CENG 384 - Signals and Systems for Computer Engineers
Spring 2023
Homework 3

Hasan Ege, Meyvecioğlu
e2449783@ceng.metu.edu.tr

Enes, Şanlı
e2375749@ceng.metu.edu.tr

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1.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^j k w_0 t \tag{1}$$

$$y(t) = \int_{-\infty}^{t} x(s) ds = \int_{-\infty}^{t} \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 s} ds \tag{2}$$

$$y(t) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{t} a_k e^{jkw_0 s} ds \tag{3}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k \frac{e^{jkw_0 t}}{jkw_0} \tag{4}$$

$$y(t) = \sum_{k=-\infty}^{\infty} \frac{a_k}{jkw_0} e^{jkw_0 t} \tag{5}$$

Now it is in the synthesis equation form

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} \frac{a_k}{jkw_0} e^{jkw_0 t} \tag{6}$$

$$b_k = \frac{a_k}{jkw_0} \tag{7}$$

$$w_0 = \frac{2\pi}{T} \tag{8}$$

$$b_k = \frac{a_k}{jk(\frac{2\pi}{T})} \tag{9}$$

2. (a)

$$x(t) \leftrightarrow a_k \tag{10}$$

$$x(t)x(t) \leftrightarrow a_k * a_k \text{ multiplication in time : convolution in freq. domain} \tag{11}$$

$$x(t)x(t) \leftrightarrow \sum_{l=-\infty}^{\infty} a_l a_{k-l} \tag{12}$$

(b)

$$Ev\{x(t)\} = \frac{1}{2}(x(t) + x(-t))$$

$$(x(t) + x(-t)) \leftrightarrow a_k + a_{-k} (\text{ From time reversal property})$$

$$Ev\{x(t)\} = \frac{1}{2}(x(t) + x(-t)) \leftrightarrow \frac{1}{2}(a_k + a_{-k}) \text{ (From linearity property)} \tag{13}$$

(c)

$$x(t) \leftrightarrow a_k \tag{14}$$

$$x(t + t_0) \leftrightarrow e^{jkw_0 t_0} a_k \tag{15}$$

$$x(t - t_0) \leftrightarrow e^{-jkw_0 t_0} a_k \tag{16}$$

$$x(t + t_0) + x(t - t_0) \leftrightarrow e^{jkw_0 t_0} a_k + e^{-jkw_0 t_0} a_k \tag{17}$$

$$\leftrightarrow (e^{jkw_0 t_0} + e^{-jkw_0 t_0}) a_k \tag{18}$$

3. This function is periodic and T = 4, then take x 0 to 4

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t} \tag{19}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt \tag{20}$$

$$a_k = \frac{1}{4}\int_0^1 2e^{-jkw_0 t}dt + \frac{1}{4}\int_2^3 -2e^{-jkw_0 t}dt \tag{21}$$

$$a_k = \frac{1}{2}(\frac{e^{-jkw_0} - 1}{-jkw_0} + \frac{-e^{3jkw_0} + e^{2jkw_0}}{-jkw_0}) \tag{22}$$

$$w_0 = \frac{2\pi}{T} = \frac{\pi}{2} \tag{23}$$

$$a_k = \frac{-1 + e^{-jk\pi/2} + e^{-2jk\pi/2} - e^{-3jk\pi/2}}{-jk\pi} \tag{24}$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{-1 + e^{-jk\pi/2} + e^{-2jk\pi/2} - e^{-3jk\pi/2}}{-jk\pi} e^{jk(\pi/2)t} \tag{25}$$

4. (a)

$$x(t) = 1 + sin(\omega_0 t) + 2cos(\omega_0 t) + cost(2\omega_0 t + \frac{\pi}{4})$$

Let us represent this signal with complex exponentials:

$$x(t) = 1 + \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t}) + (e^{j\omega_0 t} + e^{-j\omega_0 t}) + \frac{1}{2}(e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)})$$

If we regroup the terms:

$$x(t) = 1 + (1 + \frac{1}{2j})e^{j\omega_0 t} + (1 - \frac{1}{2j})e^{-j\omega_0 t} + (\frac{1}{2}e^{j(\pi/4)})e^{j2\omega_0 t} + (\frac{1}{2}e^{-j(\pi/4)})e^{-j2\omega_0 t}$$

So, the coefficients are:

$$a_0 = 1$$

$$a_1 = (1 + \frac{1}{2j})$$

$$a_{-1} = (1 - \frac{1}{2j})$$

$$a_2 = \frac{1}{2}e^{j(\pi/4)}$$

$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)}$$

$$a_k = 0, k > 2 \text{ or } k < -2$$

Magnitudes and phases are as follows:

