MULTIVARIABLE AND VECTOR ANALYSIS

W W L CHEN

Chapter 1

FUNCTIONS OF SEVERAL VARIABLES

1.1. Basic Definitions

In this chapter, we consider functions of the form

$$f : A \to \mathbb{R}^m : \mathbf{x} \mapsto f(\mathbf{x}), \tag{1}$$

where the domain $A \subseteq \mathbb{R}^n$ is a set in the n-dimensional euclidean space, and where the codomain $\mathbb{R}^m$ is the m-dimensional euclidean space. For each $\mathbf{x} \in A$, we can write

$$\mathbf{x} = (x_1, \ldots, x_n),$$

where $x_1, \ldots, x_n \in \mathbb{R}$. In other words, we think of the function (1) as a function of n real variables $x_1, \ldots, x_n$. If $n > 1$, then we say that the function (1) is a function of several (real) variables. On the other hand, we can write

$$f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})),$$

where $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}) \in \mathbb{R}$. We say that the function (1) is a vector valued function. If $m = 1$, then we also say that the function (1) is a real valued function.

Example 1.1.1. We are familiar with the case $n = m = 1$ (real valued functions of a real variable) and the case $n = 2$ and $m = 1$ (real valued functions of two real variables).

Example 1.1.2. The area of a rectangular box in 3-dimensional euclidean space can be given by a function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^3$ consists of all triples $\mathbf{x} = (x, y, z)$, with real numbers $x, y, z \geq 0$ describing the lengths of the sides of the rectangular box. Here $f(\mathbf{x}) = xyz$.

EXAMPLE 1.1.3. To describe the flow of air, we may consider a function of the form $f : A \to \mathbb{R}^3$, where $A \subseteq \mathbb{R}^4$ consists of all 4-tuples $\mathbf{x} = (x, y, z, t)$, with the triple (x, y, z) describing the position and the real number t describing time. Here the image $f(\mathbf{x})$ describes the velocity of air at position (x, y, z) and time t.

EXAMPLE 1.1.4. We can define $f : \mathbb{R}^7 \to \mathbb{R}^5$ by writing

$$f(x_1, \ldots, x_7) = (x_1 + x_2, x_3x_4x_5, x_2x_6, x_1x_7, x_2x_3 + x_5)$$

for every $(x_1, \ldots, x_7) \in \mathbb{R}^7$.

1.2. Open Sets

In the next section, we shall develop the concept of continuity. To do this, we need the notion of a limit. However, to understand limits, we must first study open sets.

DEFINITION. For every $\mathbf{x}_0 \in \mathbb{R}^n$ and every real number $r > 0$, the set

$$D(\mathbf{x}_0, r) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}_0\| < r\}$$

is called the open disc or open ball with centre $\mathbf{x}_0$ and radius r. Here $\|\mathbf{x} - \mathbf{x}_0\|$ denotes the euclidean distance between $\mathbf{x}$ and $\mathbf{x}_0$.

REMARK. More precisely, for every $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, the quantity $\|\mathbf{y}\|$ denotes the norm of the vector $\mathbf{y}$, and is given by

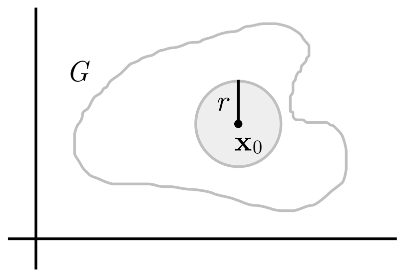
$$\|\mathbf{y}\| = \sqrt{y_1^2 + \ldots + y_n^2}.$$

EXAMPLE 1.2.1. Suppose that $x_0 \in \mathbb{R}$. Then $D(x_0, r)$ denotes the open interval $(x_0 - r, x_0 + r)$.

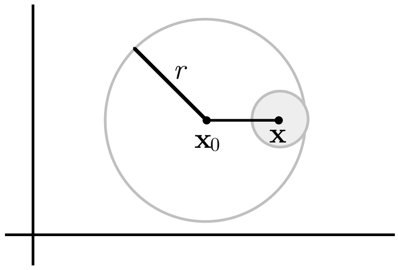
EXAMPLE 1.2.2. For $n = 2$, $D(\mathbf{0}, r)$ is the open disc $\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r^2\}$.

EXAMPLE 1.2.3. For $n = 3$, $D(\mathbf{0}, r)$ is the open ball $\{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 < r^2\}$.

