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Q1 - a)

$$P(x \mid a) = a(1-x)^{a-1}$$

① Likelihood $(L(a|x)) = \prod_{i=1}^{n} a(1-x_i)^{a-1}$

② Take the ln of Likelihood: $\ln(L(a|x)) = \sum_{i=1}^{n} \left(\ln a + (a-1)\ln(1-x_i) \right)$

③ Take the derivative of $\ln(L(a|x))$: $\sum_{i=1}^{n} \left(\frac{d}{da} \ln a + \frac{d}{da}(a-1)\ln(1-x_i) \right)$

$$= \frac{n}{a} + \sum_{i=1}^{n} \ln(1-x_i)$$

④ Set this value to zero to find $\hat{a}_{MLE}$: $\frac{n}{a} + \sum_{i=1}^{n} \ln(1-x_i) = 0$,

$$a = \hat{a}_{MLE} = \frac{-n}{\sum_{i=1}^{n} \ln(1-x_i)}$$

Q1 - b)

$$P(a \mid x) = \frac{P(x|a)P(a)}{P(x)}, \quad \hat{a}_{MAP} = \underset{a}{\operatorname{argmax}}\, P(a|x):$$

① $\underbrace{a(1-x)^{a-1}}_{P(x|a)} \cdot \underbrace{\lambda a^{\lambda-1} e^{-\lambda a}}_{P(a)} = \lambda a^{\lambda}(1-x)^{a-1} e^{-\lambda a} = P(x|a)\,P(a)$

② Take the ln of it: $\ln\lambda + \lambda \ln a + (a-1)\ln(1-x) - \lambda a$

③ Take the derivative: $\frac{d}{da}\left(\ln\lambda + \lambda\ln a + (a-1)\ln(1-x) - \lambda a\right) = \frac{\lambda}{a} + \ln(1-x) - \lambda$

④ Set this value to zero to find $\hat{a}_{MAP}$: $\frac{\lambda}{a} + \ln(1-x) - \lambda = 0$,

$$\hat{a}_{MAP} = a = \frac{-\lambda}{\ln(1-x) - \lambda}$$