Biostatistics
Week III

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ACIBADEM
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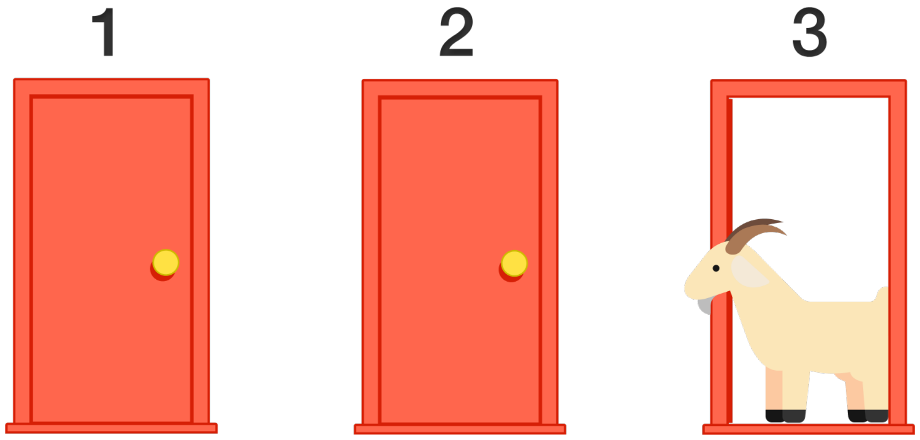
ÜNİVERSİTESİ

Monty Hall Problem



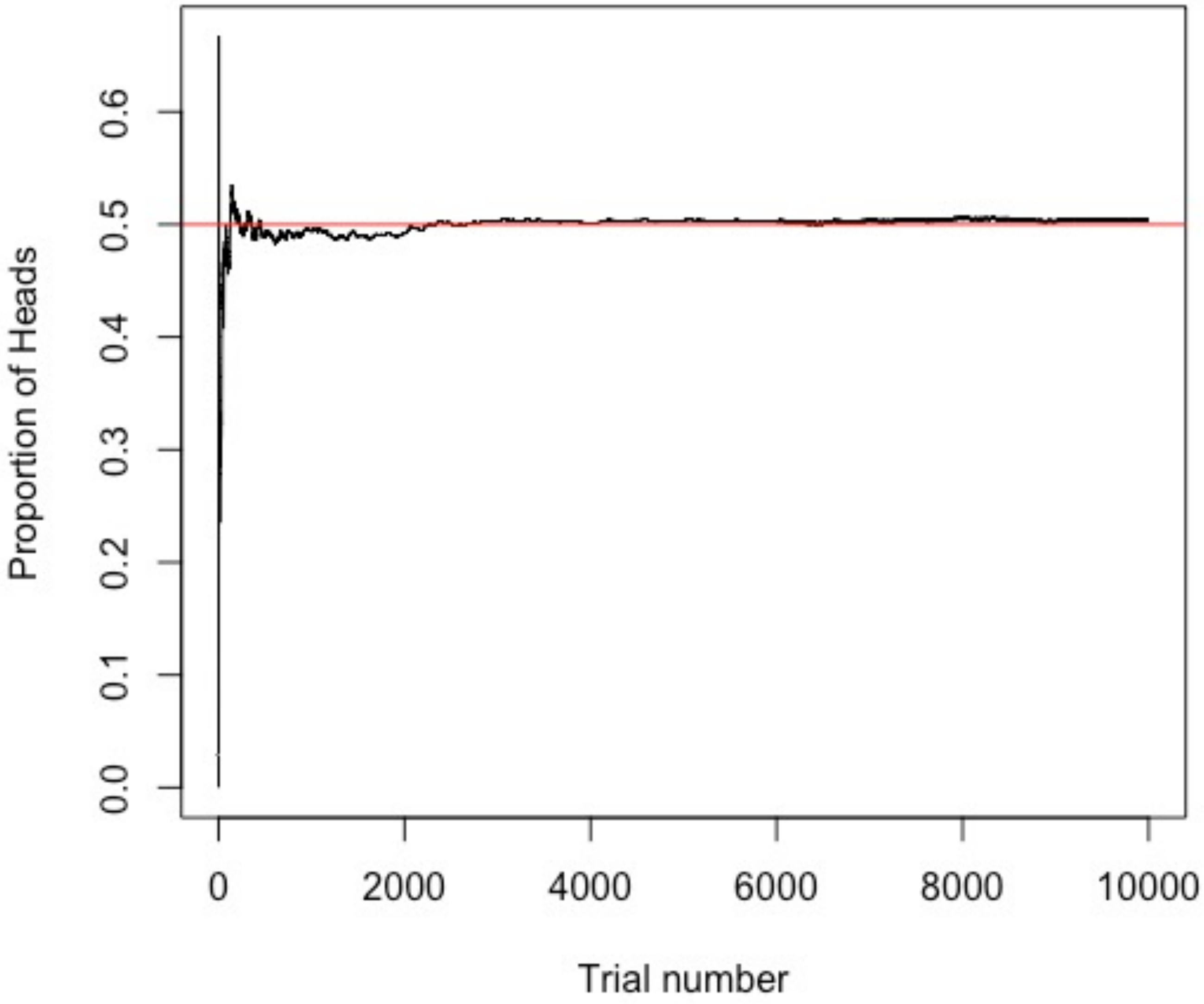
- Suppose there are three doors,
  - behind one there is a car,
  - behind the others: goats
- You select one door without knowing what's behind
- Then, one of the doors behind which is a goat is opened
- In the end, there are two closed doors, behind one is a car, behind the other is a goat
- Would you
  - **stick** to the door you selected previously, or
  - **switch** to the other door?

