PHYS 326 Lecture 8 (March 11)

#phys326 #phys326/lecturenotes

MT 1: March 25 13:00
Continuing with asymptotic analysis

Laplace Method

(...)

$$I(\alpha) \sim e^{\alpha} \int_{-\infty}^{\infty} d\theta e^{-\alpha\frac{\theta^2}{2}} e^{-\alpha(\frac{\theta^4}{4!}-\frac{\theta^6}{6!})+\dots}$$

(...)

Stirling Approximation

We have the Gamma Function:

$$\Gamma(x+1) = \int_0^{\infty} dt t^x e^{-t} \sim ? \text{ as } x \to \infty$$

$$= \int_0^{\infty} dt e^{\ln t^{\alpha}} e^{-t} = \int_0^{\infty} dt e^{\alpha \ln t - t}$$

Let $t = \alpha u$ so $dt = \alpha du$ then,

$$= \int_0^{\infty} du \alpha e^{\alpha \ln \alpha u - \alpha u}$$

$$= \int_0^{\infty} du \alpha e^{\alpha \ln \alpha} e^{\alpha(\ln u - u)} = \alpha e^{\alpha \ln \alpha} \int_0^{\infty} du e^{\alpha(\ln u - u)}$$

Exercise 1:
Show that $\int_{1+\epsilon}^{\infty} du e^{\alpha(\ln u + u)}$ is exponentially small.
Show that $\int_0^{1-\epsilon} du e^{\alpha(\ln u + u)}$ is exponentially small.

Continuing...

$$\Gamma(\alpha+1) \sim e^{\alpha \ln \alpha} \int_{1-\epsilon}^{1+\epsilon} du e^{\alpha[\ln u - u]}$$

$$f(u) = \ln u - u = -1 - \frac{1}{2!}(u-1)^2 + O((u-1)^3)$$

$$f(1) = -1$$

$f'(u) = \frac{1}{u} - 1$, $f'(1) = 0$
$f''(u) = \frac{1}{u^2}$, $f''(1) = -1$

$$\Gamma(\alpha+1) \sim e^{\alpha \ln \alpha} \int_{1-\epsilon}^{1+\epsilon} du e^{-\alpha \frac{(u-1)^2}{2} - \alpha} \left[1 + O((u-1)^3)\right]$$

$$\sim \alpha e^{\alpha \ln \alpha} \int_0^\infty du e^{-\alpha \frac{(u-1)^2}{2}}$$

Let $v = u - 1$

$$\sim \alpha e^{\alpha \ln \alpha} \int_0^\infty dv e^{-\alpha \frac{v^2}{2}}$$

$$\int_0^\infty dv e^{-\alpha \frac{v^2}{2}} = \int_{-\infty}^{-1} dv e^{-\alpha \frac{v^2}{2}} + \int_{-1}^\infty dv e^{-\alpha \frac{v^2}{2}}$$

Exercise: Show that this is exponentially small

Then,

$$\sim \alpha e^{\alpha \ln \alpha - \alpha} \int_{-1}^\infty dv e^{-\alpha \frac{v^2}{2}} \sim \alpha e^{\alpha \ln \alpha - \alpha} \int_{-\infty}^\infty dv e^{-\alpha \frac{v^2}{2}}$$

$$\sim \alpha e^{\alpha \ln \alpha - \alpha} \sqrt{\frac{2\pi}{\alpha}} = \sqrt{2\pi\alpha} e^{\alpha \ln \alpha - \alpha}$$

This is the stirling approximation.

Then,

$$\Gamma(\alpha+1) \sim \sqrt{2\pi\alpha} e^{\alpha \ln \alpha - \alpha}$$

$$\sim e^{\alpha \ln \alpha - \alpha + \frac{1}{2}\ln(2\pi\alpha)}$$

$$\sim e^{\alpha \ln \alpha - \alpha + \frac{1}{2}\ln(\alpha) + \frac{1}{2}\ln(2\pi)}$$

Consider this integral:

$$I(\alpha) = \int_0^\infty dx e^{-\alpha x^p} x^q = ?$$

$$p > 0, q \geq 0$$

We will make use of the gamma function.

