Fully Homomorphic Encryption Using Ideal Lattices

Presenter: Alison Tsai-Yin Lin

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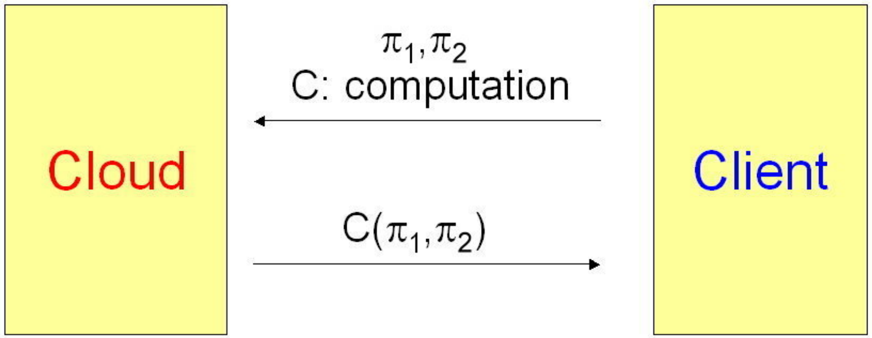
Cloud computing problem



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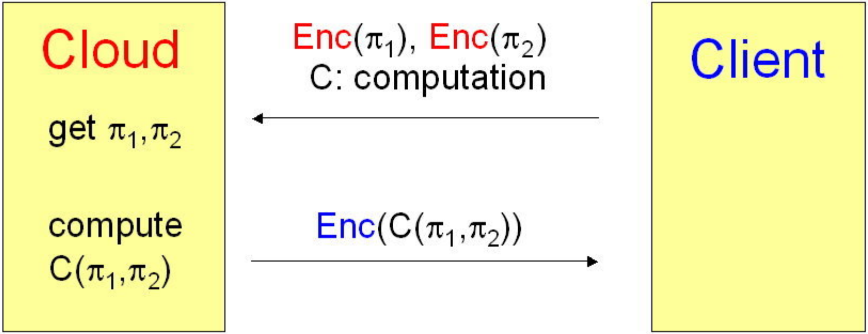

- How to keep $\pi_1, \pi_2, C(\pi_1, \pi_2)$ private from both others and Cloud?

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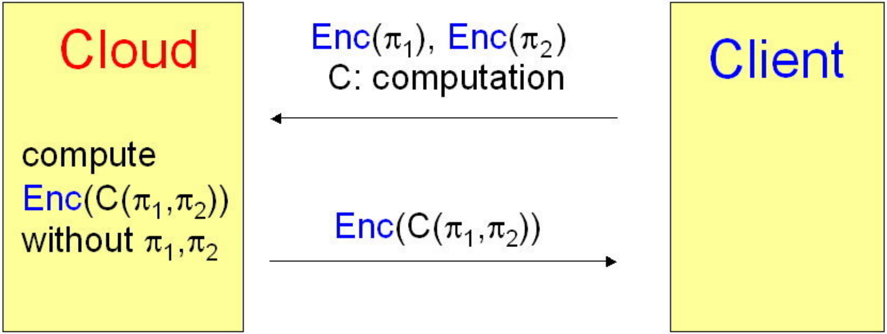

- Use **homomorphic encryption scheme** to do this.

- Example: RSA is a multiplicatively homomorphic encryption scheme, i.e.,
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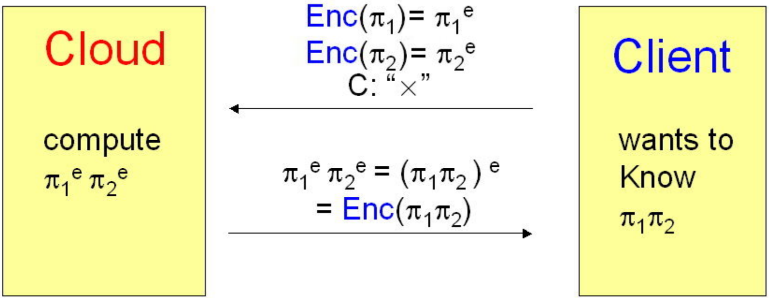

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- A cryptosystem which supports both " $+$ " and " $\times$ " is called a fully homomorphic encryption scheme
- Idea: Use mod. mod is homomorphic under " $+$ " and " $\times$ ".

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  Evaluate[pk, C, Enc(pk,π_1),... , Enc(pk,π_t)]
  = Enc[pk, C(π_1,..., π_t)], for all circuit C.

