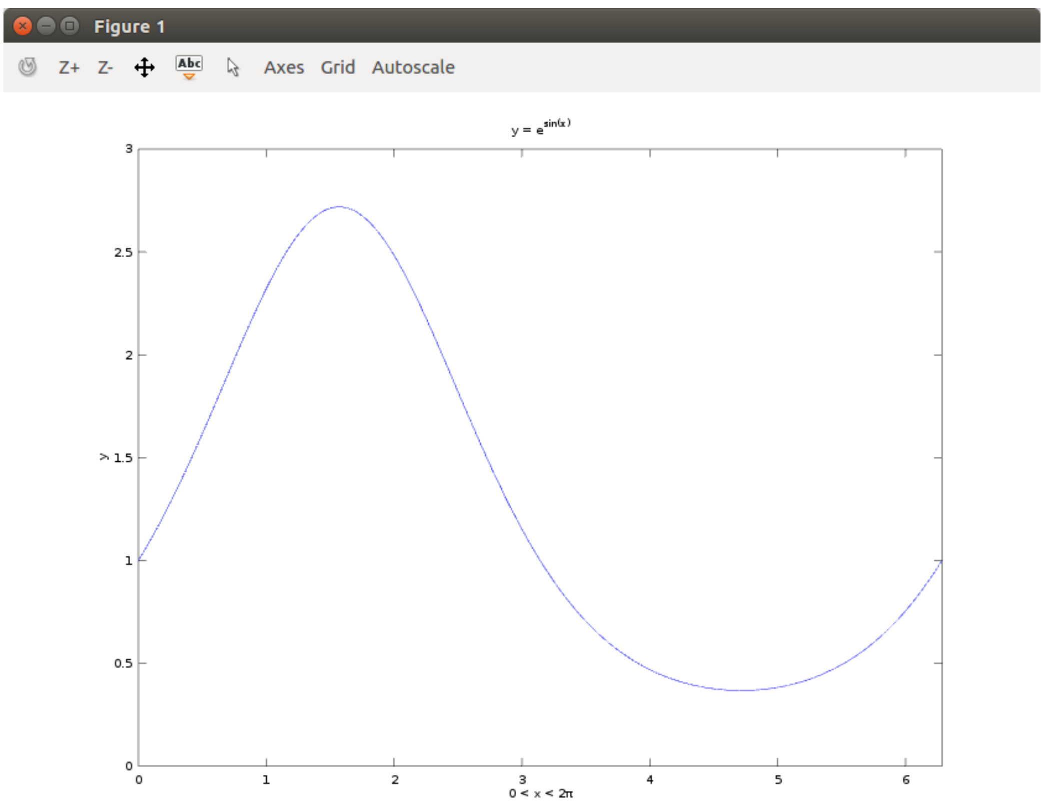
Numerical Integration II

Erin Goeke

March 4, 2017

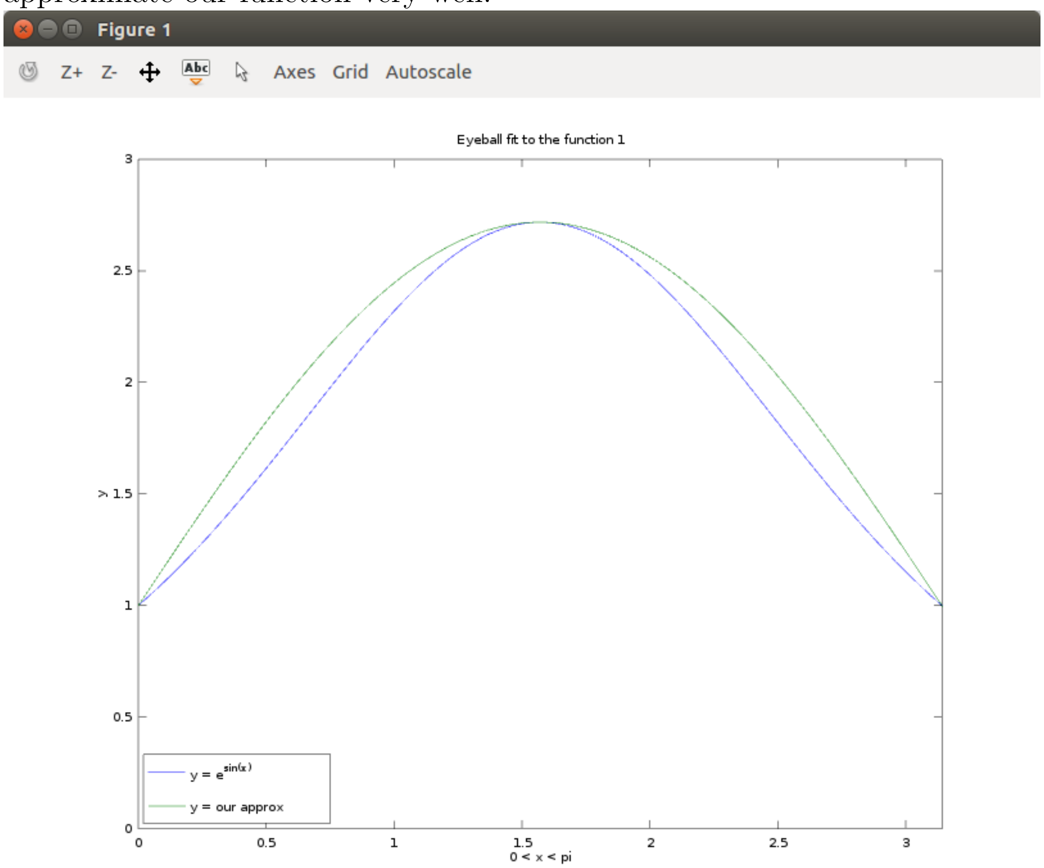
1 $\int_0^{2\pi} e^{\sin(x)} dx$



This function is quite beautiful and just glancing at the graph it seems deceptively easy to integrate. Of course, that is not the case.

After about 45 minutes I figured out that I was *not* going to be able to solve this integral using the tricks I learned in calculus II. So I thought to approximate the function using terms which were much easier to integrate. At first I thought a parabolic approximation might be nice. But parabolic functions' derivatives continue to increase as they get further from the vertex, however $e^{\sin(x)}$'s derivative drastically lowers as x approaches $k\pi$. Then it hit me! I could use sin to approximate the function!

I did a pretty good job eyeballing the first function and came up with $y = (e-1)\sin(x) + 1$, however as you can see in the graph below it didn't approximate our function very well.



Then Fourier whispered into my ear that multiple frequencies can be used to approximate any weird function; whether using his transformation or not. I decided to create a "trick" of my own. I call it the ***Goeke Trick***. I don't know what all it can approximate, but it is by no means general. Also its less of a trick and more of a little bit of tedious linear algebra. Behold the holy grail of our first integral:

$$A\widetilde{x} = \widetilde{b} \tag{1}$$

