amsmath

Proofs for Conditional Entropy

Chain Rule for Entropy

We aim to prove that:

$$H(Y|X) = H(X,Y) - H(X)$$

Given:

$$H(Y|X) = -\sum_x \sum_y p(x,y) \log p(y|x)$$

$$H(X,Y) = -\sum_x \sum_y p(x,y) \log p(x,y)$$

$$H(X) = -\sum_x p(x) \log p(x)$$

By definition:

$$p(y|x) = \frac{p(x,y)}{p(x)}$$

Substitute $p(y|x)$ into $H(Y|X)$ based on Bayes rule:

$$H(Y|X) = -\sum_x \sum_y p(x,y) \log \left(\frac{p(x,y)}{p(x)} \right)$$

$$= -\sum_x \sum_y p(x,y) \log p(x,y) + \sum_x \sum_y p(x,y) \log p(x)$$

We know $\sum_y p(x,y) = p(x)$ based on above, thus:

$$= H(X,Y) + \sum_x p(x) \log p(x)$$

$$= H(X,Y) - H(X)$$

Hence proved the Chain Rule for Entropy.

Bayes Rule for Conditional Entropy

We aim to prove that:

$$H(Y|X) = H(X|Y) - H(X) + H(Y)$$

Using the chain rule for entropy, we know:

$$H(X,Y) = H(Y|X) + H(X)$$

$$H(X,Y) = H(X|Y) + H(Y)$$

Thus, by equating the two expressions for $H(X,Y)$:

$$H(Y|X) + H(X) = H(X|Y) + H(Y)$$

Rearranging the terms gives us:

$$H(Y|X) = H(X|Y) - H(X) + H(Y)$$