)^n$$

→ Remember the contour trick!

$$\oint_C dz\, f(z) = \oint_{C_\epsilon} dz\, f(z)$$

C_ϵ : Circle with radius ϵ center z_0 lying inside the disk of convergence of the Laurent expansion.

Write the Laurent Expansion:

$$\oint_{C_\epsilon} dz\, f(z) = \oint_{C_\epsilon} dz \sum_{n=-N}^{\infty} a_n (z-z_0)^n = \boxed{\overset{2\pi i\, a_{-1}}{\oint_{C_\epsilon} dz\, a_{-1} (z-z_0)^{-1}}} + \underbrace{\int_{C_\epsilon} dz \sum_{n=-N}^{-2} a_n (z-z_0)^n}_{0 \;(n \neq -1)}$$

$$+ \underbrace{\oint dz \sum_{n=0}^{\infty} a_n (z-z_0)^n}_{=0}$$

↳ See this is an analytic function. (Tay. Exp.) So, by Cauchy-Goursat thm it's $=0$.

From last week:

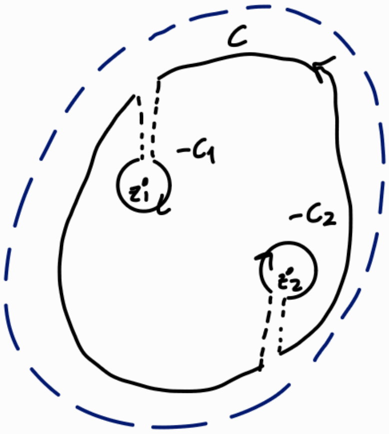
$$\rightarrow \int_{C_\epsilon} dz (z-z_0)^n = 0 \quad \text{if } n \neq -1$$
$$= 2\pi i \quad \text{if } n = -1$$

So, :

$$\Rightarrow \oint_C dz\, f(z) = \oint_{C_\epsilon} dz\, f(z) = \boxed{2\pi i\, a_{-1}}$$

↳ Residue of $f(z)$

Now let's have 2 poles



Now I can do the exact calculation for both of them.

$$\oint_C dz f(z) = \oint_{C_1} dz\, f(z) + \int_{C_2} dz f(z)$$

$$= 2\pi i\, a_{-1, z_1} + 2\pi i\, a_{-1, z_2}$$

Fundamental Theorem of Calculus of Residues

$$\boxed{\oint_C dz f(z) = 2\pi i \sum_{k=1}^{r} a_{-1, z_k}}$$

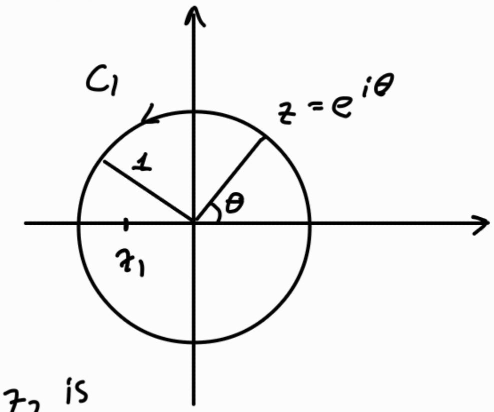
$z_1, z_2, \ldots, z_r$ are the poles of $f(z)$ enclosed by C

→ may or may not work for essential singularities Why?

We don't know if Laurent series is uniformly convergent or not.

Example:

$|\epsilon| < 1$



z_2 is outside

$$z = e^{i\theta}$$

$$dz = de^{i\theta} = (i d\theta) e^{i\theta} \qquad d\theta = -i \frac{dz}{e^{i\theta}} = -i e^{-i\theta} dz$$

$$= -i \frac{1}{z}\, dz$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$$

$$I = \int_0^{2\pi} d\theta \, \frac{1}{1+\epsilon\cos\theta} = \oint_{C_1} dz \, \frac{-i}{z} \, \frac{1}{1+\epsilon\left(\frac{z+\frac{1}{z}}{2}\right)} = \oint_{C_1} dz \, \frac{-i}{\frac{\epsilon}{2}z^2+z+\frac{\epsilon}{2}}$$

Exercise: To check, start from here and go back.

$$= \oint_{C_1} dz \, \frac{-i}{\frac{\epsilon}{2}(z-z_1)(z-z_2)}$$

z_1 & z_2 are the roots of:

$$az^2+bz+c = a(z-z_1)(z-z_2)$$

$$\frac{\epsilon}{2}z^2 + z + \frac{\epsilon}{2} = 0$$

$$z_{1,2} = \frac{-1 \pm \sqrt{1-\epsilon^2}}{\epsilon} = -\frac{1}{\epsilon} \pm \frac{\sqrt{1-\epsilon^2}}{\epsilon}$$

$$z_2 = -\frac{1}{\epsilon} - \frac{\sqrt{1-\epsilon^2}}{\epsilon} < -1$$

or

$$z_2 > 1$$

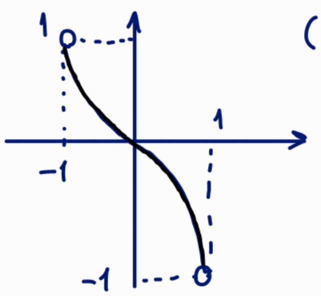
in either case it is outside the circle.

$$z_1 = -\frac{1}{\epsilon} + \frac{\sqrt{1-\epsilon^2}}{\epsilon}$$

z_1 is inside the circle.

