1 Basic algebra and arithmetic

1.1 Elementary arithmetic

The first mathematical skill that we learn as children is arithmetic: starting with counting on our fingers, we soon progress to adding, taking away, multiplying, and dividing. All this seems quite straightforward until we encounter an expression like $2+6\times 3$; does this mean that we add two to six and multiply the sum by three (to get 24), or should we multiply six by three and then add two (giving 20)? To avoid such ambiguities, an order ranking the priorities of the various basic arithmetical operations has been agreed; this convention can be remembered by the acronym BODMAS.

According to BODMAS, things occurring inside a bracket must be evaluated first; the next most important operation is 'of', as in half-of-six, and is rarely used; then it's division, multiplication, addition, and subtraction, in that order. Thus our simple example above can be made explicit as follows

Brackets
Of
Divide
Multiply
Add
Subtract

$$2+6\times 3 = 2+(6\times 3) \neq (2+6)\times 3$$

While brackets are not necessary if the expression on the left is meant to stand for the one in the middle, they are essential for the one on the right.

Before moving on, we should also remind ourselves about the rule for multiplying out brackets; that is

$$(a+b)(c+d) = ac+ad+bc+bd \tag{1.1}$$

where a, b, c, and d could be any arbitrary numbers (and a small space between quantities is equivalent to a multiplication, as in $ac = a\times c$).

1.2 Powers, roots and logarithms

If a number, say a, is multiplied by itself, then we can write it as a^2 and call it a-squared. A triple product can be written as a^3 and is called a-cubed. In general, an N-times self-product is denoted by a^N where the superscript, or *index*, N is referred to as the *power* of a.

$$\begin{aligned} a^1 &= a \\ a^2 &= a\times a \\ a^3 &= a\times a\times a \\ a^4 &= a\times a\times a\times a \end{aligned}$$

Although the meaning of 'a to the power of N' is obvious when N is a positive integer (i.e. $1,2,3,\ldots$), what happens when it's zero or negative? This question is easily answered once we notice that the procedure for going from a power of N to $N-1$ involves a division by a. Thus if a^0 is a^1 divided by a, then a to the power of nought must be one (or *unity*); similarly, if a^{-1} is a^0 divided by a, then it must be equal to one over a; and, in general, a^{-N} is equivalent to the *reciprocal* of a^N. Thus, we have

$$a^0 = 1 \qquad \text{and} \qquad a^{-N} = \frac{1}{a^N} \tag{1.2}$$

The basic definition of powers leads immediately to the formula for adding the indices M and N when the numbers a^M and a^N are multiplied

$$a^M a^N = a^{M+N} \tag{1.3}$$

While our discussion has so far focused only on the case of integer powers, suppose that we were to legislate that eqn (1.3) held for all values of M and N. Then, we would be led to the interpretation of fractional powers as *roots*. To see this, consider the case when $M = N = 1/2$; it follows from eqn (1.3) that a to the power of a half must be equal to the square root of a. Extending the argument slightly, if a to the power of a third is multiplied by itself three times then we obtain a; therefore, $a^{1/3}$ must be equal to the cube root of a. In general, the p^{th} root of a is given by

$$a^{1/2}a^{1/2} = a$$

$$a^{1/3}a^{1/3}a^{1/3} = a$$

$$a^{1/p} = \sqrt[p]{a} \tag{1.4}$$

where p is an integer. One final result on powers that we should mention is

$$(a^M)^N = a^{MN} \tag{1.5}$$

which can at least be verified readily for integer values of M and N. More complicated powers can be decomposed into a series of simpler manipulations by using the rules of eqns (1.2) – (1.5). For example

$$9^{-5/2} = \frac{1}{9^{5/2}} = \frac{1}{9^{2+1/2}} = \frac{1}{9^2\,9^{1/2}} = \frac{1}{81\sqrt{9}} = \frac{1}{243}$$

