AMA3020 : Solo Project

Deriving Paths in Gravitational Potential

Using approach from 'Classical Mechanics' by M. Benacquasta and J. Romano

We state the general orbit equation:

$$\phi - \phi_0 = \int_{u_0}^{u} \frac{d\bar{u}}{\sqrt{\frac{2m}{\ell^2}\left[E - U\left(\frac{1}{\bar{u}}\right)\right] - \bar{u}^2}}$$

The gravitational potential is given by:

$$U(r) = -\frac{\alpha}{r} \Rightarrow U(1/\bar{u}) = -\alpha\bar{u}$$

$$\phi - \phi_0 = \int_{u_0}^{u} \frac{d\bar{u}}{\sqrt{\frac{2m}{\ell^2}\left[E + \alpha\bar{u}\right] - \bar{u}^2}}$$

The quantity inside the square root is a quadratic function of $\bar{u}$:

$$\frac{2mE}{\ell^2} + \frac{2m\alpha\bar{u}}{\ell^2} - \bar{u}^2$$

Define $\beta \equiv \frac{\ell^2}{\alpha m}$

Rewrite quadratic as:

$$\frac{2E}{\alpha\beta} + \frac{2\bar{u}}{\beta} - \bar{u}^2 = \frac{1}{\beta^2}\Big[\underbrace{\frac{2E\beta}{\alpha} + 2\beta\bar{u} - \beta^2\bar{u}^2}_{\text{complete the square}}\Big]$$

$$= \frac{1}{\beta^2}\left[\frac{2E\beta}{\alpha} + 1 - (\beta\bar{u} - 1)^2\right]$$

Define $e^2 \equiv \frac{2E\beta}{\alpha} + 1$

Also, change variable $\bar{x} \equiv \beta \bar{u} - 1$

$$d\bar{x} = \beta \, d\bar{u}$$

$$\phi - \phi_0 = \int_{x_0}^{x^u} \frac{\beta \, d\bar{u}}{\sqrt{e^2 - x^2}}$$

$$= \int_{x_0}^{x} \frac{d\bar{x}}{\sqrt{e^2 - x^2}}$$

$$\phi - \phi_0 = \arcsin\left(\frac{x}{e}\right) - \arcsin\left(\frac{x_0}{e}\right)$$

Choose co-ordinate system such that:

$\phi_0 = 0$ when $x_0 = e$

$$\phi = \arcsin\left(\frac{x}{e}\right) - \frac{\pi}{2}$$

$$x = e \sin(\phi + \pi/2) = e\cos\phi$$

$$\beta \bar{u} - 1 = e \cos\phi$$

$$\beta \bar{u} = 1 + e\cos\phi$$

$$\frac{\beta}{r} = 1 + e\cos\phi$$

$$\Rightarrow \boxed{r = \frac{\beta}{1 + e\cos\phi}} \qquad \text{where } \beta = \frac{\ell^2}{\alpha M}$$

This equation gives the trajectory of a particle under gravitational potential

To understand the solutions, we must investigate behaviour for differing e.

$$e^2 \equiv \frac{2E\beta}{\alpha} + 1$$

We take α and β to be constant, and hence e is varied by varying energy, E.

The trajectory equation is the equation of an ellipse in polar co-ordinates.

For $0 \leq e < 1$, corresponding to $E < 0$, it can be shown that

$$r = \frac{\beta}{1 + e\cos\phi} \quad \text{satisfies} \quad \frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$

i.e. bound orbits ($E<0$, $0 \leq e < 1$) are ellipses

Consider the critical case $E=0$. $\Rightarrow e = 1$.

$$r = \frac{\beta}{1 + \cos\phi}$$

This is the equation of a parabola with closest approach $\beta/2$.

If $E > 0 \Rightarrow e > 1$.
In this case, the orbit is a hyperbola.

Note:
A <u>parabola</u> is a set of points which are all equidistant from a focus and a directrix.
A <u>hyperbola</u> is when the difference of distances between a set of points in a plane to two fixed points is a positive constant.

Ellipses, parabolas and hyperbolas are all 'conic sections'. Curves generated from the intersection of a plane and a right circular cone.