Isn't that *beautiful.* Now I don't hate myself enough to calculate more than four terms because I don't want to find the inverse of a matrix larger than 4x4. The goal is to be able to describe $f(x)$ where $x\epsilon(0,\pi)$ (which I shall call function 1) and $f(x)$ where $x\epsilon(\pi, 2\pi)$ (which I shall call function 2) with two different variations of (2). Then I will integrate the approximations over their respective domains and add them together for the total integral.

$$\begin{pmatrix} a & b*\sin(x_1) & c*\sin(2x_1) & d*\sin(3x_1) \\ a & b*\sin(x_2) & c*\sin(2x_2) & d*\sin(3x_2) \\ a & b*\sin(x_3) & c*\sin(2x_3) & d*\sin(3x_3) \\ a & b*\sin(x_4) & c*\sin(2x_4) & d*\sin(3x_4) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \tag{2}$$

1.1 Function 1

$$\begin{pmatrix} a & b*\sin(0) & c*\sin(0) & d*\sin(0) \\ a & b*\sin(\frac{\pi}{4}) & c*\sin(\frac{\pi}{2}) & d*\sin(\frac{3\pi}{4}) \\ a & b*\sin(\frac{\pi}{2}) & c*\sin(\pi) & d*\sin(\frac{3\pi}{2}) \\ a & b*\sin(\frac{3\pi}{4}) & c*\sin(\frac{3\pi}{2}) & d*\sin\frac{9\pi}{4}) \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ a & \frac{\sqrt{2}}{2}b & c & \frac{\sqrt{2}}{2}d \\ a & b & 0 & -d \\ a & \frac{\sqrt{2}}{2}b & -c & \frac{\sqrt{2}}{2}d \end{pmatrix}$$

Now that we have plugged in our x values into our matrix we can simply pull out our unknowns, plug in our y values and rewrite our equation as shown below. The points chosen are $(0,1),(\frac{\pi}{4}, e^{\frac{\sqrt{2}}{2}}),(\frac{\pi}{2}, e),(\frac{3\pi}{4}, e^{\frac{\sqrt{2}}{2}})$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2} \\ 1 & 1 & 0 & -1 \\ 1 & \frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ e^{\frac{\sqrt{2}}{2}} \\ e \\ e^{\frac{\sqrt{2}}{2}} \end{pmatrix} \tag{3}$$

