# Türev Yardımıyla Limit Problemlerinin Çözümü:

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, $0.\infty$, $0^0$, $\infty^0$ ve $1^\infty$ ifadeleri limitte belirsiz ifadelerdir.

## $\frac{0}{0}$ ve $\frac{\infty}{\infty}$ Belirsizlik Hali:

**L' Hospital Kuralı:** f ve g, a da sürekli, a'nın a noktası dışındaki bir komşuluğunda türevli iki fonksiyon ve bu komşuluktaki her x için $g'(x)\neq 0$ olsun.

a) $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$ ise, $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ olur.

b) $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=\infty$ ise, $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ olur

**Uyarı:** L' Hospital kuralı sadece $x\to a$ için değil aynı zamanda $x\to\infty$ ve $x\to-\infty$ içinde geçerlidir.

## Örnekler:

1) $\lim_{x\to 0}\frac{x-\sin x}{x^3}=?$

Çözüm: $\lim_{x\to 0}\frac{x-\sin x}{x^3}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{1-\cos x}{3x^2}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{\sin x}{6x}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$

2) $\lim_{x\to 1}\frac{1+\cos(\pi x)}{x^2-2x+1}=?$

Çözüm: $\lim_{x\to 1}\frac{1+\cos(\pi x)}{x^2-2x+1}\overset{\frac{0}{0}}{=}\lim_{x\to 1}\frac{-\pi\sin(\pi x)}{2x-2}\overset{\frac{0}{0}}{=}\lim_{x\to 1}\frac{-\pi^2\cos(\pi x)}{2}=\frac{\pi^2}{2}$

3) $\lim\limits_{x\to 0} \dfrac{\ln(\cos 3x)}{\ln(\cos 2x)} = ?$

Çözüm: $\lim\limits_{x\to 0} \dfrac{\ln(\cos 3x)}{\ln(\cos 2x)} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{\frac{-3\sin 3x}{\cos 3x}}{\frac{-2\sin 2x}{\cos 2x}} = \dfrac{3}{2} \lim\limits_{x\to 0} \dfrac{\tan 3x}{\tan 2x}$

$$\overset{\frac{0}{0}}{=} \frac{3}{2} \lim_{x\to 0} \frac{3(1+\tan^2 3x)}{2(1+\tan^2 2x)} = \frac{9}{4}$$

4) $\lim\limits_{x\to 0} \dfrac{e^{2x} - 2e^{x} + 1}{\cos 3x - 2\cos 2x + \cos x} = ?$

Çözüm: $\lim\limits_{x\to 0} \dfrac{e^{2x} - 2e^{x} + 1}{\cos 3x - 2\cos 2x + \cos x} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{2e^{2x} - 2e^{x}}{-3\sin 3x + 4\sin 2x - \sin x}$

$$\overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{4e^{2x} - 2e^{x}}{-9\cos 3x + 8\cos 2x - \cos x} = \frac{2}{-2} = -1$$

5) $\lim\limits_{x\to 0} \dfrac{\sin x - \arctan x}{x^2 \ln(x+1)} = ?$

Çözüm:

$\lim\limits_{x\to 0} \dfrac{\sin x - \arctan x}{x^2 \ln(x+1)} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{\cos x - \frac{1}{1+x^2}}{2x\ln(x+1) + \frac{x^2}{x+1}} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{-\sin x + \frac{2x}{(1+x^2)^2}}{2\ln(x+1) + \frac{2x}{x+1} + \frac{x^2+2x}{(x+1)^2}}$

$$\overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{-\cos x + \frac{2(1+x^2)^2 - 8x^2(1+x^2)}{(1+x^2)^4}}{\frac{2}{x+1} + \frac{2}{(x+1)^2} + \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2+2x)}{(x+1)^4}}$$

$$= \frac{1}{6}$$

6) $\lim\limits_{x\to \infty} \dfrac{\sin\left(\frac{1}{x}\right)}{\frac{x}{1+x^2}} = ?$