DEFINITION. A set $G \subseteq \mathbb{R}^n$ is said to be an open set if, for every $\mathbf{x}_0 \in G$, there exists $r > 0$ such that the open disc $D(\mathbf{x}_0, r) \subseteq G$. In other words, a set G is open if every point of G is the centre of some open disc contained in G.



EXAMPLE 1.2.4. For every $\mathbf{x}_0 \in \mathbb{R}^n$ and $r > 0$, the open disc $D(\mathbf{x}_0, r)$ is open. To see this, we shall show that for every $\mathbf{x} \in D(\mathbf{x}_0, r)$, we can find some $s > 0$ such that $D(\mathbf{x}, s) \subseteq D(\mathbf{x}_0, r)$. The picture below in the case $n = 2$ should convince you that $s = r - \|\mathbf{x} - \mathbf{x}_0\|$ is a suitable choice.



To prove that $D(\mathbf{x}, s) \subseteq D(\mathbf{x}_0, r)$, note that for every $\mathbf{y} \in D(\mathbf{x}, s)$, we have

$$\|\mathbf{y} - \mathbf{x}_0\| = \|\mathbf{y} - \mathbf{x} + \mathbf{x} - \mathbf{x}_0\| \le \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{x}_0\| < s + \|\mathbf{x} - \mathbf{x}_0\|,$$

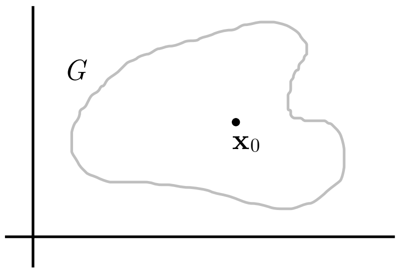
using the Triangle inequality for vectors in $\mathbb{R}^n$. It now follows from our choice of s that $\|\mathbf{y} - \mathbf{x}_0\| < r$, so that $\mathbf{y} \in D(\mathbf{x}_0, r)$.

EXAMPLE 1.2.5. The set $G = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1 \text{ and } |x_2| < 1\}$ in $\mathbb{R}^2$ is open.

EXAMPLE 1.2.6. The set $G = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ in $\mathbb{R}^3$ is open.

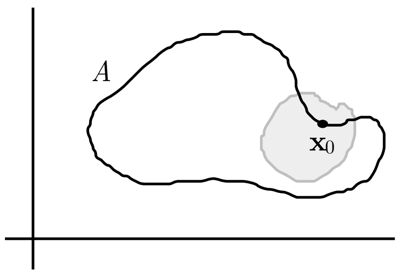
Suppose that $\mathbf{x}_0 \in \mathbb{R}^n$ is given. On many occasions, we do not need to consider open discs centred at $\mathbf{x}_0$. Very often, it may be sufficient to consider some open set containing $\mathbf{x}_0$. We therefore make the following definition for convenience.

DEFINITION. Suppose that $\mathbf{x}_0 \in \mathbb{R}^n$. Then any open set G such that $\mathbf{x}_0 \in G$ is called a neighbourhood of the point $\mathbf{x}_0$. In other words, any open set containing $\mathbf{x}_0$ is a neighbourhood of $\mathbf{x}_0$.



To complete our preparation before introducing the idea of a limit, we make two more definitions.

DEFINITION. Suppose that $A \subseteq \mathbb{R}^n$ is given. A point $\mathbf{x}_0 \in \mathbb{R}^n$ is said to be a boundary point of A if every neighbourhood of $\mathbf{x}_0$ contains a point of A as well as a point not in A.



REMARK. Note that a boundary point of a set A does not necessarily belong to A.

DEFINITION. Suppose that $A \subseteq \mathbb{R}^n$ is given. The set $\overline{A}$, containing precisely all the points of A and all the boundary points of A, is called the closure of A.

EXAMPLE 1.2.7. In $\mathbb{R}$, the intervals $(0,1)$, $(0,1]$, $[0,1)$ and $[0,1]$ all have boundary points 0 and 1, and closure $[0,1]$.

EXAMPLE 1.2.8. The boundary points of an open disc in $\mathbb{R}^2$ are precisely all the points of a circle. The closure of an open disc in $\mathbb{R}^2$ is the open disc together with its boundary circle.