Factor Ring and Lattice

- Let $f(x) \in \mathbb{Z}[x]$ with degree n.
- Let $R := \mathbb{Z}[x]/f(x)$.
- Identify R with n-dim integer lattice $\mathbb{Z}^n$ by

$$a_0 + \cdots + a_{n-1}x^{n-1} \in \mathbb{Z}[x]/f(x) \longleftrightarrow (a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n$$

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- additive subgroup of R $\longleftrightarrow$ additive subgroup of $\mathbb{Z}^n$, i.e., sublattice of $\mathbb{Z}^n$
- Let $I \subseteq R$ be an ideal. $I \longleftrightarrow$ sublattice of $\mathbb{Z}^n$.

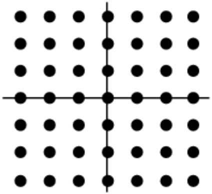
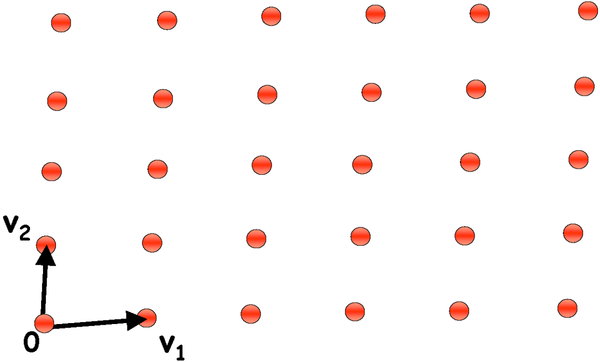


Figure: $\mathbb{Z}^2$ lattice

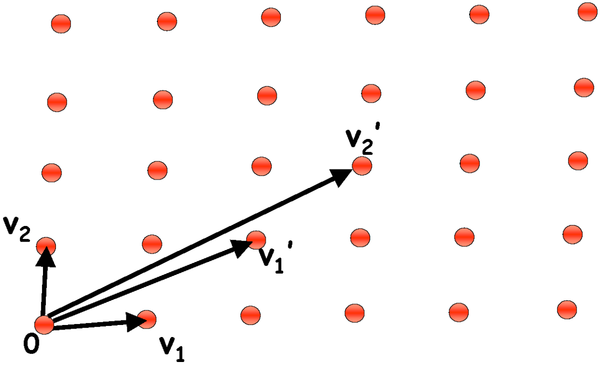
Sublattice of $\mathbb{Z}^n$ and basis

- $B = \{v_1, v_2\}$ is a basis of the lattice $L \subseteq \mathbb{Z}^n$.



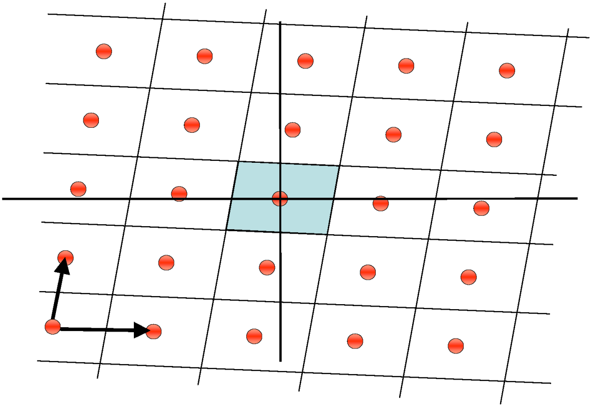
Sublattice of $\mathbb{Z}^n$ and basis

- $B_{sk} = \{v_1, v_2\}$ and $B_{pk} = \{v_1', v_2'\}$ are both bases of the lattice L.



Sublattice of $\mathbb{Z}^n$ and basis

- Let $\mathcal{P}(B_{sk})$ be the parallelepiped of $B_{sk} = \{v_1, v_2\}$, i.e., $\mathcal{P}(B_{sk}) := \{c_1 v_1 + c_2 v_2 \mid c_1, c_2 \in [-\frac{1}{2}, \frac{1}{2})\}$



