Math/Physics 320
Fourier Analysis 1
Due 17 March 2017

1. Finish the code we designed in class (add your own comments) to calculate the Fourier coefficients from a known function $f(x)$ that is defined at evenly spaced points over a given range of x. Pick three functions to test this code. One should be a function youve solved analytically in your homework, and two should be from the equations below (Equations 2 through 8). When you read these equations, the first interval given represents the total interval you should consider. Within that interval, the function may be piecewise defined as explained after the functional form.

Explain what results you should get, and compare with your code's results. (When I say "explain," that means describe generally what properties you would expect the answers to have. This might mean comparing to your homework, this might mean looking at even/odd behavior, etc. It does not mean you need to work them all out, analytically.) Include plots of amplitude vs. frequency for your output and explain these graphs. Use your output to reconstruct the original function and create plots that show both curves in different colors.

2. Write a program to generate a column of numbers. This column should be the sum of three or four sinusoidal functions (sine or cosine) at different frequencies as well as some randomly-generated "noise." Something like

$$f(x) = 12.6\sin(10x) - 5.8\cos(4x) + \texttt{randn}(\texttt{size}(x)) \tag{1}$$

would do nicely. Choose amplitudes for the sine waves that are much bigger than the noise (> factor of four) and choose frequencies that are integers less than 20. The point here is just that the amplitudes and frequencies are unknown to another student, not that they're mean or tricky.

Write this column of data to a file. Give this file to another student (who gives a file to whom is specified on the course web page) without telling him or her what function you used to generate it. Take a similar file from another student. Use your Fourier code to deduce what amplitudes and frequencies the other student used. Compare your answer with what they actually used (make sure you keep a record of what you used, so you can tell him or her after they've done their analysis).

$$[-\pi, 3\pi] \rightarrow f(x) = \sin x \ (0 < x < 2\pi); 0 \text{ otherwise} \tag{2}$$

$$[-2\pi, 2\pi] \rightarrow f(t) = 6e^{-i4.67t}(-\pi < t < \pi); 0 \text{ otherwise} \tag{3}$$

$$[-2, 2] \rightarrow f(x) = e^{-x^2}\cos 5x \tag{4}$$

$$[-1, 1] \rightarrow f(x) = e^{-|x|} \tag{5}$$

$$[-15, 20] \rightarrow f(x) = -x^2 + 100(-10 < x < 10);\ 0 \text{ otherwise} \tag{6}$$

$$[-\pi, \pi] \to f(x) = \cos x \ (-\pi/2 < x < \pi/2); \ 0 \text{ otherwise} \tag{7}$$

$$[0, 5] \to f(x) = \frac{\sin(10x)}{x} \tag{8}$$

3. You should also by this point **tell me which project you want to do for part 2**!!!! This way, I can make sure you have everything you need to make it a productive and interesting experience.

Math/Physics 320
Fourier Analysis 2
Due 31 March 2017

1. Modify your Fourier Transform program to read audio files[1], plot amplitude vs. frequency in one graph, and the original audio signal in another graph. Your plot function should give the user a means to choose what to plot and how to plot it.

Record an audio file with the sound from two or three tuning forks. Analyze the Fourier spectrum of the signal and verify that you can identify the frequencies of the forks. Think about how many ns you should have (how many frequencies – real and aliased) and explain your logic. Show me one figure and explain it, just to demonstrate that your code seems reliable.

2. Choose one of the following five topics to explore (talk to me before making a final decision):

A. **Astronomy** An X-ray binary happens when a normal star is orbiting too closely around a compact object like a neutron star or a black hole. The gas at the surface of the regular star is pulled down onto the compact object, but because of angular momentum, it forms a swirling disc on its way in. The inner parts of this disc get so hot that they give off x-rays. Sometimes these x-rays are blocked by the regular star passing between us and the compact object, and sometimes the disc itself can wobble, blocking the x-rays from view. Since orbits tend to follow more or less stable periodicities, on time scales of human lifetimes, orbital effects can embed periodic signals in the time history of the x-ray brightness of these sources.

I have downloaded the X-ray light curve for the X-ray binary Her X-1 from the MIT All-Sky Monitor data archive. The file is in my home directory on the Unix cluster. Copy it into your own directory.

Bin the data into evenly spaced bins of 0.5 days. Run your Fourier program on the resulting data stream. How many signals do you find? What do you think you might be seeing? Include a plot of the X-ray intensity vs. time for some representative time interval (do not try to plot the entire dataset that would just look like a mess), and also plot your Fourier amplitudes. You may want to include zoomed-in graphs of particular regions of interest.

