KISMİ TÜREVLER

Tanım: $A \subset \mathbb{R}^2$, $f: A \to \mathbb{R}$ bir fonksiyon ve $(a,b) \in A$ olsun. Eğer

$$\lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

limiti var ise, bu limite f nin x değişkenine göre (a,b) noktasındaki kısmi türevi denir ve $\frac{\partial f}{\partial x}(a,b)$, $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$, $f_x(a,b)$ sembollerinden biri ile gösterilir.

$$\lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

limiti var ise, bu limite f nin y değişkenine göre (a,b) noktasındaki kısmi türevi denir ve $\frac{\partial f}{\partial y}(a,b)$, $\left.\frac{\partial f}{\partial y}\right|_{(a,b)}$, $f_y(a,b)$ sembollerinden biri ile gösterilir.

Örnek: $f(x,y) = x^2 - 3xy + y^2$ ise, $f_x(1,2)$ ve $f_y(1,2)$ türevlerini bulunuz.

Çözüm: I. Yol:

$$f_x(1,2) = \lim_{h \to 0} \frac{f(1+h,2) - f(1,2)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^2 - 3(1+h).2 + 4 - (1 - 3.2 + 4)}{h}$$

$$= \lim_{h\to 0} \frac{h^2+2h+1-6-6h+4+1}{h}$$

$$= \lim_{h\to 0} \frac{h^2-4h}{h}$$

$$= \lim_{h\to 0} (h-4) = -4$$

<u>II. Yol:</u> $f_x(x,y) = 2x-3y$

$$f_x(1,2) = 2.1 - 3.2 = -4$$

$$f_y(1,2) = \lim_{k\to 0} \frac{f(1,2+k)-f(1,2)}{k}$$

$$= \lim_{k\to 0} \frac{1-3(2+k)+(2+k)^2-(-1)}{k}$$

$$= \lim_{k\to 0} \frac{k^2+k}{k}$$

$$= \lim_{k\to 0} (k+1)$$

$$= 1$$

veya $f_y(x,y) = -3x+2y \Rightarrow f_y(1,2) = 1$ olur.

Örnek: $f(x,y) = \begin{cases} \dfrac{x^2y^3}{x^2+4y^3} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$ ise,

a) $f_x(0,0)$ ve $f_y(0,0)$ türevlerini bulunuz.

b) $f_x(1,1)$ ve $f_y(1,1)$ türevlerini bulunuz.

Çözüm: a) $$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{0}{h^2}-0}{h}$$

$$=\lim_{h\to 0} 0 \quad = 0$$

$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}$$

$$=\lim_{k\to 0}\frac{\frac{0}{4k^3}-0}{k}=\lim_{k\to 0} 0 = 0$$

b) $(x,y)\neq(0,0)$ için

$$f_x(x,y)=\frac{2xy^3(x^2+4y^3)-2x(x^2y^3)}{(x^2+4y^3)^2}=\frac{8xy^6}{(x^2+4y^3)^2}$$

$\Rightarrow f_x(1,1)=\frac{8}{25}$ olur.

$(x,y)\neq(0,0)$ için

$$f_y(x,y)=\frac{3x^2y^2(x^2+4y^3)-12y^2(x^2y^3)}{(x^2+4y^3)^2}=\frac{3x^4y^2}{(x^2+4y^3)^2}$$

$\Rightarrow f_y(1,1)=\frac{3}{25}$ olur.

Örnek: $$f(x,y)=\begin{cases}\frac{(x-1)^2y}{2x-y}, & (x,y)\neq(1,2)\\ 0, & (x,y)=(1,2)\end{cases}$$ ise

$f_x(1,2)$ ve $f_y(1,2)$ türevlerini bulunuz.

Çözüm: $f_x(1,2) = \lim_{h\to 0} \dfrac{f(1+h,2) - f(1,2)}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{(1+h-1)^2 \cdot 2}{2(1+h)-2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{2h^2}{2h}}{h} = 1$$

$$f_y(1,2) = \lim_{k\to 0} \frac{f(1,2+k) - f(1,2)}{k}$$

$$= \lim_{k\to 0} \frac{\dfrac{0.(2+k)}{2-(2+k)} - 0}{k} = \lim_{k\to 0} 0 = 0$$

Örnek: $f(x,y) = \arctan\left(\frac{y}{x}\right)$ fonksiyonunun $(x,y) \neq (0,0)$ için $f_x(x,y)$ ve $f_y(x,y)$ türevlerini bulunuz.