In the morning I often wake up and think to myself *I really wish I could find the inverse of a matrix right now.* I feel very blessed right now to have the opportunity to find A^{-1}.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & \frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2} & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & \frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2} & -1 & 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} & -1 & 0 & 0 & 1 \end{array}\right) =$$

$$
\left(\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & \sqrt{2} & \frac{\sqrt{2}-2}{2} & 1 & -\frac{\sqrt{2}}{2} & 0 \\
0 & 0 & -1 & \sqrt{2} & \frac{\sqrt{2}-2}{2} & 0 & -\frac{\sqrt{2}}{2} & 1
\end{array}\right) = \left(\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & \sqrt{2} & \frac{\sqrt{2}-2}{2} & 1 & -\frac{\sqrt{2}}{2} & 0 \\
0 & 0 & 0 & 2\sqrt{2} & \sqrt{2}-2 & 1 & -\sqrt{2} & 1
\end{array}\right) =
$$

$$
\left(\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & \sqrt{2} & \frac{\sqrt{2}-2}{2} & 1 & -\frac{\sqrt{2}}{2} & 0 \\
0 & 0 & 0 & 1 & \frac{\sqrt{2}-2}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2\sqrt{2}}
\end{array}\right) = \left(\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \frac{\sqrt{2}-2}{2\sqrt{2}} - 1 & \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\
0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 1 & \frac{\sqrt{2}-2}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2\sqrt{2}}
\end{array}\right)
$$

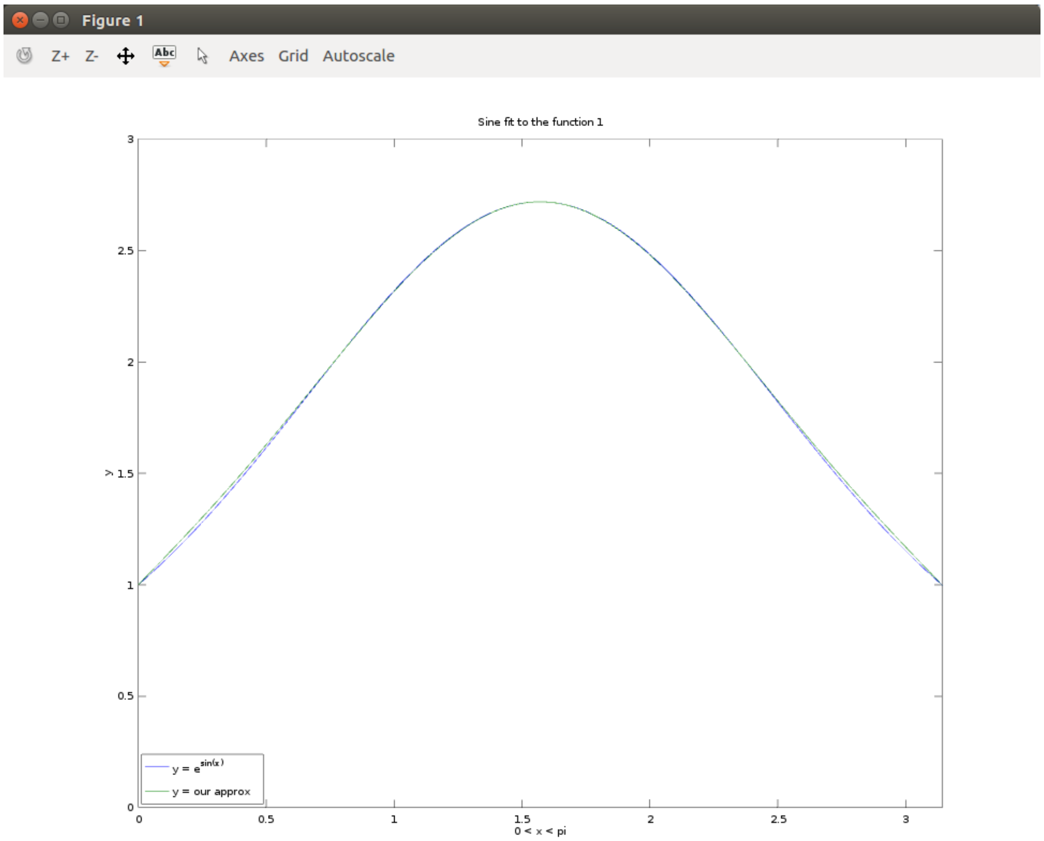
Through simplification we get:

$$
A^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-\frac{1+\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & \frac{1}{2} & \frac{\sqrt{2}}{4} \\
0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
\frac{1-\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & -\frac{1}{2} & \frac{\sqrt{2}}{4}
\end{pmatrix}
$$

