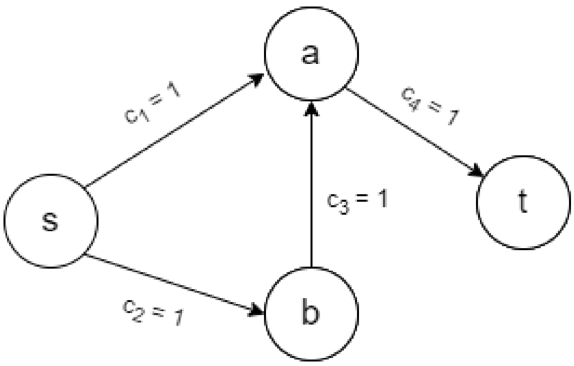
CS301 Algorithms - Homework 4

Ege Demirci - 28287

Sabanci University, Spring 2022-2023, Algorithms

1. Question - 1

In the flow network $G = (V, E, s, t, c)$; we can show at least two different max–flow functions and state the value of the max–flow, here is the example flow network:



We can define flow function $f1$ as follows:

$f_1(s, a) = 1$

$f_1(s, b) = 0$

$f_1(b, a) = 0$

$f_1(a, t) = 1$

$0, otherwise$

Total flow out of the source s: 1

Total flow into the target t: 1

Let's also consider if there is a flow from b as well:

$$s \to b \to a \quad \text{(Flow: } x\text{)}$$

If there is any flow x, it means that there must be a positive flow entering node a. However, since the edge (s, a) already carries a flow of 1, there can't be any additional flow entering node a (because capacity of outflow a is 1 so inflow cannot be 2). Therefore, we found that in this case there is no option for additional flow.

So the flow configuration described by Max-Flow Function 1 is indeed the max flow for this network, with a flow value of 1.

We can define flow function f_2 as follows:

$f_2(s, a) = 0$

$f_2(s, b) = 1$

$f_2(b, a) = 1$

$f_2(a, t) = 1$

$0, otherwise$

Total flow out of the source s: 1

Total flow into the target t: 1

Let's also consider if there is a flow from s as well:

$$s \to a \quad \text{(Flow: } x\text{)}$$

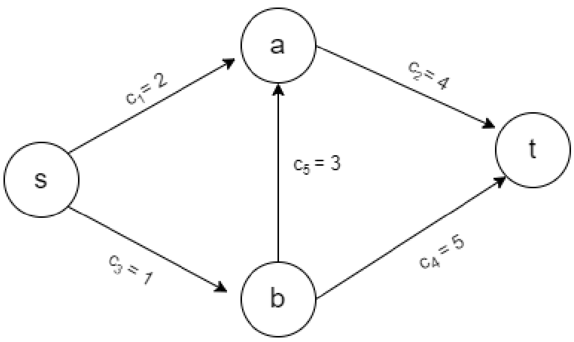
If there is any flow x, it means that there must be a positive flow entering node a. However, since the edge (b, a) already carries a flow of 1, there can't be any additional flow entering node a (because capacity of outflow a is 1 so inflow cannot be 2). Therefore, we found that in this case there is no option for additional flow.

So the flow configuration described by Max-Flow Function 2 is indeed the max flow for this network, with a flow value of 1.

As a result both f_1 and f_2 described above are certainly valid max-flow functions, as they utilize the available capacities of the network and we showed two different max-flow functions.

2. Question - 2

Given statement A is **false**. We can show why the statement is false by counter-example, here is the example flow network:



We can define flow function f_1 as follows:

$f_1(s, a) = 2$

$f_1(b, a) = 1$

$f_1(a, t) = 3$

$f_1(s, b) = 1$

$f_1(b, t) = 0$

$0, otherwise$

Total flow out of the source s: 3

Total flow into the target t: 3

We fully use the capacities s, a and s, b, and b's inflow is 1, so its outflow must be 1. Since we have a capacity of 3 between b and a, we can use 1 unit of capacity here. Additionally, since a's inflow is 3, its outflow must be 3, and since the capacity between a and t is 4, we can achieve a usage of 3 here. Between b and t, we can't provide any flow since we have already used 1 unit of capacity between b and a for outflow. Therefore, the maximum flow value = 3 is assured by flow function f_1.

We can define flow function f_2 as follows:

$f_2(s, a) = 2$

$f_2(b, a) = 0$

$f_2(a, t) = 2$

$f_2(s, b) = 1$

$f_2(b, t) = 1$

$0, otherwise$

Total flow out of the source s: 3

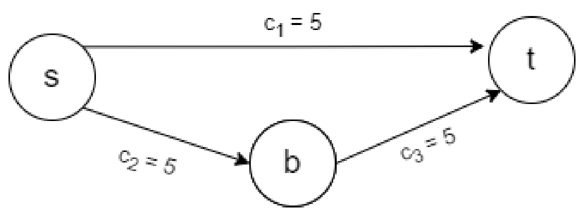
Total flow into the target t: 3

Again we fully use the capacities s, a and s, b, and b's inflow is 1, so its outflow must be 1. Since we have a capacity of 5 between b and t, we can use 1 unit of capacity here. Additionally, since a's inflow is 2, its outflow must be 2, and since the capacity between a and t is 2, we can achieve a usage of 2 here. Between b and a, we can't provide any flow since we have already used 1 unit of capacity between b and t for outflow. Therefore, the maximum flow value = 3 is assured by flow function f_2.

For both of the max-flow functions, f_1 and f_2, the value of the max-flow, i.e., the total net flow leaving the source node s, is 3. Since there are two max-flow functions for such a network with the given property in the question, Claim A is proven to be false by a counterexample.

3. Question - 3

For this question, let's use following flow network:



Since there is no maximum constraint for the flow functions, we only need to care about capacities.

We can define flow function f_1 as follows:

$f_1(s, b) = 2$

$f_1(b, t) = 2$

$f_1(s, t) = 2$

$0, otherwise$

We can define flow function f_2 as follows:

$f_2(s, b) = 4$

$f_2(b, t) = 4$

$f_2(s, t) = 4$

$0, otherwise$

Lastly flow function F as follows:

$F(s, b) = f_1(s, b) + f_2(s, b) = 6$

$F(b, t) = f_1(b, t) + f_2(b, t) = 6$

$F(s,t) = f_1(s,t) + f_2(s,t) = 6$

Both functions ensure the capacity constraint:

$$\forall u, v \in V : F(u,v) \leq c(u,v)$$

But in our network, while the capacities of all the edges are 5; last flow function cannot ensure the capacity constraint since the flow values of the last function we defined are 6. In other words, all flows exceed the given capacity. Capacity constraint is not provided so **so F is not guaranteed to be a flow on G.**