30 Magnetic Fields Due to Currents

This is the way we presently launch materials into space. However, when we begin mining the Moon and the asteroids, where we will not have a source of fuel for such conventional rockets, we shall need a more effective way. Electromagnetic launchers may be the answer. A small prototype, the *electromagnetic rail gun*, can accelerate a projectile from rest to a speed of 10 km/s (36 000 km/h) within 1 ms.

How can such rapid acceleration possibly be accomplished?

The answer is in this chapter.

30-1 Introduction

When people first began to study magnetism scientifically (say, starting from Gilbert's *Treatise de Magnete* in 1600), they focused on the properties of magnets. For example, they studied lodestones (pieces of iron naturally magnetized by the Earth's magnetic field) and found that magnets interact with other magnets through an "action-at-a-distance" force that we now call magnetism. Magnetism was found to be a third distinct noncontact force to add to the list of the two already known: gravity and electricity.

As we learned in the previous chapter, *stationary* electric charges and magnets do not interact (except for the polarization effects that stationary charges can induce in all objects). However, *moving* electric charges do experience a force in the presence of a magnet. Since magnets can exert forces on other magnets, could it be that moving charges behave like magnets?

We have postulated the existence of an entity called the magnetic field in order to introduce a magnetic force law that provides a mathematical description of the force that a permanent magnet can exert on moving electrical charges. Newton's Third Law states that whenever one object exerts a force on another object, the latter object exerts an equal and opposite force on the former. So, if a magnet exerts a force on a current-carrying wire, shouldn't the wire exert an equal and opposite force on the magnet? The symmetry demanded by Newton's Third Law leads us to predict that if moving charges feel forces as they pass through magnetic fields, then they should be capable of exerting forces on the sources of these magnetic fields. In the early 19th century the Danish physicist Hans Christian Oersted demonstrated that an electric current does indeed exert forces on a magnet in its vicinity.

In this chapter we describe how to determine the magnetic fields associated with current-carrying wires and the forces they exert on other wires and magnets. We begin with a summary of Oersted's observations of magnetic phenomena associated with current-carrying wires. We also discuss the work of Biot and Savart, two French scientists. Biot and Savart made a series of careful observations to formulate a mathematical expression describing the magnetic field from a short segment of current-carrying wire, doing for magnetism what Coulomb did for electricity. Next we show how the Biot–Savart law and an alternative law known as Ampère's law (much as Gauss' law was an alternative to Coulomb's) can be used to calculate the magnetic fields and forces associated with various configurations of current-carrying wires. The ability to make such calculations has had a tremendous impact on the design of devices ranging from electric toothbrushes to gigantic particle accelerators.

30-2 Magnetic Effects of Currents—Oersted's Observations

The Earth has a relatively weak magnetic field that interacts with magnets. This phenomenon was exploited for navigational purposes through the development of the compass—a small bar magnet suspended so it pivots freely. Hence a compass is a sensitive magnetic field detector. Oersted and other scientists used the orientation of a compass to detect magnetic fields and determine their directions. By convention, the north-seeking pole of a magnet points in the direction of the magnetic field at its location.

In 1820, H. C. Oersted reported on a famous experiment connecting magnetism with electric currents. He placed a conducting wire along the north–south line of the Earth's magnetic field and laid a compass on top of the wire. The needle pointed along the wire (and the Earth's north–south line). When Oersted connected the ends of the wire to the terminals of a battery, the compass needle swung *perpendicular* to the wire as shown in Fig. 30-1, demonstrating that moving charges in a wire affect a compass in the same way a magnet does. Oersted also noticed that when the direction of the current is reversed, the compass needle flips so it points in the opposite direction.