$u = \alpha x^p$ $x = \left(\frac{u}{\alpha}\right)^{1/p}$

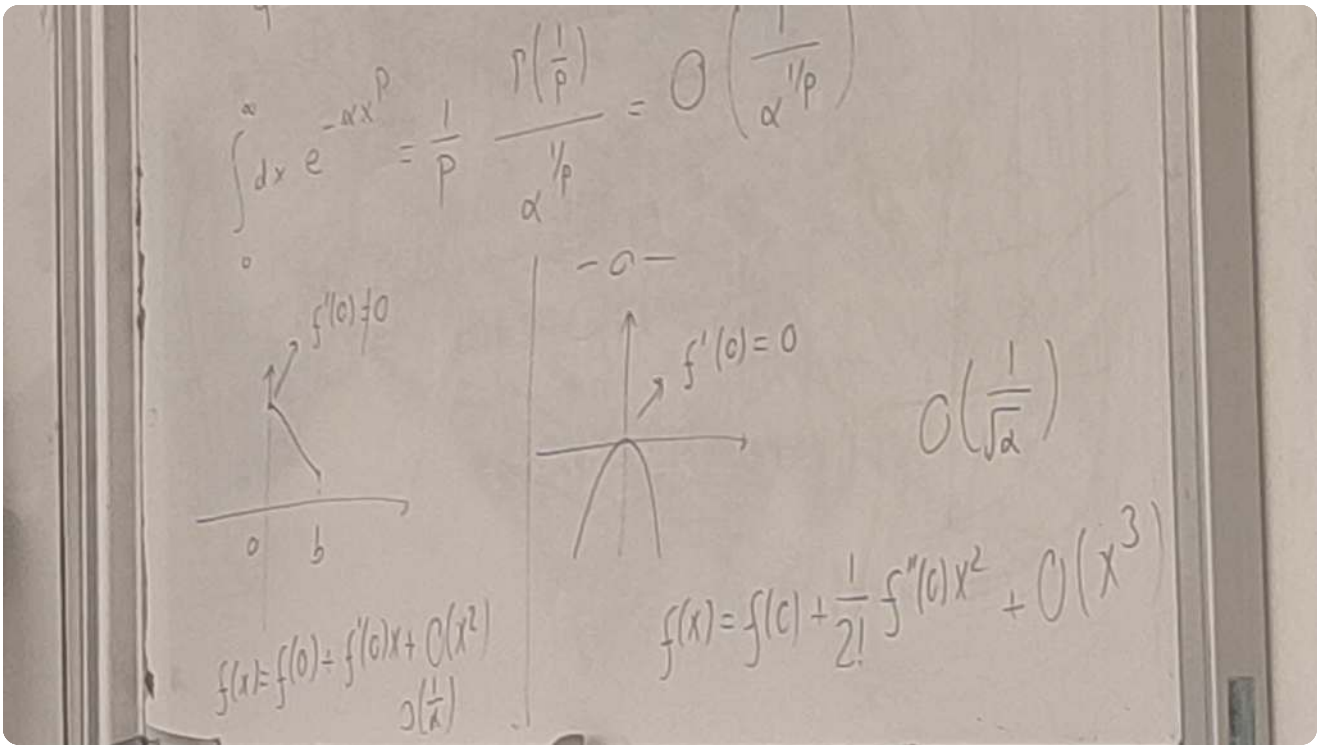
$dx = \frac{\frac{1}{p} u^{\frac{u}{p}-1} du}{\alpha^{1/p}}$

$$I(\alpha) = \int_0^\infty du \frac{\frac{1}{p} u^{\frac{u}{p}-1} du}{\alpha^{1/p}} e^{-u} \frac{u^{q/p}}{\alpha^{\frac{q}{p}}}$$

$$= \frac{1}{p} \frac{1}{\alpha^{q+1/p}} \int_0^\infty du u^{(q+1/p)-1} e^{-u} = \frac{1}{p} \frac{\Gamma\left(\frac{q+1}{p}\right)}{\alpha^{(q+1)/p}}$$

Special case:
$q = 0$

$$\int_0^\infty dx e^{-\alpha x^p} = \frac{1}{p} \frac{p\left(\frac{1}{p}\right)}{\alpha^{1/p}} = O\left(\frac{1}{\alpha^{1/p}}\right)$$



Leading contribution will come from local maximum (critical point → derivative is zero)