Physics and the Art of Scientific Modeling
Module C: Circuits – RC, LC and RLC

v.1.5 © pHn 2016

C.1 RC circuits

Charging RC circuit

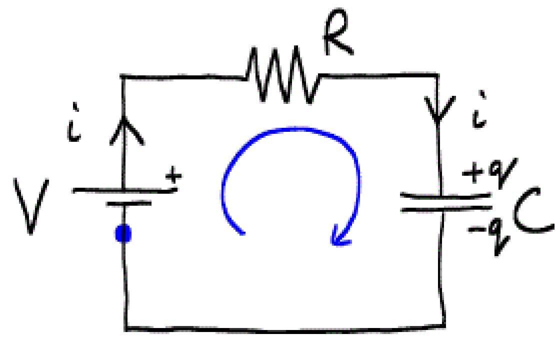

Figure C.1 Circuit diagram of a charging RC circuit

Q.C.01 By analyzing the RC circuit in Figure C.1 using Kirchhoff's loop rule, starting at the blue dot and going in the direction indicated, *show that* the current is given by

$$i = \frac{V}{R} - \frac{q}{\tau} \tag{C.1}$$

where the **time constant** τ of the RC circuit is defined to be

$$\tau = RC \tag{C.2}$$

The current i in the circuit of Figure C.1, is related to the charge on the capacitor by

$$i = +\frac{\delta q}{\delta t} \tag{C.3}$$

The plus sign in equation (C.3) indicates that the conventional current is produced by positive charge δq arriving on the top plate of the capacitor. Rearranging equation (C.3) gives the finite difference (FD) equation

$$\delta q = i\delta t \tag{C.4}$$

Q.C.02 **(a)** Using equations (C.1) and (C.4) *write out* a complete FD algorithm (including unit checks for τ, δq, and i) to calculate how the charge q and current i change

as a function of time if the initial charge on the capacitor is $q_0 = 0$ μC, with $V = 5$ V $R = 10\ \Omega$, $C = 1$ μF and $\delta t = 8$ μs.
Hint: The current can be calculated using equation (C.1). (A model answer to this question is posted on Canvas.)
(b) By hand, *calculate* steps 0, 1, and 2 of your finite difference algorithm and write your answer in the form of an output table.
Hint: As usual, you should do parts (a) and (b) of this question together. It's easier that way. (A model answer to this question is posted on Canvas.)

Implement your algorithm in a copy of the preformatted spreadsheet Phys_Q.C.03.xlsx and check that it generates exactly the same sequence that you calculated in Q.C.02(b). Then extend your numerical method to show at least 60 microseconds and make δt smaller until you're sure that your graphs of current and charge versus time are accurate.

Q.C.03 *Record* your graphs of charge and current versus time.

Discharging RC circuit

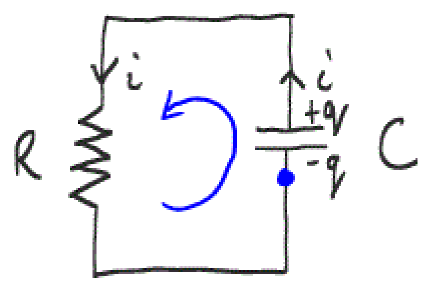

Figure C.2 Circuit diagram of a discharging RC circuit

Q.C.04 By analyzing the RC circuit in Figure C.2 using Kirchhoff's loop rule, starting at the blue dot and going in the direction indicated, *show that* the current is given by

$$i = \frac{q}{\tau} \tag{C.5}$$

where the **time constant** τ of the RC circuit is defined to be

$$\tau = RC \tag{C.6}$$

The current i in the circuit of Figure C.1, is related to the charge on the capacitor by

$$i = -\frac{\delta q}{\delta t} \tag{C.7}$$

The minus sign in equation (C.7) indicates that the conventional current is produced by positive charge δq leaving the top plate of the capacitor. Rearranging equation (C.7) gives the finite difference (FD) equation

$$\delta q = -i\delta t \tag{C.8}$$

Q.C.05 **(a)** Using equations (C.5) and (C.8) *write out* a complete FD algorithm (including unit checks for τ, i, and δq) to calculate how the charge q and current i change as a function of time if the initial charge on the capacitor is $q_0 = 5\ \mu\text{C}$, with $R = 10\ \Omega$, $C = 1\ \mu\text{F}$ and $\delta t = 8\ \mu\text{s}$.
Hint: The current can be calculated using equation (C.5).
(b) By hand, *calculate* steps 0, 1, and 2 of your finite difference algorithm and write your answer in the form of an output table.