Oersted found that moving charged particles, such as a current in a wire, create magnetic fields. Oersted's observation was especially surprising because this was the first known instance in which the force on an object (in this case the compass) was not observed to act along a line connecting it with the source of the force (in this case the wire). Within a week of the time that Oersted announced his observations, a French physicist, André Marie Ampère, began to refine them. Ampère noted that the magnetic field lines lay in concentric circles around the wire. His careful observations revealed that a long current-carrying wire sets up a magnetic field that orients small compass magnets so they are tangent to a circle centered on the wire that lies in a plane perpendicular to the wire. The alignment of iron filings, which act like small compasses, is shown in Fig. 30-2. Drawing the direction of the compass needle alignments at many different points that completely surround the wire results in an image of concentric circles like those shown in Fig. 30-3.

Ampère also developed a graphic way of relating the direction of conventional current (that is, traveling from the positive to the negative terminal of a battery) and the orientation of the magnetic field, which is indicated by the direction of the north pole of a compass needle. Ampère stated his **right-hand rule** as follows:

> Encircle the wire with the fingers of the right hand, thumb extended in the direction of positive current. The fingers then point in the direction of deflection of the north pole.

This right-hand rule is shown graphically in Fig. 30-4.

You can easily replicate the following observations made by Oersted, Ampère, and many others in the early 19th century using a battery, wire, a piece of cardboard, and one or more small compasses:

- The compass needles are more strongly deflected when they are close to the wire than when they are far from the wire.
- For a given current, the amount of needle deflection depends only on the needle's radial distance from the wire.
- At a given radial distance from the wire, increasing the current in the wire increases the needle deflection.
- The direction of the needle deflection flips (change by 180°) if you reverse the direction of the current flow.
- Drawing the directions of the needle orientations at many different points that completely surround the wire results in an image of concentric circles like those shown in Fig. 30-3.

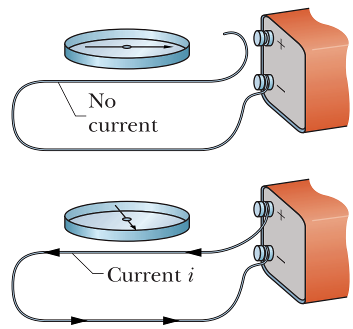


FIGURE 30-1 ■ Oersted's experiment showing how a compass needle becomes aligned in a direction that is perpendicular to the direction of the current in a length of wire.

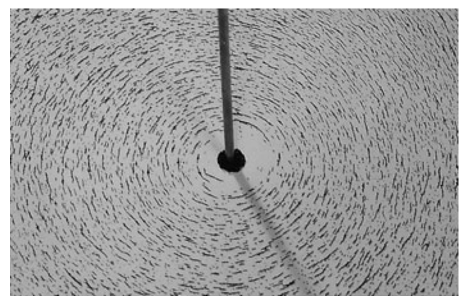


FIGURE 30-2 ■ Iron filing slivers that have been sprinkled onto cardboard collect in concentric circles when a strong current is sent through the central wire. The filings are magnetized and align themselves like tiny compasses in the direction of the magnetic field produced by the current.

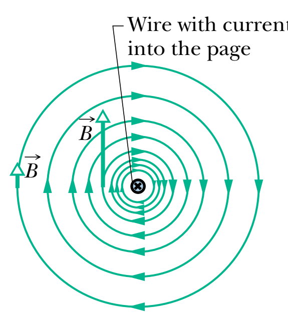
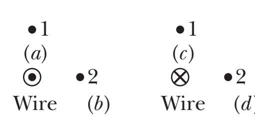


FIGURE 30-3 ■ The magnetic field lines produced by a current in a long straight wire form concentric circles around the wire. Here the current is into the page, as indicated by the ×. The field lines are farther apart as the distance from the wire increases, signifying a decrease in the magnitude of the field with distance.

READING EXERCISE 30-1: In each of the following situations, assume that the magnetic field associated with a current-carrying wire can point up, down, left, right, into the page, or out of the page. (a) If the direction of the conventional current in the wire is out of the page, what is the direction of the magnetic field it generates at point 1? (b) At point 2? (c) If the direction of the conventional current in the wire is into the page, what is the direction of the magnetic field it generates at point 1? (d) At point 2?