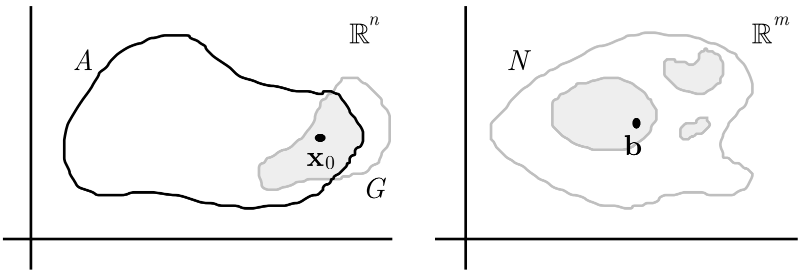
1.3. Limits and Continuity

In this section, we shall use the idea of neighbourhoods to study limits. The interested reader may wish also to study the ϵ-δ approach discussed in the next section.

DEFINITION. Consider a function of the form $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$. Suppose that $\mathbf{x}_0 \in \overline{A}$ and $\mathbf{b} \in \mathbb{R}^m$. We say that the function f has a limit $\mathbf{b}$ as $\mathbf{x}$ approaches $\mathbf{x}_0$, denoted by $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$ or

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b},$$

if, given any neighbourhood N of $\mathbf{b}$, there exists a neighbourhood G of $\mathbf{x}_0$ such that $f(\mathbf{x}) \in N$ for every $\mathbf{x} \neq \mathbf{x}_0$ satisfying $\mathbf{x} \in G \cap A$.



EXAMPLE 1.3.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 0 & \text{if } x < 2, \\ 1 & \text{if } x \geq 2. \end{cases}$$

Then f has no limit as x approaches 2.

EXAMPLE 1.3.2. Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 2, \\ 1 & \text{if } x = 2. \end{cases}$$

Then $f(x) \to 0$ as $x \to 2$.

EXAMPLE 1.3.3. In Example 1.3.1, if we change the domain of the function to $A = (-\infty, 2)$, then $f(x) \to 0$ as $x \to 2$.

Below we state three results. The interested reader may refer to the next section for the proofs.

THEOREM 1A. *Suppose that $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$. Suppose further that $\mathbf{x}_0 \in \overline{A}$, and that $f(\mathbf{x}) \to \mathbf{b}_1$ and $f(\mathbf{x}) \to \mathbf{b}_2$ as $\mathbf{x} \to \mathbf{x}_0$, where $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^m$. Then $\mathbf{b}_1 = \mathbf{b}_2$. In other words, the limit, if it exists, is unique.*

THEOREM 1B. *Suppose that $f : A \to \mathbb{R}^m$ and $g : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$. Suppose further that $\mathbf{x}_0 \in \overline{A}$. Then as $\mathbf{x} \to \mathbf{x}_0$,*
(a) if $f(\mathbf{x}) \to \mathbf{b}$, then $(cf)(\mathbf{x}) \to c\mathbf{b}$ for every $c \in \mathbb{R}$;
(b) if $f(\mathbf{x}) \to \mathbf{b}_1$ and $g(\mathbf{x}) \to \mathbf{b}_2$, then $(f+g)(\mathbf{x}) \to \mathbf{b}_1 + \mathbf{b}_2$; and
(c) if $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ for every $\mathbf{x} \in A$, then $f(\mathbf{x}) \to \mathbf{b}$ if and only if $f_i(\mathbf{x}) \to b_i$ for every $i = 1, \ldots, m$, where $\mathbf{b} = (b_1, \ldots, b_m)$.

THEOREM 1C. *Suppose that $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$. Suppose further that $\mathbf{x}_0 \in \overline{A}$. Then as $\mathbf{x} \to \mathbf{x}_0$,*
(a) if $f(\mathbf{x}) \to b_1$ and $g(\mathbf{x}) \to b_2$, then $(fg)(\mathbf{x}) \to b_1 b_2$; and
(b) if $f(\mathbf{x}) \to b \neq 0$, then $(1/f)(\mathbf{x}) \to 1/b$.

We can now define continuity in terms of limits.

DEFINITION. Consider a function of the form $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$. The function f is said to be continuous at $\mathbf{x}_0 \in A$ if $f(\mathbf{x}) \to f(\mathbf{x}_0)$ as $\mathbf{x} \to \mathbf{x}_0$. Furthermore, we say that f is continuous in A if f is continuous at every $\mathbf{x}_0 \in A$.

Corresponding to Theorems 1B and 1C, we deduce immediately the following two results.