Probability

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

- P(E): probability of event E,
  - e.g., head in tossing a coin
- n(E): number of times event E occurs (out of n)
- n: number of trials

Probability – Tossing a (fair) coin



Probability - Definitions

- **Experiment:** a process that produces an outcome/outcomes
  Sample Space (Ω): the set of all possible outcomes from an experiment
- **Event:** any set of outcomes of an experiment

Probability - Definitions

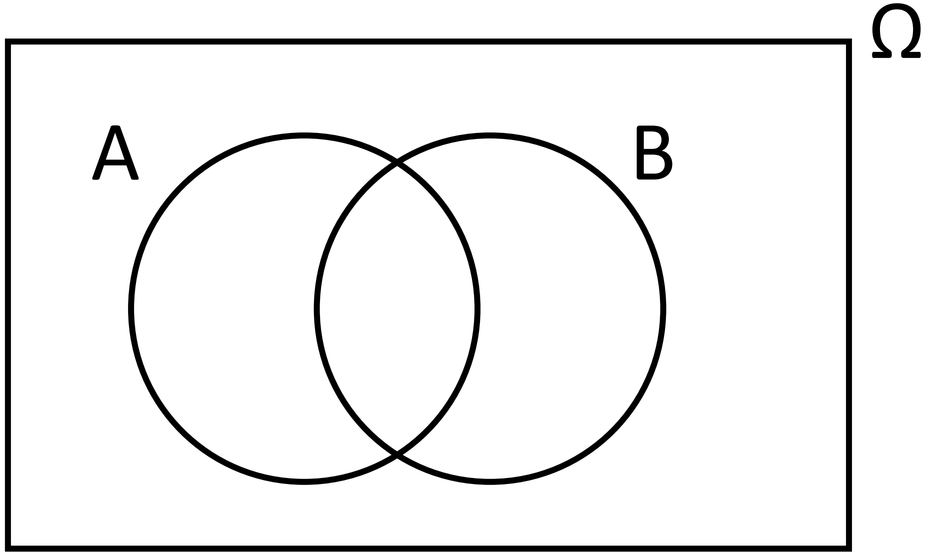
- **Experiment:** flipping a coin and rolling a die at the same time
- **Sample Space:**