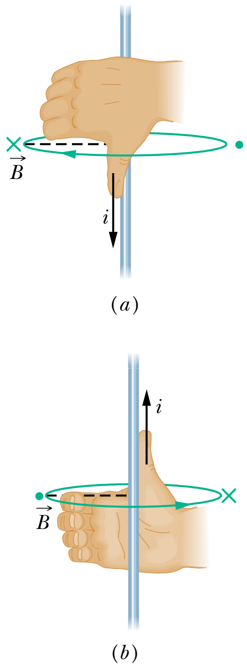


FIGURE 30-4 ■ Ampère's right-hand rule gives the direction of the magnetic field relative to the conventional current in a wire. (a) The situation of Fig. 30-3, seen from the side. The magnetic field $\vec{B}$ at any point to the left of the wire is perpendicular to the dashed radial line and directed into the page, in the direction of the fingertips, as indicated by the $\times$. (b) If the current is reversed, $\vec{B}$ at any point to the left is still perpendicular to the dashed radial line but now is directed out of the page, as indicated by the dot.

30-3 Calculating the Magnetic Field Due to a Current

It is very useful to be able to compute the net magnetic field created by a current-carrying wire. We would also like to be able to do this either for long straight wires or for any wire no matter how it bends around.

Two French physicists named Biot and Savart (rhymes with "Leo and bazaar") were able to develop a mathematical description of the magnetic field in the vicinity of a short segment of current-carrying wire. To do this, these investigators made a set of very clever experimental measurements:

- First, the two investigators positioned magnets around their experimental setup in order to cancel out the local magnetic field of the Earth.
- Next they placed sharp bends in a current-carrying wire so they could observe the approximate effect that an "isolated" short element of wire would have.
- Then they ran a known current through the wire and measured the direction of the magnetic field produced by the small wire segment at various locations using the final orientation of the suspended compass needles.
- Finally, they measured the relative magnitude of the torque on the suspended compass needles before they reached their final orientation and thus the relative force applied to the needles. In doing this, they were actually making measurements of the strength of the field at various locations.

Given what we know of the observations summarized in the previous section, it is not surprising that Biot and Savart found that the magnitude of the magnetic field contribution $|d\vec{B}| = dB$ is directly proportional to the amount of the current $|i|$ and the length of the small segment of wire. They also found that the magnitude of the magnetic field at a point P in space decreases as the inverse square of the distance between the segment of wire and point P. The two investigators proposed that the magnitude of the field contribution $d\vec{B}$ produced at a point P by a segment of wire $d\vec{s}$ carrying a current i is

$$dB = \frac{\mu_0}{4\pi}\frac{|i\,ds|\sin\phi}{r^2}, \tag{30-1}$$

where $d\vec{s}$ is a vector of magnitude ds equal to the length of the piece of wire and direction given by the direction of the current. ϕ is the angle between the directions of $d\vec{s}$ and $\vec{r}$, where $\vec{r}$ is the vector that extends from $d\vec{s}$ to point P. (See Figure 30-5b.) The symbol μ_0 is called the *magnetic constant* (or permeability). By definition its value in SI units is exactly

$$\mu_0 \equiv 4\pi \times 10^{-7}\,\mathrm{T\cdot m/A} \approx 1.26 \times 10^{-6}\,\mathrm{T\cdot m/A} \qquad \text{(magnetic constant)}. \tag{30-2}$$

Equation 30-1 is similar in many ways to that found for the differential electric field from a small segment of wire holding static charge described by Eq. 23-21. However, the perpendicular relationship between the direction of a segment of wire and the magnetic field it produces is a new phenomenon. Fortunately, it turns out that a vector crossproduct can be used to find the direction of the magnetic field contribution. The direction of $d\vec{B}$, shown as being into the page in Fig. 30-5b, is the same as that given by the cross product $d\vec{s} \times \vec{r}$. We can therefore recast Eq. 30-1 in vector form as

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s} \times \vec{r}}{r^3} \qquad \text{(Biot–Savart law)}. \tag{30-3}$$

This vector equation is known as the **Biot–Savart law.** The law, which was experimentally deduced, is an inverse-square law (the exponent in the denominator of Eq. 30-3 is 3 only because of the factor $\vec{r}$ in the numerator). How can we use this law to calculate the net magnetic field $\vec{B}$ produced at a point by various distributions of current?