$$|a_0| = 1$$

$$\angle a_0 = 0$$

$$|a_1| = \frac{\sqrt{5}}{2} = 1.11$$

$$\angle a_1 = \arctan(-1/2) = -0.46$$

$$|a_{-1}| = \frac{\sqrt{5}}{2} = 1.11$$

$$\angle a_{-1} = \arctan(1/2) = 0.46$$

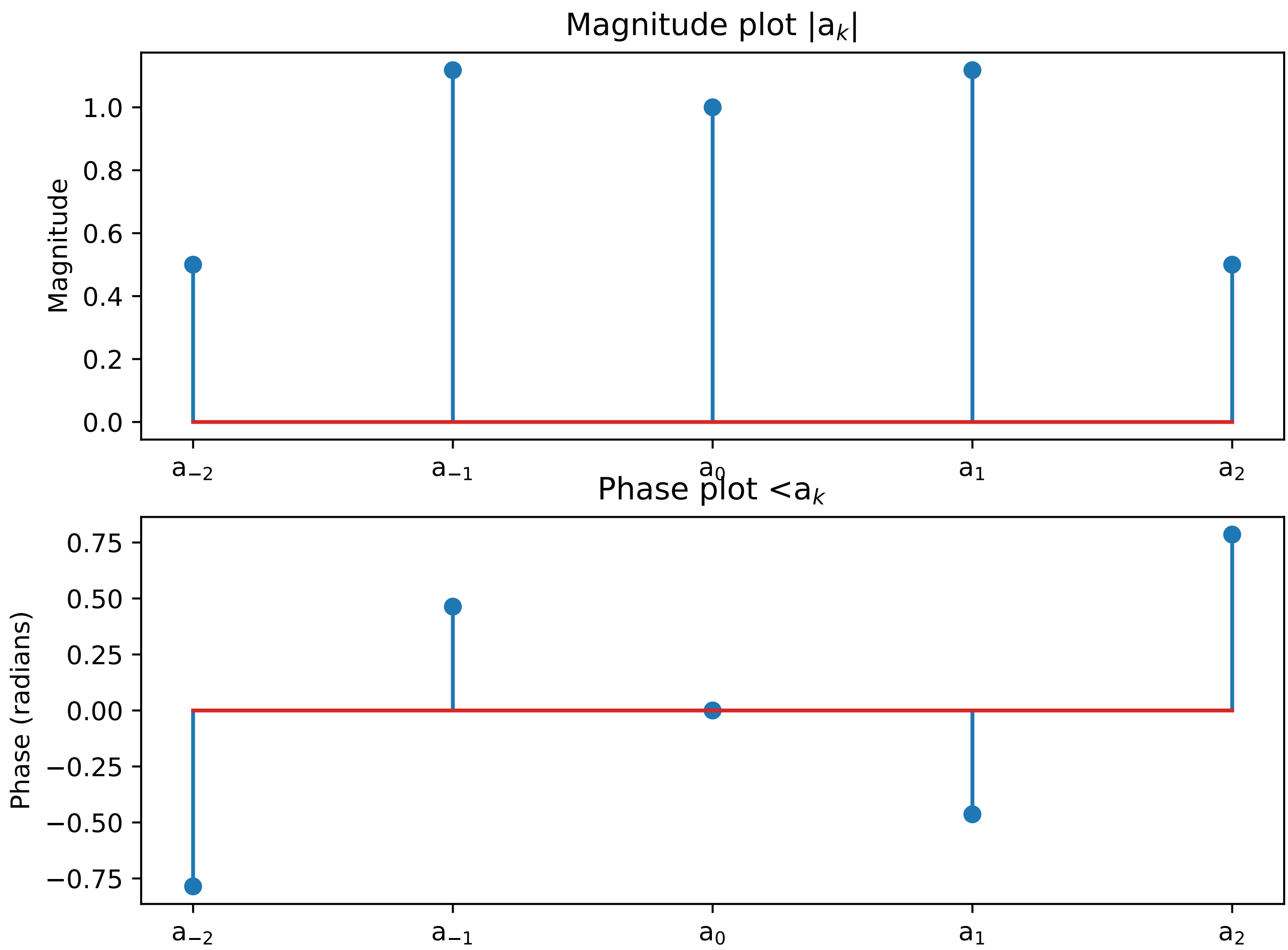
$$|a_2| = \frac{1}{2} = 0.5$$

$$\angle a_2 = \arctan(1) = 0.78$$

$$|a_{-2}| = \frac{1}{2} = 0.5$$

$$\angle a_{-2} = \arctan(-1) = -0.78$$

We can plot them as:



(b) Let us take the Laplace transform of the original equation:

$$\mathscr{L}\{y'(t)\} + \mathscr{L}\{y(t)\} = \mathscr{L}\{x(t)\}$$

$$sY(s) + Y(s) = X(s)$$

The transfer function is equal to the ratio of output and input signals:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)}{(s+1)Y(s)} = \frac{1}{s+1} \tag{26}$$

Since transfer function and eigenvalue both refer to $H(s)$, we have found them.
Another way, let say

$$x(t) = e^{jwt} \tag{27}$$

$$y(t) = H(jw)e^{jwt} \tag{28}$$

$$H(jw)jwe^{jwt} + H(jw)e^{jwt} = e^{jwt} \tag{29}$$

$$H(jw) = \frac{1}{1+jw} \tag{30}$$

Both are equal

(c)

$$b_k = a_k * H(jw_0 k) \tag{31}$$

$$b_0 = a_0 * H(0) = 1 * 1 = 1 \tag{32}$$

$$b_1 = a_1 * H(jw_0 1) = (1 + \frac{1}{2j}) * \frac{1}{1 + jw_0} = \tag{33}$$

$$b_{-1} = a_{-1} * H(jw_0(-1)) = (1 - \frac{1}{2j}) * \frac{1}{1 - jw_0} = \tag{34}$$

$$b_2 = a_2 * H(jw_0 2) = (\frac{1}{2}e^{j\pi/4}) * \frac{1}{1 + 2jw_0} = \tag{35}$$

$$b_{-2} = a_{-2} * H(jw_0(-2)) = (\frac{1}{2}e^{-j\pi/4}) * \frac{1}{1 - 2jw_0} = \tag{36}$$

other b_k's equal to 0

$$|b_0| = |a_0 * H(0)| = 1 \tag{37}$$

$$|b_1| = |a_1| * |H(jw_0 1)| = \frac{\sqrt{5}}{2} * \sqrt{\frac{1}{1 + w_0^2}} = \sqrt{\frac{5}{4 + 4w_0^2}} = 0.175 \tag{38}$$

