PHYS 326 - Lecture 2

13.02.25

→ I was late to class!

$$(R^{-1})^i{}_k e_i = (R^{-1})^i{}_k R^j{}_i \, e'_j$$

$$= R^j{}_i (R^{-1})^i{}_k \, e'_j \qquad \text{(Definition of matrix multiplication)}$$

$$= (RR^{-1})^j{}_k \, e'_j = \delta^j_k \, e'_j = e'_k$$

→ I can write "this" in a slightly different form:

$$e'_k = (R^{-1})^i{}_k \, e_i \qquad \text{"this"}$$

$$= \left[(R^{-1})^T\right]_k{}^i \, e_i$$

→ $v = v^i e_i = \underbrace{v^i R^j{}_i}_{v'^j} e'_j$

$$v'^j = R^j{}_i v^i$$

"The components transform this way"

<u>Exercise:</u> Get v^i in terms of v'^j

<u>Hint:</u> • Try to multiply both sides with the inverse but be careful with indices

→ Let's Look at the Dual Basis

$$e^j(e_i) = \delta^j_i \qquad e'^l(e'_k) = \delta^l_k$$

$$\overset{?}{=}$$

Let's start with $e^k(e_i) = \delta^k_i$

(Derivation is tricky! Be careful →)

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$$\boxed{v'^j = R^j_{\ i}\, v^i}$$

Derivation:

$$e^k(e_i) = \delta^k_i$$

$$e^k(R^j_{\ i}\, e'_j) = R^j_{\ i}\, e^k(e'_j) = \delta^k_j$$

$$R^l_{\ k}\, R^j_{\ i}\, e^k(e'_j) = R^l_{\ k}\, \delta^k_i \quad \text{(See: Summation over } k\text{)}$$

$$= R^l_{\ i} \quad \text{(Because of sumation \& } \delta\text{)}$$

Now, multiply both sides with R^{-1}:

$$(R^{-1})^i_{\ n}\, R^l_{\ k}\, R^j_{\ i}\, e^k(e'_j) = R^l_{\ i}\, (R^{-1})^i_{\ n}$$

$$R^l_{\ k} \underbrace{R^j_{\ i}\, (R^{-1})^i_{\ n}}_{(RR^{-1})^j_{\ n} \,=\, \delta^j_n} e^k(e'_j) = \underbrace{R^l_{\ i}\, (R^{-1})^i_{\ n}}_{(RR^{-1})^l_{\ n} \,=\, \delta^l_n}$$

Then, $R^l_{\ k}\, \delta^j_n\, e^k(e'_j) = \delta^l_n$

Notice the summation over j. Then we have:

$$[R^l_{\ k}\, e^k](e'_n) = \delta^l_n \rightarrow e'^l = R^l_{\ k}\, e^k$$

Remember: we are after finding $e'^l(e'_n) = \delta^l_n$

→ So this is how the dual basis changes.

$$\boxed{e'^j = R^j_{\ i}\, e^i}$$

<u>Exercise:</u> Go over this calculation. ("index gymnastics")

How The Components of A Covector Changes:

$$\alpha = \alpha_i e^i = \alpha'_j e'^j = \alpha'_j R^j{}_i e^i$$

$$\alpha_i = \alpha'_j R^j{}_i \longrightarrow (R^{-1})^i{}_k \alpha_i = \alpha'_j R^j{}_i (R^{-1})^i{}_k$$

$$= \alpha'_j (RR^{-1})^j{}_k$$

$$= \alpha'_j \delta_k^j = \alpha'_k$$

$$\rightarrow \alpha'_k = (R^{-1})^i{}_k \alpha_i = [(R^{-1})^T]^i{}_k \alpha_i$$

$$\boxed{\alpha'_k = [(R^{-1})^T]^i{}_k \alpha_i}$$

! "The objects with ~~same~~ lower subscript ~~upper index~~ transform the same way" → But they are of different nature.

Our list so far:

$v'^j = R^j{}_i v^i$	$\alpha'_k = [(R^{-1})^T]^i{}_k \alpha_i$
$e'^j = R^j{}_i e^i$	$e'_k = [(R^{-1})^T]^i{}_k e_i$

Tensors (Characthesistic Property of Tensors)

$$T = T^{j_1 \cdots j_n}{}_{l_1 \cdots l_m} \, e_{j_1} \otimes \cdots \otimes e_{j_n} \otimes e^{l_1} \otimes \cdots \otimes e^{l_m}$$

$$= T'^{i_1 \cdots i_n}{}_{k_1 \cdots k_m} \, e'_{i_1} \otimes \cdots \otimes e'_{i_n} \otimes e'^{k_1} \otimes \cdots \otimes e'^{k_m}$$