Implement your algorithm in a copy of the preformatted spreadsheet Phys_Q.C.06.xlsx and check that it generates exactly the same sequence that you calculated in Q.C.05(b). Then extend your numerical method to show at least 60 microseconds and make δt smaller until you're sure that your graphs of current and charge versus time are accurate.

Q.C.06 *Record* your graphs of charge and current.

Q.C.07 Change the y-axis of your charge versus time graph to a log scale and *record* your semi-log graph.
Hint: To display your graph in this format, you'll need to select the **Format Axis…** option for the q-axis and check the box for **Logarithmic scale**.

Q.C.08 **DISCUSSION QUESTION** What can you conclude from the shape of the semi-log graph?

About what you discovered: proportional change

Combining equations (C.5) and (C.8) can be rewritten as

$$\delta q = -\frac{1}{\tau} q \delta t \tag{C.9}$$

Equation (C.9) is another example of a quantity q that changes by an amount δq that is proportional to itself (i.e. the change is proportional how big q is). If q is twice as big – the change is also twice as big, or conversely, if q is twice as small, then the change is twice as small. This is another example of the **proportional change**. This type of situation occurs in an amazingly wide range of situations ranging from unconstrained population growth to drug elimination, radioactive decay, the isothermal atmosphere and is encoded in the Boltzmann factor of statistical physics and, as we have just seen, it also applies to a discharging RC circuit.

The minus sign in equation (C.9) means that the change always makes q smaller. This means that as time goes by, the system will tend towards a state in which q is zero. As you just discovered, this produces an **exponential decay** in q that is characteristic of many processes.

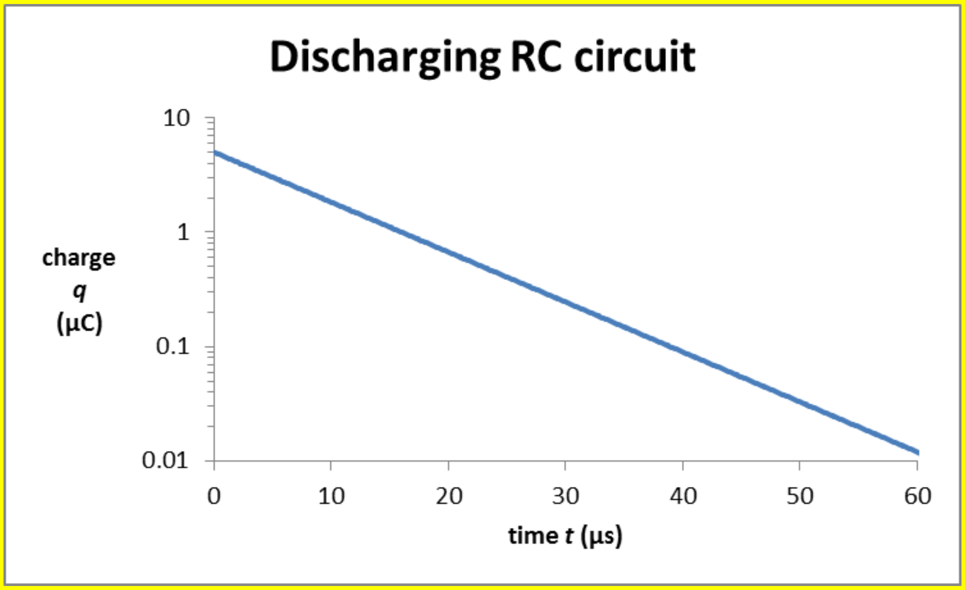

Figure C.3 Semi-log Excel 2010 chart of a discharging RC circuit with $q_0 = 5$ μC, $R = 10$ Ω, $C = 1$ μF and $\delta t = 8$ μs.