$$|b_{-1}| = |a_{-1}| * |H(jw_0(-1))| = \frac{\sqrt{5}}{2} * \sqrt{\frac{1}{1 + w_0^2}} = \sqrt{\frac{5}{4 + 4w_0^2}} = 0.175 \tag{39}$$

$$|b_2| = |a_2| * |H(jw_0 2)| = \frac{1}{2} * \sqrt{\frac{1}{1 + 4w_0^2}} = \sqrt{\frac{1}{4 + 16w_0^2}} = 0.039 \tag{40}$$

$$|b_{-2}| = |a_{-2}| * |H(jw_0(-2))| = \frac{1}{2} * \sqrt{\frac{1}{1 + 4w_0^2}} = \sqrt{\frac{1}{4 + 16w_0^2}} = 0.039 \tag{41}$$

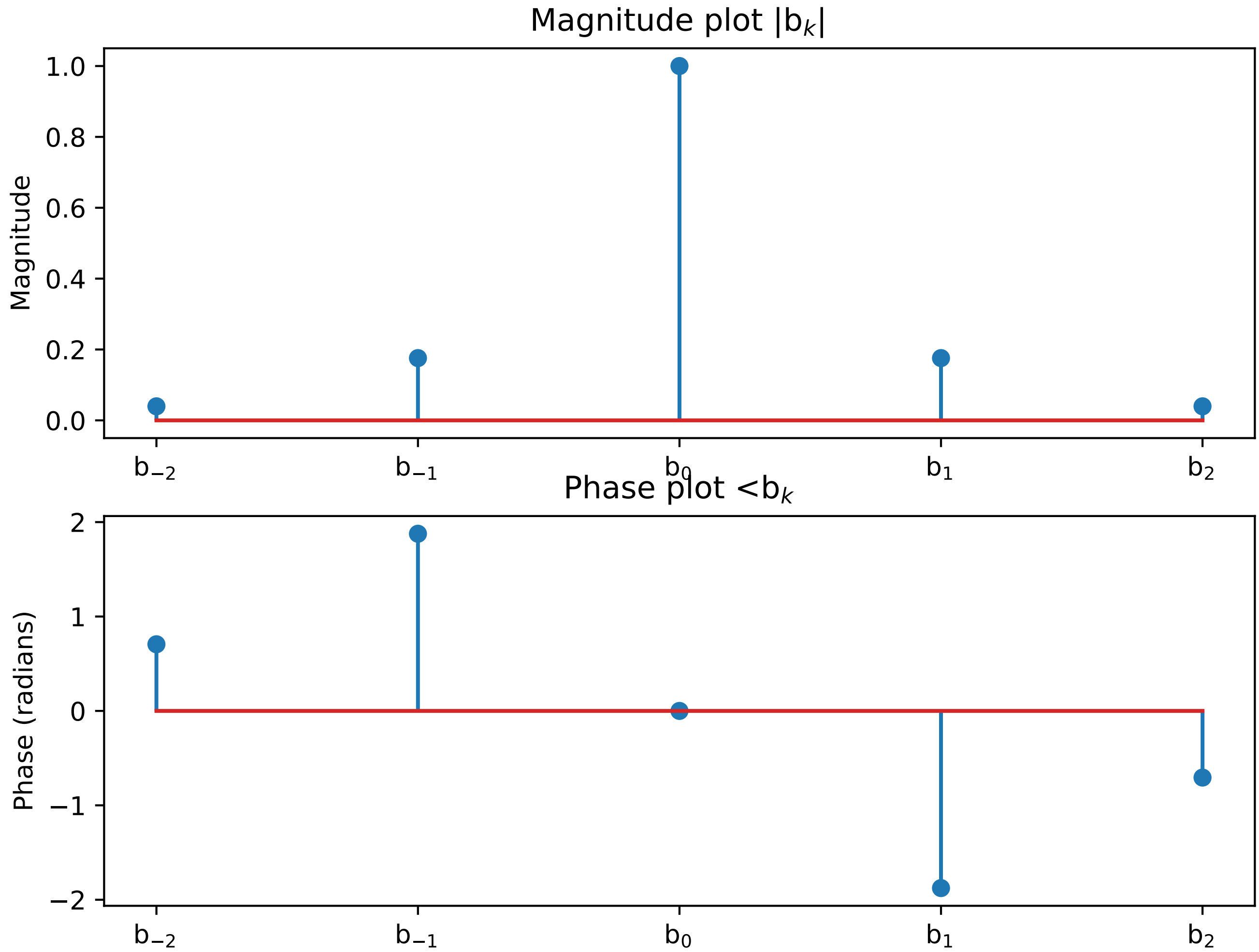
$$\angle b_0 = \angle(a_0 * H(jw_0 1)) = 0 + 0 = 0 \tag{42}$$

$$\angle b_1 = \angle(a_1 * H(jw_0 1)) = \arctan(-1/2) + \arctan(-w_0) = -1.876 rad \tag{43}$$

$$\angle b_{-1} = \angle(a_{-1} * H(jw_0 - 1)) = \arctan(1/2) + \arctan(w_0) = 1.876 rad \tag{44}$$

$$\angle b_2 = \angle(a_2 * H(jw_0 2)) = \arctan(1) + \arctan(-2w_0) = -0.705 rad \tag{45}$$

$$\angle b_{-2} = \angle(a_{-2} * H(jw_0 - 2)) = \arctan(-1) + \arctan(2w_0) = 0.705 rad \tag{46}$$