Once you've come up with your own guesses as to what is going on, you may Google "Her X-1 X-ray Binary" (or equivalent keywords) to learn how the experts interpret what

[1] Although you could have the program read the file directly, it is probably a better idea to read in the file outside the program, so you can cut out segments of sound that you want to focus on, rather than using the whole file. Of course, if you were *very* clever, you could write a program that would read in a sound file, plot it, give the user the ability to select out a segment of the sound, and then return the numbers for that segment of sound as an output. This output could then be fed into the FourTrans program.

you are seeing. How does your theory compare with theirs?

B. **Biology** Many biological systems exhibit periodic behaviors that are masked by noise. Biologists use Fourier Analysis to study many aspects of organism behavior. Particularly the way they emit sounds. For example, you can find at this page many recordings of bat sounds: `http://www.partnersinrhyme.com/soundfx/batsounds.shtml`

I don't have a specific question for you to answer. If you choose to do this project, you will have to formulate your own question. I would suggest you try to use Fourier analysis to answer questions like: can you distinguish individual bats by the frequencies they use? Do they use different frequencies to locate prey than to communicate? I have not looked at the wav files to verify that they will yield clean signals that are fruitful for analysis. Remember to consider processing time relative to length of file, and that you want to choose segments of sound that are as uniform as possible.

You should talk to me before Spring Break if you want to try to analyze the bat sounds. If you want to address a different Biological question that involves Fourier analysis (not bats), I am open to suggestions, but you should clear it with me before Spring Break.

C. **Quantum Mechanics** I mentioned in class that there are times that a transform is easier to deal with than the actual function. The quantum mechanics of a free particle is one of those cases. In quantum mechanics, a particle is represented by a "wave function", the square of which tells you the probability of finding the particle in any particular spot. However, if you consider a free particle, it is equally likely to be found anywhere, which means the integral of the wave function technically diverges (equally likely means a uniform/constant distribution, but the limits of the integral over all space are infinite, so you can't actually work out the integral). However, most particles usually have a more or less well-defined momentum, which depends on the wave number, or spacial frequency of the particle. The transform tells you about the probability distribution in frequency space, which can be normalized (probability of getting infinite or zero momentum goes to zero, so integral is bounded).

So let's assume that you can know $g(\alpha)$, while $f(x)$ is trickier to define. The time-dependent wave function is given by

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\alpha) e^{i(\alpha x - \frac{\hbar \alpha^2}{2m} t)} d\alpha \tag{9}$$

You can see that if you know something about the initial distributions of α, you can say something about where you are likely to find the particle at later times.

1. Try picking some initial distribution of α, like

$$g(\alpha) \propto e^{-\left(\frac{\alpha-\alpha_0}{\sigma^2}\right)^2}, \tag{10}$$

and then pick some reasonable values of α_0 and σ (assume $\hbar = m = 1$, and make sure you figure out the normalization constant), and make some plots to see how Ψ changes with time. Remember $\Psi^*\Psi$ is the probability of detecting the particle at some location at some time. Given that it's hard to plot complex numbers, $|\Psi|^2$ might be better to plot than Ψ itself. Given that you are looking at time evolution, it might be most useful to make a mesh plot or a movie.

If you look at Equation 9, you can see that it gets a little simpler (and hopefully familiar) if $t = 0$:

$$\Psi(x, t = 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\alpha) e^{i\alpha x} d\alpha. \tag{11}$$

This means that if you know something about where the particle is at $t = 0$, you can make predictions about where it will be at later times, because you can do Fourier transforms!

Let's assume that you've made a measurement that puts the particle inside a box at time $t = 0$. You know that the particle is somewhere between $\pm a$, but not any more than that. This means that $\Psi(x, t = 0) = A$, if $|x| < a$ and 0 otherwise. (You should find A in terms of a.) This in turn means that

$$g(\alpha) = \frac{A}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\alpha x} dx, \tag{12}$$

which is an integral you can do.

Once you have $g(\alpha)$, you can plug it back into Equation 9 and figure out where you might find the particle at some later time, if you know it is within $\pm a$ at time $t = 0$. This formula cannot be solved analytically, but you can solve it numerically.

2. Make a movie or mesh plot that shows $|\Psi(x, t)|^2$ and explain what this means for where you think you might detect the particle at later times.