THEOREM 1D. *Suppose that $f : A \to \mathbb{R}^m$ and $g : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$. Suppose further that $\mathbf{x}_0 \in \overline{A}$.*
(a) If f is continuous at $\mathbf{x}_0$, then cf is also continuous at $\mathbf{x}_0$.
(b) If f and g are continuous at $\mathbf{x}_0$, then $f + g$ is also continuous at $\mathbf{x}_0$.
(c) If $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ for every $\mathbf{x} \in A$, then f is continuous at $\mathbf{x}_0$ if and only if $f_1, \ldots, f_m$ are all continuous at $\mathbf{x}_0$.

THEOREM 1E. *Suppose that $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$. Suppose further that $\mathbf{x}_0 \in \overline{A}$.*
(a) If f and g are continuous at $\mathbf{x}_0$, then fg is also continuous at $\mathbf{x}_0$.
(b) If f is continuous at $\mathbf{x}_0$ and $f(\mathbf{x}_0) \neq 0$, then $1/f$ is also continuous at $\mathbf{x}_0$.

EXAMPLE 1.3.4. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$, given by

$$f(x_1, x_2, x_3) = \left(x_1^2 x_2 + x_3, \frac{x_2 x_3}{1 + x_1^2} \right).$$

We shall show that f is continuous in $\mathbb{R}^3$. Using Theorem 1D(c), it suffices to show that both components

$$(x_1, x_2, x_3) \mapsto x_1^2 x_2 + x_3 \qquad \text{and} \qquad (x_1, x_2, x_3) \mapsto \frac{x_2 x_3}{1 + x_1^2} \tag{2}$$

are continuous in $\mathbb{R}^3$. Clearly all three functions

$$(x_1, x_2, x_3) \mapsto x_1 \qquad \text{and} \qquad (x_1, x_2, x_3) \mapsto x_2 \qquad \text{and} \qquad (x_1, x_2, x_3) \mapsto x_3,$$

as well as the constant function $(x_1, x_2, x_3) \mapsto 1$, are continuous in $\mathbb{R}^3$. It follows from Theorem 1E(a) and Theorem 1D(b) that the function on the left hand side of (2) is continuous in $\mathbb{R}^3$. It follows from Theorem 1E and Theorem 1D(b) that the function on the right hand side of (2) is also continuous in $\mathbb{R}^3$.

We also have the following result on compositions of functions.

THEOREM 1F. *Suppose that $f : A \to \mathbb{R}^m$ and $g : B \to \mathbb{R}^p$, where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. Suppose further that $f(A) \subseteq B$, so that $g \circ f : A \to \mathbb{R}^p$ is well defined. If f is continuous at $\mathbf{x}_0 \in A$ and g is also continuous at $\mathbf{y}_0 = f(\mathbf{x}_0) \in B$, then $g \circ f$ is continuous at $\mathbf{x}_0$.*

1.4. Limits and Continuity: Proofs

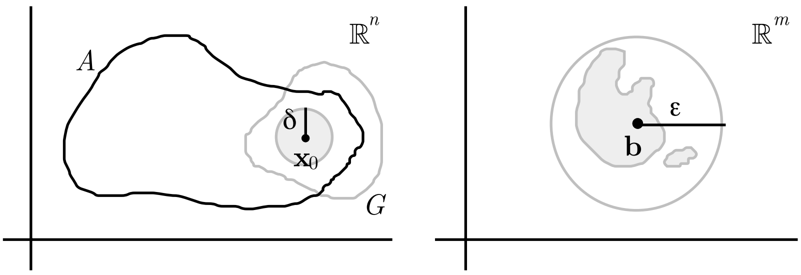
The purpose of this section is to illustrate the equivalence between the neighbourhood approach and the ϵ-δ approach, and to establish Theorems 1A, 1B, 1C and 1F. The material is optional for students not proceeding beyond the current unit of study. However, other students are advised to study the proofs carefully.