If our goal is to calculate the magnetic field that is produced by a given current *distribution* based on the field produced by *segments* of the distribution, perhaps we should use the same basic procedure we used in Chapter 23 to calculate the electric field produced by a given distribution of charged particles. Let us quickly review that basic procedure. We first mentally divide the charge distribution into charge elements dq, as is done for a charge distribution of arbitrary shape in Fig. 30-5*a*. We then calculate the field $d\vec{E}$ produced at some point P by a single charge element. Because the electric fields contributed by different elements can be superimposed, we calculate the net field $\vec{E}$ at P by summing, via integration, the contributions $d\vec{E}$ from all the elements.

Recall that we express the magnitude of $d\vec{E}$ as

$$|dE| = k\frac{|dq|}{r^2}, \qquad (30\text{-}4)$$

in which r is the distance between the charge element dq and point P. For a positively charged element, the direction of $d\vec{E}$ is that of $\vec{r}$, where $\vec{r}$ is the vector that extends from the charge element dq to the point P. Using $\vec{r}$, we can rewrite Eq. 30-4 in vector form as

$$d\vec{E} = k\frac{dq}{r^3}\vec{r}, \qquad (30\text{-}5)$$

which indicates that the direction of the vector $d\vec{E}$ produced by a positively charged element is the direction of the vector $\vec{r}$. Note that just as is the case for the Biot-Savart law this is an inverse-square law ($d\vec{E}$ depends on inverse r^2) in spite of the r^3 term in the denominator. This is because the $\vec{r}$ term in the numerator cancels one of the r's in the denominator.

We can use the same basic procedure to calculate the magnetic field due to a current. Figure 30-5*b* shows a wire of arbitrary shape carrying a current i. We want to find the magnetic field $\vec{B}$ at a nearby point P. We first mentally divide the wire into differential elements ds and then define for each element a length vector $d\vec{s}$ that has length ds and whose direction is the direction of the current in ds. We can then define a differential *current-length element* to be $i\,d\vec{s}$; we wish to calculate the field $d\vec{B}$ produced at P by a single current-length element. From experiment we find that magnetic fields, like electric fields, can be superimposed to find a net field. Thus, we can calculate the net field $\vec{B}$ at P by summing contributions for discrete sources or by integrating the contributions $d\vec{B}$ from all the current-length elements in a continuous source. However, this summation (or integration) is more challenging than the process associated with electric fields because of a complexity. The charge element dq that produces an electric field is a scalar, but a current-length element $i\,d\vec{s}$ that produces a magnetic field is the product of a scalar and a vector.

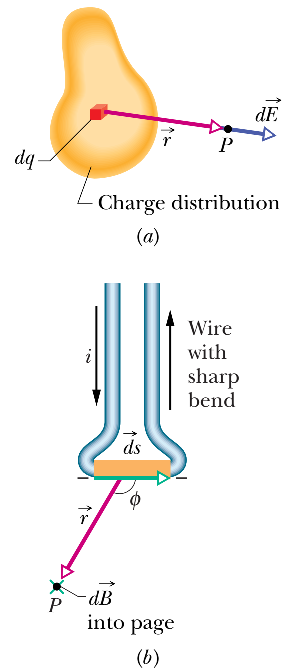


FIGURE 30-5 ■ (*a*) A charge element dq produces a differential electric field $d\vec{E}$ at point P. (*b*) A current-length element $i\,d\vec{s}$, isolated by sharp bends in the wire, produces a differential magnetic field $d\vec{B}$ at point P. The × (the tail of an arrow) at the dot for point P indicates that $d\vec{B}$ is directed into the page there for the special case where i and $d\vec{s}$ are parallel. If a small magnetic compass needle is used to detect the magnetic field, then its north pole points into the page.