3. If you change a (make it much larger or much smaller), what effect does this have on your answer? Does it match what you expect?

If you make a movie, you might want to turn in your final report in an electronic format like powerpoint, where you can embed the movie so I can see it, rather than a PDF file or printout. See the "movie" command in the help file, or "mesh" for ways you can make these kinds of plots.

D. **Optics** As you saw in class, light passing through an aperture can be thought of as performing a fourier transform. In principle, one could reconstruct the shape of any aperture by inverse transforming the intensity pattern observed far from the aperture. This project choice isn't finished yet, but it would involve picking an aperture shape, with 0 to indicate a barrier and 1 to indicate open, then performing the 2D fourier transform on it to find out what the intensity pattern on the focal plane would look like. This is very similar to the Image Processing project, only you create your own "image" of the aperture and then transform it.

E. **Image Processing** An image is just a two-dimensional array of data trains. Intensity can be broken down into Fourier components, and then inverse transformed back into an image. Figure 1 shows what image transforms can look like[2], and Figure 2 shows how the properties of the 2D transform are similar to those of a 1D transform. This process is used in radio astronomy, medical imaging, pattern recognition, and image processing. By changing the transform and then inverse transforming, one can filter out periodic artifacts, as shown in Figure 3. Patterns in the image can be identified by structure in the transform, as shown in Figure 4.

What you will do in this project is build and test a 2-D Discrete Fourier Transform program. You should run it on some “fake” images that you make yourself with `imwrite`. Make images with periodic stripes, horizontal, vertical, and diagonal. Make sure the transform makes sense. Change the width of the stripes and make sure the transform behaves as you expect. For the purposes of this project, you should only use black and white images.

Once you believe that your program works, take an image with a lot of noise in it. You might, for example, use a raw astronomical image, like one of the supernova at `www.facebook.com/guilfordobservatory`. Alternately, you can find your own image of interest. Transform it, identify the noise in the spectrum, zero that out, and then inverse transform it. This should work like the example in Figure 3.

Next, you should take a few images of your choice (as you might guess, `imread` will read in images) and do both low-pass and high-pass filtering on it, just to see how that changes the look of the image.

Finally, take an image you think is interesting and use your 2D DFT code to find something interesting about its structure, kind of like the fly’s eye in Figure 4. But it could be anything. A tree in leaf, the petals of a flower, the shell of a nautilus, a fractal, what ever you want. But there should be some kind of repeating structure in it, or the DFT won’t tell you much you can’t see by eye.

Make sure in your writeup you address the issue of aliasing and how it shows up in the 2D transform. How is it similar to or different from the aliasing in a 1D transform?

F. **Music**. To do this project, you will need access to several different musical instruments. The basic question here is: why does a violin sound different than a clarinet, if they are both playing the same note? To answer this question, you will need to make recordings of each instrument playing several different notes (the same notes on each instrument). You should also compare to tuning fork sounds at the same note. (Make sure you consider the effects of tuning the instrument!) Remember that a note has an attack and a fade, but you want to perform your FT on the middle part, when the note is the most stable. Also, remember that the height of the peaks is correlated with loudness, and it’s very difficult to control for that from recording to recording, so don’t put too much emphasis on comparing the height of the peaks from recording to recording – relative heights (like if you divided the peak heights by the height of the fundamental, or

[2] These screen captures are taken from
`http://research.stowers-institute.org/efg/Report/FourierAnalysis.pdf`