Çözüm: $\lim\limits_{x\to \infty} \dfrac{\sin\left(\frac{1}{x}\right)}{\frac{x}{1+x^2}} \overset{\frac{0}{0}}{=} \lim\limits_{x\to \infty} \dfrac{-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)}{\frac{1+x^2-2x^2}{(1+x^2)^2}} = \underbrace{\lim\limits_{x\to \infty} \cos\left(\frac{1}{x}\right)}_{=1} \cdot \underbrace{\lim\limits_{x\to \infty} \dfrac{-(1+x^2)^2}{x^2(1-x^2)}}_{=1}$

$$= 1$$

7) $\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)} = ?$

Çözüm: $\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{1+\tan^2 x}{\tan x}} = \lim_{x\to 0^+} \frac{1}{1+\tan^2 x} = 1$

8) $\lim_{x\to 0^+} \frac{\ln(\tan 2x)}{\ln(\tan 3x)} = ?$

Çözüm: $\lim_{x\to 0^+} \frac{\ln(\tan 2x)}{\ln(\tan 3x)} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0^+} \frac{\frac{2(1+\tan^2 2x)}{\tan 2x}}{\frac{3(1+\tan^2 3x)}{\tan 3x}}$

$= \frac{2}{3} \cdot \underbrace{\lim_{x\to 0^+} \frac{1+\tan^2 2x}{1+\tan^2 3x}}_{1} \cdot \lim_{x\to 0^+} \frac{\tan 3x}{\tan 2x} \overset{\frac{0}{0}}{=} \frac{2}{3} \lim_{x\to 0^+} \frac{3(1+\tan^2 3x)}{2(1+\tan^2 2x)}$

$= 1$

8) $\lim_{x\to \infty} \frac{\ln x}{\sqrt[3]{x}} = ?$

Çözüm: $\lim_{x\to \infty} \frac{\ln x}{\sqrt[3]{x}} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to \infty} \frac{\frac{1}{x}}{\frac{1}{3\sqrt[3]{x^2}}} = \lim_{x\to \infty} \frac{3}{\sqrt[3]{x}} = 0$

## $0.\infty$ Belirsizlik Hali:

$u.v = \frac{u}{\frac{1}{v}}$ dönüşümü yardımıyla $0.\infty$ belirsizliği $\frac{0}{0}$ veya $\frac{\infty}{\infty}$ belirsizliğine dönüştürülür.

Örnek: $\lim_{x\to \frac{\pi}{2}} (2x-\pi)\sec x = ?$

Çözüm: $\lim_{x\to \frac{\pi}{2}} (2x-\pi)\sec x \overset{0.\infty}{=} \lim_{x\to \frac{\pi}{2}} \frac{2x-\pi}{\cos x} \overset{\frac{0}{0}}{=} \lim_{x\to \frac{\pi}{2}} \frac{2}{-\sin x} = -2$

Örnek: $\lim_{x\to 0} (1-\cos x)\cot x = ?$

Çözüm: $\lim_{x\to 0} (1-\cos x)\cot x \overset{0.\infty}{=} \lim_{x\to 0} \frac{(1-\cos x)\cos x}{\sin x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{-\sin x + 2\cos x \sin x}{\cos x} = 0$

**Örnek:** $\lim_{x\to 0^+} x^2 \ln x = ?$

**Çözüm:** $\lim_{x\to 0^+} x^2.\ln x \overset{0.\infty}{=} \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x^2}} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0^+}\left(\dfrac{-x^2}{2}\right) = 0$

**Örnek:** $\lim_{x\to \frac{\pi}{4}} (1-\tan x).\sec 2x = ?$

**Çözüm:** $\lim_{x\to \frac{\pi}{4}} (1-\tan x)\sec 2x \overset{0.\infty}{=} \lim_{x\to \frac{\pi}{4}} \dfrac{1-\tan x}{\cos 2x} \overset{\frac{0}{0}}{=} \lim_{x\to \frac{\pi}{4}} \dfrac{-(1+\tan^2 x)}{-2\sin 2x} = 1$

## $\infty - \infty$ Belirsizlik Hali:

$u - v = \dfrac{\frac{1}{v} - \frac{1}{u}}{\frac{1}{u.v}}$ dönüşümü yardımıyla $\infty-\infty$ belirsizliği $\frac{0}{0}$ belirsizliğine dönüştürürler.