Magnetic Field Due to a Current in a Long Straight Wire

Shortly we shall use the law of Biot and Savart to prove that the magnitude of the magnetic field at a perpendicular distance R from a long (infinite) straight wire carrying a current i is given by

$$B = \frac{\mu_0 |i|}{2\pi R} \quad \text{(long straight wire).} \qquad (30\text{-}6)$$

The field magnitude $B = |\vec{B}|$ in Eq. 30-6 depends only on the amount of current and the perpendicular distance R of the point from the wire. We shall show in our derivation that the field lines of $\vec{B}$ form concentric circles around the wire, as Fig. 30-3 shows and as the iron filings in Fig. 30-2 suggest. The increase in the spacing of the lines in Fig. 30-3 with increasing distance from the wire represents the $1/R$ decrease in the magnitude of $\vec{B}$ predicted by $B = \mu_0|i|/2\pi R$ (Eq. 30-6). The lengths of the two vectors $\vec{B}$ in Fig. 30-3 also show the $1/R$ decrease when we use Ampère's right-hand rule for finding the direction of the magnetic field set up by a current-length element, such as a section of a long wire. What we are really doing is describing the orientation of concentric circles centered on the wire. A careful review of Fig. 30-3 yields two additional points that are often quite useful in solving magnetic field problems. Namely, the magnetic field $\vec{B}$ due to a current-carrying wire at any point is *tangent to a magnetic field line* and it is *perpendicular to a dashed radial line connecting the point and the current.*

Proof of Equation 30-6

Figure 30-6, which is just like Fig. 30-5*b* except that now the wire is straight and of infinite length, illustrates the task at hand; we seek the field $\vec{B}$ at point P, a perpendicular distance R from the wire. The magnitude of the differential magnetic field produced at P by the current-length element $|i d\vec{s}\,|$ located a distance r from P is given by Eq. 30-1:

$$|d\vec{B}\,| = \frac{\mu_0}{4\pi}\frac{|i\,d\vec{s}\,|\sin\phi}{r^2}.$$

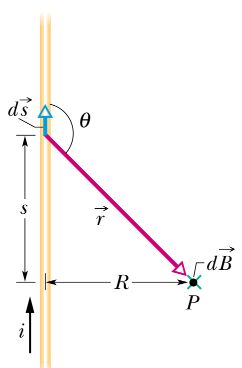


FIGURE 30-6 ■ Calculating the magnetic field produced by a current i in a long straight wire. Using either Ampère's right-hand rule or $d\vec{s} \times \vec{r}$, we find $d\vec{B}$ at P is directed into the page as shown.

Since the direction of $d\vec{s}$ is always in the direction of the current, we find that the direction of $d\vec{B}$ in Fig. 30-6 (given by $d\vec{s} \times \vec{r}$) is into the page.

Note that $d\vec{B}$ at point P has this same direction (into the page) for all the current-length elements into which the wire can be divided. Thus, we can find the magnitude of the magnetic field produced at P by the current-length elements in the upper half of the infinitely long wire by integrating dB in Eq. 30-1 from 0 to ∞.

Now consider a current-length element in the lower half of the wire, one that is as far below P as $d\vec{s}$ is above P. By Eq. 30-6, the magnetic field produced at P by this current-length element has the same magnitude and direction as that from $i\,d\vec{s}$ in Fig. 30-6. Further, the magnetic field produced by the lower half of the wire is exactly the same as that produced by the upper half. To find the magnitude of the total magnetic field $\vec{B}$ at P, we need only multiply the result of our integration by 2. We get

$$B = 2\int_0^\infty dB = \frac{\mu_0|i|}{2\pi}\int_0^\infty \frac{|\sin\phi\,ds|}{r^2}. \qquad (30\text{-}7)$$