❑

Rearranging equation (C.9) and taking the limit that $\delta t \to 0$ gives the differential equation

$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{q}{\tau} \tag{C.10}$$

Q.C.09 CALCULUS QUESTION Using the methods outlined in the "calculus solution for $u(t)$" AWYD in the extract from **BPM CHAPTER 3** posted on Canvas, *solve* the differential equation (C.10) for q as a function of time, to *show that* the **analytical solution** for the charge q on the capacitor is

$$q = q_0 e^{-t/\tau} \tag{C.11}$$

where the **initial condition** is that $q = q_0$ at time $t = 0$.

Q.C.10 CALCULUS QUESTION Differentiate equation (C.11) with respect to time to *show that* the analytical solution for the current i in the discharging RC circuit is

$$i = i_0 e^{-t/\tau} \tag{C.12}$$

where from equation (C.5) $i_0 = q_0/\tau$.

Q.C.11 **DISCUSSION QUESTION** *Compare* the predictions of equations (C.11) and (C.12) with your FD method and report your results and conclusion. I.e. test the hypothesis that equations (C.11) and (C.12) predict the results of your FD method and *report* your results in an appropriate scientific manner.

Back to charging RC circuits

Substituting the definition of current into equation (C.1) and taking the limit that $\delta t \to 0$ gives the differential equation

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{V}{R} - \frac{q}{\tau} \qquad \text{(C.13)}$$

Q.C.12 **CALCULUS QUESTION** Using calculus, *solve* the differential equation (C.13) for q as a function of time, to *show that* the **analytical solution** for the charge q on the capacitor is

$$q = q_\infty\left(1 - e^{-t/\tau}\right) \qquad \text{(C.14)}$$

where $q_\infty = VC$ and the **initial condition** is that $q = 0$ at time $t = 0$.

Q.C.13 **CALCULUS QUESTION** Differentiate equation (C.14) with respect to time to *show that* the analytical solution for the current i in the charging RC circuit is

$$i = i_0 e^{-t/\tau} \qquad \text{(C.15)}$$

where from equation (C.1) $i_0 = V/R$.

Q.C.14 **DISCUSSION QUESTION** *Compare* the predictions of equations (C.14) and (C.15) with your FD method and report your results and conclusion. I.e. test the hypothesis that equations (C.14) and (C.15) predict the results of your FD method and *report* your results in an appropriate scientific manner.

Q.C.15 **CHALLENGE CALCULUS QUESTION** Using equations (C.1) and (C.13) *derive* a differential equation for the current $i(t)$ and show that equation (C.15) is a solution to that differential equation.

LC circuits

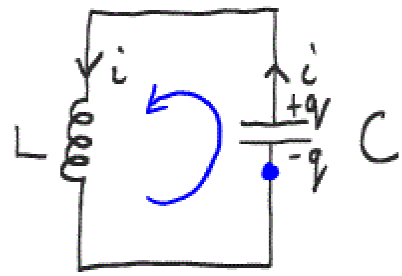

Figure F.4 Circuit diagram of an LC circuit

Figure F.4 shows a circuit somewhat similar to the one shown in Figure C.2 except the resistor has been replaced with an **inductor** (labelled with inductance L [=] H [=] $\Omega \cdot$ s in the diagram). The circuit symbol looks like a spring because and inductor is a coil of wire. For the purposes of this module, all you need to know about an inductor is that, like a resistor, if you travel along a circuit you'll see a voltage drop, but instead of it being given by Ohm's law ($\Delta V_R = -iR$) the voltage drop through an inductor ΔV_L is given by

$$\Delta V_L = -L\frac{\delta i}{\delta t} \tag{C.16}$$

where δt is a timestep that is short enough that the voltage ΔV_L does not depend on the actual value of δt. The unit of inductance is the Henry [=] H [=] $\Omega \cdot$ s.