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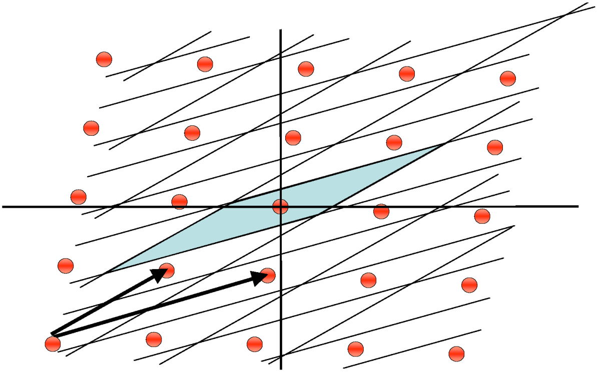
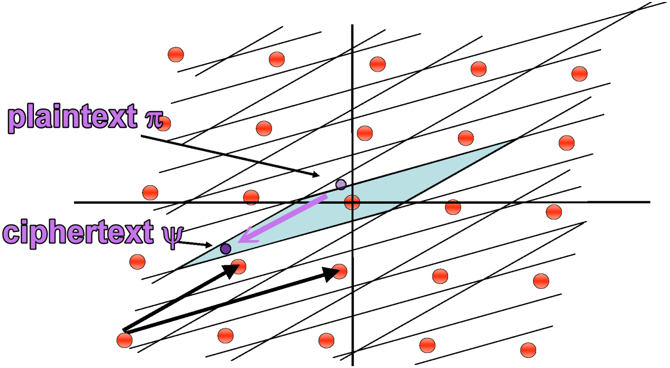
- 

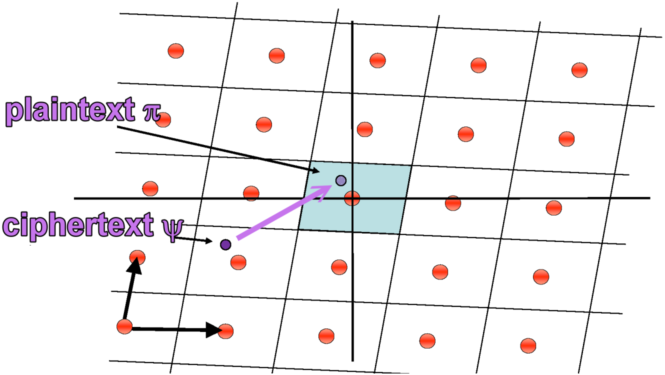
Sketch of the Encryption Scheme

- Enc: $\pi \mapsto \psi = (\pi \mod B_{pk})$
- Enc is homomorphic under " $+$ "," $\times$ "



Sketch of the Encryption Scheme

- Enc: $\pi \mapsto \psi = (\pi \mod B_{pk})$
- Dec: $\psi \mapsto \pi = (\psi \mod B_{sk})$



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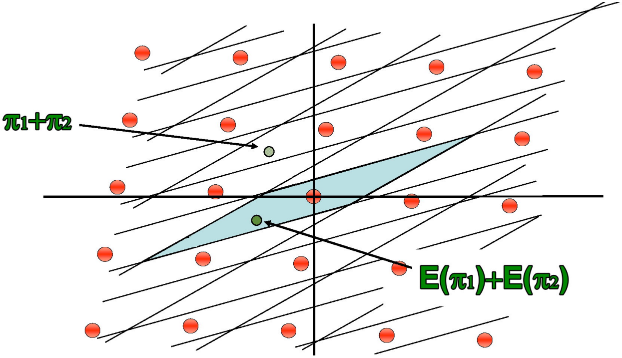

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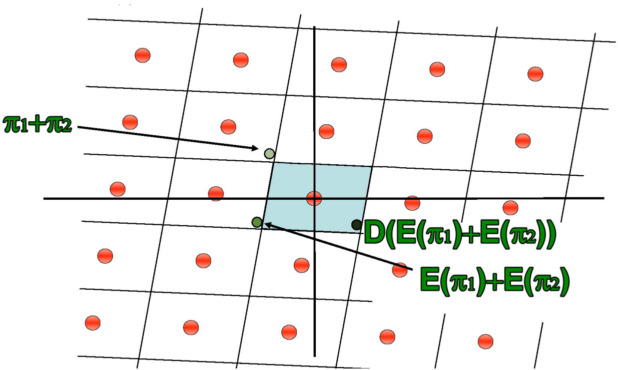

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- We call the encryption scheme is **bootstrappable** if Dec$\in \mathcal{C}$

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- Q: How long can a plaintext be before being inputted to a circuit C? (r_{ENC})
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- Let r_{ENC}, r_{DEC} be radii s.t.

$$\mathcal{C} = \{C : R \rightarrow R \mid \forall \pi_i \in B(r_{ENC}), C(\pi_1, ..., \pi_t) \in B(r_{DEC})\}$$

Analyze the depth of permitted circuit

- To analyze r_{ENC}, r_{DEC}, d, let's see how "$+$" and "$\times$" increse the length of vectors

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- Now we have ideas of circuit depth of the permitted circuits. We are going to reduce the decryption circuit. Let's see the encryption scheme more concretely.