(d)

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jkw_0t} \tag{47}$$

$$y(t) = \frac{(\frac{1}{2}e^{-j\pi/4})}{1-2jw_0}e^{-2jw_0t} + \frac{(1-\frac{1}{2j})}{1-jw_0}e^{-jw_0t} + 1 + \frac{(1+\frac{1}{2j})}{1+jw_0}e^{jw_0t} + \frac{(\frac{1}{2}e^{j\pi/4})}{1+2jw_0}e^{2jw_0t} \tag{48}$$

5. (a)

$$x[n] = sin(\frac{\pi}{2}n) = \frac{e^{j\pi n/2} - e^{-j\pi n/2}}{2i} \tag{49}$$

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n} \tag{50}$$

We know $sin(n\pi/2)$ period is 4, so N=4

$$a_{-1} = \frac{-1}{2i} = \frac{i}{2} \tag{51}$$

$$a_1 = \frac{1}{2i} = \frac{-i}{2} \tag{52}$$

Because it is periodic

$$a_{-1} = a_{-1+N} = a_3 \tag{53}$$

$$a_0 = 0, a_1 = \frac{-i}{2}, a_2 = 0, a_3 = \frac{i}{2} \tag{54}$$

(b)

$$y[n] = 1 + cos(\frac{\pi}{2}n) = 1 + \frac{e^{j\pi n/2} + e^{-j\pi n/2}}{2} \tag{55}$$

$$y[n] = \sum_{k=<N>} b_k e^{jk\frac{2\pi}{N}n} \tag{56}$$

We know $1 + cos(n\pi/2)$ period is 4, so N=4

$$= b_0 + b_1 e^{j\pi n/2} + b_2 e^{j\pi n} + b_{-1} e^{-j\pi n/2} \tag{57}$$

$$b_0 = 1 \tag{58}$$

$$b_1 = \frac{1}{2} \tag{59}$$

$$b_{-1} = \frac{1}{2} \tag{60}$$

Because it is periodic

$$b_{-1} = b_{-1+N} = b_3 \tag{61}$$

$$b_0 = 1, b_1 = \frac{1}{2}, b_2 = 0, b_3 = \frac{1}{2} \tag{62}$$

(c)

$$x[n]y[n] \rightarrow c_k \tag{63}$$

$$c_k = \sum_{l=<N>} a_l b_{k-l} \tag{64}$$

$$c_0 = a_1 b_{-1} + a_3 b_{-3} \tag{65}$$

$$c_0 = -i/2 + i/2 = 0 \tag{66}$$

$$c_1 = a_1 b_0 + a_3 b_{-2} \tag{67}$$

$$c_1 = -i/2 \tag{68}$$

$$c_2 = a_1 b_1 + a_3 b_{-1} \tag{69}$$

$$c_2 = 0 \tag{70}$$

$$c_3 = a_1 b_2 + a_3 b_0 \tag{71}$$

$$c_3 = i/2 \tag{72}$$

(d)

$$x[n]y[n] = sin((\pi/2)n) + \frac{sin(\pi n)}{2} \tag{73}$$

$$\frac{sin(\pi n)}{2} = 0 \tag{74}$$

Because n is always an integer

$$x[n]y[n] = \frac{e^{j\pi n/2} - e^{-j\pi n/2}}{2i} \tag{75}$$

$$x[n]y[n] = \sum_{k=<N>} c_k e^{jk2\frac{\pi}{N}n} \tag{76}$$

$$c_0 = 0 \tag{77}$$

$$c_1 = \frac{-i}{2} \tag{78}$$

$$c_2 = 0 \tag{79}$$

$$c_{-1} = c_3 = \frac{i}{2} \tag{80}$$

Part c and Part d is equal.

6. (a) The period N of this function is equal to 4. We can use the analysis equations for the discrete time Fourier Series coefficients:

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{4} \sum_{n=0}^{n=3} x[n] e^{-jk\frac{\pi}{2}n}$$

