Homework 4

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1 Introduction

In order to graph the functions the Fourier Transform modeled I wrote a function which creates piecewise functions. The version of the script in this homework set is the version which created the piecewise function for Boas 7.5.7.

2 Boas 7.5.2

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < x < \pi. \end{cases}$$

2.1 Integral evaluation

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx =$$

$$\frac{1}{\pi}[\int_{-\pi}^{0} f(x)\cos(nx)dx + \int_{0}^{\frac{\pi}{2}} f(x)\cos(nx)dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\cos(nx)dx] =$$

$$\frac{1}{\pi}[\int_{-\pi}^{0} 0 * \cos(nx)dx + \int_{0}^{\frac{\pi}{2}} 1 * \cos(nx)dx + \int_{\frac{\pi}{2}}^{\pi} 0 * \cos(nx)dx] =$$

$$\frac{1}{\pi}[\int_{0}^{\frac{\pi}{2}} \cos(nx)dx] = \frac{1}{n\pi}\sin(n\frac{\pi}{2})$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx =$$

$$\frac{1}{\pi}[\int_{-\pi}^{0} f(x)\sin(nx)dx + \int_{0}^{\frac{\pi}{2}} f(x)\sin(nx)dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\sin(nx)dx] =$$

$$\frac{1}{\pi}[\int_{-\pi}^{0} 0 * \sin(nx)dx + \int_{0}^{\frac{\pi}{2}} 1 * \sin(nx)dx + \int_{\frac{\pi}{2}}^{\pi} 0 * \sin(nx)dx] =$$

$$\frac{1}{\pi}[\int_{0}^{\frac{\pi}{2}} \sin(nx)dx] = -\frac{1}{n\pi}\cos(n\frac{\pi}{2}) + \frac{1}{n\pi}$$

2.2 Average value

$$\frac{1}{2\pi}[\int_{-\pi}^{0} 0dx + \int_{0}^{\frac{\pi}{2}} 1dx + \int_{\frac{\pi}{2}}^{\pi} 0dx] =$$

$$\frac{\pi}{2} * \frac{1}{2\pi} = \frac{1}{4}$$

2.3 Constants

$$a_n = \begin{cases} \frac{1}{4}, & n = 0, \\ 0, & n \mod 2 = 0, \\ (\sqrt{-1})^{n+3}\frac{1}{n\pi}, & n \mod 2 \neq 0. \end{cases}$$

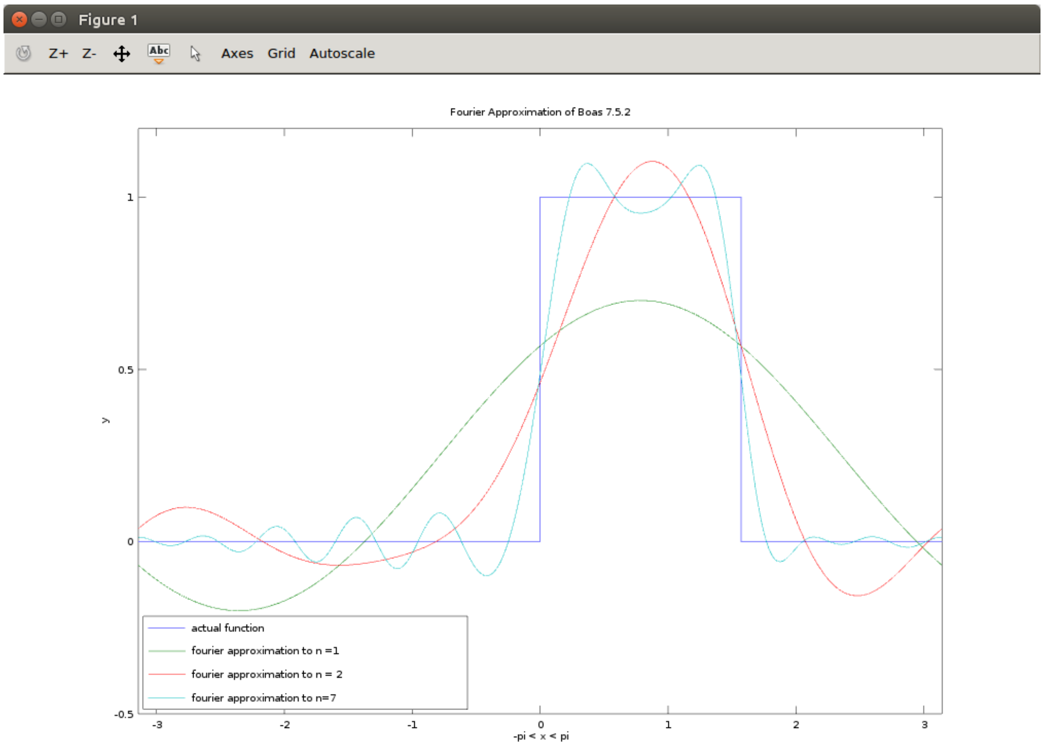
$$b_n = \begin{cases} 0, & n = 0, \\ \frac{1}{n\pi}(1 - \cos(n\frac{\pi}{2})), & n \neq 0. \end{cases}$$