Çözüm:

$$f_x(x,y) = \frac{\left(\frac{y}{x}\right)'_x}{1+\left(\frac{y}{x}\right)^2} = \frac{-\dfrac{y}{x^2}}{\dfrac{x^2+y^2}{x^2}} = \frac{-y}{x^2+y^2}$$

$$f_y(x,y) = \frac{\left(\frac{y}{x}\right)'_y}{1+\left(\frac{y}{x}\right)^2} = \frac{\dfrac{1}{x}}{\dfrac{x^2+y^2}{x^2}} = \frac{x}{x^2+y^2}$$

Örnek: $f(x,y,z) = \ln\left(\frac{xy}{z}\right)$ fonksiyonunun f_x, f_y, f_z türevlerini bulunuz.

Çözüm:

$$f_x(x,y,z) = \frac{\left(\frac{xy}{z}\right)'_x}{\frac{xy}{z}} = \frac{\frac{y}{z}}{\frac{xy}{z}} = \frac{1}{x}$$

$$f_y(x,y,z) = \frac{\left(\frac{xy}{z}\right)'_y}{\frac{xy}{z}} = \frac{\frac{x}{z}}{\frac{xy}{z}} = \frac{1}{y}$$

$$f_z(x,y,z) = \frac{\left(\frac{xy}{z}\right)'_z}{\frac{xy}{z}} = \frac{-\frac{xy}{z^2}}{\frac{xy}{z^2}} = \frac{-1}{z}$$

İkinci Mertebeden Kısmi Türevler

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Özellik: Eğer f_x, f_y, f_{xy}, f_{yx} türevleri (a,b) noktasını içeren bir açık bölgede tanımlı ve (a,b) noktasında sürekli iseler $f_{xy}(a,b) = f_{yx}(a,b)$ olur.

Örnek: $f(x,y)=xe^{y/x}$ fonksiyonu için $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ türevlerini hesaplayınız.

Çözüm: $f_x(x,y)=e^{y/x}-\frac{y}{x^2}e^{y/x}\cdot x=\left(1-\frac{y}{x}\right)e^{y/x}$

$f_y(x,y)=e^{y/x}$

$$f_{xx}(x,y)=\frac{y}{x^2}e^{y/x}-\frac{y}{x^2}e^{y/x}\cdot\left(1-\frac{y}{x}\right)=\frac{y^2}{x^3}e^{y/x}$$

$$f_{xy}(x,y)=-\frac{1}{x}e^{y/x}+\frac{1}{x}e^{y/x}\left(1-\frac{y}{x}\right)=-\frac{y}{x^2}e^{y/x}$$

$$f_{yx}(x,y)=\frac{-y}{x^2}e^{y/x}$$

$$f_{yy}(x,y)=\frac{1}{x}e^{y/x}$$

Örnek: $f(x,y)=\begin{cases}\frac{x^3y-xy^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ fonksiyonu için

$f_{xy}(0,0)\neq f_{yx}(0,0)$ olduğunu gösteriniz.

Çözüm: $f_x(0,0)=\lim\limits_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim\limits_{h\to0}\frac{\frac{0}{h^2}-0}{h}=0$

$$f_y(0,0)=\lim_{k\to0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to0}\frac{\frac{0}{k^2}-0}{k}=0$$

$f_{xy}(0,0)=\lim\limits_{k\to0}\frac{f_x(0,k)-f_x(0,0)}{k}$ hesaplamak için

$f_x(0,k)$ türevini bulmalıyız.

$(x,y)\neq(0,0)$ için

$f_x(x,y)=\frac{(3x^2y-y^3)(x^2+y^2)-2x(x^3y-xy^3)}{(x^2+y^2)^2}$ olduğundan

$f_x(0,k)=\frac{-k^3.k^2-0}{(k^2)^2}=-k$ olur.

O halde

$$f_{xy}(0,0)=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}=\lim_{k\to 0}\frac{-k-0}{k}=-1$$

olur. Şimdi $f_{yx}(0,0)$ değerini bulalım.