Q.C.16 By analyzing the LC circuit in the direction indicated in Figure F.4 (starting at the blue dot) *show that* the change in the current δi during a short time interval δt is given by

$$\delta i = \frac{q}{LC}\delta t \tag{C.17}$$

Q.C.17 **(a)** Using equations (C.17) and (C.4) *write out* a complete FD algorithm to calculate how the current i and charge q change as a function of time if the initial charge on the capacitor is $q_0 = 5\ \mu\text{C}$ and the initial current is $i_0 = 0$, with $L = 4\ \mu\text{H}$, $C = 10\ \mu\text{F}$, and $\delta t = 5\ \mu\text{s}$. You should use the **midpoint algorithm** (**Chapter N**) for δq using equation (C.18)

$$\delta q^{\text{new}} = -0.5 * \left(i^{\text{old}} + i^{\text{new}}\right) * \delta t \tag{C.18}$$

(b) By hand, *calculate* steps 0, 1, and 2 of your finite difference algorithm and write your answer in the form of an output table.

Implement your algorithm in a *blank* spreadsheet and check that it generates exactly the same sequence that you calculated in Q.C.17(b). Then extend your numerical method to show at least

60 microseconds and make δt smaller until you're sure that your graphs of current and charge versus time are accurate.

Q.C.18 *Record* your graphs on a single chart.

Q.C.19 Using scientific terminology briefly describe your graphs.

Q.C.20 **DISCUSSION QUESTION** By systematically changing L and C find the functional form of the period T of the circuit.
Hint: The period of the circuit described in Q.C.17 should be about $T = 40\ \mu\text{s}$. Also recall that $\omega = 2\pi f = 2\pi/T$, where f is frequency and ω is angular frequency. Your answer to this question should contain at least two graphs – one relating T and L and one relating T and C.

Q.C.21 **CHALLENGE CALCULUS QUESTION** By combining equations (C.17) and (C.4) and taking the limit that $\delta t \to 0$ derive analytical expressions for the charge q on the capacitor and the current i in the circuit of Figure F.4.

Connection with simple harmonic motion

In **CHAPTER M** we analyzed a mass on a spring – see Figure F.5.

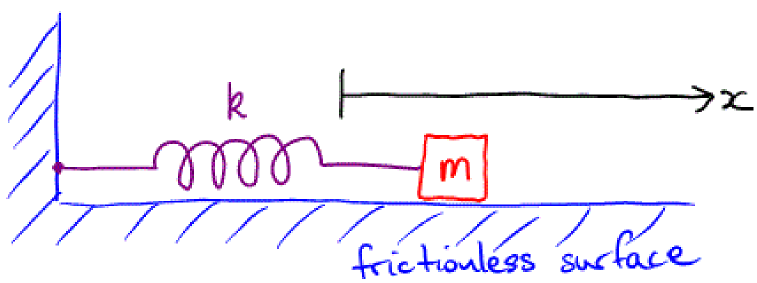

Figure F.5 Mass on a spring. The position of the mass is given by x, which is measured from the equilibrium position of the spring so that the net force on the mass is $F = -kx$.

The FD equations for the motion of the mass on a spring can be written as

$$\delta v = -\frac{kx}{m}\delta t \tag{C.19}$$

and

$$\delta x = v\delta t \tag{C.20}$$

As we saw in **CHAPTER M** , these equations successfully predicted **simple harmonic motion** (**SHM**).

Q.C.22 **DISCUSSION QUESTION** By comparing equations (C.19) and (C.20) with equations (C.17) and (C.4) *briefly discuss* how an LC circuit is analogous with SHM.

Hint: You should be specific about how variable and parameters match up. Would the correspondence be clearer if we reversed the direction of positive current in Figure F.4?

