$$\phi - \phi_0 = \int_{r_0}^{r} \frac{L}{r^2 \sqrt{2m\left[-U(r)\right] - \frac{L^2}{r^2}}} \, dr$$

For $M = 0$: $\quad U(r) = -\dfrac{UR^2}{2r^2}$

Let $\quad u = \dfrac{1}{r} \qquad U(u) = -\dfrac{UR^2 u^2}{2}$

$du = -r^{-2}$

$$\phi - \phi_0 = \int_{u_0}^{\frac{1}{v}} \frac{L}{\sqrt{MUR^2 u^2 - L^2 u^2}} \, du$$

$$\phi - \phi_0 = \int_{u_0}^{v} \frac{L}{u\sqrt{MUR^2 - L^2}} \, du$$

$$\phi - \phi_0 = \frac{L}{\sqrt{MUR^2 - L^2}} \int_{u_0}^{v} \frac{1}{u} \, du$$

$$\phi - \phi_0 = \frac{L}{\sqrt{MUR^2 - L^2}} \left[\ln\left(\frac{1}{u}\right) - \ln\left(\frac{1}{u_0}\right) \right]$$

let $\quad u = \dfrac{1}{r}$

$$\phi - \phi_0 = \frac{L}{\sqrt{MUR^2 - L^2}} \ln\left(\frac{r}{r_0}\right)$$

Let $\phi_0 = 0$: $\quad \phi = \dfrac{L}{\sqrt{MUR^2 - L^2}} \ln\left(\dfrac{r}{r_0}\right)$