The variables ϕ, s, and r in this equation are not independent but (see Fig. 30-6) are related by

$$r = \sqrt{s^2 + R^2}$$

and

$$\sin\phi = \sin(\pi - \phi) = \frac{R}{\sqrt{s^2 + R^2}}.$$

Using these substitutions along with the solution to integral 19 in Appendix E, Eq. 30-7 describing the magnitude of the magnetic field becomes

$$B = \frac{\mu_0|i|}{2\pi}\int_0^\infty \frac{R\,ds}{(s^2 + R^2)^{3/2}} = \frac{\mu_0|i|}{2\pi R}\left[\frac{s}{(s^2 + R^2)^{1/2}}\right]_0^\infty.$$

Substituting the limits in the expression above gives a B-field magnitude of

$$B = \frac{\mu_0 |i|}{2\pi R}, \qquad \text{(infinite straight wire)}, \qquad (30\text{-}8)$$

which is the relation we set out to prove. Note that the magnitude of the magnetic field at P due to either the lower half or the upper half of the infinite wire in Fig. 30-6 is half this value; that is,

$$B = \frac{\mu_0 |i|}{4\pi R} \qquad \text{(semi-infinite straight wire)}. \qquad (30\text{-}9)$$

Magnetic Field Due to a Current in a Circular Arc of Wire

To find the magnetic field produced at a point by a current in a curved wire, we would again use Eq. 30-1 to write the magnitude of the field produced by a single current-length element, and we would again integrate to find the net field produced by all the current-length elements. That integration can be difficult, depending on the shape of the wire; it is fairly straightforward, however, when the wire is a circular arc and the point is the center of curvature.

Figure 30-7*a* shows such an arc-shaped wire with central angle ϕ_C, radius R, and center C, carrying current i. At C, each current-length element $i\,d\vec{s}$ of the wire produces a magnetic field element of magnitude dB given by Eq. 30-1. Moreover, as Fig. 30-7*b* shows, no matter where the element is located on the wire, the angle ϕ between the vectors $d\vec{s}$ and $\vec{r}$ is 90°; also, $r = R$. Thus, by substituting R for r and 90° for ϕ, we obtain from Eq. 30-1,

$$dB = \frac{\mu_0}{4\pi} \frac{|i|\,ds \sin 90^\circ}{R^2} = \frac{\mu_0}{4\pi} \frac{|i|\,ds}{R^2}. \qquad (30\text{-}10)$$

The field at C due to each current-length element in the circular arc has this same magnitude.

An application of the right-hand rule anywhere along the wire (as in Fig. 30-7*c*) will show that all the differential fields $d\vec{B}$ have the same direction at C—directly out of the page. Thus, the total field at C is simply the sum (via integration) of all the fields $d\vec{B}$. We use the identity $ds = R\,d\phi$ to change the variable of integration from ds to $d\phi$ and obtain, from Eq. 30-10, a magnitude of

$$B = \int dB = \int_0^{\phi_C} \frac{\mu_0}{4\pi} \frac{|i|\,R\,d\phi}{R^2} = \frac{\mu_0 |i|}{4\pi R} \int_0^{\phi_C} d\phi.$$

Integrating, we find that

$$B = \frac{\mu_0 |i| \phi_C}{4\pi R} \qquad \text{(at center of circular arc)}. \qquad (30\text{-}11)$$

Note that this equation gives us the magnitude of the magnetic field *only* at the center of curvature of a circular arc of current. When you insert data into the equation, you must be careful to express ϕ_C in radians rather than degrees. For example, to find the magnitude of the magnetic field at the center of a full circle of current,

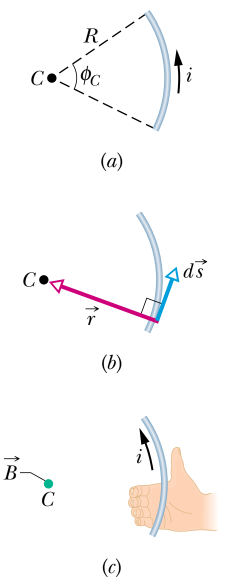
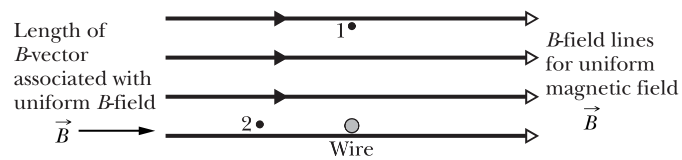