THEOREM 1G. *Suppose that $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$. Suppose further that $\mathbf{x}_0 \in \overline{A}$ and $\mathbf{b} \in \mathbb{R}^m$. Then $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$ if and only if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$ for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$.*

PROOF. ($\Rightarrow$) Suppose that $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$. Then given any neighbourhood N of $\mathbf{b}$, there exists a neighbourhood G of $\mathbf{x}_0$ such that $f(\mathbf{x}) \in N$ for every $\mathbf{x} \neq \mathbf{x}_0$ satisfying $\mathbf{x} \in G \cap A$. Let $\epsilon > 0$ be given. Clearly

$$N = D(\mathbf{b}, \epsilon) = \{\mathbf{y} \in \mathbb{R}^m : \|\mathbf{y} - \mathbf{b}\| < \epsilon\},$$

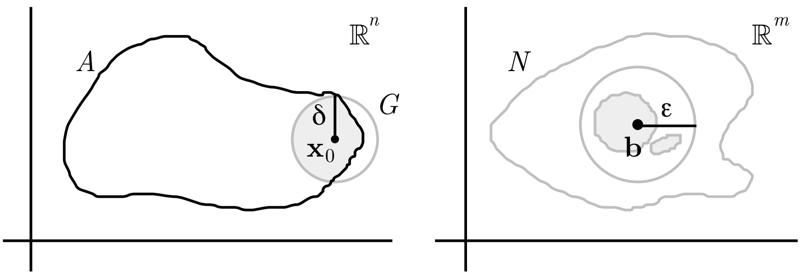
the open disc with centre $\mathbf{b}$ and radius ϵ, is a neighbourhood of $\mathbf{b}$. It follows that there exists a neighbourhood G of $\mathbf{x}_0$ such that $f(\mathbf{x}) \in D(\mathbf{b}, \epsilon)$ for every $\mathbf{x} \neq \mathbf{x}_0$ satisfying $\mathbf{x} \in G \cap A$; in other words, $\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$ for every $\mathbf{x} \neq \mathbf{x}_0$ satisfying $\mathbf{x} \in G \cap A$.



Since $G \subseteq \mathbb{R}^n$ is an open set, there exists $\delta > 0$ such that $D(\mathbf{x}_0, \delta) \subseteq G$. We therefore conclude that $\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$ for every $\mathbf{x} \neq \mathbf{x}_0$ satisfying $\mathbf{x} \in D(\mathbf{x}_0, \delta) \cap A$; in other words, for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$.

($\Leftarrow$) Suppose that given any $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$ for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. Let N be a neighbourhood of $\mathbf{b}$. Since $N \subseteq \mathbb{R}^m$ is an open set, there exists $\epsilon > 0$ such that

$$D(\mathbf{b}, \epsilon) = \{\mathbf{y} \in \mathbb{R}^m : \|\mathbf{y} - \mathbf{b}\| < \epsilon\} \subseteq N.$$



It follows that there exists $\delta > 0$ such that $\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$ for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. Let

$$G = B(\mathbf{x}_0, \delta) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}_0\| < \delta\}.$$

Then clearly $f(\mathbf{x}) \in D(\mathbf{b}, \epsilon) \subseteq N$ for every $\mathbf{x} \neq \mathbf{x}_0$ satisfying $\mathbf{x} \in G \cap A$. It follows that $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$. ◯

PROOF OF THEOREM 1A. Since $f(\mathbf{x}) \to \mathbf{b}_1$ and $f(\mathbf{x}) \to \mathbf{b}_2$ as $\mathbf{x} \to \mathbf{x}_0$, it follows from Theorem 1G that given any $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\|f(\mathbf{x}) - \mathbf{b}_1\| < \epsilon \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1,$$

and

$$\|f(\mathbf{x}) - \mathbf{b}_2\| < \epsilon \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then by the Triangle inequality,

$$\|\mathbf{b}_1 - \mathbf{b}_2\| \le \|f(\mathbf{x}) - \mathbf{b}_1\| + \|f(\mathbf{x}) - \mathbf{b}_2\| < 2\epsilon$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. Since $\epsilon > 0$ is arbitrary and $\|\mathbf{b}_1 - \mathbf{b}_2\|$ is independent of $\mathbf{x}$, we must have $\|\mathbf{b}_1 - \mathbf{b}_2\| = 0$, so that $\mathbf{b}_1 = \mathbf{b}_2$. ◯

PROOF OF THEOREM 1B. We use Theorem 1G to enable us to use the ϵ-δ approach.

(a) The result is trivial if $c = 0$, so we shall assume that $c \neq 0$. Given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|f(\mathbf{x}) - \mathbf{b}\| < \frac{\epsilon}{c} \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta.$$

It follows that

$$\|(cf)(\mathbf{x}) - c\mathbf{b}\| = c\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$.