- Parameters: Ring $R = \mathbb{Z}[x]/(f(x))$, basis B_I of ideal lattice I, radii r_{DEC} and r_{ENC}, "$+$" and "$\times$" in R.

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- Add(B_{pk}, ψ_1, ψ_2): Output $\psi \leftarrow \psi_1 + \psi_2 \mod B_{pk}$
- Mult(B_{pk}, ψ_1, ψ_2): Output $\psi \leftarrow \psi_1 \cdot \psi_2 \mod B_{pk}$

Reduce the decryption circuit - Effort 1

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  $= \psi - \lfloor v_{sk} \times \psi \rceil \mod B_I$
  for some $v_{sk} \in \mathbb{Q}[x]/(f(x))$.

  Remark v_{sk} and $v_{sk} \times \psi \in \mathbb{Q}[x]/(f(x)) \approx \mathbb{Q}^n$

Reduce the decryption circuit - Effort 2

- Recall Dec: $\psi \mapsto \psi - \lfloor v_{sk} \times \psi \rceil \mod B_I$
- Hide v_{sk} in $\{t_1, ..., t_K\}$
- secret $S \subseteq \{1, ..., K\}$ s.t. $v_{sk} = \sum_{i \in S} t_i \mod B_I$. So $v_{sk} \times \psi = \sum_{i \in S}(t_i \times \psi)$

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- Encryptor: Compute $t_i \times \psi$ for all $i = 1, ..., K$ and sends those to Decryptor
- Decryptor: Just **add up** $\sum_{i \in S}(t_i \times \psi) = v_{sk} \times \psi$ without doing multiplication
- Theorem bootstrappable when $|S| \leq \frac{\log(r_{DEC}/m)}{\alpha \cdot 2^c \cdot \log(\gamma_R \cdot r_{ENC})}$

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- Lemma For $a_1, \ldots, a_t \in \mathbb{Q}$ with $\sum_i a_i \mod 1 \in [-\frac{1}{4}, \frac{1}{4}]$, $\exists\ z_1, \ldots, z_{t+1} \in \mathbb{Z}$ s.t. $\lfloor \sum_1^t a_i \rceil = \sum_1^{t+1} z_i$

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  Proof Let $T := \lceil log_2 t \rceil + 2$. i.e., $2^{-T} \leq \frac{1}{4t}$.

Reduce the decryption circuit - Effort 3

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$$\vdots$$

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$$
\begin{array}{ll}
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a_2 = \cdots a_2^{(1)} a_2^{(0)} . a_2^{(-1)} a_2^{(-2)} \cdots\cdots & z_2 = \boxed{\cdots a_2^{(1)} a_2^{(0)}} \; a_2^{(-1)} \cdots a_2^{(-T)} \quad = z_{t+1} \\
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\end{array}
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\vdots & \vdots \\
a_t = \cdots a_t^{(1)} a_t^{(0)} . a_t^{(-1)} a_t^{(-2)} \cdots\cdots & z_t = \boxed{\cdots a_t^{(1)} a_t^{(0)}}
\end{array}
\quad
\left.\begin{array}{|c|}
a_1^{(-1)} \cdots a_1^{(-T)} \\
a_2^{(-1)} \cdots a_2^{(-T)} \\
\\
a_t^{(-1)} \cdots a_t^{(-T)}
\end{array}\right. = z_{t+1}
$$

  Remark If $x \mod 1 \in [-\frac{1}{4}, \frac{1}{4}]$ then $\lfloor x \rceil = \lfloor x + \epsilon \rceil$ for any $|\epsilon| < \frac{1}{4}$

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  Remark If $x \mod 1 \in [-\frac{1}{4}, \frac{1}{4}]$ then $\lfloor x \rceil = \lfloor x + \epsilon \rceil$ for any $|\epsilon| < \frac{1}{4}$
  $|a_1^{(-T-1)} \cdot 2^{(-T-1)} + a_1^{(-T-2)} \cdot 2^{(-T-2)} + \cdots| < 2^{-T} \leq \frac{1}{4t}$

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$$\Rightarrow \sum_{j=-T-1}^{\infty} a_i^{(j)} < t \cdot \frac{1}{4t} = \frac{1}{4}$$

Summary

- "mod an ideal" in a ring is well-defined but "mod a basis of a lattice" causes some problem for long vectors.
- So we need to pull those vector back by bootsrapping.
- In order to be bootstrappable, we analyze the restriction of DEC circuit. And then reduce it.