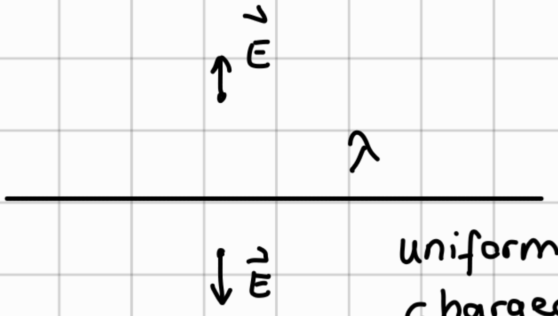
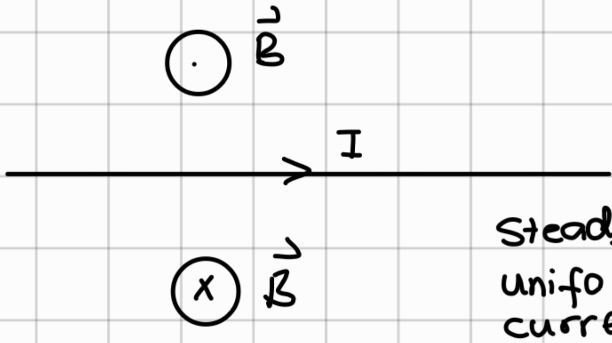
PHYS 311 Lecture 6 09.10.2024



uniformly charged line



steady uniform current

Propose: J = 4-current (current 4-vector)

Dimensions

$$j^0 = \rho \times ? \qquad \frac{\text{Coulomb}}{L^3}\, c \;\Rightarrow\; j^0 = \rho \cdot c$$

$$\vec{j} = \vec{J} \qquad \frac{\text{Coul.}}{T}\,\frac{1}{L^2}$$

Lorentz Transformation

$\vec{v} = \beta c \hat{\imath}$

$$j^{0'} = \gamma(j^0 - \beta j^1)$$

$$j^{1'} = \gamma(j^1 - \beta j^0)$$

$$j^{2'} = j^2, \quad j'^3 = j^3$$

For G say $j^0 = \rho c$ $\vec{j} = 0$

$$j^{0'} = \gamma(j^0 = \gamma j^0 = \gamma \rho c = \rho' c \qquad \rho' = \frac{\rho}{\sqrt{1 - v^2/c^2}}$$

$$j^{1'} = -\gamma\beta J^0 = -\gamma\beta\rho c = -\beta\rho' c = -v\rho'$$

→ Any current in SR can be written as $-v\rho'$. ("Mass current" too...)

Tensor Product

Say A is a 4-vector, B is a 4-vector $\Big]$ $A \otimes B$

E.g: 3-dim

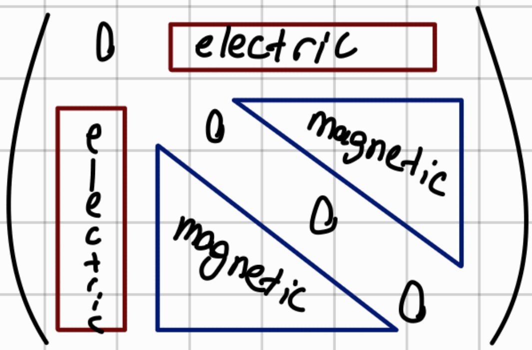
v^i $\qquad \tilde{v}^i = \sum_j O^{ji} v^i$ (How v^i transforms, we know)

$v^i v^j$ $\qquad \tilde{v}^i \tilde{v}^j = \sum_a \sum_b O^{ia} v^a O^{jb} v^b$

I will call anything that transforms like this, is an element of this space (M^{ij})

→ Doesn't have to be symmetric

(*) I'm asking for a 4x4 matrix that holds 6 degrees: Antisymmetric



M^{00}, M^{0i}, M^{ij}

M^{0i} → transforms as a vector

M^{ij} → transforms as a pseudovector

Don't forget the vector character of this object ($\vec{B}$ and $\vec{E}$)

$\vec{B}$ → pseudo vector!

Remember: $\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$

Newton's 2nd law "forces" F to be a vector
($\vec{a}$ comes from the position vector)

"in a mirror"

- Consider parity transformation (Not a rigid transformation)
(everything goes to $\ominus$ itself)
↳ $\vec{F} \to -\vec{F}$ $\vec{v} \to -\vec{v}$ so $\vec{B}$ shouldn't.

<u>Def:</u> Pseudovector: "Something" that doesn't change under parity transformation

$$\tilde{M}^{0i} = \tilde{E}^{i}$$

$$\tilde{M}^{ij} = \frac{1}{2}\epsilon^{ijk}\tilde{B}^{k} \quad \text{sum over } k$$

$\epsilon^{123} = +1$ (epsilon tensor)

"Electromagnetic Tensors"

Electric - Magnetic fields in relativity are one object

$j \to$ 4 vector:

Invariant: $$j^{0\,2} - \vec{j}\cdot\vec{j} = (j'^{0})^2 - \vec{j}'\cdot\vec{j}'$$

You can classify these
>0
$=0$
<0

>0 means charge densities are stronger than currents.
$=0$ means point particle moves with c. (no mass)
<0 can't exist as single particles (possible for collective

There are no particles with charge who are massless.

If there is a density and it's motion it must obey: (No sources)

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

Local conservation law

"money doesn't obey this, gold does" :)

You can deduce a global conservation law.