(b) Given any $\epsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\|f(\mathbf{x}) - \mathbf{b}_1\| < \frac{\epsilon}{2} \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1,$$

and

$$\|g(\mathbf{x}) - \mathbf{b}_2\| < \frac{\epsilon}{2} \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then by the Triangle inequality,

$$\|(f+g)(\mathbf{x}) - (\mathbf{b}_1 - \mathbf{b}_2)\| \le \|f(\mathbf{x}) - \mathbf{b}_1\| + \|g(\mathbf{x}) - \mathbf{b}_2\| < \epsilon$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$.

(c) Suppose first of all that $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$. Given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|f(\mathbf{x}) - \mathbf{b}\| < \epsilon \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta.$$

For every $i = 1, \ldots, m$, it is clear that

$$|f_i(\mathbf{x}) - b_i| \le \|f(\mathbf{x}) - \mathbf{b}\|,$$

so it follows easily that $|f_i(\mathbf{x}) - b_i| < \epsilon$ for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. Hence $f_i(\mathbf{x}) \to b_i$ as $\mathbf{x} \to \mathbf{x}_0$ for every $i = 1, \ldots, m$. Suppose now that $f_i(\mathbf{x}) \to b_i$ as $\mathbf{x} \to \mathbf{x}_0$ for every $i = 1, \ldots, m$. Then given any $\epsilon > 0$ and any $i = 1, \ldots, m$, there exists $\delta_i > 0$ such that

$$|f_i(\mathbf{x}) - b_i| < \frac{\epsilon}{m} \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_i.$$

Let $\delta = \min\{\delta_1, \ldots, \delta_m\}$. Then

$$\|f(\mathbf{x}) - \mathbf{b}\| \le \sum_{i=1}^{m} |f_i(\mathbf{x}) - b_i| < \epsilon$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. Hence $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$. ◯

Proof of Theorem 1C. We again use Theorem 1G to enable us to use the ϵ-δ approach.

(a) We use the inequality

$$|f(\mathbf{x})g(\mathbf{x}) - b_1b_2| = |f(\mathbf{x})g(\mathbf{x}) - f(\mathbf{x})b_2 + f(\mathbf{x})b_2 - b_1b_2| \le |f(\mathbf{x})||g(\mathbf{x}) - b_2| + |b_2||f(\mathbf{x}) - b_1|.$$

Since $f(\mathbf{x}) \to b_1$ as $\mathbf{x} \to \mathbf{x}_0$, it follows that (with $\epsilon = 1$) there exists $\delta_1 > 0$ such that

$$|f(\mathbf{x}) - b_1| < 1 \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1.$$

It follows from the Triangle inequality that

$$|f(\mathbf{x})| \le |f(\mathbf{x}) - b_1| + |b_1| < 1 + |b_1| \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1.$$

Let $\epsilon > 0$ be given. Since $f(\mathbf{x}) \to \mathbf{b}_1$ as $\mathbf{x} \to \mathbf{x}_0$, it follows that there exists $\delta_2 > 0$ such that

$$|f(\mathbf{x}) - b_1| < \frac{\epsilon}{2(1 + |b_2|)} \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_2.$$

Since $g(\mathbf{x}) \to b_2$ as $\mathbf{x} \to \mathbf{x}_0$, it follows that there exists $\delta_3 > 0$ such that

$$|g(\mathbf{x}) - b_2| < \frac{\epsilon}{2(1 + |b_1|)} \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_3.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then it is easily seen that $|f(\mathbf{x})g(\mathbf{x}) - b_1b_2| < \epsilon$ for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$.

(b) We use the identity

$$\left| \frac{1}{f(\mathbf{x})} - \frac{1}{b} \right| = \frac{|f(\mathbf{x}) - b|}{|f(\mathbf{x})||b|}.$$

Since $b \ne 0$ and $f(\mathbf{x}) \to b$ as $\mathbf{x} \to \mathbf{x}_0$, it follows that (with $\epsilon = \frac{1}{2}|b|$) there exists $\delta_1 > 0$ such that

$$|f(\mathbf{x}) - b| < \frac{1}{2}|b| \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1.$$

It follows from the Triangle inequality that

$$|f(\mathbf{x})| \ge +|b| - |f(\mathbf{x}) - b| > \frac{1}{2}|b| \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1.$$