$|z_1| < 1$ for $|\epsilon| < 1$

$$\left| -\frac{1}{x} + \frac{\sqrt{1-x^2}}{x} \right| < 1$$



(Take the limit for $x=0$)

Therefore we are only interested in z_1

Practicle way of Getting the Residues

Suppose: $f(z) = \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$

$$(z-z_0) f(z) = a_{-1} + a_0(z-z_0) + a_1 (z-z_0)^2 + \cdots$$

Take the limit:

$$\boxed{\lim_{z \to z_0} (z-z_0) f(z_0) = a_{-1, z_0}}$$

laurent expansion with one term

We don't consider z_2.

Partial function of $= \oint_{C_1} dz \frac{-i}{\frac{\epsilon}{2}(z-z_1)(z-z_2)} \sim \frac{A}{z-z_1} + \frac{B}{z-z_2}$

Can taylor expand (+) powers

$$a_{-1,z_1} = \lim_{z \to z_1} (z-z_1) \frac{-i \frac{2}{\epsilon}}{(z-z_1)(z-z_2)} = -i \frac{2}{\epsilon} \frac{1}{z_1 - z_2}$$

$$= \frac{-i}{\sqrt{1-\epsilon^2}}$$

$$z_{1,2} = -\frac{1}{\epsilon} \pm \frac{\sqrt{1-\epsilon^2}}{\epsilon}$$

So: $\oint_{C_1} dz \frac{-i}{\frac{\epsilon}{2}(z-z_1)(z-z_2)} = 2\pi i \frac{-i}{\sqrt{1-\epsilon^2}} = \frac{2\pi}{\sqrt{1-\epsilon^2}}$

$$\lim (z-z_0) f(z) = a_{-1,z_0}$$

Let's say ...

$$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + (z-z_0) a_1 + \cdots$$

$$(z-z_0)^2 f(z) = a_{-2} + (z-z_0) a_{-1} + \cdots$$

$$\frac{d(z-z_0)^2 f(z)}{dz} = a_{-1} + 2(z-z_0)a_0 + 3(z-z_0)^2 a_1 + \cdots$$

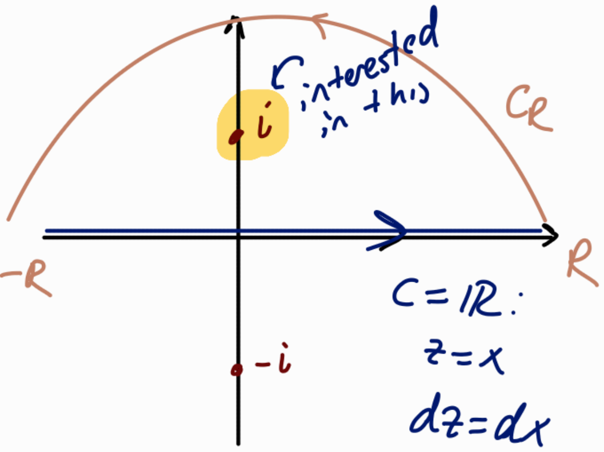
Therefore I can write:

$$\lim_{z\to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = a_{-1,\,z_0}$$

EXERCISE:
- What if pole is of 3^{rd} order.
- What is the general case?

(from Artken)

Ex: $\displaystyle\int_{-\infty}^{\infty} dx \frac{1}{x^2+1} = \int_C dz \overbrace{\frac{1}{1+z^2}}^{f(z)}$ (Again, you can go backward)



$C = \mathbb{R}$:
$z = x$
$dz = dx$

C is an open contour. Now what?:

$$\int_{-\infty}^{\infty} dx \frac{1}{x^2+1} = \lim_{R\to\infty} \int_{-R}^{R} dx \frac{1}{x^2+1}$$

$$= \int dz \frac{1}{(z+i)(z-i)}$$

C_R = semi circle of Radius R, center 0.

EXERCISE:
You can repeat this calculation with the lower circle

$C_T = C \cup C_R$
close

$$\overbrace{\int_{C_T} dz\, f(z)}^{\pi} = \underbrace{\int_C dz\, f(z)}_{?} + \int_{C_R} dz\, f(z)$$

$$\int_{C_T} dz\, f(z) = 2\pi i \frac{1}{2i} = \pi$$

$$a_{-1,i} = \lim_{z\to i} (z-i)\frac{1}{(z-i)(z+i)}$$

- We will show that:

$$\lim_{R\to\infty} \int_{C_R} dz\, f(z) = 0$$

on C_R: $f(z) = \frac{1}{z^2+1} = \frac{1}{R^2 e^{2i\theta}+1}$

$$z = Re^{i\theta} \quad (0 \leq \theta \leq \pi)$$

We had triangle inequality: $\underbrace{\big|\, |z_1| - |z_2| \,\big|}_{\text{exercise! Try to get this}} \leq |z_1 + z_2| \leq |z_1| + |z_2|$

$$\left| \int_{C_R} dz\, f(z) \right| \leq \left| \int_0^\pi R\, i\, e^{i\theta} d\theta \frac{1}{R^2 e^{2i\theta}+1} \right|$$

$$\leq \int_0^\pi d\theta \left| R\, i\, e^{i\theta} \frac{1}{R^2 e^{i\theta}+1} \right|$$

$$(*) \quad = \int_0^\pi d\theta\, R\, 1 \cdot 1 \cdot \frac{1}{|R^2 e^{2i\theta}+1|}$$

So: $\underbrace{R^2-1}_{\text{very large}} \leq |R^2 e^{2i\theta}+1|$

Therefore $\frac{1}{|R^2 e^{2i\theta}+1|} \leq \frac{1}{R^2-1} + \dots$

$$(*) \leq \int_0^\pi d\theta \, \frac{R}{R^2-1} = \pi \frac{R}{R^2-1}$$

So:

$$\boxed{\int_{-\infty}^{\infty} dx \, \frac{1}{1+x^2} = \pi}$$

I add an extra contour, apply calculus of residues, Then show the extra has no contribution at all.