CENG222

Statistical Methods for Computer Engineering

Spring 2021-2022

Homework 2

Due: April 24th, 2022, Sunday 23:59

Question 1

a) They are dependent. This is clearly seen from the changing range of one random variable with the choice of the other. For $X = \pm 1$, we know for sure $Y = 0$, whereas, say, for $X = 0$, Y can be any value in $[-1, 1]$.

b) For $f(x)$ we need to integrate $f_{(X,Y)}(x, y)$ over y

$$f(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi}\sqrt{1 - x^2}.$$

X and Y are symmetric, implying

$$f(y) = \frac{2}{\pi}\sqrt{1 - y^2}.$$

c) We can find the expected value just by examining the domain: it is a circle centered at origin with a uniform distribution on it. So $\mathbf{E}(X) = 0$. More rigorously,

$$\mathbf{E}(X) = \frac{2}{\pi}\int_{-1}^{1} x\sqrt{1 - x^2} dx = 0,$$

since the domain is symmetric and the integrant is an odd funciton.

d) $\mathrm{Var}(X) = \mathbf{E}(X^2) - \mathbf{E}(X)^2 = \mathbf{E}(X^2)$. Then

$$\begin{aligned}\mathbf{E}(X^2) &= \frac{2}{\pi}\int_{-1}^{1} x^2\sqrt{1 - x^2} dx = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \sin^2\theta \cos^2\theta d\theta = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \sin^2 2\theta d\theta \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi} \sin^2\theta' d\theta' = \frac{1}{4}\end{aligned}$$

Question 2

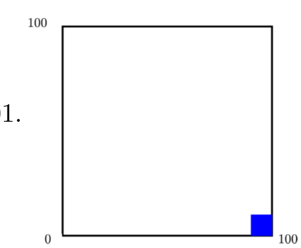
a) $f(t_A) = \dfrac{1}{100}$ and $f(t_B) = \dfrac{1}{100}$ since we have a uniform distribution. Using the independence of T_A and T_B we have

$$f_{(T_A,T_B)}(t_A, t_B) = \frac{1}{10^4} \qquad \text{and} \qquad F_{(T_A,T_B)}(t_A, t_B) = \frac{t_A\, t_B}{10^4}.$$

b)

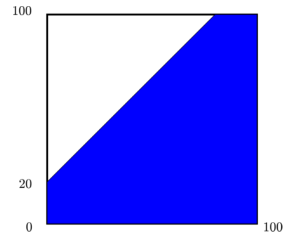
$$\mathbf{P}\{T_A \leq 10 \cap 90 \leq T_B \leq 100\} = \mathbf{P}\{T_A \leq 10\}\,\mathbf{P}\{90 \leq T_B \leq 100\} = \frac{1}{10} \times \frac{1}{10} = 0.01.$$

An alternative way to think about this is in terms of the ratio of the region of the asked event to the whole domain. This is shown in the figure in the right.



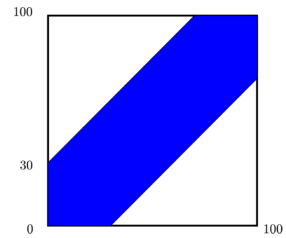
c) We think in terms of the ratio of the region of the asked event. The ratio of the shaded area in the right is the asked probability:

$$1 - \frac{80 \times 80/2}{10^4} = 0.68$$



d) Again we think in terms of the ratio of the region of the asked event. The ratio of the shaded area in the right is the asked probability:

$$1 - \frac{70 \times 70}{10^4} = 0.51$$



Question 3

a) $\mathbf{P}\{T \geq t\}$ corresponds to $\mathbf{P}\{X_1 \geq t \cap X_2 \geq t \cap \cdots \cap X_N \geq t\}$. By the independence of variables we have

$$\mathbf{P}\{T \geq t\} = \prod_{n=1}^{N} \mathbf{P}\{X_n \geq t\} = \prod_{n=1}^{N} 1 - (1 - e^{-\lambda_n t}) = \exp\Big\{ - \sum_{n=1}^{N} \lambda_n t \Big\}.$$

Then the cdf of T is

$$F_T(t) = \mathbf{P}(T \leq t) = 1 - \exp\Big\{ - \sum_{n=1}^{N} \lambda_n t \Big\}.$$

Note that the resulting cdf is Exponential $\left(\sum_{n=1}^{N} \lambda_n \right)$.

b) We can directly use the result of part **a)** with $\lambda_n = \dfrac{n}{10}$ for n in range $1, \ldots, 10$. The λ of T is $\sum_{n=1}^{10} \dfrac{n}{10} = 5.5$. Since T also has an exponential distribution $\mathbf{E}(T) = \dfrac{1}{\lambda} \approx 0.18$ years.

Question 4

For both of the following we can use Normal approximation to Binomial distribution.

a) We calculate the expected value and variance in the selected population as $\mathbf{E}(X_1) = 100 \times 0.74 = 74$ and $\mathrm{Var}(X_1) = 100 \times 0.74 \times 0.26 = 19.24$. We can model X_1 as

$$X_1 \approx \text{Normal}(74, \sqrt{19.24}).$$

With the continuity correction

$$\mathbf{P}(X_1 \geq 69.5) = \mathbf{P}\left(\frac{X_1 - 74}{\sqrt{19.24}} \geq \frac{69.5 - 74}{\sqrt{19.24}} \right) \approx \mathbf{P}(Z \geq -1.02) = \mathbf{P}(Z \leq 1.02) \approx 0.8461.$$

We have rounded the limiting value of Z to look up from the textbook's appendix. A more accurate answer would be 0.8475. Both answers are accepted. The results that do not deploy continuity correction, *i.e.*, those calculating

$$\mathbf{P}(X_1 \geq 70) \approx 0.8186$$

will also be accepted.

b) $\mathbf{E}(X_2) = 100 \times 0.10 = 10$ and $\mathrm{Var}(X_2) = 100 \times 0.1 \times 0.9 = 9$. Then

$$X_2 \approx \text{Normal}(10, 3).$$

The asked quantity can be approximated as

$$\mathbf{P}(X_2 \leq 5.5) = \mathbf{P}\left(\frac{X_2 - 10}{3} \leq \frac{5.5 - 10}{3} \right) \approx \mathbf{P}(Z \leq -1.5) \approx 0.0668.$$

Again, answers calculating $\mathbf{P}(X_2 \leq 5) \approx 0.0475$ are also accepted.