PHYS325 Lecture 17

28.11.24

Dominated Convergence Theorem

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a pointwise convergent sequence of functions on $[a,b] \subseteq \mathbb{R}$

$$f(x) := \lim_{n\to\infty} f_n(x)$$

Assume there exists an integrible function $h(x)$ on $[a,b]$ $\int_a^b dx\, h(x) < \infty$

such that:

$$|f_n(x)| \leq h(x) \qquad \forall x \in [a,b]$$

Example:

$$D_n(x) = \sqrt{\frac{n}{\pi}}\, e^{-nx^2}$$

$g(x)$: test function, assume continuous and

$$\lim_{x\to\pm\infty} g(x) = 0$$

in particular g is a boundary function.

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} dx \sqrt{\frac{n}{\pi}}\, e^{-nx^2} g(x) = \lim_{n\to\infty}\left[\int_0^{\infty} dx \sqrt{\frac{n}{\pi}}\, e^{-nx^2} g(x) + \int_{-\infty}^{0} dx \sqrt{\frac{n}{\pi}}\, e^{-nx^2} g(x)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{2\sqrt{\pi}}\int_0^{\infty} dy\, \frac{g\left(\frac{\sqrt{y}}{\sqrt{n}}\right)}{\sqrt{y}}\, e^{-y} + \frac{1}{2\sqrt{\pi}}\int_0^{\infty} dy\, \frac{g\left(\frac{\sqrt{y}}{\sqrt{n}}\right)}{\sqrt{y}}\, e^{-y}\right]$$

$$\frac{g(0)}{2\sqrt{\pi}}\int_0^\infty dy\,\frac{e^{-y}}{\sqrt{y}} + \frac{g(0)}{2\sqrt{\pi}}\int_0^\infty dy\,\frac{e^{-y}}{\sqrt{y}} = \frac{g(0)}{\sqrt{\pi}}\int_0^\infty dy\,\frac{e^{-y}}{\sqrt{y}} = \frac{g(0)}{\sqrt{\pi}}\int_0^\infty du\,2u\,\frac{e^{-u^2}}{u}$$

$$y = u^2 \qquad dy = 2u\,du$$

$$= \frac{2g(0)}{\sqrt{\pi}}\,\frac{\sqrt{\pi}}{2} = g(0)$$

[redacted] $= g(0)$

$$|g(x)| < M \quad \forall x \in \mathbb{R}$$

$$\lim_{n\to\infty}\int_a^b dx\, f_n(x) = \int_a^b dx \lim_{n\to\infty} f(x)$$

- We must find an integrible $h(x)$ which dominates it.

$$\lim_{n\to\infty} \frac{g\left(\frac{\sqrt{y}}{\sqrt{n}}\right)}{\sqrt{y}}\, e^{-y} = \frac{g(0)}{\sqrt{y}}\, e^{-y}$$

! See how this is similar to M test!

$$\underbrace{\left|\frac{g\left(\frac{\sqrt{y}}{\sqrt{n}}\right)}{\sqrt{y}}\, e^{-y}\right|}_{f_n(y)} \leq \underbrace{M\,\frac{e^{-y}}{\sqrt{y}}}_{h(y)}$$

More Properties of Generalized Functions

Let $\chi(x) = \lim_{n\to\infty} f_n(x)$ (distributional sense)

$$\chi'(x) := \lim_{n\to\infty} f_n'(x)$$

Schwartz Space of Test Functions

"very fast decaying functions"

$= \{ g : \mathbb{R} \to \mathbb{R} :$ g is smooth (i.e. derivatives of all orders exist)
and $\lim_{x\to\pm} \left| x^n \frac{d^m g(x)}{dx^m} \right| = 0 \quad \forall n, m \in \mathbb{Z}_{\geq 0} \}$

$$\int_{-\infty}^{\infty} dx\, \chi'(x) g(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} dx\, f_n'(x) g(x) = \lim_{n\to\infty} \left[\underbrace{\cancel{f_n(x) g(x) \Big|_{-\infty}^{\infty}}}_{0} - \int_{-\infty}^{\infty} dx\, f_n(x) g'(x) \right]$$

$$= - \lim_{n\to\infty} \int_{-\infty}^{\infty} dx\, f_n(x) g'(x) = - \int_{-\infty}^{\infty} dx\, \chi(x) g'(x)$$

Example: Application to Dirac Delta

$$\int_{-\infty}^{\infty} dx\, \delta'(x) g(x) = - \int_{-\infty}^{\infty} dx\, \delta(x) g'(x) = -g'(0)$$

$$\int_{-\infty}^{\infty} dx\, \delta''(x) g(x) = \int_{-\infty}^{\infty} dx\, \delta(x) g''(x) = g''(0)$$

$$\int_{-\infty}^{\infty} dx\, \chi''(x) g(x)$$

$$= \int_{-\infty}^{\infty} dx\, \chi(x) g''(x)$$

You can also;

For dirac delta:

$a < 0 < b$

$$\int_{a}^{b} dx\, \delta(x) g(x) = g(0)$$

Example: Heaviside Step Function

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} dx\, \theta(x) g(x) = \int_{-\infty}^{\infty} dx\, g(x)$$

$$\int_{-\infty}^{\infty} dx\, \theta'(x) g(x) = -\int_{-\infty}^{\infty} dx\, \theta(x)\, g'(x)$$

$$= -\int_{0}^{\infty} dx\, g'(x) = -\left[g(x) \Big|_{0}^{\infty} \right]$$

$$= -(-g(0)) = g(0)$$

So: $\theta'(x) = \delta(x)$

- Yet another property (*) (will use in greens)

$$D_n(x) = \sqrt{\frac{n}{\pi}}\, e^{-nx^2} = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{itx}\, e^{-\frac{t^2}{4n}}$$

Exercise: Take this integral by completing it to a square and taking contour integration.

$$\int_{-\infty}^{\infty} dx \sqrt{\frac{n}{\pi}}\, e^{-nx^2} g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt\, e^{itx - \frac{t^2}{4n}} g(x)$$

(Fubini's Thrm)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{-\frac{t^2}{4n}} \int_{-\infty}^{\infty} dx\, e^{itx} g(x) \quad (*)$$

Fact: $\lim_{n\to\infty} \frac{e^{-\frac{t^2}{4n}}}{2\pi} = 1$ generalized function.