  Ω = {(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6),
  (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6),}
- **Event:**

  A: {rolling an even number} P(A) = 6 / 12

  B: {getting heads and an odd number} P(B) = 3 / 12

Probability - Properties



- $P(\Omega) = 1$

- $0 \leq P(A) \leq 1$
- $P(A^c) = 1 - P(A)$

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

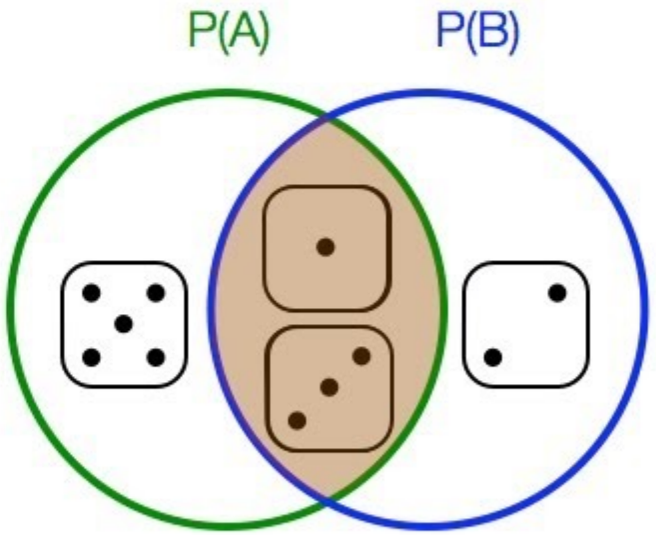
- If $A \cap B$ is an empty set (i.e., if A and B do not occur at the same time), A and B are called disjoint (mutually-exclusive)

Conditional Probability

- P(birth weight > 3500 g)
- P(birth weight > 3500 g | Male)
- P(birth weight > 3500 g | Female)

Conditional Probability

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$



We're rolling a fair die

- Given that the value is odd
- What is the probability that the value is less than 4?

$$P(B|A) = \frac{2/6}{3/6} = \frac{2}{3}$$

https://www.youtube.com/watch?v=H02B3aMNKzE

Independence

- If knowing that event A occurred doesn't change the probability of event B, A and B are **independent** events

$$P(A \cap B) = P(A) \times P(B)$$
$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$

- Independence does not necessarily indicate $P(A \cap B) = \emptyset$

Independence - Example

- Suppose that we roll a pair of fair dice, so each of the 36 possible outcomes is equally likely
  - Let A denote the event that the first die lands on 3
  - Let B be the event that the sum of the dice is 8, and
  - Let C be the event that the sum of the dice is 7
- Are A and B independent?
- Are A and C independent?

Independence - Example

$A : \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}$
$B : \{(2,6),(6,2),(3,5),(5,3),(4,4)\}$
$C : \{(3,4),(4,3),(2,5),(5,2),(1,6),(6,1)\}$

$P(A) = 6/36,\ P(B) = 5/36,\ P(C) = 6/36$
$P(A \cap B) = 1/36,\ P(A \cap C) = 1/36$
$P(A \cap B) \neq P(A)P(B),\ P(A \cap C) = P(A)P(C)$

Bayes' Theorem

- A formula that describes how to update the probabilities of hypotheses (H) when given evidence (E)

$$P(H \mid E) = \frac{P(E \mid H)P(H)}{P(E)}$$

- It can be used to powerfully reason about a wide range of problems involving belief updates

Bayes’ Theorem

$$P(H \mid E) = \frac{P(E \mid H)P(H)}{P(E)}$$

- $P(H)$ is the **prior probability**
- $P(H \mid E)$ is the **posterior probability**
- $\frac{P(E|H)}{P(E)}$ is the **likelihood ratio**
- $P(E)$ is the **evidence**