$f_{yx}(0,0)=\lim\limits_{h\to 0}\dfrac{f_y(h,0)-f_y(0,0)}{h}$ hesaplamak için

$f_y(h,0)$ türevini bulmalıyız. $(x,y)\neq(0,0)$ için

$$f_y(x,y)=\frac{(x^3-3xy^2)(x^2+y^2)-2y(x^3y-xy^3)}{(x^2+y^2)^2}$$ olduğundan

$$f_y(h,0)=\frac{h^3.h^2-0}{(h^2)^2}=h$$ olur. Böylece

$$f_{yx}(0,0)=\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}=\lim_{h\to 0}\frac{h-0}{h}=1$$

elde edilir. O halde,

$$f_{xy}(0,0)=-1\neq 1=f_{yx}(0,0)$$

olur.

Zincir Kuralı

$z=f(x,y)$ şeklinde tanımlanan $f:B\to\mathbb{R}$ fonksiyonu verilmiş olsun. f, f_x ve f_y fonksiyonları B üzerinde sürekli ve $x=g(u,v)$, $y=h(u,v)$ fonksiyonlarının u ve v değişkenlerine göre kısmi türevleri var ise, $z=f(g(u,v),h(u,v))$

fonksiyonunun da u ve v değişkenlerine göre kısmi türevleri vardır. Bu türevler

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

ve

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

olur.

Eğer $w=f(x,y,z)$ ve $x=g(u,v)$, $y=h(u,v)$, $z=k(u,v)$ ise,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

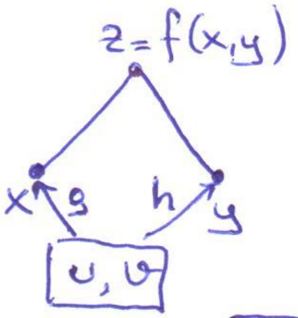
ve

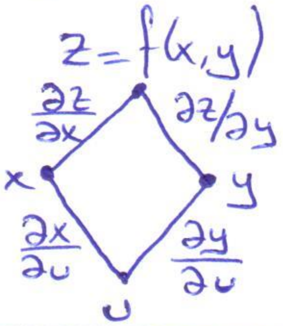
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

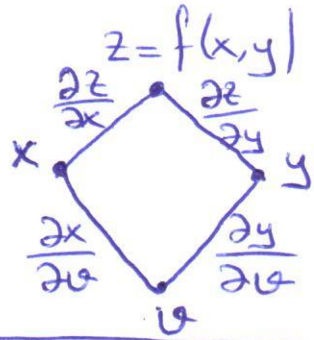
olur.

Kısmi türevleri şematik olarak gösterelim.

$z=f(x,y)$ ve $x=g(u,v)$, $y=h(u,v)$ ise,



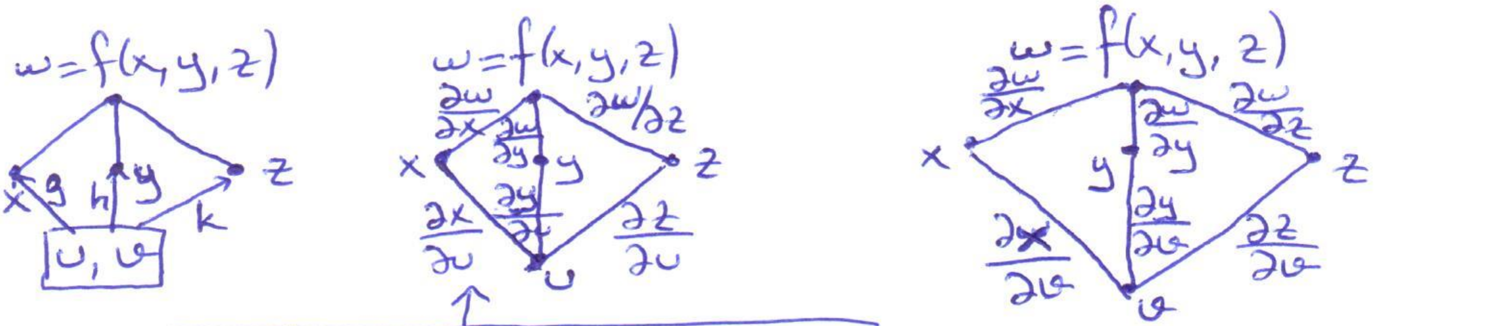




$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}}$$

$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}}$$

$w=f(x,y,z)$ ve $x=g(u,v)$, $y=h(u,v)$, $z=k(u,v)$ ise



$$\boxed{\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}\cdot\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}}$$

$$\boxed{\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}}$$

Örnek: $w=\ln(x^2+y^2)$, $x=e^u\cdot\cos v$, $y=e^u\sin v$ ise, $\frac{\partial w}{\partial u}$ ve $\frac{\partial w}{\partial v}$ türevlerini hesaplayınız.