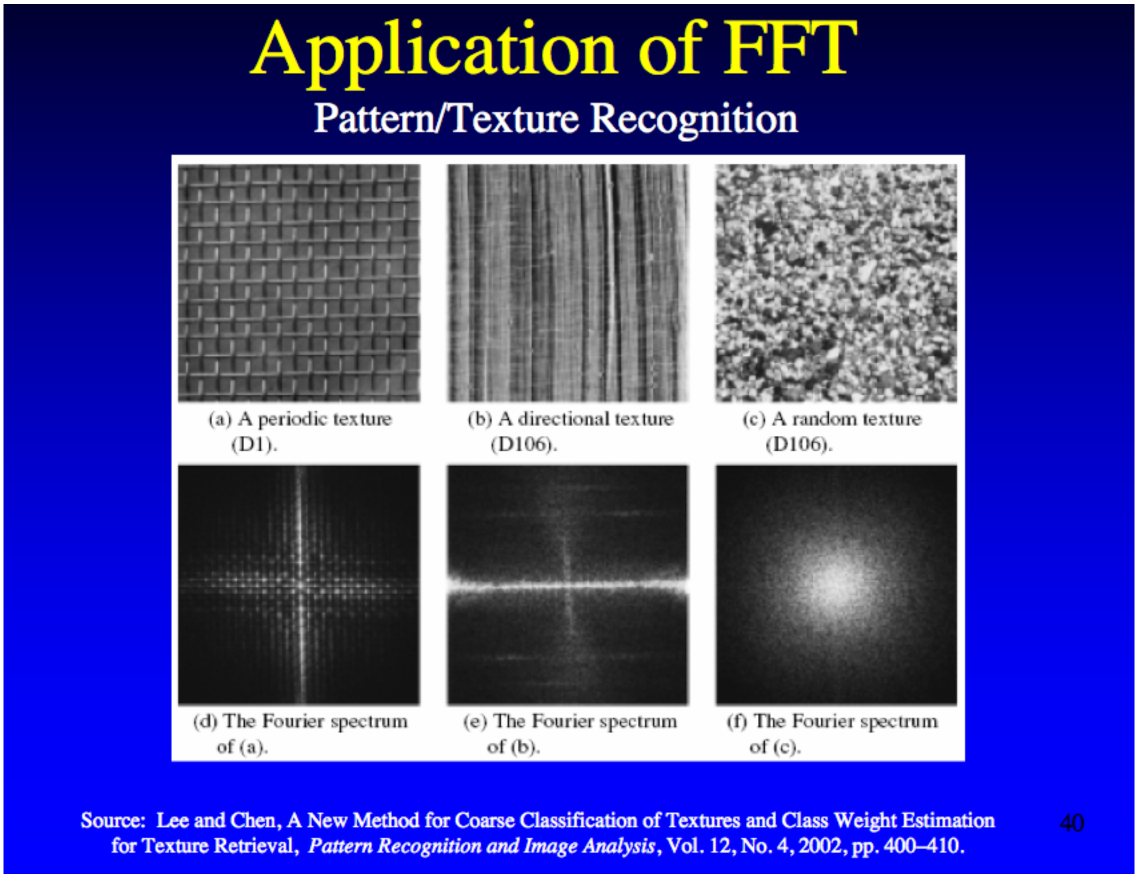

Figure 1: Examples of a Fourier Transform of an Image

normalized them) might be a useful comparison. But mostly, you should be looking at the frequency values of the peak locations, rather than the amplitudes.