→ Relate primed coeffs. to unprimed coeff.s

$$* \; T'^{i_1, \dots i_n}{}_{k_1 \dots k_m} = T(e'^{i_1}, \dots, e'^{i_n}, e'_{k_1}, \dots, e'_{k_m}) =$$

$$= T\left(R^{i_1}{}_{j_1} e^{j_1}, \dots, R^{i_n}{}_{j_n} e^{j_n}, \left[(R^{-1})^T\right]^{l_1}{}_{k_1} e_{l_1}, \dots \right.$$

$$\left. \dots, \left[(R^{-1})^T\right]^{l_m}{}_{k_m} e_{l_m}\right)$$

$$= R^{i_1}{}_{j_1} \cdots R^{i_n}{}_{j_n} \left[(R^{-1})^T\right]^{l_1}{}_{k_1} \cdots \left[(R^{-1})^T\right]^{l_m}{}_{k_m} T^{j_1 \cdots j_n}{}_{l_1 \cdots l_m}$$

→ Very important result! (General Relativity)

→ There's a well defined relation between components of different coordinate systems (basis')

Almost like a dictionary

Let's Discuss This in a geometric setting.

$\mathbb{R}^n$

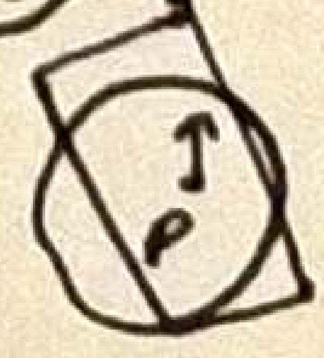
$\cdot P$

Partial Differentials

$$\left.\frac{\partial}{\partial x^1}\right|_P, \left.\frac{\partial}{\partial x^2}\right|_P, \cdots \left.\frac{\partial}{\partial x^n}\right|_P$$

They form a vector space

& I can think of these as linear operators



• Sphere isn't a vector space but the tangent plane is!

→ Assign a vector space to each point in space

The vector space they form is called the Tangent Space

$$\underbrace{T_p \mathbb{R}^n}_{\text{tangent space to } \mathbb{R}^n \text{ at } p \in \mathbb{R}^n} = \text{span}\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

Are they linearly independent?

Lets just apply the definition:

we want to show that:

$$\alpha_1 \left.\frac{\partial}{\partial x^1}\right|_p + \alpha_2 \left.\frac{\partial}{\partial x^2}\right|_p + \cdots \alpha_n \left.\frac{\partial}{\partial x^n}\right|_p = 0$$

$$\text{iff} \quad \alpha_1, \dots, \alpha_n = 0$$

Proof: Let $f(x) = x^1$

$$\alpha_1 \left.\frac{\partial x^1}{\partial x^1}\right|_p + \alpha_2 \left.\frac{\partial x^1}{\partial x^2}\right|_p + \cdots + \alpha_n \left.\frac{\partial x^1}{\partial x^n}\right|_p = 0$$

$$\alpha_1 + 0 + \cdots + 0 = 0 \qquad \alpha_1 = 0$$

You can show similarly for any α_i. Thus they are linearly independent. ■

Dual Space of This (Cotangent Space)

$\mathbb{R}^n$ dual space: $(T_p\mathbb{R}^n)^* = T_p^*\mathbb{R}^n$

$= \text{span}\{dx_p^1, dx_p^2, \dots, dx_p^n\}$

is called the cotangent space to $\mathbb{R}^n$ at point $p \in \mathbb{R}^n$

(we will stop writing p from now on)

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$$

- Let us choose a new coordinate system.

$x'^j = f^i(x^1, \dots, x^n)$ (new coordinate)

! This relation must be invertible

$$x^j = g^j(x'^1, x'^2, \dots, x'^n)$$

- Differential Operators in the New Coordinate System.

$$\frac{\partial}{\partial x'^j} \underbrace{\qquad\qquad}_{\text{Relating these two}} \frac{\partial}{\partial x^j} \to \text{CHAIN RULE!}$$

$$\frac{\partial}{\partial x'^i} = \underbrace{\frac{\partial x^j}{\partial x'^i}}_{\frac{\partial g^j}{\partial x'^i}} \frac{\partial}{\partial x^j}$$

$\to$ How do the dualspacechange?

continued $\to$

$$dx'^k = \frac{\partial x'^k}{\partial x^l} \partial x^l$$

Let's check: $dx'^k \left(\frac{\partial}{\partial x'^i} \right) \overset{?}{=} \delta_i^k$ ✓

$$= \frac{\partial x'^k}{\partial x^l} dx^l \left(\frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j} \right) =$$

$$\frac{\partial x'^k}{\partial x^l} \frac{\partial x^j}{\partial x'^i} dx^l \left(\frac{\partial}{\partial x^j} \right)$$

(Chain rule)

$$\rightarrow \frac{\partial x'^k}{\partial x^l} \frac{\partial x^j}{\partial x'^i} \delta_j^l = \frac{\partial x'^k}{\partial x^j} \frac{\partial x^j}{\partial x'^i} = \frac{\partial x'^k}{\partial x'^i} = \delta^k_i$$

Checked ✓

Transformation Rules So far:

$$\boxed{dx'^k = \frac{\partial x'^k}{\partial x^l} dx^l}$$

$$\boxed{\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j}}$$