2.4 Series

$$f(x) = \frac{1}{4} + \frac{1}{\pi}\big(cos(x) - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} - \frac{\cos(7x)}{7} + ...\big)$$

$$+\frac{1}{\pi}\big(\sin(x) + \frac{2\sin(2x)}{2} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{2\sin(6x)}{6} + ...\big)$$

2.5 Plot



Everything looks good! We can see that as we add more terms to our Fourier series our function converges to the original piecewise!

3 Boas 7.5.4

$$f(x) = \begin{cases} -1, & -\pi < x < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < x < \pi. \end{cases}$$

3.1 Integral evaluation

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx =$$

$$\frac{1}{\pi} (\int_{-\pi}^{\frac{\pi}{2}} f(x) \cos(nx) dx + \int_{\frac{\pi}{2}}^{\pi} f(x) \cos(nx) dx) =$$

$$\frac{1}{\pi}\Big(\int_{-\pi}^{\frac{\pi}{2}} -\cos(nx)dx + \int_{\frac{\pi}{2}}^{\pi} \cos(nx)dx\Big) =$$

$$\frac{1}{\pi}\Big(-\frac{\sin(nx)}{n}\Big|_{-\pi}^{\frac{\pi}{2}} + \frac{\sin(nx)}{n}\Big|_{\frac{\pi}{2}}^{\pi}\Big) = \frac{1}{n\pi}\Big(-\sin(n\frac{\pi}{2}) + \sin(-n\pi) + \sin(n\pi) - \sin(n\frac{\pi}{2})\Big) =$$

$$\frac{1}{n\pi}\Big(-2\sin(n\frac{\pi}{2})\Big)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx =$$

$$\frac{1}{\pi}\Big(\int_{-\pi}^{\frac{\pi}{2}} f(x)\sin(nx)dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\sin(nx)dx\Big) =$$

$$\frac{1}{\pi}\Big(\int_{-\pi}^{\frac{\pi}{2}} -\sin(nx)dx + \int_{\frac{\pi}{2}}^{\pi} \sin(nx)dx\Big) =$$

$$\frac{1}{\pi}\Big(\frac{\cos(nx)}{n}\Big|_{-\pi}^{\frac{\pi}{2}} - \frac{\cos(nx)}{n}\Big|_{\frac{\pi}{2}}^{\pi}\Big) = \frac{1}{n\pi}\Big(\cos(n\frac{\pi}{2}) - \cos(-n\pi) - \cos(n\pi) + \cos(n\frac{\pi}{2})\Big) =$$

$$\frac{1}{n\pi}\Big(2\cos(n\frac{\pi}{2}) - 2\cos(n\pi)\Big)$$

3.2 Average value

$$\frac{1}{2\pi}\Big(\int_{-\pi}^{\frac{\pi}{2}} -1dx + \int_{\frac{\pi}{2}}^{\pi} 1dx\Big) =$$

$$\frac{1}{2\pi}\Big(-x\Big|_{-\pi}^{\frac{\pi}{2}} + x\Big|_{\frac{\pi}{2}}^{\pi}\Big)\Big) = \frac{1}{2\pi}\Big(-\frac{\pi}{2} - \pi + \pi - \frac{\pi}{2}\Big) = -\frac{1}{2}$$