FIGURE 30-7 ■ (*a*) A wire in the shape of a circular arc with center C carries current i. (*b*) For any element of wire along the arc, the angle between the directions of $d\vec{s}$ and $\vec{r}$ is 90°. (*c*) Determining the direction of the magnetic field at the center C due to the current in the wire; the field is out of the page, in the direction of the fingertips, as indicated by the colored dot at C.

you would substitute 2π for ϕ_C in Eq. 30-11, finding

$$B = \frac{\mu_0 |i| (2\pi)}{4\pi R} = \frac{\mu_0 |i|}{2R} \quad \text{(at center of full circle).} \tag{30-12}$$

READING EXERCISE 30-2: A uniform magnetic field is directed toward the right in the plane of the paper as shown in the diagram that follows. A wire oriented perpendicular to the plane of the paper carries a current i. Suppose that the resultant magnetic field at point 1 due to a superposition of the uniform magnetic field of magnitude $|\vec{B}|$ and the magnetic field of the wire at point 1 is zero. (a) Is the direction of the current in the wire into or out of the paper? Explain how you arrived at your conclusion. (b) Assume that point 2 lies at the same distance from the center of the wire as point 1 and that the length of the vector assigned to represent the magnitude of the uniform external magnetic field is that shown to the left. Construct a vector arrow showing the length and direction of the resultant magnetic field vector at point 2. Explain how you deduced what the vector should be. (Adapted from A. Arons, *Homework and Test Questions for Introductory Physics Teaching,* John Wiley and Sons, 1947.)



TOUCHSTONE EXAMPLE 30-1: An Arc and Two Straight Lines

The wire in Fig. 30-8a carries a current i and consists of a circular arc of radius R and central angle $\pi/2$ rad, and two straight sections whose extensions intersect the center C of the arc. What magnetic field $\vec{B}$ does the current produce at C?

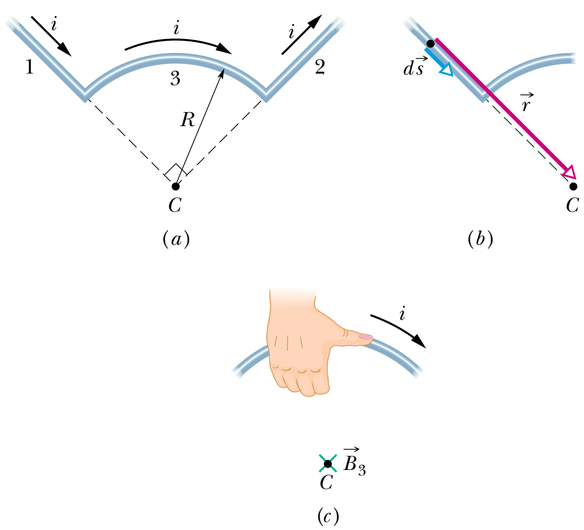


FIGURE 30-8 (a) A wire consists of two straight sections (1 and 2) and a circular arc (3), and carries current i. (b) For a current-length element in section 1, the angle between $d\vec{s}$ and $\vec{r}$ is zero. (c) Determining the direction of magnetic field $\vec{B}_3$ at C due to the current in the circular arc; the field is into the page there.

SOLUTION One **Key Idea** here is that we can find the magnetic field $\vec{B}$ at point C by applying the Biot–Savart law of Eq. 30-3 to the wire. A second **Key Idea** is that the application of Eq. 30-3 can be simplified by evaluating $\vec{B}$ separately for the three distinguishable sections of the wire—namely, (1) the straight section at the left, (2) the straight section at the right, and (3) the circular arc.

