H_2^+ Solution Techniques

AMA3020 Pairs Project - Supplementary Material

- Consider $$\frac{d}{d\mu}\left((1-\mu^2)\frac{dM}{d\mu}\right) + \left[\beta - \frac{m^2}{1-\mu^2} - p^2\mu^2\right]M = 0$$

Compare this to the general Legendre equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_L^m(x)}{dx}\right] + \left[L(L+1) - \frac{m^2}{1-x^2}\right]P_L^m(x) = 0$$

where the solution is in the form of the associated Legendre polynomials

$$P_L^m(x) = \frac{(-1)^m}{2^L L!}(1-x^2)^{m/2}\frac{d^{L+m}(x^2-1)^L}{dx^{L+m}}$$

Hence, when $\beta = L(L+1)$ and $p^2 \to 0$, we have a solution in terms of the associated Legendre polynomials

There is an essential singularity at $\mu = 1$.
We also note
$\mu \in [-1, +1]$ based on prolate spheroidal co-ordinate solutions
Also $\lambda \in [+1, \infty)$.

- Factorisation to get General Solutions

Consider $M(\mu)$ eqn.

Near $\mu = 1$, let $\mu = 1 - \varepsilon$

$1 - \mu^2 \sim 2\varepsilon$

$\varepsilon M'' + M' - \frac{\mu^2 M}{\varepsilon} \approx 0$

$M(\mu) \sim (1-\mu^2)^{|m|/2}$

$\Rightarrow M(\mu) = (1-\mu^2)^{|m|/2} S(\mu)$

where $S(\mu)$ is a smooth function to be calculated.

General solution techniques:
- Expand $S(\mu)$
- Use recurrence relation

$$\frac{\partial}{\partial\lambda}\left((\lambda^2-1)\frac{\partial\Lambda}{\partial\lambda}\right) + \left[\beta - p^2\lambda^2 + 2qR\lambda - \frac{\alpha}{\lambda^2-1}\right] = 0$$

Need exponential decay as $\lambda \to \infty$ to ensure finiteness

At large λ : $\Lambda'' - p^2\Lambda = 0$

$\Lambda(\lambda) \sim e^{-p\lambda}$

Near $\lambda = 1$ (essential singularity):

$\Lambda(\lambda) \sim (\lambda^2-1)^{|M|/2}$ as before

$\Rightarrow \Lambda(\lambda) = (\lambda^2-1)^{|M|/2} e^{-p\lambda} F(\lambda)$

again $F(\lambda)$ is smooth

In general, use:
- recurrance relations
- shooting methods
- continued fractions