3.3 Constants

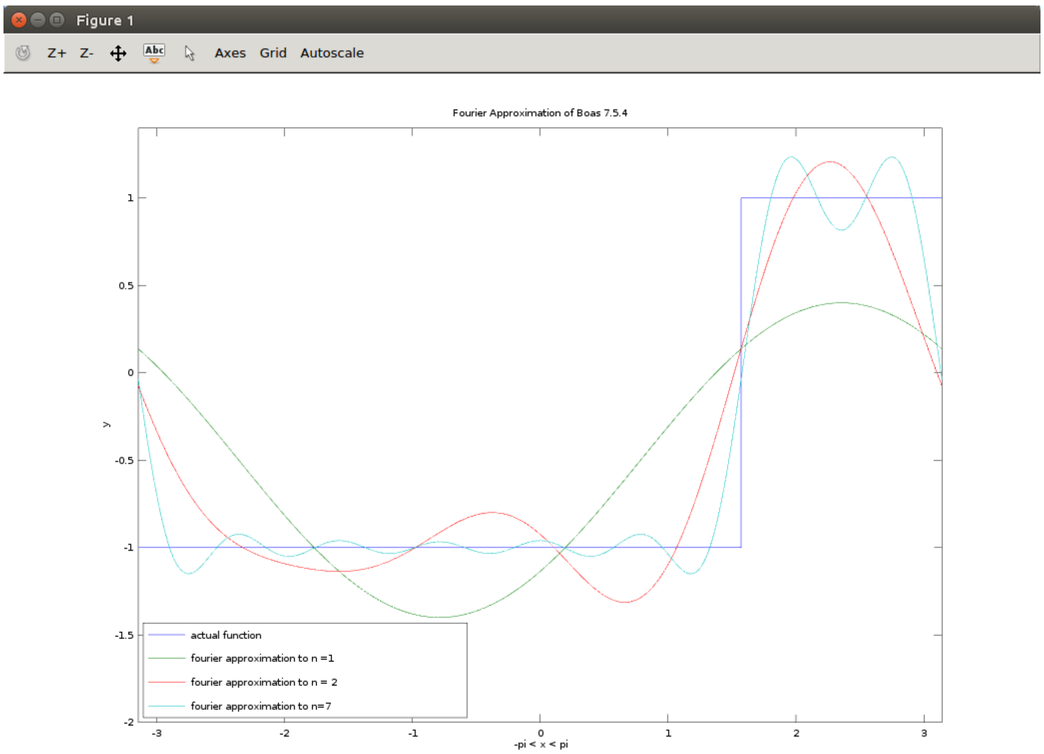
$$a_n = \begin{cases} -\frac{1}{2}, & n = 0, \\ \frac{1}{n\pi}\Big(-2\sin(n\frac{\pi}{2})\Big), & n \neq 0. \end{cases}$$

$$b_n = \begin{cases} 0, & n = 0, \\ \frac{1}{n\pi}\Big(2\cos(n\frac{\pi}{2}) - 2\cos(n\pi)\Big), & n \neq 0. \end{cases}$$

3.4 Series

$$f(x) = -\frac{1}{2} + \frac{1}{\pi}\Big(-2\cos(x) + \frac{2\cos(3x)}{3} - \frac{2\cos(5x)}{5} + \frac{2\cos(7x)}{7} + ...\Big)$$
$$+\frac{1}{\pi}\big(2\sin(x) - \frac{4\sin(2x)}{2} + \frac{2\sin(3x)}{3} + \frac{2\sin(5x)}{5} - \frac{4\sin(6x)}{6} + ...\big)$$

3.5 Plot



Look at that symmetry of our n = 7 Fourier transform! Its beautiful! Our series approximates the piecewise function wonderfully and it clearly converges as more terms are added.

4 Boas 7.5.7

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$$

4.1 Integral evaluation

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx =$$

$$\frac{1}{\pi}[\int_{-\pi}^{0} f(x)\cos(nx)dx + \int_{0}^{\pi} f(x)\cos(nx)dx] =$$

$$\frac{1}{\pi}[\int_{-\pi}^{0} 0 * \cos(nx)dx + \int_{0}^{\pi} x\cos(nx)dx] = \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)dx =$$

Integration by parts

$$u = \frac{1}{\pi}x \qquad dv = \cos(nx)dx$$

$$du = \frac{1}{\pi}dx \qquad v = \int \cos(nx)dx = \frac{\sin(nx)}{n}$$

$$\int_{0}^{\pi} udv = v*u - \int_{0}^{\pi} vdu = \frac{1}{\pi}\Big(\frac{x\sin(nx)}{n}\Big|_{0}^{\pi} - \int_{0}^{\pi}\frac{\sin(nx)}{n}dx\Big) = -\frac{1}{\pi}\int_{0}^{\pi}\frac{\sin(nx)}{n}dx$$

$$\frac{-\cos(nx)}{\pi n^2}\Big|_{0}^{\pi} = \frac{-\cos(n\pi)+\cos(0)}{\pi n^2} = \frac{1}{\pi n^2}(1-\cos(n\pi))$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx =$$

$$\frac{1}{\pi}[\int_{-\pi}^{0} f(x)\sin(nx)dx + \int_{0}^{\pi} f(x)\sin(nx)dx] =$$

$$\frac{1}{\pi}[\int_{-\pi}^{0} 0 * \sin(nx)dx + \int_{0}^{\pi} x\sin(nx)dx] = \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)dx =$$