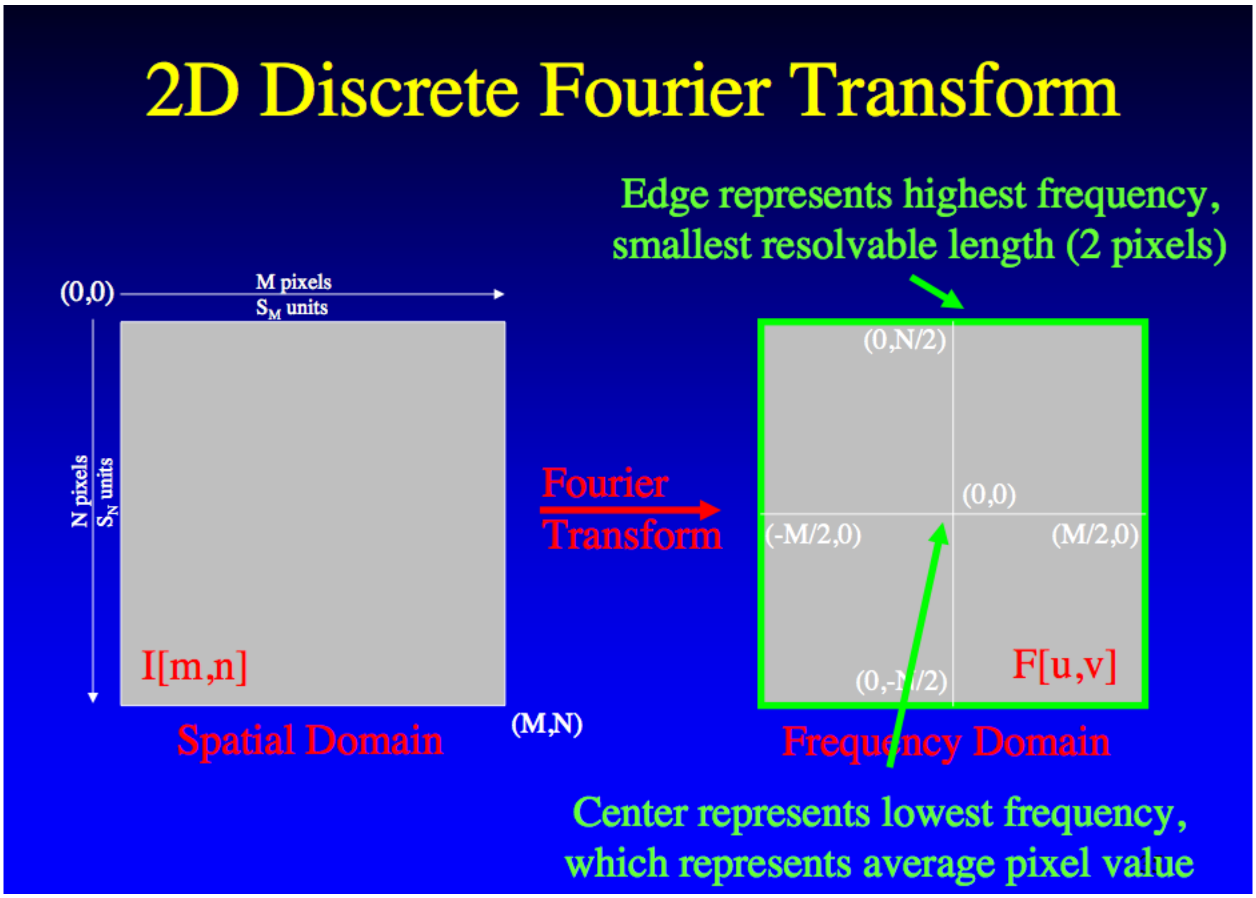

Figure 2: Properties of image Fourier Transforms

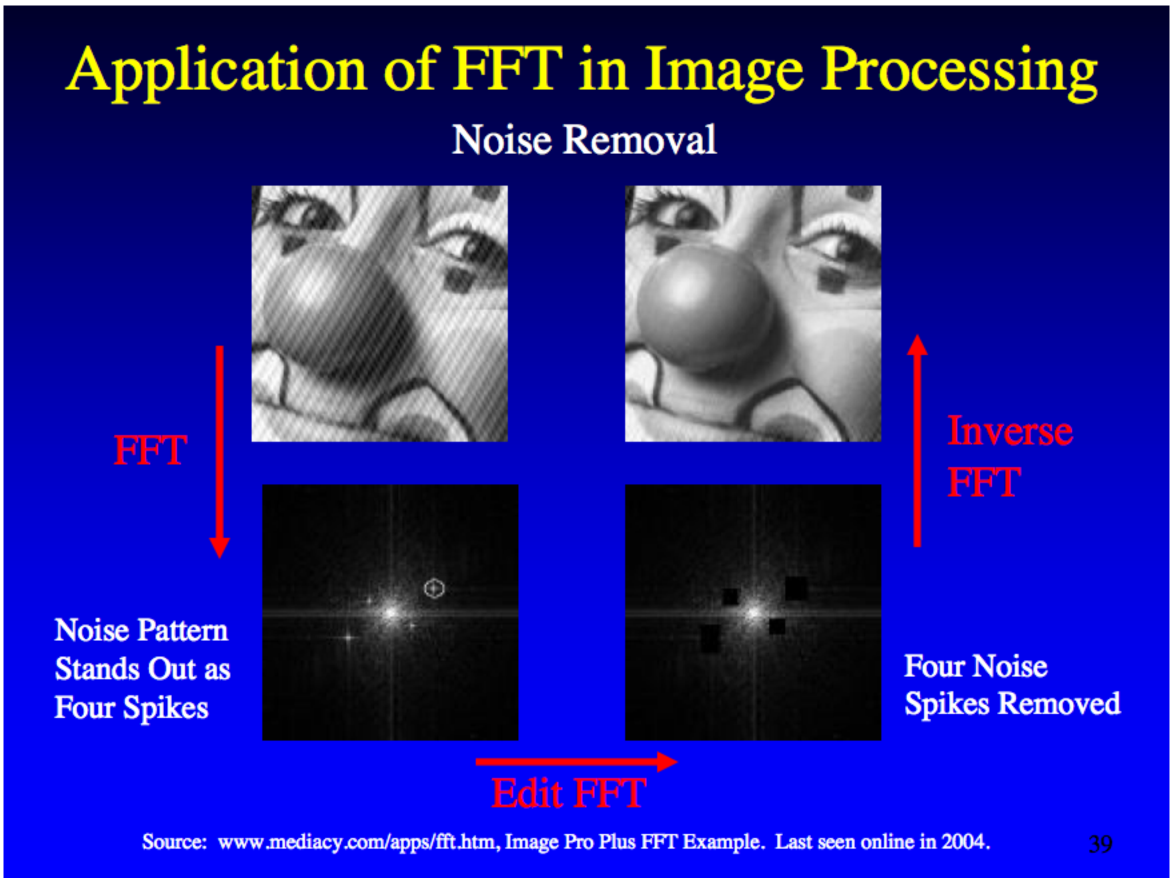

Figure 3: Example of image