CMSC 25300 / 35300
Homework 1: Vectors and Matrices

1. Matrix multiplication. *Settlers of Catan* is an awesome game. In this game, participants build roads, settlements, cities, and development cards by using resources such as wood, bricks, wheat, sheep, and ore. The number of resources required for each building project are reflected in Figure 1. (Ignore the text at the bottom about cities replacing already-built settlements.)

Figure 1: Building costs in Settlers of Catan

a) Write the information about how many of what resources are required to build roads, settlements, cities, or development cards in a matrix. What do the rows represent? What do the columns represent?

b) Departing from the game somewhat, suppose resources cost \$1 for each unit of wood, \$2 for brick, \$3 for sheep, \$5 for wheat, and \$8 for ore. Write this information in a vector. Write out a matrix-vector multiplication that calculates the total cost of buying roads, settlements, and cities. (Ignore the fact that a city had to replace a settlement.)

c) Suppose you want to build a city (over an already-built settlement), two settlements, and six road lengths connecting them. Again using matrix multiplication, find the resources required to fill the order. I.e. how many bricks, how many wood, etc.?

d) Calculate the total cost for the order (using, you guessed it, matrix multiplication)

e) Get up and running with either Matlab or Python. In your language of choice, write a script that computes the matrix multiplications in the previous parts of this problem.

2. You are given a matrix

$$\begin{bmatrix} 8 & 0 & 1 & 1 \\ 9 & 2 & 9 & 4 \\ 1 & 5 & 9 & 9 \\ 9 & 9 & 4 & 7 \\ 6 & 9 & 8 & 9 \end{bmatrix} \in \mathbb{R}^{5\times 4}.$$

a) What is a vector $y \in \mathbb{R}^5$ so that $y^\top X$ is the fourth row of X?

b) Given a number $k \in \{1, 2, 3, 4, 5\}$, how would you construct a vector $y \in \mathbb{R}^5$ so that $y^\top X$ is the k-th row of X?

c) How would you construct a vector $y \in \mathbb{R}^5$ so that $y^\top X$ is a times the k-th row of X plus b times the j-th row of X for some $a, b \in \mathbb{R}$ and $j, k \in \{1, \dots, 5\}$?

d) What is a vector $w \in \mathbb{R}^4$ so that Xw is the third column of X?

e) Given a number $k \in \{1, 2, 3, 4\}$, how would you construct a vector $w \in \mathbb{R}^4$ so that Xw is the k-th column of X?

f) How would you construct a vector $w \in \mathbb{R}^4$ so that Xw is a times the k-th column of X plus b times the j-th column of X for some $a, b \in \mathbb{R}$ and $j, k \in \{1, \dots, 4\}$?

g) Write a Python (or Matlab or R or Julia) script to verify your answers above.

3. This matrix has rank = 1:

$$\begin{bmatrix} 15 & 27 & 6 & 18 \\ 5 & 9 & 2 & 6 \\ 20 & 36 & 8 & 24 \\ 5 & 9 & 2 & 6 \end{bmatrix}.$$

How can you tell? (Hint: you don't need any tools beyond what was discussed in class.)

4. You learn a model that says:

$$\hat{y}_i = w_1 x_{i,1} + w_2 x_{i,2} + w_3$$
$$\text{predicted label } = \begin{cases} +1, & \hat{y}_i > 0 \\ -1, & \text{otherwise} \end{cases}.$$

For $\mathbf{w} = \begin{bmatrix} 3 & 5 & -2 \end{bmatrix}^\top$, draw a plot and indicate for which $\mathbf{x}$ your model would predict the label $+1$.

5. Polynomials using linear models. Suppose we observe pairs of scalar points (z_i, y_i), $i = 1, \dots, n$. Imagine these points are measurements from a scientific experiment. The variables z_i are the experimental conditions and the y_i correspond to the measured response in each condition. Suppose we wish to fit a degree $d < n$ polynomial to these data. In other words, we want to find the coefficients of a degree d polynomial p so that $p(z_i) \approx y_i$ for $i = 1, 2, \ldots, n$. We want to use a linear model.

a) Suppose p is a degree d polynomial. Write the general expression for $p(z) = y$.

b) Express the $i = 1, \ldots, n$ equations as a system in matrix form $\boldsymbol{X}\boldsymbol{w} = \boldsymbol{y}$. Specifically, what is the form/structure of $\boldsymbol{X}$ in terms of the given x_i.

c) Write a Matlab or Python script to generate a plot of a polynomial given $\mathbf{w}$ and points $z_1, \ldots, z_n$. Here is some starter code:

```
import numpy as np
import scipy.io as sio
import matplotlib.pyplot as plt

# n = number of points
# z = points where polynomial is evaluated
# p = array to store the values of the interpolated polynomials
n = 100
z = np.linspace(-1, 1, n)

d = 3 # degree
w = np.random.rand(d)
X = np.zeros((n,d))

# generate X-matrix

# evaluate polynomial at all points z, and store the result in p
# do NOT use a loop for this

# plot the datapoints and the best-fit polynomials
plt.plot(z, p, linewidth=2)
plt.xlabel('z')
plt.ylabel('y')
plt.title('polynomial with coefficients w = %s' %w )
plt.show()
```