$$a_k = \frac{e^{-jk\pi/2} + 2e^{-jk2\pi/2} + e^{-jk3\pi/2}}{4} \tag{81}$$

$$a_0 = 4/4 = 1 \tag{82}$$

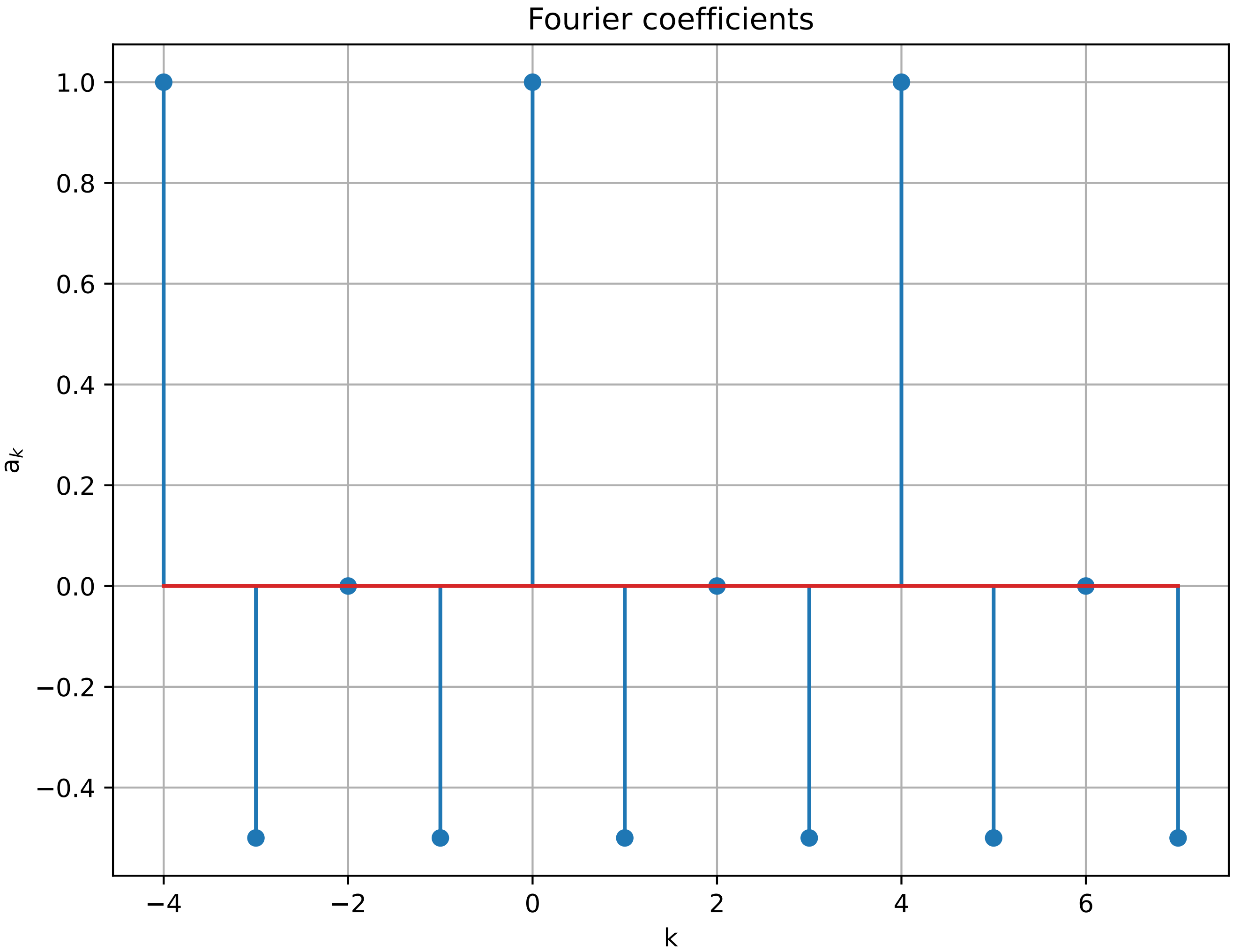
$$a_1 = \frac{e^{-j\pi/2} + 2e^{-j2\pi/2} + e^{-j3\pi/2}}{4} = \frac{-1}{2} \tag{83}$$

$$a_2 = \frac{e^{-j\pi} + 2e^{-j2\pi} + e^{-j3\pi}}{4} = 0 \tag{84}$$

$$a_3 = \frac{e^{-j3\pi/2} + 2e^{-j6\pi/2} + e^{-j9\pi/2}}{4} = \frac{-1}{2} \tag{85}$$

$$a_{0+N} = 1, a_{1+N} = -1/2, a_{2+N} = 0, a_{3+N} = -1/2 \tag{86}$$

The plot is:



(b) y can be expressed as below:

$$y[n] = x[n] - \sum_{m=-\infty}^{\infty} \delta[n + mN + 1] = x[n] - \sum_{m=-\infty}^{\infty} \delta[n + 4m + 1)]$$

since delta function will only equal to 1 at $n = 1 + N$, this will give us the $y[n]$.
Let us find the coefficients of $\sum_{m=-\infty}^{\infty} \delta[n + 4m + 1]$: If we look at 1 period starting from 0, this will be:

$$\sum_{m=-\infty}^{\infty} \delta[n + 4m + 1] = \delta[n + 4 * (-1) + 1] = \delta[n - 3]$$

We will look for the coefficients b_k of $\delta[n - 3]$

$$b_k = \frac{1}{4} \sum_{n=0}^{3} \delta[n - 3] e^{-jk\frac{\pi}{2}n} = \frac{1}{4} e^{-3jk\frac{\pi}{2}}$$

$$b_{0+N} = \frac{1}{4} \tag{87}$$

$$b_{1+N} = \frac{1}{4} e^{-3j\frac{\pi}{2}} \tag{88}$$

$$b_{2+N} = \frac{1}{4} e^{-6j\frac{\pi}{2}} \tag{89}$$

$$b_{3+N} = \frac{1}{4} e^{-9j\frac{\pi}{2}} \tag{90}$$

Using the linearity property of Discrete-Time Fourier series, the coefficients c_k for $y[n]$ will be:

$$c_k = a_k + b_k \tag{91}$$

$$c_{0+N} = a_{0+N} + b_{0+N} = 1 + \frac{1}{4} = \frac{5}{4}, \angle c_{0+N} = 0 \tag{92}$$

$$c_{1+N} = a_{1+N} + b_{1+N} = \frac{-1}{2} + \frac{1}{4}e^{-3j\frac{\pi}{2}}, \angle c_{1+N} = \frac{\pi}{2} \tag{93}$$