Çözüm:

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$=\frac{2x}{x^2+y^2}\cdot e^u\cdot\cos v+\frac{2y}{x^2+y^2}\cdot e^u\sin v$$

$$=\frac{2e^u\cos v}{e^{2u}}\cdot e^u\cos v+\frac{2e^u\sin v}{e^{2u}}\cdot e^u\sin v$$

$$=2$$

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$=\frac{2x}{x^2+y^2}\cdot(-e^u\sin v)+\frac{2y}{x^2+y^2}\cdot e^u\cdot\cos v$$

$$=\frac{2e^u\cos v}{e^{2u}}(-e^u\sin v)+\frac{2e^u\sin v}{e^{2u}}\cdot e^u\cdot\cos v$$

$$=0$$

Örnek: $u = \ln(x^2+y^2+z^2)$, $x = \cos t$, $y = \sin t$, $z = t$ ise,

$$\left.\frac{du}{dt}\right|_{t=0} = ?$$

Çözüm: $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$\frac{du}{dt} = \frac{2x}{x^2+y^2+z^2}\cdot(-\sin t) + \frac{2y}{x^2+y^2+z^2}\cdot\cos t + \frac{2z}{x^2+y^2+z^2}\cdot 1$$

$$= \frac{-2\cos t\sin t + 2\sin t\cos t + 2t}{\cos^2 t + \sin^2 t + t^2}$$

$$= \frac{2t}{1+t^2}$$

$$\left.\frac{du}{dt}\right|_{t=0} = \frac{2.0}{1+0^2} = 0$$

Örnek: $z = e^{xy}.\ln^2(x^2+y^2)$ ise $\frac{\partial z}{\partial x} = ?$ $\frac{\partial z}{\partial y} = ?$

Çözüm: $u = e^{xy}$, $v = \ln(x^2+y^2)$ ise $z = u.v^2$ olur.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= v^2.\,y e^{xy} + 2uv\cdot\frac{2x}{x^2+y^2}$$

$$= y e^{xy}.\ln^2(x^2+y^2) + \frac{4x}{x^2+y^2}\cdot e^{xy}.\ln(x^2+y^2)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= v^2.\,x e^{xy} + 2uv\cdot\frac{2y}{x^2+y^2}$$

$$= x e^{xy}.\ln^2(x^2+y^2) + \frac{4y}{x^2+y^2}\cdot e^{xy}.\ln(x^2+y^2)$$

Örnek: $f(x,y)=(x+y)^{xy}$ ise $\frac{\partial f}{\partial x}=?$ $\frac{\partial f}{\partial y}=?$

Çözüm: I. Yol $u=x+y$, $v=xy$ ise $f=u^v$ olur.

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$=v.u^{v-1}.1+u^v.\ln u\cdot y$$
$$=xy\,(x+y)^{xy-1}+(x+y)^{xy}.\ln(x+y).y$$
$$=(x+y)^{xy}\left[\frac{xy}{x+y}+y\cdot\ln(x+y)\right]$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$
$$=v.u^{v-1}.1+u^v.\ln u\cdot x$$
$$=xy\cdot(x+y)^{xy-1}+(x+y)^{xy}.\ln(x+y).x$$
$$=(x+y)^{xy}\left[\frac{xy}{x+y}+x\cdot\ln(x+y)\right]$$

II. Yol: $\ln f(x,y)=xy\ln(x+y)$

$$\frac{f_x}{f}=y\ln(x+y)+\frac{xy}{x+y}$$
$$f_x=(x+y)^{xy}\left[y\ln(x+y)+\frac{xy}{x+y}\right]$$

$\ln f(x,y)=xy\ln(x+y)$

$$\frac{f_y}{f}=x\ln(x+y)+\frac{xy}{x+y}$$
$$f_y=(x+y)^{xy}\left[x\ln(x+y)+\frac{xy}{x+y}\right]$$

Örnek: $\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$ denklemini $u=x$, $v=x^2+y^2$ eşitlikleri ile verilen yeni u, v değişkenlerine göre yazınız.