Bayes' Theorem

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

$$= \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)}$$

Bayes' Theorem – Example

- Given the following statistics, what is the probability that a woman over 50 has cancer if she has a positive mammogram result?
  - One percent of women over 50 have breast cancer
  - Ninety percent of women who have breast cancer test positive on mammograms
  - Eight percent of women will have false positives

Bayes' Theorem – Example

- C: breast cancer, H: healthy, M⁺ : positive mammogram result

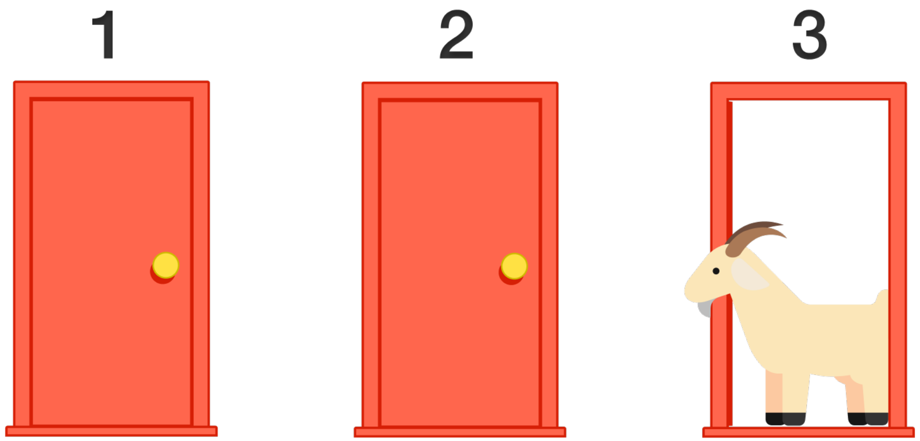
$$P(C) = 0.01$$
$$P(H) = 0.99$$
$$P(M^{+}|C) = 0.9$$
$$P(M^{+}|H) = 0.08$$

$$P(C|M^{+}) = \frac{P(M^{+}|C)P(C)}{P(M^{+}|C)P(C) + P(M^{+}|H)P(H)}$$

$$= \frac{0.9 \times 0.01}{0.9 \times 0.01 + 0.08 \times 0.99} \approx 0.1$$

Monty Hall Problem Revisited

- A: prize is behind the **selected** door
- B: prize is behind the **opened** door
- C: prize is behind the **remaining** door
- H_B: host opens door B



$$P(A) = P(B) = P(C) = \frac{1}{3}$$

We can also conclude that:

$$P(H_B|A) = \frac{1}{2} \qquad P(H_B|B) = 0 \qquad P(H_B|C) = 1$$

Monty Hall Problem Revisited

$$P(A) = P(B) = P(C) = \frac{1}{3}$$

$$P(H_B|A) = \frac{1}{2} \qquad P(H_B|B) = 0 \qquad P(H_B|C) = 1$$

$$P(A|H_B) = \frac{P(H_B|A)P(A)}{P(H_B|A)P(A)+P(H_B|B)P(B)+P(H_B|C)P(C)} = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+0\cdot\frac{1}{3}+1\cdot\frac{1}{3}} = \frac{1}{3}$$

$$P(C|H_B) = \frac{P(H_B|C)P(C)}{P(H_B|A)P(A)+P(H_B|B)P(B)+P(H_B|C)P(C)} = \frac{1\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3}+0\cdot\frac{1}{3}+1\cdot\frac{1}{3}} = \frac{2}{3}$$

Therefore, it is probabilistically advantageous to SWITCH

Brief Summary

- Probability can be defined as the extent to which an event is likely to occur
  - measured by the ratio of the favorable cases to the whole number of cases possible
- Conditional probability allows one to update of the probability of an event based on new information
- Bayes’ theorem is a simple formula for determining conditional probability