An alternative method of describing a number as a 'power of something' is to use *logarithms*. That is to say, if y is written as a to the power of x then x is the logarithm of y to the *base a*

$$y = a^x \qquad \Longleftrightarrow \qquad x = \log_a(y) \tag{1.6}$$

where the double-headed arrow indicates an equivalence, so that the expression on the left implies the one on the right and vice versa. Since we talk in powers of ten in everyday conversations (e.g. hundreds, thousands, millions), the use of $a = 10$ is most common; this gives rise to the name *common* logarithm for $\log_{10}$, often abbreviated to just log (but this can be ambiguous). Other bases that are encountered frequently are 2 and 'e' (2.718 to 3 decimal places); we will meet the latter in more detail in later chapters, but state here that $\log_e$, or ln, is called the *natural* logarithm.

$$\log_{10}(10^0) = 0$$
$$\log_{10}(10^1) = 1$$
$$\log_{10}(10^2) = 2$$

$$\ln(e^0) = 0$$
$$\ln(e^1) = 1$$
$$\ln(e^2) = 2$$

By combining the definition of the logarithm in eqn (1.6) with the rule of eqn (1.3), it can be shown that the 'log of a product is equal to the sum of the logs' (to any base); and, in conjunction with eqn (1.2), that the 'log of a quotient is equal to the difference of the logs'

$$\log(AB) = \log(A) + \log(B) \qquad \text{and} \qquad \log(A/B) = \log(A) - \log(B) \tag{1.7}$$

Similarly, eqn (1.6) allows us to rewrite eqn (1.5) in terms of the log of a power and to derive a formula for changing the base of a log (from a to b)

$$\log(A^\beta) = \beta\log(A) \qquad \text{and} \qquad \log_b(A) = \log_a(A) \times \log_b(a) \tag{1.8}$$

Thus $\ln(x) = 2.3026\,\log_{10}(x)$, where the numerical prefactor is $\ln(10)$ to four decimal places, and so on.

1.3 Quadratic equations

The simplest type of equation involving an 'unknown' variable, say x, is one that takes the form $ax + b = 0$, where a and b are (known) constants. Such *linear* equations can easily be rearranged according to the rules of elementary algebra, 'whatever you do to one side of the equation, you must do exactly the same to the other', to yield the solution $x = -b/a$.

A slightly more complicated situation, which is met frequently, is that of a *quadratic* equation; this takes the general form

$$ax^2 + bx + c = 0 \tag{1.9}$$

The crucial difference between this and the linear case is the occurrence of the x^2 term, which makes it far less straightforward to work out which values of x satisfy the equation. If we could rewrite eqn (1.9), albeit divided by a, as

$$(x - x_1)(x - x_2) = 0$$

$$x_1 x_2 = c/a$$

$$x_1 + x_2 = -b/a$$

where x_1 and x_2 are constants, then the solutions are obvious: either $x = x_1$ or $x = x_2$, because the product of two numbers can only be zero if either one or the other (or both) is nought. While such a *factorization* may not be easy to spot, it's not too difficult to rearrange eqn (1.9) into the form

$$(x + \alpha)^2 - \beta = 0$$

$$\alpha = \frac{b}{2a}, \quad \beta = \frac{b^2}{4a^2} - \frac{c}{a}$$

a procedure called 'completing the square', where α and β can be expressed in terms of the constants a, b, and c (but don't involve x). This leads to the following general formula for the two solutions of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.10}$$

and is equivalent to $x = -\alpha + \sqrt{\beta}$ and $x = -\alpha - \sqrt{\beta}$. Since the square of any number, positive or negative, is always greater than or equal to zero, we require that $b^2 \geq 4ac$ for eqn (1.10) to yield 'real' values of x.

1.4 Simultaneous equations

There are many situations in which an equation will contain more than one variable; taking the case of just two, say x and y, a simple example would be $x + y = 3$. On its own, this relationship does not determine the values of x and y uniquely. Indeed, there are an infinite number of solutions which satisfy the equation: $x = 0$ and $y = 3$, or $x = 1$ and $y = 2$, or $x = p$ and $y = 3 - p$, to name but a few. To pin down x and y to a single possibility, we need one more equation to constrain them; e.g. $x - y = 1$. The two conditions can only be satisfied at the same time if $x = 2$ and $y = 1$, and are an example of solving *simultaneous* equations.