Now that we know A^{-1} we can rearrange (1) such that $\widetilde{x} = A^{-1}\widetilde{b}$ and plug in our knowns for A^{-1} and $\widetilde{b}$

$$
\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-\frac{1+\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & \frac{1}{2} & \frac{\sqrt{2}}{4} \\
0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
\frac{1-\sqrt{2}}{2} & \frac{\sqrt{2}}{4} & -\frac{1}{2} & \frac{\sqrt{2}}{4}
\end{pmatrix} \begin{pmatrix} 1 \\ e^{\frac{\sqrt{2}}{2}} \\ e \\ e^{\frac{\sqrt{2}}{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{e+\sqrt{2}e^{\frac{\sqrt{2}}{2}}-1-\sqrt{2}}{2} \\ 0 \\ \frac{-e+\sqrt{2}e^{\frac{\sqrt{2}}{2}}+1-\sqrt{2}}{2} \end{pmatrix} \approx \begin{pmatrix} 1 \\ 1.5861 \\ 0 \\ -0.13215 \end{pmatrix} \tag{4}
$$

Here's the finished result of our fitting. I think it looks pretty good; maybe not the best, but certainly we can get two significant figures out of it.



Now its time to integrate part one of our function!

$$\int_0^\pi (a + b\sin(x) + c\sin(2x) + d\sin(3x))dx =$$

$$ax - b\cos(x) - \frac{c\cos(2x)}{2} - \frac{d\cos(3x)}{3}\Big|_0^\pi =$$

$$a\pi + b - \frac{c}{2} + \frac{d}{3} - (0 - b - \frac{c}{2} - \frac{d}{3})$$

$$a\pi + 2b + \frac{2d}{3} = \pi + 2(1.5861) + \frac{2(-0.13215)}{3} \approx 6.22$$

1.2 Function 2

For Function 2 I have chosen the points $(\pi, 1), (\frac{5\pi}{4}, e^{\frac{-\sqrt{2}}{2}}), (\frac{3\pi}{2}, e^{-1}), (\frac{7\pi}{4}, e^{\frac{-\sqrt{2}}{2}})$ to do our approximation.

$$\begin{pmatrix} 1 & \sin(\pi) & \sin(2\pi) & \sin(3\pi) \\ 1 & \sin(\frac{5\pi}{4}) & \sin(\frac{5\pi}{2}) & \sin(\frac{15\pi}{4}) \\ 1 & \sin(\frac{3\pi}{2}) & \sin(3\pi) & \sin(\frac{9\pi}{2}) \\ 1 & \sin(\frac{7\pi}{4}) & \sin(\frac{7\pi}{2}) & \sin(\frac{21\pi}{4}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -\frac{\sqrt{2}}{2} & 1 & -\frac{\sqrt{2}}{2} \\ 1 & -1 & 0 & 1 \\ 1 & -\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -\frac{\sqrt{2}}{2} & 1 & -\frac{\sqrt{2}}{2} & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 1 & -\frac{\sqrt{2}}{2} & -1 & 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -1 & -\frac{\sqrt{2}}{2} & -1 & 0 & 0 & 1 \end{array}\right) =$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\sqrt{2} & \frac{\sqrt{2}-2}{2} & 1 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & -1 & -\sqrt{2} & \frac{\sqrt{2}-2}{2} & 0 & -\frac{\sqrt{2}}{2} & 1 \end{array}\right) = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\sqrt{2} & \frac{\sqrt{2}-2}{2} & 1 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & -2\sqrt{2} & \sqrt{2}-2 & 1 & -\sqrt{2} & 1 \end{array}\right) =$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\sqrt{2} & \frac{\sqrt{2}-2}{2} & 1 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{2-\sqrt{2}}{2\sqrt{2}} & -\frac{\sqrt{2}}{4} & \frac{1}{2} & -\frac{\sqrt{2}}{4} \end{array}\right) = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1+\sqrt{2}}{2} & -\frac{\sqrt{2}}{4} & -\frac{1}{2} & -\frac{\sqrt{2}}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{\sqrt{2}-1}{2} & -\frac{\sqrt{2}}{4} & \frac{1}{2} & -\frac{\sqrt{2}}{4} \end{array}\right)$$