Let $\epsilon > 0$ be given. Since $f(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{x}_0$, it follows that there exists $\delta_2 > 0$ such that

$$|f(\mathbf{x}) - b| < \frac{1}{2}|b|^2\epsilon \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then it is easily seen that

$$\left|\frac{1}{f(\mathbf{x})} - \frac{1}{b}\right| < \epsilon$$

for every $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. ◯

PROOF OF THEOREM 1F. Since g is continuous at $\mathbf{y}_0 = f(\mathbf{x}_0)$, it follows that given any $\epsilon > 0$, there exists $\eta > 0$ such that

$$\|g(\mathbf{y}) - g(f(\mathbf{x}_0))\| = \|g(\mathbf{y}) - g(\mathbf{y}_0)\| < \epsilon \qquad \text{for every } \mathbf{y} \in B \text{ satisfying } \|\mathbf{y} - f(\mathbf{x}_0)\| < \eta.$$

Since f is also continuous at $\mathbf{x}_0$, it follows that given any $\eta > 0$, there exists $\delta > 0$ such that

$$\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \eta \qquad \text{for every } \mathbf{x} \in A \text{ satisfying } \|\mathbf{x} - \mathbf{x}_0\| < \delta.$$

Suppose now that $\mathbf{x} \in A$ satisfies $\|\mathbf{x} - \mathbf{x}_0\| < \delta$. Then it follows that $f(\mathbf{x}) \in f(A) \subseteq B$ and satisfies $\|f(\mathbf{x}) - \mathbf{y}_0\| < \eta$, so that

$$\|(g \circ f)(\mathbf{x}) - (g \circ f)(\mathbf{x}_0)\| = \|g(f(\mathbf{x})) - g(f(\mathbf{x}_0))\| < \epsilon$$

as required. ◯

PROBLEMS FOR CHAPTER 1

1. Draw each of the following sets in $\mathbb{R}^2$, and determine heuristically whether it is open:
   a) $A = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$
   b) $A = \{(x, y) \in \mathbb{R}^2 : 0 \leq x^2 + y^2 < 4\}$
   c) $A = \{(x, y) = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 : 0 \leq \theta < \pi/4 \text{ and } \theta^2 < r < \theta\}$

2. Suppose that G and H are both neighbourhoods of a point $\mathbf{x}_0 \in \mathbb{R}^n$. Show that both $G \cap H$ and $G \cup H$ are neighbourhoods of $\mathbf{x}_0$.
   [REMARK: You may assume that both $G \cap H$ and $G \cup H$ are open. Alternatively, you may take it as a challenge to prove these two assertions first.]

3. Use the arithmetic of limits to evaluate

$$\lim_{(x,y,z)\to(0,0,0)} \left(\frac{xy}{x^2 + y^2 + 2}, \frac{x^2 + 3y^2 + 4yz}{x + y + 1}\right).$$

4. Use the ϵ-δ definition of a limit to show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2 + y^2}} = 0.$$

5. For each of the following two functions, investigate the behaviour on each of the two lines given, and explain why the limit does not exist as $(x,y) \to (0,0)$:

   a) $f(x,y) = \dfrac{x^2}{x^2+y^2}$; lines $x=0$ and $y=0$

   b) $f(x,y) = \dfrac{(x-y)^2}{x^2+y^2}$; lines $y=x$ and $y=-x$

6. Consider the function $h : \mathbb{R}^3 \to \mathbb{R}$, given by

$$h(x,y,z) = \begin{cases} \dfrac{1-\cos(x+y+z)}{(x+y+z)^2} & \text{if } x+y+z \neq 0, \\ \dfrac{1}{2} & \text{if } x+y+z = 0. \end{cases}$$

   By writing down suitable functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ which are continuous at $(0,0,0)$ and $f(0,0,0)$ respectively, use Theorem 1F to show that

$$\lim_{(x,y,z)\to(0,0,0)} h(x,y,z) = \frac{1}{2}.$$

7. Use Theorem 1F to show that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1.$$

8. Consider the open disc $D(\mathbf{0},1) \in \mathbb{R}^2$, centred at the origin $\mathbf{0} = (0,0)$ and of radius 1. Suppose that $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in D(\mathbf{0},1)$. Show that $f(\mathbf{x}) \to 0$ as $\mathbf{x} \to (1,0)$.
   [Remark: Note that $(1,0) \notin D(\mathbf{0},1)$.]

9. Find a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(0,0) = 1$, $f(1,0) = 0$, and $0 \le f(x,y) \le 1$ for every $(x,y) \in \mathbb{R}^2$.