PHYS 311 Lecture 15 18.11.2024

Quantum Harmonic Oscillator

Inner Product

- Let V be a vector space
- An innerproduct is a map: Let ψ and ϕ be two elements of V one has:

$$(\psi, \phi) \in \mathbb{C}$$

$$(\phi, \psi) = (\psi, \phi)^*$$

$$(\psi, \underbrace{\overset{\in \mathbb{C}}{\alpha}\phi_1, \beta\phi_2}_{\in V}) = \alpha(\psi, \phi_1) + \beta(\psi, \phi_2)$$

$(\psi, \psi) \geq 0$

$= 0$ iff $\psi = 0$

Let A be an operator acting on V:

$$(\psi, A^\dagger \phi) \equiv (A\psi, \phi)$$

See: $(\phi, \psi) = \int_{-\infty}^{\infty} dx \; \phi^*(x)\, \psi(x)$

Function spaces can be vector spaces.

x = hermitian $\qquad (\psi, x^\dagger \phi) = (x\psi, \phi)$

P = hermitian? $\qquad (\psi, P^\dagger \phi) = (P\psi, \phi)$

$P\psi = -i\hbar \dfrac{d\psi}{dx} \qquad (P\psi)^* = i\hbar \dfrac{d\psi^*}{dx}$

$$\int_{-\infty}^{\infty} dx\, \psi^* (P^\dagger \phi) = \int_{-\infty}^{\infty} dx\, i\hbar \frac{d\psi^*}{dx}\phi$$

$$= \int_{-\infty}^{\infty} dx\, i\hbar \frac{d}{dx}(\psi^* \phi) + \int_{-\infty}^{\infty} dx\, \psi^* (P\phi)$$

Let $\psi, \phi \to 0$ as $x \longrightarrow \pm\infty$

often ignored!

So P is hermitian when $\psi, \phi \to 0$ as $x \to \infty$, Thus the hamiltonian is hermitian.

(See, we work with $|\psi|^2$ s, so $\psi, \phi \to 0$ as $x \to \infty$ is often satisfied)

- $$i\hbar \frac{\partial \psi}{\partial t} = \left(\frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2\right)\Psi = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 x^2\right)\Psi$$

Stationary States $\Psi = e^{\frac{-iEt}{\hbar}} \Psi_E(t)$

$$E\,\psi_E(x) = \underbrace{\left(\frac{P^2}{2m} + \frac{1}{2} m\omega^2 x^2\right)}_{H \text{ (Hamiltonian)}} \Psi_E(x)$$

Let a be:

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{P}{m\omega}\right) \longrightarrow a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{P}{m\omega}\right)$$

$$[p, x] = -i\hbar \longrightarrow [a, a^\dagger] = 1$$

$[A,B] = AB - BA = -[B,A]$

$[A+B, C] = [A,C] + [B,C]$

$$a a^\dagger = \frac{m\omega}{2\hbar}\left(x + \frac{ip}{m\omega}\right)\left(x - \frac{ip}{m\omega}\right) = \frac{m\omega}{2\hbar}\left(x^2 + \frac{p^2}{m^2\omega^2} + \frac{ipx}{m\omega} - \frac{ipx}{m\omega}\right)$$

$(\cdots)$

$$a a^\dagger = \frac{1}{2} + \frac{1}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2\right)$$

$$\boxed{a a^\dagger = \frac{1}{2} + \frac{H}{\hbar\omega}} \quad \& \quad \boxed{a^\dagger a = -\frac{1}{2} + \frac{H}{\hbar\omega}}$$

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \qquad [AB, C] = A[B,C] + [A,C]B$$

$$[H, a] = [\hbar\omega(a^\dagger a + 1/2), a] = \hbar\omega[a^\dagger a, a]$$

Numbers commute.

$$\boxed{[H, a] = -\hbar\omega a} \qquad \boxed{[H, a^\dagger] = +\hbar\omega a^\dagger}$$

- $H\psi_E = E\psi_E$ $\qquad (\psi_E, \psi_{E'}) = \begin{cases} 0 & \text{if } E \neq E' \\ 1 & \text{if } E = E \end{cases}$

$$(Ha - aH)\psi_E = -\hbar\omega a\psi_E$$

$$(Ha\psi_E - Ea\psi_E) = -\hbar\omega a\psi_E$$

$$H(a\psi_E) = (E - \hbar\omega)(a\psi_E)$$

a new eigenvector & eigenvalue

$$H(a^\dagger \psi_E) = (E+\hbar\omega)(a^\dagger \psi_E)$$ eigenvector

Condition:

ψ_{ground} exists with $E_{ground} > 0$ s.t.