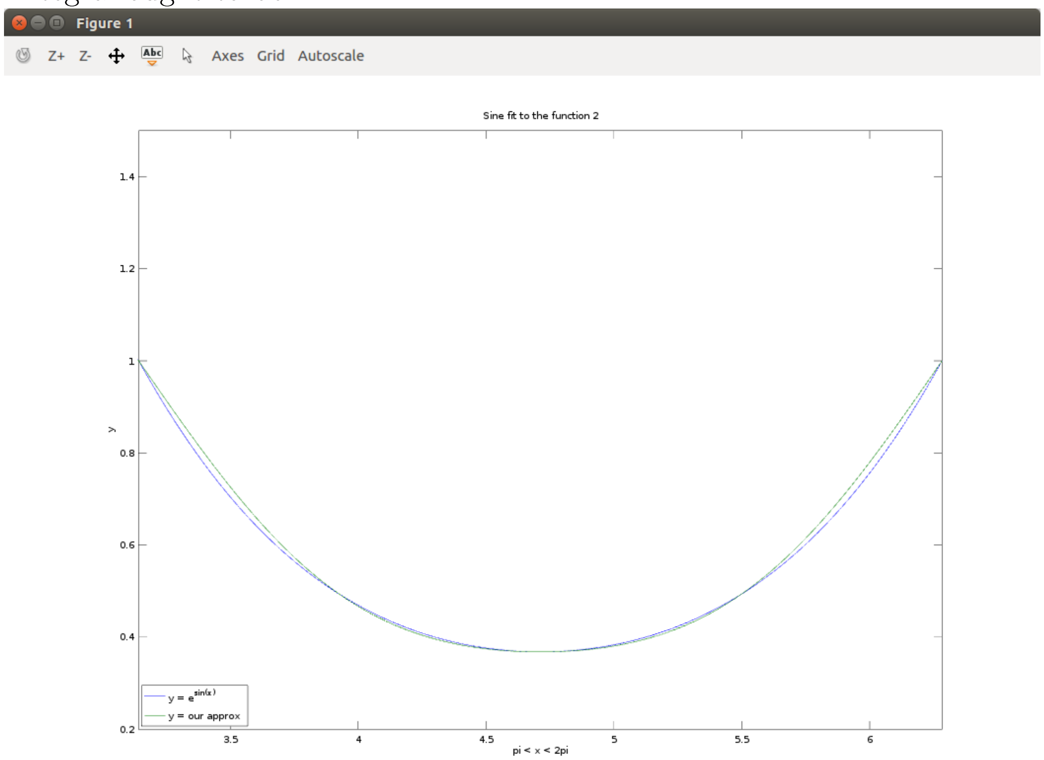
I am proud to say that finding the inverse of this matrix was *much* easier than the first (thanks to the practice I got). Feel free to send homework/project challenges my way :^) I shall slay them with the mighty powers of linear algebra, without need for an extension.

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1+\sqrt{2}}{2} & -\frac{\sqrt{2}}{4} & -\frac{1}{2} & -\frac{\sqrt{2}}{4} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{\sqrt{2}-1}{2} & -\frac{\sqrt{2}}{4} & \frac{1}{2} & -\frac{\sqrt{2}}{4} \end{pmatrix}$$

Again, we rearrange (1) such that $\widetilde{x} = A^{-1}\widetilde{b}$ and plug in our knowns for function 2's A^{-1} and $\widetilde{b}$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1+\sqrt{2}}{2} & -\frac{\sqrt{2}}{4} & -\frac{1}{2} & -\frac{\sqrt{2}}{4} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{\sqrt{2}-1}{2} & -\frac{\sqrt{2}}{4} & \frac{1}{2} & -\frac{\sqrt{2}}{4} \end{pmatrix} \begin{pmatrix} 1 \\ e^{-\frac{\sqrt{2}}{2}} \\ e^{-1} \\ e^{-\frac{\sqrt{2}}{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1+\sqrt{2}}{2} - \frac{\sqrt{2}}{2e^{\frac{\sqrt{2}}{2}}} - \frac{1}{2e} \\ 0 \\ \frac{\sqrt{2}-1}{2} - \frac{\sqrt{2}}{2e^{\frac{\sqrt{2}}{2}}} + \frac{1}{2e} \end{pmatrix} \approx \begin{pmatrix} 1 \\ .67451 \\ 0 \\ .042394 \end{pmatrix} \tag{5}$$

Not as great as the first fitting, however I think I'm merely spoiled by how great the first fit was, and that this will do plenty fine for estimating what our integral ought to be.