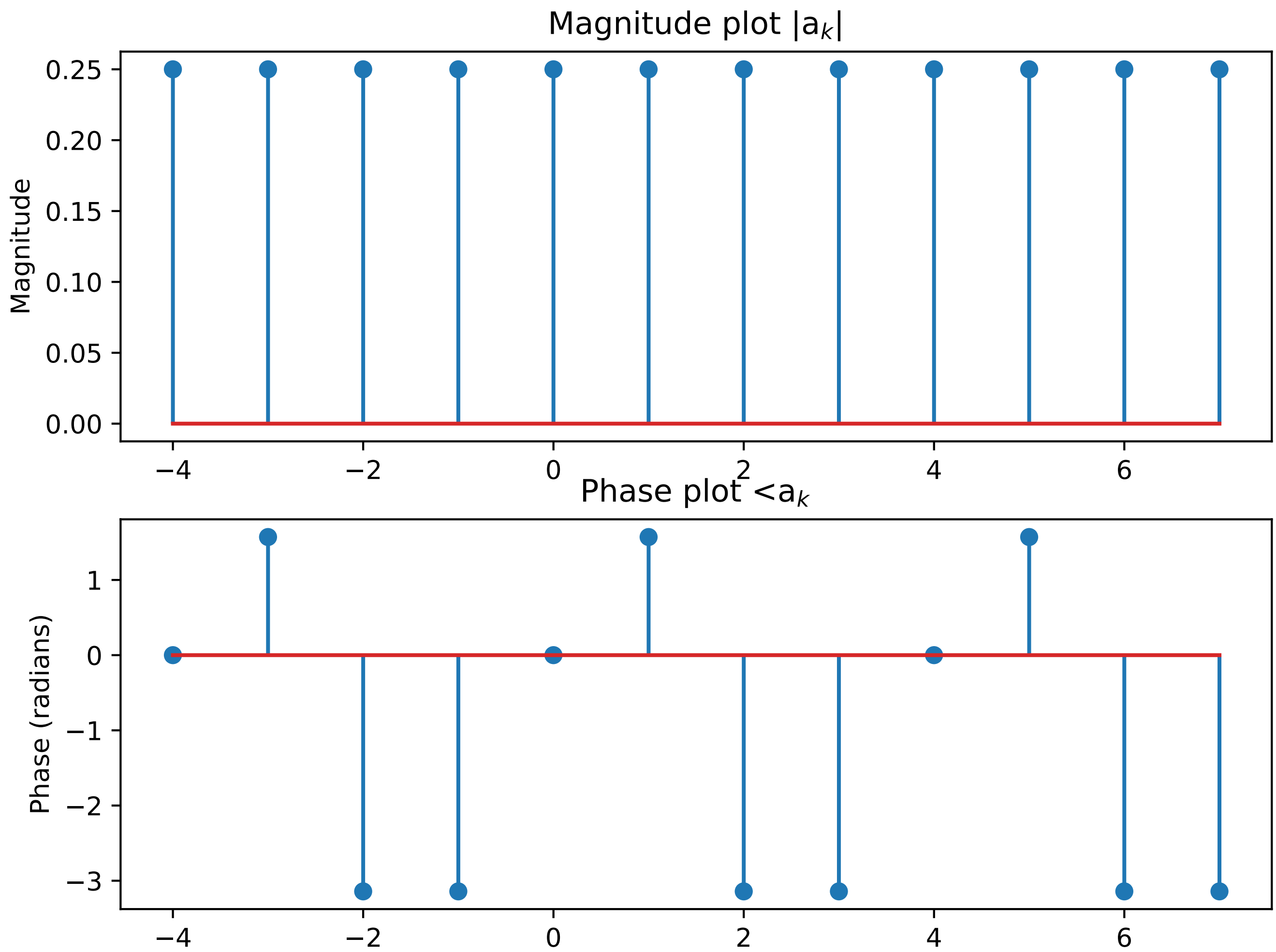
$$c_{2+N} = a_{2+N} + b_{2+N} = 0 + \frac{1}{4}e^{-6j\frac{\pi}{2}}, \angle c_{2+N} = -\pi \tag{94}$$

$$c_{3+N} = a_{3+N} + b_{3+N} = \frac{-1}{2} + \frac{1}{4}e^{-9j\frac{\pi}{2}}, \angle c_{3+N} = -\pi \tag{95}$$

And for all k:

$$|c_k| = \frac{1}{4}$$

These are the phase and magnitude plots of c_k:



7. (a) Remember the below equation:

$$b_k = H(jw_0k)a_k \tag{96}$$

If $y(t) = x(t)$, then in order to satisfy above equation, coefficients b_k and a_k must be the same for all of the k values, and this is possible if $H(jw_0k) = 1$ for all k. We know that $|w| \leq 80$, so $|w_0k| \leq 80$