Çözüm: $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$= \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}\cdot 2x$$
$$= \frac{\partial z}{\partial u} + 2u\frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$
$$= \frac{\partial z}{\partial u}\cdot 0 + \frac{\partial z}{\partial v}\cdot 2y$$
$$= \mp 2\sqrt{v-u^2}\cdot\frac{\partial z}{\partial v}$$

$$v = u^2+y^2 \Rightarrow y^2 = v-u^2$$
$$\Rightarrow y = \mp\sqrt{v-u^2}$$

$$\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0 \Longrightarrow \left(\frac{\partial z}{\partial u} + 2u\frac{\partial z}{\partial v}\right) - u\left(\mp 2\sqrt{v-u^2}\cdot\frac{\partial z}{\partial v}\right) = 0$$
$$\Longrightarrow \frac{\partial z}{\partial v} + 2u\cdot\frac{\partial z}{\partial v}\left(1 \pm \sqrt{v-u^2}\right) = 0$$

Örnek: $\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = 0$ denklemini $x = r\cos\theta$, $y = r\sin\theta$ eşitlikleri ile verilen yeni x, y değişkenlerine göre yazınız.

Çözüm: $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$
$$= \frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}\cdot(r\cos\theta)$$

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left[\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\sin\theta\cdot\cos\theta+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta\right]$$

$$+\frac{1}{r^2}\left[\left(\frac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta-2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}r^2\sin\theta\cos\theta+\left(\frac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta\right]$$

$$=\left(\frac{\partial z}{\partial x}\right)^2(\cos^2\theta+\sin^2\theta)+\left(\frac{\partial z}{\partial y}\right)^2(\cos^2\theta+\sin^2\theta)$$

$$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$$

olduğundan denklemin yeni hali

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=0$$

olur.

Örnek: $z_{xx}-m^2 z_{yy}=0$ denklemini $\begin{cases} u=mx+y \\ v=-mx+y \end{cases}$ eşitlikleriyle verilen yeni u ve v değişkenlerine göre yazınız. Burada z, sürekli kısmi türevlere sahip fonksiyondur.

Çözüm: $z_x=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\overbrace{\dfrac{\partial u}{\partial x}}^{=m}+\dfrac{\partial z}{\partial v}\cdot\overbrace{\dfrac{\partial v}{\partial x}}^{=-m}$

$$=mz_u-mz_v=m(z_u-z_v)$$

$$z_{xx}=\frac{\partial z_x}{\partial x}=\frac{\partial z_x}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z_x}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=m(z_{uu}-z_{vu})\cdot m+m(z_{uv}-z_{vv})\cdot(-m)$$

$$=m^2(z_{uu}-2z_{uv}+z_{vv})$$

$$z_y = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\overbrace{\frac{\partial u}{\partial y}}^{=1} + \frac{\partial z}{\partial v}\cdot\overbrace{\frac{\partial v}{\partial y}}^{=1} = z_u + z_v$$

$$z_{yy} = \frac{\partial z_y}{\partial y} = \frac{\partial z_y}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z_y}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= (z_{uu} + z_{vu})\cdot 1 + (z_{uv} + z_{vv})\cdot 1$$

$$= z_{uu} + 2z_{uv} + z_{vv}$$

$$z_{xx} - m^2 z_{yy} = 0 \Longrightarrow m^2(z_{uu} - 2z_{uv} + z_{vv}) - m^2(z_{uu} + 2z_{uv} + z_{vv}) = 0$$

$$\Longrightarrow -4m^2 z_{uv} = 0$$

$\Longrightarrow m \neq 0$ olduğunda $z_{uv} = 0$ denklemine dönüşür.