The only difference between integrating this function and the first one in terms of integrating is the interval.

$$\int_{\pi}^{2\pi} (a + b\sin(x) + c\sin(2x) + d\sin(3x))dx =$$

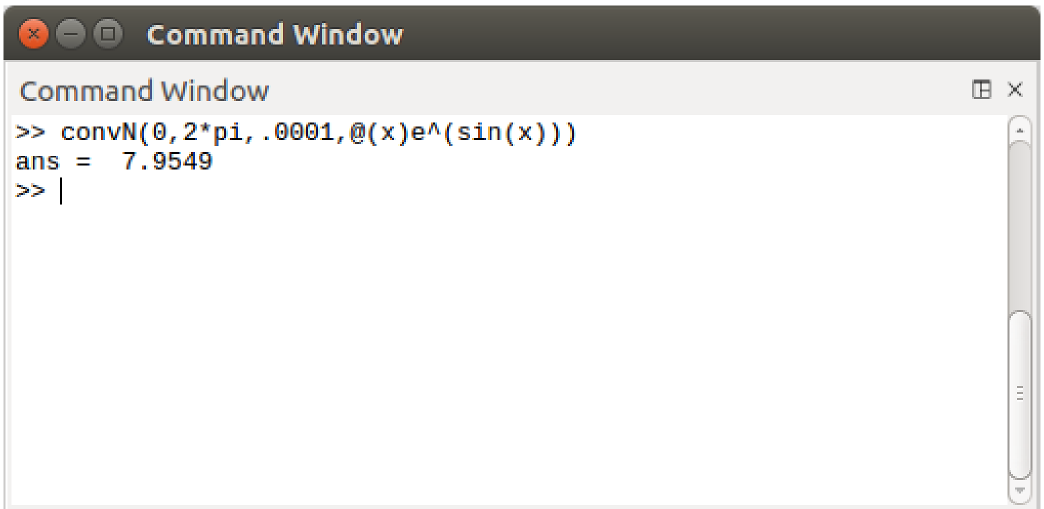
$$ax - b\cos(x) - \frac{c\cos(2x)}{2} - \frac{d\cos(3x)}{3}\Big|_{\pi}^{2\pi} =$$

$$2a\pi - b - \frac{c}{2} - \frac{d}{3} - (\pi - b + \frac{c}{2} - \frac{d}{3})$$

$$a\pi - 2b - \frac{2d}{3} = \pi + 2(.67451) + \frac{2(.042394)}{3} \approx 1.74$$

1.3 Final Reveal

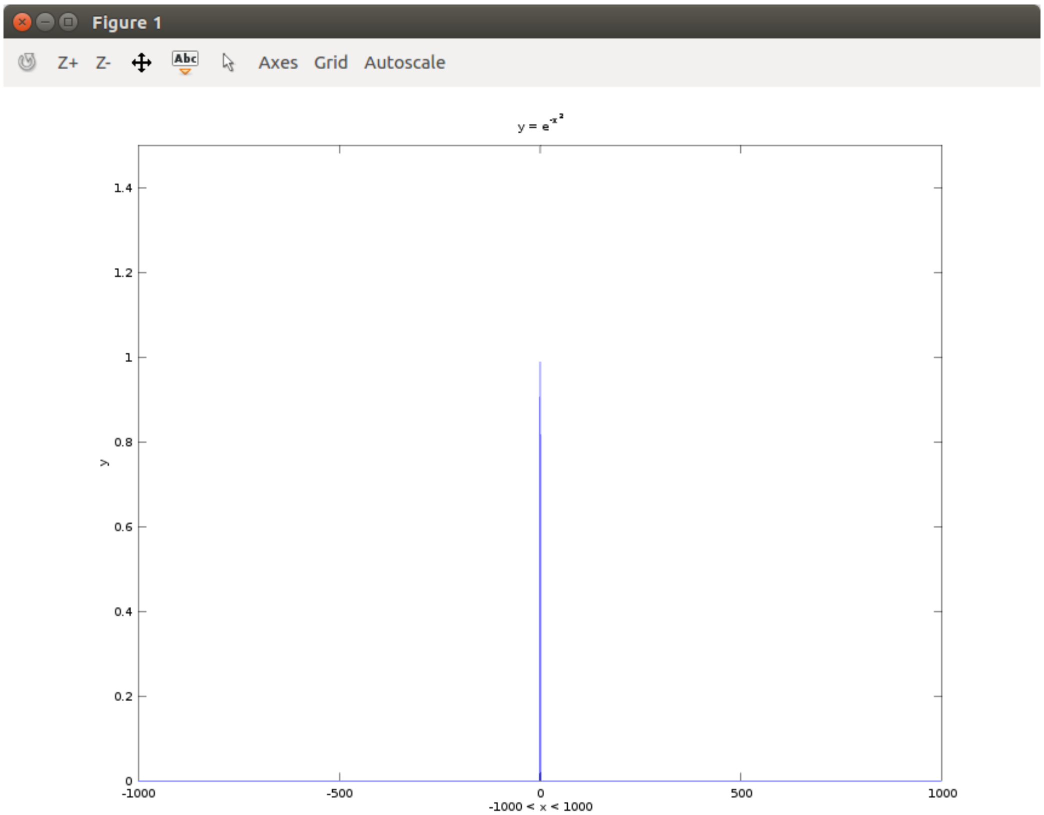
Our total analytically value is 7.98! I cannot tell to how many significant figures it is accurate to without testing more terms, but just by looking at the fit, I'd say its an alright approximation of the real value. However we can use our friend, Mr. Octave, to cross check our work, and use our work to cross check his.