$$w_0 = 2\pi/T = \frac{2\pi}{\pi/K} = 2K \tag{97}$$

$$|2K * k| \leq 80 \tag{98}$$

$$|k| \leq \frac{40}{K} \tag{99}$$

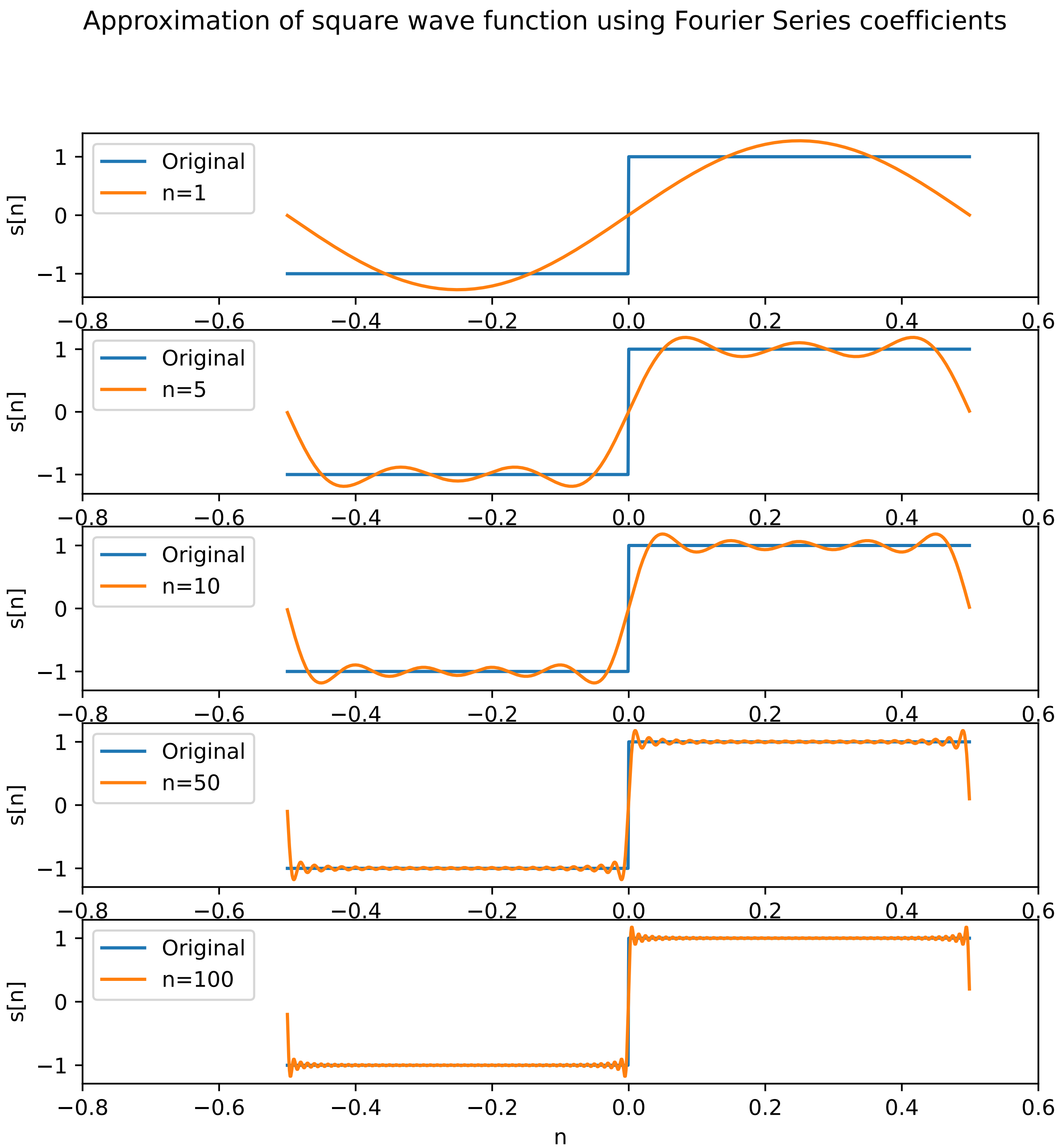
Therefore, the system have the Fourier Series coefficients $a_k = b_k$ in the range of k found above. And because of these functions equal, b_k is 0 when k is out of the range, because of that a_k must be equal to 0 when k out of the range Since the input is directly transmitted as output, this is a pass-band filter.

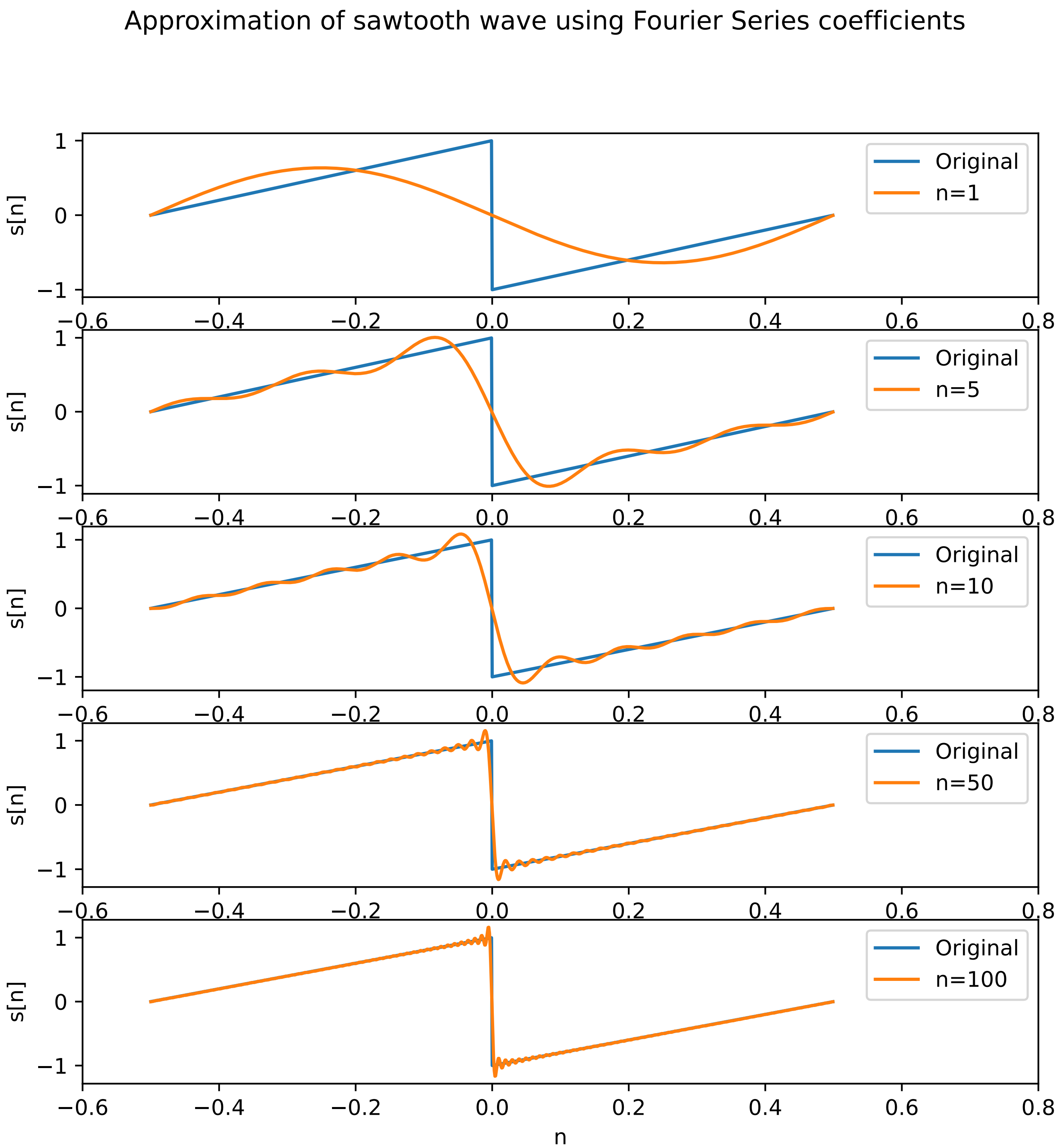
(b) If $y(t) \neq x(t)$, then the system will effect the input coefficients. Specifically, for the coefficients b_k, if k is in the same range found in part a, then it will be the same as a_k. However, if k is out of range, the system will suppress a_k. As a result, b_k for that k values will be equal to 0. And we cannot know a_k. Therefore, this is a low-pass filter, permitting signals with a frequency lower than a value.

