Math 541: Written Homework 5
Due: 03/13

Theory : So in class, we saw how using the points x_0, $x_0 + h$, and $x_0 - h$, we were able to write the *centered difference* approximation to $f'(x_0)$ in the form

$$f'(x_0) \sim \frac{1}{2h}\left(f(x_0+h) - f(x_0-h)\right).$$

First, using Taylor series arguments, show that we can write the centered difference approximation to the second derivative as

$$f''(x_0) \sim \frac{1}{h^2}\left(f(x_0+h) - 2f(x_0) + f(x_0-h)\right).$$

What is the order of the error in this approximation? Repeat your derivation of this approximation using Lagrange interpolation. Could you use three points to approximate a third derivative of $f(x)$? Why or why not?

Coding : So as we saw from last week's homework, for the function

$$f(x) = \sin(\lambda x) + \cos(\lambda^2 x),$$

using both Lagrange and Hermite interpolation, we were not able to make λ very large before either method broke down. Now, starting with the mesh $(0, .1, .2, \cdots, 1)$, generate a cubic spline approximation to $f(x)$ for $\lambda = 1$, 2, and 5. Plot the absolute error of the spline approximation and turn in these plots. Discuss how this is different than using Lagrange and Hermite interpolation. For $\lambda = 5$, how small do you need to make the interpolation mesh size, say δx, in order to get the error close to 10^{-2}? Find the step size necessary to get the error close to 10^{-2} for $\lambda = 10$, 15, and 20. Plot δx as a function of λ and turn in this plot. Describe the plot and argue why splines are better for rapidly oscillating functions than Lagrange or Hermite interpolation.