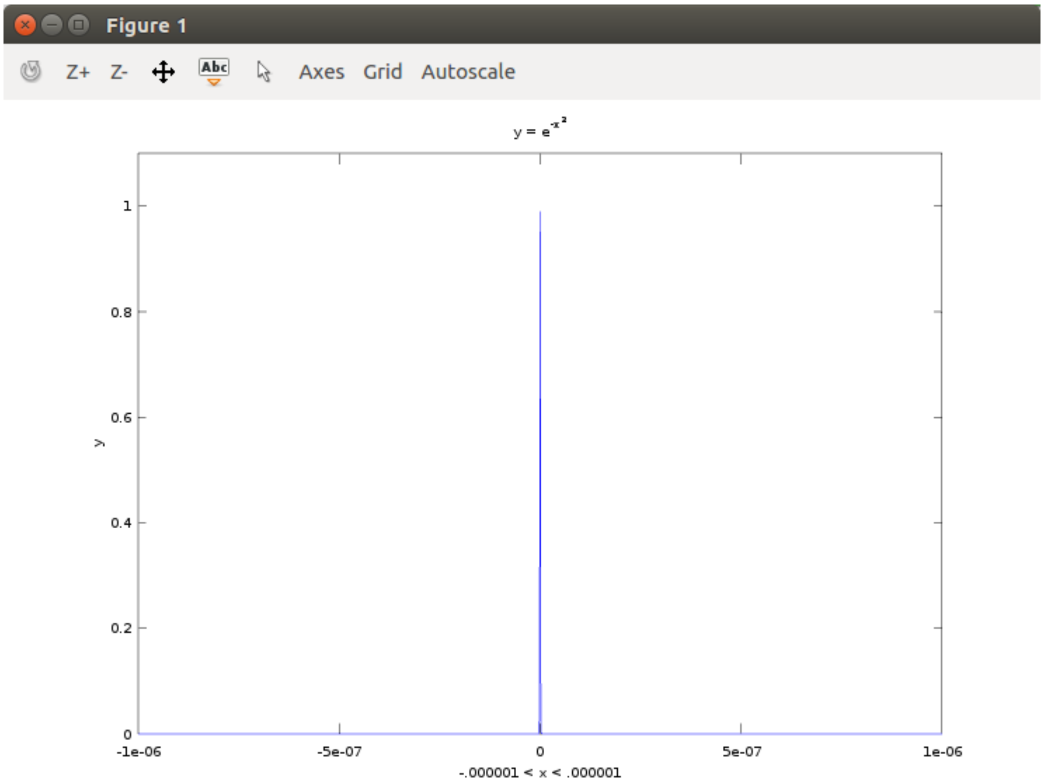
Setting the tolerance to four significant figures for our computational solution we can see that our analytical and computational solutions are very similar! However I don't believe this is enough to say that the computer has given us the right answer. After all, I could have messed up my math somewhere in the analytical solution and there is a very small chance that the computer too messed up in way to give a similar erroneous solution.

However I have used this exact program (under a different name) to calculate integrals in other homework sets. Those calculated integrals matched my analytical derivations. The nature of the Simpson's Rule program is to approximate the function in very small intervals and then sum the area under that approximation. Because of this, it is capable of finding the integral of any function thats domain and range only contain numbers within Octave's maximum and minimum limit (between 2.2251e-308 and 1.7977e+308).