RLC circuits

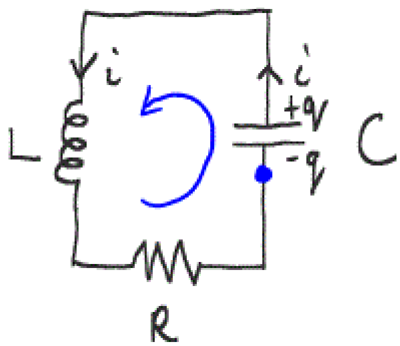

Figure F.6 Circuit diagram of an RLC circuit

Q.C.23 By analyzing the RLC circuit in the direction indicated in Figure F.6 (starting at the blue dot) *show that* the change in the current δi during a short time interval δt is given by

$$\delta i = \left(\frac{q}{LC} - i\frac{R}{L}\right)\delta t \tag{C.21}$$

Q.C.24 **(a)** Using equations (C.21) and (C.18) *write out* a complete FD algorithm to calculate how the current i and charge q change as a function of time if the initial charge on the capacitor is $q_0 = 5\ \mu\text{C}$ and the initial current is $i_0 = 0$, with $L = 4\ \mu\text{H}$, $C = 10\ \mu\text{F}$, $R = 0.2\ \Omega$ and $\delta t = 5\ \mu\text{s}$.
(b) By hand, *calculate* steps 0, 1, and 2 of your finite difference algorithm and write your answer in the form of an output table.

Implement your algorithm in a *blank* spreadsheet and check that it generates exactly the same sequence that you calculated in Q.C.24(b). Then extend your numerical method to show at least 60 microseconds and make δt smaller until you're sure that your graphs of current and charge versus time are accurate.

Q.C.25 *Record* your graphs on a single chart.

Q.C.26 Use scientific terminology to *briefly describe* your graphs.

Q.C.27 **Discussion Question** *Briefly describe* what happens when you systematically change the resistance R?

Q.C.28 **Challenge Calculus Question** By combining equations (C.21) and (C.4) and taking the limit that $\delta t \to 0$ derive analytical expressions for the charge q on the capacitor and the current i in the RLC circuit of Figure F.6.

Q.C.29 **DISCUSSION QUESTION** *Briefly discuss* how an RLC circuit is analogous with the motion of a real mass on a spring.
Hint: You should be specific about how variables and parameters match up.

About what you discovered: RLC circuits

Your answer to Q.C.25 should look something like Figure F.7.

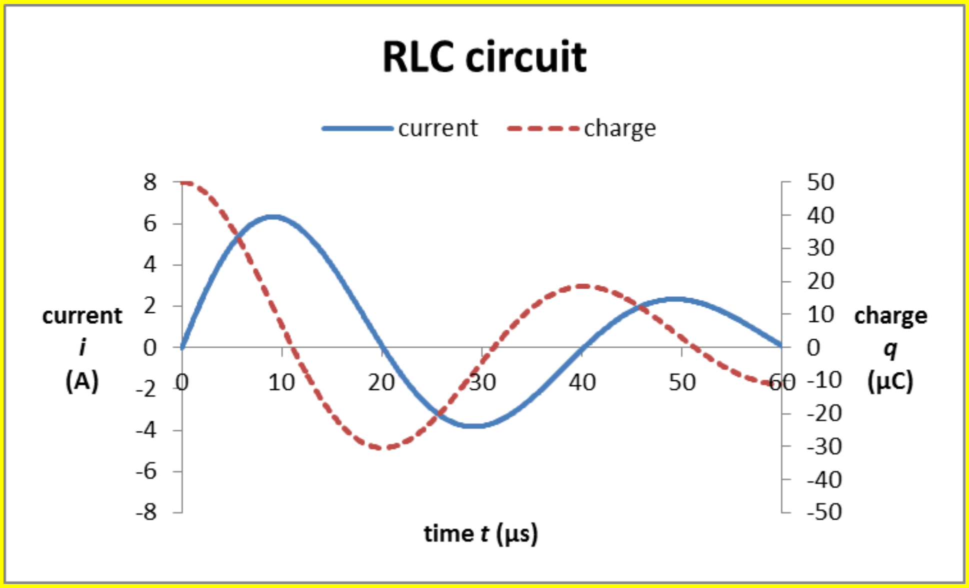

Figure F.7 Excel 2010 chart of an FD model of an RLC circuit with $q_0 = 50$ μC, $i_0 = 0$, $L = 4$ μH, $C = 10$ μF and $R = 0.2$ Ω.

❑

This material is based upon work supported by the National Science Foundation under Grant No. 0836833. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.