**Örnek:** $\lim_{x\to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right) = ?$

**Çözüm:** $\lim_{x\to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right) \overset{\infty-\infty}{=} \lim_{x\to 1} \dfrac{x\ln x - x + 1}{(x-1).\ln x} \overset{\frac{0}{0}}{=} \lim_{x\to 1} \dfrac{\ln x + \cancel{1} - \cancel{1}}{\ln x + \frac{x-1}{x}}$

$\overset{\frac{0}{0}}{=} \lim_{x\to 1} \dfrac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \dfrac{1}{2}$

**Örnek:** $\lim_{x\to 0} \left(\cot x - \dfrac{1}{x}\right) = ?$

**Çözüm:** $\lim_{x\to 0} \left(\cot x - \dfrac{1}{x}\right) \overset{\infty-\infty}{=} \lim_{x\to 0} \left(\dfrac{\cos x}{\sin x} - \dfrac{1}{x}\right) = \lim_{x\to 0} \dfrac{x\cos x - \sin x}{x\sin x}$

$\overset{\frac{0}{0}}{=} \lim_{x\to 0} \dfrac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{\sin x + x\cos x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \dfrac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x}$

$= 0$

**Örnek:** $\lim_{x\to \infty} (x - \ln x) = \lim_{x\to \infty} \left(\ln \dfrac{e^x}{x}\right) = \ln\left(\lim_{x\to \infty} \dfrac{e^x}{x}\right) \overset{\frac{\infty}{\infty}}{=} \ln\left(\lim_{x\to \infty} \dfrac{e^x}{1}\right) = \infty$

## $\infty^0, 0^0, 1^\infty$ Belirsizlik Halleri:

$$y = (u(x))^{v(x)} \Rightarrow \ln y = \ln (u(x))^{v(x)} \Rightarrow \ln y = v(x).\ln(u(x))$$

$$\Rightarrow \lim(\ln y) = \lim v(x).\ln(u(x))$$

$$\Rightarrow \lim y = e^{\lim v(x).\ln(u(x))}$$

**Örnekler:**

1) $\lim\limits_{x\to 0^-} (1-e^x)^{\sin x} = ?$ $(0^0)$

Çözüm: $y = (1-e^x)^{\sin x} \Rightarrow \ln y = \sin x.\ln(1-e^x)$

$$\Rightarrow \lim_{x\to 0^-} (\ln y) = \lim_{x\to 0^-} \sin x.\ln(1-e^x)$$

$$\Rightarrow \ln\left(\lim_{x\to 0^-} y\right) = \lim_{x\to 0^-} \frac{\ln(1-e^x)}{\frac{1}{\sin x}} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0^-} \frac{\frac{-e^x}{1-e^x}}{\frac{-\cos x}{\sin^2 x}}$$

$$= \underbrace{\lim_{x\to 0} \frac{e^x}{\cos x}}_{\substack{\shortparallel \\ 1}} . \lim_{x\to 0^-} \frac{\sin^2 x}{1-e^x} \overset{\frac{0}{0}}{=} \lim_{x\to 0^-} \frac{2\sin x\cos x}{-e^x}$$

$$= 0$$

$$\Rightarrow \lim_{x\to 0^-} y = e^0 = 1$$

2) $\lim\limits_{x\to 0^+} (\sin x)^{\frac{1}{\ln x}} = ?$ $(0^0)$

Çözüm: $y = (\sin x)^{\frac{1}{\ln x}} \Rightarrow \ln y = \frac{1}{\ln x}.\ln(\sin x)$