Örnek: $z = f(u, v, w)$ türevlenebilir bir fonksiyon ve $u = x - y$, $v = y - z$, $w = z - x$ ise,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = ?$$

Çözüm: $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\overbrace{\frac{\partial u}{\partial x}}^{=1} + \frac{\partial f}{\partial v}\cdot\overbrace{\frac{\partial v}{\partial x}}^{=0} + \frac{\partial f}{\partial w}\cdot\overbrace{\frac{\partial w}{\partial x}}^{=-1}$$

$$= \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\underbrace{\frac{\partial u}{\partial y}}_{=-1} + \frac{\partial f}{\partial v}\cdot\underbrace{\frac{\partial v}{\partial y}}_{=1} + \frac{\partial f}{\partial w}\cdot\underbrace{\frac{\partial w}{\partial y}}_{=0} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\cdot\underbrace{\frac{\partial u}{\partial z}}_{=0} + \frac{\partial f}{\partial v}\cdot\underbrace{\frac{\partial v}{\partial z}}_{=-1} + \frac{\partial f}{\partial w}\cdot\underbrace{\frac{\partial w}{\partial z}}_{=1} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}\right) + \left(-\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) + \left(-\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}\right) = 0$$

Örnek: $x=\frac{1}{t}$ dönüşümü ile

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\frac{a^2}{x^2}y=0$$

denkleminin alacağı yeni şekli bulunuz.

$$\boxed{\begin{aligned} t&=\frac{1}{x}\\ \frac{dt}{dx}&=\frac{-1}{x^2}=-t^2\end{aligned}}$$

Çözüm: $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=-t^2\cdot\frac{dy}{dt}$

$$\frac{d^2y}{dx^2}=\frac{dy'}{dx}=\frac{dy'}{dt}\cdot\frac{dt}{dx}$$

$$=\left(-2t\frac{dy}{dt}-t^2\frac{d^2y}{dt^2}\right)(-t^2)$$

$$=2t^3\frac{dy}{dt}+t^4\frac{d^2y}{dt^2}$$

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\frac{a^2}{x^2}y=0$$

$$\Longrightarrow \frac{1}{t^2}\left(2t^3\frac{dy}{dt}+t^4\frac{d^2y}{dt^2}\right)+\frac{2}{t}\left(-t^2\frac{dy}{dt}\right)+a^2t^2y=0$$

$$\Longrightarrow t^2\left(\frac{d^2y}{dt^2}+a^2y\right)=0$$

$\Longrightarrow t\neq 0$ ise $\frac{d^2y}{dt^2}+a^2y=0$ olur.

Örnek: $z=f(\cos(x-y))$ fonksiyonu için $z_{xx}+2z_{xy}+z_{yy}=?$

Çözüm: $u=\cos(x-y)$

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}=-\sin(x-y)\cdot\frac{\partial f}{\partial u}$$

$$z_{xx}=\frac{\partial z_x}{\partial x}=-\cos(x-y)\cdot\frac{\partial f}{\partial u}+\frac{\partial f_u}{\partial x}\cdot(-\sin(x-y))$$

$$=-\cos(x-y)\cdot f_u-\sin(x-y)\cdot\frac{df_u}{du}\cdot\underbrace{\frac{\partial u}{\partial x}}_{=-\sin(x-y)}$$

$$=-\cos(x-y)\cdot f_u+\sin^2(x-y)\cdot f_{uu}$$

$$z_{xy} = \frac{\partial z_x}{\partial y} = \cos(x-y)\cdot f_u + \frac{\partial f_u}{\partial y}\cdot\left(-\sin(x-y)\right)$$

$$= \cos(x-y)\cdot f_u - \sin(x-y)\cdot\frac{df_u}{du}\cdot\underbrace{\frac{\partial u}{\partial y}}_{=\sin(x-y)}$$

$$= \cos(x-y)\cdot f_u - \sin^2(x-y)\cdot f_{uu}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} = \sin(x-y)\cdot f_u$$

$$z_{yy} = \frac{\partial z_y}{\partial y} = -\cos(x-y)\cdot f_u + \frac{\partial f_u}{\partial y}\cdot\sin(x-y)$$

$$= -\cos(x-y)\cdot f_u + \sin(x-y)\cdot\frac{df_u}{du}\cdot\underbrace{\frac{\partial u}{\partial y}}_{=\sin(x-y)}$$

$$= -\cos(x-y)\cdot f_u + \sin^2(x-y)\cdot f_{uu}$$

$$z_{xx} + 2z_{xy} + z_{yy} = -\cos(x-y)\cdot f_u + \sin^2(x-y)\cdot f_{uu}$$
$$+ 2\cos(x-y) f_u - 2\sin^2(x-y) f_{uu}$$
$$- \cos(x-y) f_u + \sin^2(x-y) f_{uu}$$
$$= 0$$