filtering through Fourier Transforms

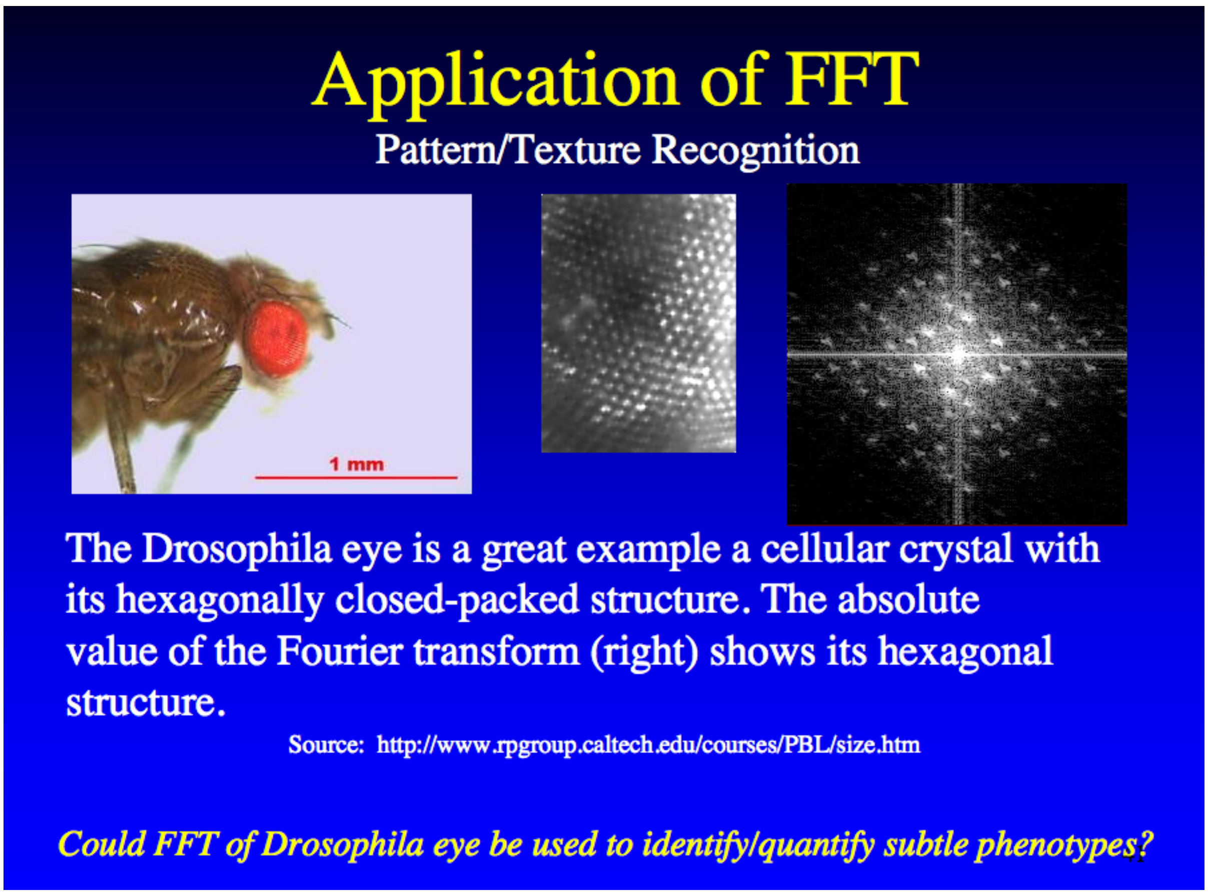

Figure 4: Example of pattern recognition though Fourier Transform of an Image