The easiest sort of simultaneous equations are linear ones; that is, where the variables only appear as separate entities to their first power (perhaps multiplied by a constant) all added together. The simplest of these is a two-by-two system, like the one above, whose general from can be written as

$$ax+by=\alpha$$
$$cx+dy=\beta$$

This can be solved by substituting for x, or y, from one of the equations into the other. For example, the first equation gives $y=(\alpha-ax)/b$; putting this in for y into the second equation yields a linear relationship for x; thus we readily obtain x, and hence y. An extension of this procedure allows us to determine the values of three variables (x, y, and z) uniquely when given three linear simultaneous equations, and so on. The only proviso is that all the equations must be genuinely distinct; that is, we cannot repeat the same one twice, or generate a third by combining any two (or more).

$$x=\frac{\alpha d-\beta b}{ad-bc}$$

$$y=\frac{\beta a-\alpha c}{ad-bc}$$

We should also note that a unique solution is only guaranteed if all the simultaneous equations are linear. If the second one in our example at the start of this section had been $x^2-y=3$, then the substitution of y from $x+y=3$ would give the quadratic equation $x^2+x-6=0$. Factorization of this as $(x+3)(x-2)=0$ tells us that either $x=2$ or $x=-3$, and hence that $y=1$ or $y=6$ respectively.

1.5 The binomial expansion

It is easily shown, by putting $c=a$ and $d=b$ in eqn (1.1), that the square of the sum of two numbers can be written as

$$(a+b)^2=a^2+2ab+b^2$$

Multiplying this by the sum again yields the following for the cube

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

When this procedure is repeated many times, a systematic pattern emerges for the N^{th} power of $a+b$; it is called the *binomial* expansion

$$(a+b)^N=\sum_{r=0}^{N}{}^N C_r\,a^{N-r}\,b^r \tag{1.11}$$

The capital Greek symbol Σ stands for a 'sum from $r=0$ to $r=N$', through the integers $r=1,2,3,\cdots,N-1$. The binomial coefficients ${}^N C_r$, which are often denoted by an alternative long bracket notation, are defined by

$$\sum_{r=0}^{N}A_r=A_0+A_1+A_2+\cdots+A_N$$

$${}^N C_r=\binom{N}{r}=\frac{N!}{r!\,(N-r)!} \tag{1.12}$$

where the *factorial* function is given by the product

$$N!=N\times(N-1)\times(N-2)\times\cdots\times3\times2\times1 \tag{1.13}$$

Although not obvious from eqn (1.13), we will see later that $0!=1$.

Another way of evaluating the coefficients in a binomial expansion, rather than using eqn (1.12), is provided by *Pascal's triangle*. In this, apart from the ones down the edges, each number is generated by adding the two closest neighbours in the line above. The third row of Pascal's triangle gives the coefficients for the square, the fourth row corresponds to the cube, and so on.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

1.6 Arithmetic and geometric progressions

Sometimes we need to work out the sum of a sequence of numbers that are generated according to a certain algebraic rule. The simplest example is that of an *arithmetic progression*, or AP, where each term is given by the previous one plus a constant

$$a+(a+d)+(a+2d)+(a+3d)+\cdots+(l-d)+l$$

If there are N terms, then the last one, l, is related to the first, a, through the *common difference*, d, by $l=a+(N-1)d$. The addition of the above series with a copy of itself written in reverse order shows that twice the sum which we seek is equal to N times $a+l$; hence, the formula for the sum of an AP is

$$1+2+3+\cdots+N=\frac{N(N+1)}{2}$$

$$\sum_{j=1}^{N} a+(j-1)d=\frac{N}{2}[2a+(N-1)d] \tag{1.14}$$

Another case which is met frequently is that of a *geometric progression*, or GP, where each term is given by the previous one times a constant