$$a\psi_g = 0$$

$$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_g(x) = 0$$

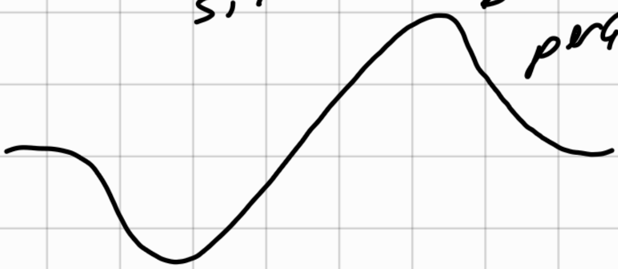
$$\psi' = -\frac{m\omega x}{\hbar}\psi \longrightarrow \frac{\psi'}{\psi} = \frac{d\ln\psi}{dx} = -\frac{m\omega x}{\hbar}$$

$$\ln\psi = -\frac{m\omega x^2}{2\hbar} \longrightarrow \boxed{\psi_g = \mathcal{N}\exp\left[-\frac{m\omega}{2\hbar}x^2\right]}$$

E of ψ_g? $\boxed{E_g = \frac{\hbar\omega}{2}}$

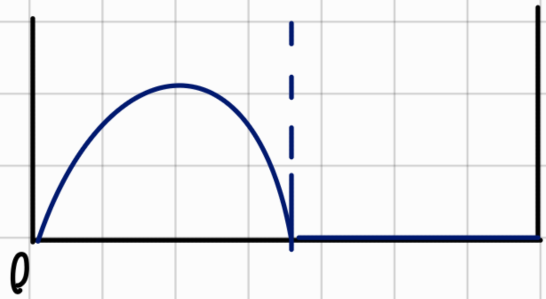
dik polinomlar silsilesinin bir parçası

ψ_g



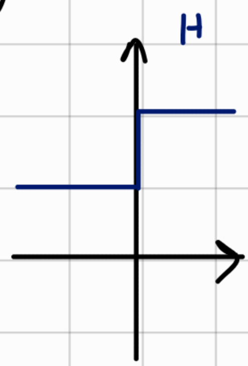
Say a particle is observed to have the groundstate energy in a box of length L

Say $L \to 2L$



Sudden Approximation

$$i\hbar \frac{\partial \psi}{\partial t} = H(t)\psi \qquad H(t) \begin{cases} H_< & t<0 \\ H_> & t>0 \end{cases}$$



Integrate between $-\epsilon < t < \epsilon$

$$i\hbar\left[\psi(\epsilon^+) - \psi(\epsilon^-)\right] = \int_{-\epsilon}^{\epsilon} dt \, H(t)\,\psi(t)$$

Sudden: (No singularities)

$$\lim_{\epsilon \to 0} RHS \to 0 \Rightarrow \psi(\epsilon^+) = \psi(\epsilon^-)$$

"ψ has no time to change)

$$= \psi_{new}(x,0) = \sum_{n=1}^{\infty} C_n e^{\frac{-iE_n^{new} t}{\hbar}} \underbrace{\psi_n^{new}(x)}_{\sqrt{\frac{2}{2L}} \sin\left(\frac{n\pi x}{2L}\right)}$$

$$E_n^{new} = \frac{\hbar^2 \pi^2 n^2}{2m\,2L}$$

$$c_1 = \int_0^{2L} dx\, \psi(x,0)\, \psi_{new}(x)$$

$$= \int_0^{L} dx \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \sqrt{\frac{2}{2L}} \sin\left(\frac{\pi x}{2L}\right)$$

$$= \int_0^{L} dx\, \frac{\sqrt{2}}{L} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi x}{2L}\right)$$

$$2 \sin\left(\frac{\pi x}{2L}\right) \cos\left(\frac{\pi x}{2L}\right)$$

$$= \int_0^{L} dx\, \frac{2\sqrt{2}}{L} \sin^2\left(\frac{\pi x}{2L}\right) \cos\left(\frac{\pi x}{2L}\right) \qquad u = \frac{\pi x}{2L}$$

$$\frac{2\sqrt{2}}{L} \frac{2L}{\pi} \int_0^{\pi/2} \sin^2(u)\, d(\sin(u)) \qquad \frac{\pi x}{2L} = u$$

$$\frac{4\sqrt{2}}{3\pi} \rightarrow \frac{32}{9\pi^2} \qquad \int_0^1 dv\, v^2 = \frac{1}{3}$$

→ Beta decay

"Can I press it suddenly?" No! (Depends on state)

Slow (adiabetic) Process

In an adiabatic process quantum numbers remain the same (if process is small enough)

$$\sqrt{\frac{2}{L(t)}} \sin\left(\frac{n\pi}{L(t)}\right) \quad \text{if} \quad \frac{\dot{L}}{L} \ll 1$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL(t)^2} \qquad \dot{E}_n = \frac{n^2\pi^2\hbar^2}{2mL^2(t)} - 2\frac{\dot{L}}{L} = -\frac{2\dot{L}}{L} E_n$$

$$\left(\bar{E} = \sum_r E_r p_r \qquad \Delta\bar{E} = \underbrace{\sum_r \Delta E_r p_r}_{\text{(Work)}} + \underbrace{\sum E_r \Delta p}_{\text{(Heat)}} \right)$$

$$\frac{dE_n}{dL} \equiv +F \qquad \frac{dE_n}{dL} = -\frac{2E_n}{L}$$