" $I = \frac{dq}{dt}$ is the worst definition of a current "

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

Integrate the equation in the region $[0, L]$

$$\frac{d}{dx} \underbrace{\int_0^L dx\, \rho}_{Q_{region}} + \underbrace{\int_0^L dx\, \frac{\partial j}{\partial x}}_{j(x=L) - j(x=0)} = 0$$

$$\frac{dQ_{region}}{dt} = \underbrace{j(x=0)}_{j_{enter}} - \underbrace{j(x=L)}_{j_{exists}}$$

See, dQ/dt isn't current.

A form of Stoke's Theorem

$$\underbrace{\int_{\text{Region}} \frac{\partial \rho}{\partial t}\, d^3x}_{\frac{dQ_{\text{region}}}{dt}} + \underbrace{\int d^3x\, \vec{\nabla}\cdot\vec{J}}_{} = 0$$

$$\underbrace{\int_{\substack{\text{Boundary}\\ \text{of region}}} d\vec{S}\cdot\vec{J}}_{\phi_J} = 0$$

I declare :

$$\circledast \quad \underbrace{\partial_0}_{\frac{\partial}{c\partial t}} J^0 + \overset{\frac{\partial}{\partial x}}{\partial_1} J^1 + \partial_2 J^2 + \partial_3 J^3 = 0 = \partial_0' J^{0'} + \partial_1' J^{1'} + \dots$$

$D \cdot J$ lorentz form.

"4 dimensional divergence"

$$\vec{\nabla}\cdot\vec{J}. = \vec{\nabla}'\cdot\vec{J}' = f$$

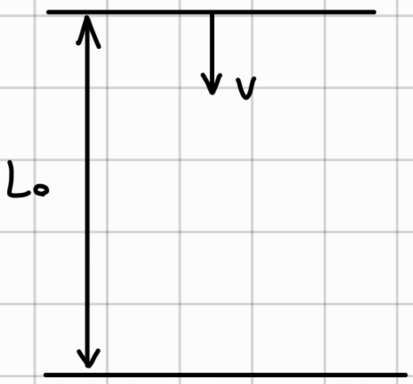
Calculate it on a rigidly rotated frame. You can show it's a Scalar.

Therefore ⊛ will be the same in all frames. So, it's closed

So, spacetime is a 4D entity ...

EXAMPLE: MUON PARADOX

To measure the lifetime of a partical, you have to measure it in the inertial frame. (They actually slow it down in a crystal)



τ_o

$$\tau_{Lab} = \frac{\tau_0}{\sqrt{1-v^2/c^2}}$$

$$v\underbrace{\tau_{Lab}}_{\frac{\tau_0}{\sqrt{1-v^2/c^2}}} \approx L_o$$

$$\underbrace{v\tau_o = L = \sqrt{1-v^2/c^2}\, L_o}_{v\tau_o = \sqrt{1-v^2/c^2}\, L_o}$$

Transforming velocity

$$U : \begin{pmatrix} U^0 = \dfrac{c}{\sqrt{1-u^2/c^2}} \\ \vec{U} = \dfrac{\vec{u}}{\sqrt{1-u^2/c^2}} \end{pmatrix} \xrightarrow{\text{Lorentz}} U' : \begin{pmatrix} \dfrac{c}{\sqrt{1-u^2/c^2}} \\ \dfrac{\vec{u}\,'}{\sqrt{1-u'^2/c^2}} \end{pmatrix}$$

Quicker Way

$$\Delta x' = \gamma(\Delta x - \beta c \Delta t)$$
$$c\Delta t' = \gamma(c\Delta t - \beta \Delta x)$$

$$\frac{\Delta x'}{c\Delta t'} = \frac{U'}{c} \Leftrightarrow \frac{\Delta x}{c\Delta t} = \frac{u}{c}$$

$$\frac{U'}{c} = \frac{\Delta x - \beta c \Delta t}{c\Delta t - \beta \Delta x}$$

$$U' = \frac{U - V}{\left(1 - \frac{vu}{c^2}\right)}$$

Notice: if $U = c$, $V' = c$

Generic Lorentz Transformation

$\vec{\beta}\cdot\vec{r}$

$$ct' = \gamma(ct - \beta r_{\parallel})$$

$$r_{\parallel}' = \gamma(r_{\parallel} - \beta ct)$$

$$\vec{r}_{\perp}' = \vec{r}_{\perp}$$

(When I say $\parallel$, means $\parallel$ to β)

- $\vec{\beta} \equiv \dfrac{\vec{V}}{c}$
- $\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$

$\hookrightarrow \vec{r}_{\parallel} = \vec{r} - \vec{r}_{\perp}$

$$\Downarrow$$

$$\vec{r}_{\parallel}' = \vec{r}' - \vec{r}_{\perp}' = \gamma(\vec{r} - \vec{r}_{\perp} - \beta ct)$$

$$\vec{r}' = \vec{r}_{\perp}(1-\gamma) + \gamma\vec{r} - \gamma\vec{\beta}ct$$

$$ct = \gamma(ct - \vec{\beta}\cdot\vec{r})$$

$$\Downarrow$$

$$\Delta\vec{r}' = \Delta\vec{r}_{\perp}(1-\gamma) + \gamma\Delta\vec{r} - \gamma\vec{\beta}c\Delta t$$

$$c\Delta t = \gamma(c\Delta t - \vec{\beta}\cdot\Delta\vec{r})$$