$$a+ar+ar^2+ar^3+\cdots+ar^{N-2}+ar^{N-1}$$

If we subtract from the above series a copy of itself that has been multiplied by the *common ratio* r, then it can be shown that $1-r$ times the sum that we seek is equal to $a-ar^N$; hence, the formula for the sum of a GP is

$$\sum_{j=1}^{N} ar^{j-1}=\frac{a(1-r^N)}{1-r} \tag{1.15}$$

If the *modulus* of the common ratio is less than unity, so that $-1<r<1$, then the sum of the GP does not 'blow up' as the number of terms becomes infinite. Indeed, since r^N becomes negligibly small as N tends to infinity, eqn (1.15) simplifies to

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots=\frac{2}{3}$$

$$\sum_{j=1}^{\infty} ar^{j-1}=\frac{a}{1-r} \quad \text{for} \quad |r|<1 \tag{1.16}$$

1.7 Partial fractions

For the final topic in this chapter, we turn to the subject of *partial fractions*. These are best explained by the use of specific examples, such as

$$\frac{5x+7}{(x-1)(x+3)}=\frac{3}{(x-1)}+\frac{2}{(x+3)}$$

The act of combining two, or more, fractions into a single entity is familiar to us, through 'finding the lowest common denominator', and is usually referred to as 'simplifying the equation'. Here we are interested in the reverse procedure of decomposing a fraction into the sum of several constituent parts. Although this may sound like a backwards step, it sometimes turns out to be a very useful manipulation.

$$\frac{1}{2}-\frac{1}{3}=\frac{3}{6}-\frac{2}{6}=\frac{1}{6}$$

The situations in which partial fractions arise involve the ratios of two *polynomials* (the sums of positive integer powers of a variable, like x), where the denominator can be written as the product of simpler components. The first thing we must check is that the *degree* (or highest power of x) of the numerator is less than that of the denominator, and 'divide out' if it is not; this is analogous to expressing a top-heavy fraction as whole number plus a proper fraction (e.g. $7/3 = 2 + 1/3$). If the numerator in our earlier example had been $2x^2 + 9x + 1$, then we would divide it by the denominator $x^2 + 2x - 3$ (when multiplied out), by 'long division' if necessary, to obtain

$$\begin{array}{r} 2 \\ x^2+2x-3\,\overline{)\,2x^2+9x+1} \\ \underline{2x^2+4x-6} \\ 5x+7 \end{array}$$

$$\frac{2x^2+9x+1}{(x-1)(x+3)} = 2 + \frac{5x+7}{(x-1)(x+3)}$$

The second part on the right can then be split up into partial fractions, as indicated at the start of this section; let's consider the mechanistic details of how this is done.

Our example constitutes the simplest case, where the denominator is a product of linear terms. This can always be expressed as

$$\frac{5x+7}{(x-1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+3)}$$

where A and B are constants. If there had been a third term on the bottom, like $2x+3$, then we would get an additional fraction on the right, $C/(2x+3)$. To evaluate A and B (and C etc.), we can rearrange the right-hand side to have the same common denominator as on the left, and then equate the numerators

$$5x+7 = A\,(x+3) + B\,(x-1)$$

For this to be satisfied, the coefficients of the various powers of x on both sides must be identical; this leads to a set of simultaneous equations for the desired constants. An alternative procedure is to substitute specific values of x into the equation, especially those that make the factors in the denominator equal to zero; thus putting $x = 1$ gives $4A = 12$, and setting $x = -3$ gives $4B = 8$. The great ease of the second method leads to a short cut way of implementing it known as the 'cover-up' rule: to evaluate A, cover up $x-1$ in the original expression and substitute $x-1=0$ in what remains; for B, cover up $x+3$ and put $x+3=0$ in the rest.

$$\begin{aligned} A+B &= 5 \\ 3A-B &= 7 \end{aligned}$$

If there had been a quadratic factor in the denominator of our example, then its partial component would require a linear term in its numerator

$$\frac{5x+7}{(x-1)(x^2+4x+3)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+4x+3)}$$