$$\Rightarrow \lim_{x\to 0^+} (\ln y) = \lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} = \underbrace{\lim_{x\to 0^+} \cos x}_{\substack{\shortparallel \\ 1}} . \underbrace{\lim_{x\to 0^+} \frac{x}{\sin x}}_{\substack{\shortparallel \\ 1}}$$

$$\Rightarrow \ln\left(\lim_{x\to 0^+} y\right) = 1$$

$$\Rightarrow \lim_{x\to 0^+} y = e^1 = e$$

3) $\lim_{x\to 0^+} (\cot x)^{\frac{1}{\ln x}} = ?$ $(\infty^0)$

<u>Çözüm:</u> $y=(\cot x)^{\frac{1}{\ln x}} \Rightarrow \ln y = \frac{1}{\ln x}\cdot \ln(\cot x)$

$\Rightarrow \lim_{x\to 0^+}(\ln y) = \lim_{x\to 0^+} \frac{\ln(\cot x)}{\ln x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0^+} \frac{\frac{-\frac{1}{\sin^2 x}}{\cos x/\sin x}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{-x}{\sin x \cos x}$

$\Rightarrow \ln\left(\lim_{x\to 0^+} y\right) = -\underbrace{\lim_{x\to 0^+}\frac{x}{\sin x}}_{=1} \cdot \underbrace{\lim_{x\to 0^+}\frac{1}{\cos x}}_{=1} = -1$

$\Rightarrow \lim_{x\to 0^+} y = e^{-1} = \frac{1}{e}$

4) $\lim_{x\to \frac{\pi}{4}} (\tan x)^{\tan 2x} = ?$ $(1^\infty)$

<u>Çözüm:</u> $y=(\tan x)^{\tan 2x} \Rightarrow \ln y = (\tan 2x).\ln(\tan x)$

$\Rightarrow \lim_{x\to \frac{\pi}{4}}(\ln y) = \lim_{x\to \frac{\pi}{4}} (\tan 2x).\ln(\tan x)$

$\Rightarrow \ln\left(\lim_{x\to \frac{\pi}{4}} y\right) = \lim_{x\to \frac{\pi}{4}} \frac{\ln(\tan x)}{\frac{1}{\tan 2x}} \overset{\frac{0}{0}}{=} \lim_{x\to \frac{\pi}{4}} \frac{\frac{1+\tan^2 x}{\tan x}}{-2(1+\cot^2 2x)} = -1$

$\cot 2x$

$\Rightarrow \lim_{x\to \frac{\pi}{4}} y = e^{-1} = \frac{1}{e}$

5) $\lim_{x\to 1} x^{\frac{1}{x-1}} = ?$ $(1^\infty)$

<u>Çözüm:</u> $y = x^{\frac{1}{x-1}} \Rightarrow \ln y = \frac{1}{x-1}\ln x \Rightarrow \lim_{x\to 1}(\ln y) = \lim_{x\to 1}\frac{\ln x}{x-1} \overset{\frac{0}{0}}{=} \lim_{x\to 1}\frac{\frac{1}{x}}{1} = 1$

$\Rightarrow \ln\left(\lim_{x\to 1} y\right) = 1 \Rightarrow \lim_{x\to 1} y = e$

6) $\lim_{x\to \infty} \sqrt[x]{x} = ?$ $(\infty^0)$

<u>Çözüm:</u> $y = x^{\frac{1}{x}} \Rightarrow \ln y = \frac{1}{x}\ln x \Rightarrow \lim_{x\to \infty}(\ln y) = \lim_{x\to\infty}\frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{1}$

$\Rightarrow \ln\left(\lim_{x\to\infty} y\right) = 0 \Rightarrow \lim_{x\to\infty} y = e^0 = 1$