8. Requested functions are implemented using the equations 3.94 and 3.95 from the lecture book. Looking at the results at part c and d, it can be clearly seen that with growing n value, the error between the original signals and their Fourier Series Representations are getting less and less. With small n values, very erroneous plots are obtained. However, the Fourier Series Representations with tens of number of coefficients are almost 100% fit the original plot. This is due to the convergence property of Fourier Series, stating that every continuous and also many discontinuous (including square vawe and sawtooth function) periodic signals have a Fourier Series Representation for which the approximation error approaches to 0 as n goes to infinity. The code and the plots are below:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.signal import sawtooth

#Part A
def fourier_coeff(signal, period, n, t):

    w_0 = 2*np.pi/period
    coeffs = np.zeros(n+1, dtype=complex)

    coeffs[0] = np.mean(signal)

    for k in range(1, n+1):
        cos_k = 0
        sin_k = 0
        for m in range(0, len(t)):
            cos_k += signal[m] * np.cos(k * w_0 * t[m])
            sin_k += signal[m] * np.sin(k * w_0 * t[m])
        cos_k /= len(t)
        sin_k /= len(t)
        coeffs[k] = cos_k - 1j * sin_k

    return coeffs

#Part B
def fourier_approx(coeffs, period, n, t):
    w_0 = 2*np.pi/period
    approximated_signal = np.zeros_like(t, dtype=complex)


    for m in range(0, len(t)):
        for k in range(0, n+1):
            approximated_signal[m] += 2* coeffs[k] * np.exp(1j * k * w_0 * t[m])

    return approximated_signal

# Part C
t = np.linspace(-0.5, 0.5, 1000, endpoint=False)
s = np.where(t < 0, -1, 1)

n_values = [1, 5, 10, 50, 100]
fig = plt.figure(figsize=(8, 8))

for i, n in enumerate(n_values):
    coeffs = fourier_coeff(s, 1, n, t)
    approximated_signal = fourier_approx(coeffs, 1, n, t)
    plt.subplot(len(n_values), 1, i+1)
    plt.plot(t, s, label="Original")
    plt.plot(t, approximated_signal, label="n={}".format(n))
    plt.legend()
    plt.xlabel('n')
    plt.ylabel('s[n]')
    plt.xlim(-0.8, 0.6)
plt.suptitle('Approximation of square wave function using Fourier Series coefficients')
plt.show()

fig.savefig('q7-partc.pdf', bbox_inches='tight')
```

```
#Part D
sawtooth_wave = sawtooth(2 * np.pi * t)
n_values = [1, 5, 10, 50, 100]

fig = plt.figure(figsize=(8, 8))

for i, n in enumerate(n_values):
    coeffs = fourier_coeff(sawtooth_wave, 1, n, t)
    approximated_signal = fourier_approx(coeffs, 1, n, t)

    plt.subplot(len(n_values), 1, i+1)
    plt.plot(t, sawtooth_wave, label="Original")
    plt.plot(t, approximated_signal, label="n={}".format(n))
    plt.legend(loc='upper right')
    plt.xlim(-0.6, 0.8)
    plt.xlabel('n')
    plt.ylabel('s[n]')

plt.suptitle('Approximation of sawtooth wave using Fourier Series coefficients')
plt.show()

fig.savefig('q7-partd.pdf', bbox_inches='tight')
```


Approximation of square wave function using Fourier Series coefficients
Original
n=1
n=5
n=10
n=50
n=100
s[n]
n


Approximation of sawtooth wave using Fourier Series coefficients
s[n]
Original
n=1
n=5
n=10
n=50
n=100
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1
0
−1
n