Integration by parts

$$u = \frac{1}{\pi}x \qquad dv = \sin(nx)dx$$

$$du = \frac{1}{\pi}dx \qquad\qquad v = \int \sin(nx)dx = \frac{-\cos(nx)}{n}$$

$$\int_0^\pi udv = v{*}u - \int_0^\pi vdu = \frac{1}{\pi}\Big(-\frac{x\cos(nx)}{n}\Big|_0^\pi - \int_0^\pi \frac{-\cos(nx)}{n}dx\Big) = \frac{1}{\pi}\Big(\frac{-\pi\cos(n\pi)}{n} - \frac{-sin(nx)}{n^2}\Big|_0^\pi\Big)$$

$$\frac{1}{\pi}\Big(\frac{-\pi\cos(n\pi)}{n} - 0\Big) = \frac{-\cos(n\pi)}{n}$$

4.2 Average value

$$\frac{1}{2\pi}\Big(\int_{-\pi}^0 0dx + \int_0^\pi xdx\Big) = \frac{1}{2\pi}\Big(\frac{x^2}{2}\Big|_0^\pi\Big) = \frac{\pi^2}{4\pi} = \frac{\pi}{4}$$

4.3 Constants

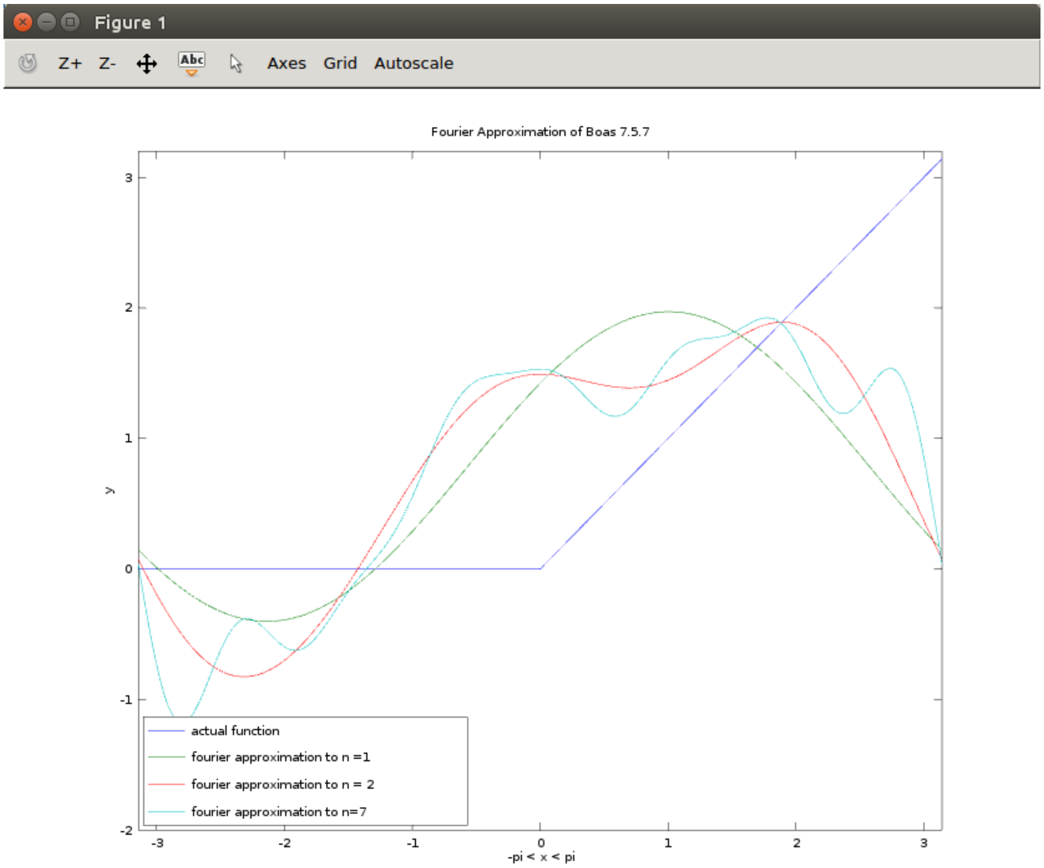
$$a_n = \begin{cases} \frac{\pi}{4}, & n = 0, \\ \frac{1}{\pi n^2}(1 - \cos(n\pi)), & n \neq 0. \end{cases}$$

$$b_n = \begin{cases} 0, & n = 0, \\ \frac{-\cos(n\pi)}{n}, & n \neq 0. \end{cases}$$

4.4 Series

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\Big(\cos(x) + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \frac{\cos(7x)}{7^2} + ...\Big)$$
$$+\Big(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \frac{\sin(5x)}{5} - -...\Big)$$

4.5 Plot



This one is a little weird and makes me nervous because as I add more terms it gets weirder. I would expect that on the left hand side of the graph our Fourier transform would approach y = 3, however this does not seem to be the case. Maybe my Fourier approximation is wrong or maybe I didn't use enough terms, however considering 7 terms were used I suggest that it is the former.

5 Matlab Code

5.1 piecewise.m

```
function [y] = piecewise (a, b)
x = linspace(a,b,10000);
y = x;
for i = 1:10000
  if(-pi <= y(i) &&  y(i) < 0)
  y(i) = 0;
  elseif(0 <= y(i))
  y(i) = x(i);
  end
end

endfunction
```