Apart from this generalisation, that an N^{th} degree factor on the bottom needs an $(N-1)^{\text{th}}$ degree polynomial on the top, the procedure for calculating the associated constants is the same as before. Rearrange the right-hand side to have the same common denominator as the left, and then equate numerators

$$5x+7 = A(x^2+4x+3) + (Bx+C)(x-1)$$

In this case, A is evaluated most easily by substituting $x=1$, giving $8A = 12$, and is equivalent to using the cover-up rule. B and C can then be ascertained readily by equating the coefficients of the x^2 and x^0 terms on both sides of the equation (and confirmed by those of x^1).

$$\begin{aligned} A+B &= 0 \\ 4A-B+C &= 5 \\ 3A-C &= 7 \end{aligned}$$

The final circumstance of note concerns the occurrence of repeated factors in the denominator, such as $(x+3)^2$. We could expand this as x^2+6x+9 and use the procedure for dealing with a quadratic discussed above, but a more useful decomposition tends to be

$$\frac{5x+7}{(x-1)(x+3)^2} = \frac{A}{(x-1)} + \frac{B}{(x+3)} + \frac{C}{(x+3)^2}$$

The related constants can be evaluated in the usual manner, by equating the numerators after the denominators have been made equal

$$5x+7 = (x+3)[A(x+3)+B(x-1)] + C(x-1)$$

A and C are given immediately by putting $x = 1$ and $x = -3$ or, equivalently, by the cover-up rule; B then follows for the coefficients of x^2, or x^1, or x^0.

Exercises

1.1 Evaluate (i) $1+2\times 6-3$, (ii) $3-1/(2+4)$, (iii) $2^3-2\times 3$.

1.2 Simplify eqn (1.1) when (i) $b=0$, (ii) $a=c$ and $b=d$, (iii) $a=c$ and $b=-d$.

1.3 Evaluate (i) $4^{3/2}$, (ii) $27^{-2/3}$, (iii) $3^2\,3^{-3/2}$, (iv) $\log_2(8)$, (v) $\log_2(8^3)$.

1.4 By using the definition of a logarithm in eqn (1.6), and substituting $A=a^M$ and $B=a^N$, show that taking $\log_a$ of both sides of eqn (1.3) leads to eqn (1.7); similarly, show that eqn (1.5) yields eqn (1.8).

1.5 By dividing eqn (1.9) by a and completing the square, derive eqn (1.10).

1.6 Solve (i) $x^2-5x+6=0$, (ii) $3x^2+5x-2=0$, (iii) $x^2-4x+2=0$.

1.7 For what values of k does $x^2+kx+4=0$ have real roots?

1.8 Solve the following simultaneous equations:

$$\text{(i)}\ \begin{matrix} 3x+2y=4 \\ x-7y=9 \end{matrix}, \quad \text{(ii)}\ \begin{matrix} x^2+y^2=2 \\ x-2y=1 \end{matrix}, \quad \text{(iii)}\ \begin{matrix} 3x+2y+5z=0 \\ x+4y-2z=9 \\ 4x-6y+3z=3 \end{matrix}$$

1.9 Expand $(2+x)^5$, $(1+x)^9$, and $(x+2/x)^6$ in powers of x.

1.10 Derive the formulae for the sums of an AP and GP.

1.11 By expressing a recurring decimal number as the sum of an infinite GP, show that $0.12121212\cdots = 4/33$. What is $0.318181818\cdots$ as a fraction?

$3.33\cdots = 3+0.3+0.03+\cdots$

1.12 Decompose the following into partial fractions:

$$\text{(i)}\ \frac{1}{x^2-5x+6}, \quad \text{(ii)}\ \frac{x^2-5x+1}{(x-1)^2(2x-3)}, \quad \text{(iii)}\ \frac{11x+1}{(x-1)(x^2-3x-2)}$$

1.13 Evaluate:

$$\text{(i)}\ \sum_{n=1}^{\infty}\frac{1}{n(n+1)}, \quad \text{(ii)}\ \sum_{n=0}^{\infty} e^{-\beta(n+1/2)}$$