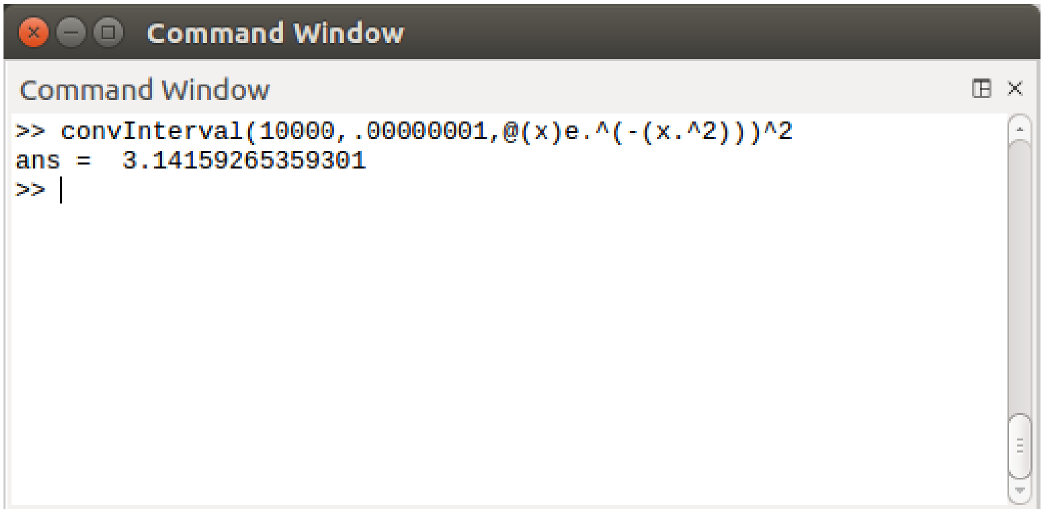
2 $[\int_{-\infty}^{\infty} e^{-x^2} dx]^2$



This function quickly drops off as it moves away from the origin, and luckily does not approach ∞ or $-\infty$ as x approaches the origin as functions such as $\frac{1}{x}$.



Even more interestingly, it looks nearly the same when the domain is greatly reduced. This gives me hope that to integrate over "infinity" will take a finite domain less than 1e100. To find a domain that was large enough, I modified the convint.m program we made in class to increase the interval being integrated over until the desired tolerance is reached.



Integrating the function with 10,000 steps to a tolerance of .00000001 (the tolerance dictates the magnitude of the interval) and the squaring it was piece of cake! (or should I say a piece of π?) To make sure that the step size was large enough I tested 1,000 and 50,0000 as well and saw no difference within five significant figures!

3 Matlab Code

3.1 convN.m

```
function [new] = convN(a,b,tol,f)
%Converges simp function to a given tolerance
% [a,b] is the interval
% tolerance is the tolerance (ie .000001)
% f is the anonymous function (ie @(x)x)
%Written by Math Phys Class Feb 6, 2017
n = 10;
old = simp(a,b,n,f);
n=n*10;
new = simp(a,b,n,f);
% compare new and old integrals until they are not
    different within
%the tolerance
while((abs(old-new) > tol)
```

```
    old = new;
    n=n*10;
    new = simp(a,b,n,f);
end
end
```

3.2 convInterval.m

```
% Author: eggoeke <eggoeke@nephele>
% Created: 2017-03-04

function [retval] = convInterval (n,tol,f)
int = 10;
old = simp(int,-int,n,f);
int = 100;
new = simp(int,-int,n,f);
% compare new and old integrals until they are not
    different within
%the tolerance
while((abs(old-new) > tol) && (int<10000000))
old = new;
int = int*10;
new = simp(int,-int,n,f);
end
retval=new;

endfunction
```

3.3 simp.m

```
function [d]=simp(a,b,n,f)
% This is a program to perform parabolic integration
% INPUTS
% a= beginning of interval
% b= end of interval
% n= number of divisions
```

```
% f= anonymous function
% OUTPUTS
% d= sum of the areas of the rectangles
% written by the MathPhys class on February 6, 2017
% continued Feburary 8,2017
% modified by Erin Goeke on February 16, 2017

%make n prime
n=n*2+1;
x=linspace(a,b,n);
%create weighting array for quadratic approximation
w = ones(size(x));
w(2:2:n-1)=4;
w(3:2:n-2)=2;
%create function array
p = 0;
for p = 1:n
    y(1,p) = f(x(p));
end
%calculate width of quadratic
deltax=x(2)-x(1);
%sum using inner product
d = y*w'*deltax/3;
```