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} dt\; e^{-\frac{t^2}{4n}} g(t) = \int_{-\infty}^{\infty} dt\, g(t)$$

Exercise: Use dominated convergence to show this.

$$(*) = \int_{-\infty}^{\infty} dx \left[\int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{itx} \right] g(x)$$

$$g(x) = \int_{-\infty}^{\infty} dx \underbrace{\left[\int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{itx} \right]}_{\delta(x)} g(x)$$

You can show: $\delta(x) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{\pm itx}$

"Fourier transform of $\delta(x) = 1$

$$\int_{-\infty}^{\infty} dx\, \delta^2(x) g(x) = \int_{-\infty}^{\infty} dx\, \delta(x)\delta(x)\, g(x) = \underbrace{\delta(0)\, g(0)}_{\text{doesn't make sense}}$$

- Renormalization Theory

Operators → will introduce "improper basis"

Operators were always finite dimensional. Not anymore.

- $\hat{p}$: momentum operator.

$$\hat{p} = -i\frac{d}{dx} \quad \text{(assume using units so } \hbar = 1\text{)}$$

It's a Hermitian operator. Let's check:

$$\langle f | \hat{p} g \rangle \stackrel{?}{=} \langle \hat{p} g | f \rangle$$

$$(*) \int_{-\infty}^{\infty} dx\, f^*(x)(-i)\frac{dg(x)}{dx} = -i\left[f^*(x)g(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx \frac{df^*(x)}{dx} g(x)\right]$$

Condition: $\int_{-\infty}^{\infty} dx\, |f(x)|^2 < \infty \longrightarrow \lim_{x\to\pm\infty} f(x) = 0$

$$(*) = \int_{-\infty}^{\infty} dx\; i\,\frac{df^*(x)}{dx} g(x) = \int_{-\infty}^{\infty} dx \left(-i\frac{df}{dx}\right)^* g(x)$$

$$= \langle \hat{p} f | g \rangle \quad \blacksquare$$

Let's look at eigenvalues/vectors of this

$$-i\frac{df(x)}{dx} = kf(x) \longrightarrow \frac{df(x)}{dx} = ik\,f(x) \longrightarrow f(x) = e^{ikx}$$

∀ Any real k is an eigenvalue!

Something is wrong with this function.

Check this condition: $\int_{-\infty}^{\infty} dx\, |f(x)|^2 < \infty \longrightarrow \lim_{x \to \pm\infty} f(x) = 0$

$$\int_{-\infty}^{\infty} dx\, |e^{ikx}|^2 = \int_{-\infty}^{\infty} dx = \infty$$

- So, we have this eigenfuncs but they are not in the hilbert space!

- But they still form a basis for Hilbert Space! → cont. Linear combination is in hilbert space (The integral below:)

n: discrete
k: continuous

improper basis (wave packet)

$c(k)e^{ikx}$

↓

$A^\dagger = A$

$Ae_n = \lambda_n e_n$

$v = \sum_n c_n e_n$

$$\psi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, c(k)\, e^{ikx}$$

inverse fourier transform

will be inside $L^2(\mathbb{R})$ if and only if $\int_{-\infty}^{\infty} |c(k)|^2 \frac{dk}{2\pi} < \infty$

if its square integrible...

"Richt Hilbert Space"

Orthogonality of This Functions:

$$e^{ikx} \quad e^{ilx} \quad \rightarrow |k\rangle, \ |l\rangle$$

$$\langle k | l \rangle = \int_{-\infty}^{\infty} dx \, e^{-ikx} e^{ilx} = \int_{-\infty}^{\infty} dx \, e^{i(l-x)x} = 2\pi \, \delta(k-l)$$

You can get rid of 2π easily:

$$\frac{e^{ikx}}{\sqrt{2\pi}} \quad \frac{e^{ilx}}{\sqrt{2\pi}} \quad \rightarrow |k\rangle, \ |l\rangle$$

$$\langle k | l \rangle = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-ikx} e^{ilx} = \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{i(l-x)x} = \delta(k-l)$$

- Spectrum: Continuous
- Improper basis

How to get coefficients? Fourier Transform

$$V_n = \sum_n c_n e_n$$

$$c_n = \langle e_n | v_n \rangle$$

$$\langle k | \psi \rangle = \int_{-\infty}^{\infty} dx \, e^{-ikx} \psi(x) = c(k)$$

$$= \int_{-\infty}^{\infty} dx \, e^{-ikx} \int_{-\infty}^{\infty} \frac{dl}{2\pi} c(l) e^{ilx}$$

$$= \int_{-\infty}^{\infty} \frac{dl}{2\pi} c(l) \underbrace{\int_{-\infty}^{\infty} dx \, e^{i(l-k)x}}_{2\pi\,\delta(l-k)} = c(k)$$