Straight sections. For any current-length element in section 1, the angle ϕ between $d\vec{s}$ and $\vec{r}$ is zero (Fig. 30-8b), so Eq. 30-1 gives us

$$|d\vec{B}_1| = \frac{\mu_0}{4\pi} \frac{|i\,d\vec{s}\,|\sin\phi}{r^2} = \frac{\mu_0}{4\pi} \frac{|i\,d\vec{s}\,|\sin 0}{r^2} = 0\ \text{T}.$$

Thus, the current along the entire length of wire in straight section 1 contributes no magnetic field at C:

$$\vec{B}_1 = 0\ \text{T}.$$

The same situation prevails in straight section 2, where the angle ϕ between $d\vec{s}$ and $\vec{r}$ for any current-length element is 180°. Thus,

$$\vec{B}_2 = 0\ \text{T}.$$

Circular arc. The **Key Idea** here is that application of the Biot–Savart law to evaluate the magnetic field at the center of a circular arc leads to Eq. 30-11 ($|B| = \mu_0 |i| \phi / 4\pi R$). Here the central angle ϕ of the arc is $\pi/2$ rad. Thus from Eq. 30-11, the magnitude of the magnetic field $\vec{B}_3$ at the arc's center C is

$$|\vec{B}_3| = \frac{\mu_0 |i| (\pi/2)}{4\pi R} = \frac{\mu_0 |i|}{8R}.$$

To find the direction of $\vec{B}_3$, we apply the right-hand rule displayed in Fig. 30-4. Mentally grasp the circular arc with your right hand as suggested in Fig. 30-8*c*, with your thumb in the direction of the current. The direction in which your fingers curl around the wire indicates the direction of the magnetic field lines around the wire. In the region of point C (inside the circular arc), your fingertips point *into the plane* of the page. Thus, $\vec{B}_3$ is directed into that plane.

Net field. Generally, when we must combine two or more magnetic fields to find the net magnetic field, we must combine the fields as vectors and not simply add their magnitudes. Here, however, only the circular arc produces a magnetic field at point C. Thus, we can write the magnitude of the net field $\vec{B}$ as

$$|\vec{B}| = |\vec{B}_1 + \vec{B}_2 + \vec{B}_3| = 0 + 0 + \left|\frac{\mu_0 i}{8R}\right| = \left|\frac{\mu_0 i}{8R}\right|. \qquad \text{(Answer)}$$

The direction of $\vec{B}$ is the direction of $\vec{B}_3$—namely, into the plane of Fig. 30-8.

TOUCHSTONE EXAMPLE 30-2: Two Long Parallel Wires

Figure 30-9*a* shows two long parallel wires carrying currents i_1 and i_2 in opposite directions. What are the magnitude and direction of the net magnetic field at point P? Assume the following values: $i_1 = 15$ A, $i_2 = 32$ A, and $d = 5.3$ cm.

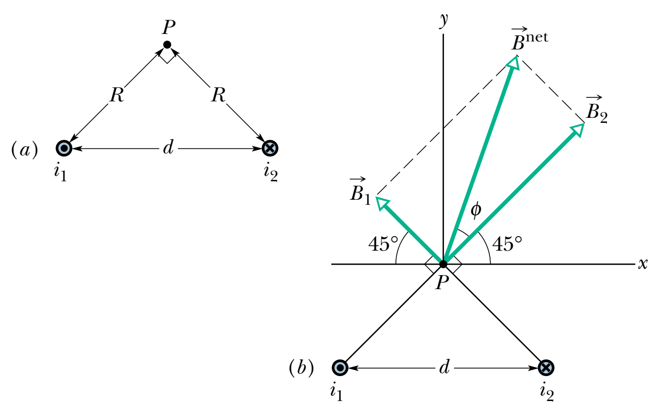


FIGURE 30-9 ■ (*a*) Two wires carry currents i_1 and i_2 in opposite directions (out of and into the page). Note the right angle at P. (*b*) The separate fields $\vec{B}_1$ and $\vec{B}_2$ are combined vectorially to yield the net field $\vec{B}^{\text{net}}$.

SOLUTION ■ One **Key Idea** here is that the net magnetic field $\vec{B}$ at point P is the vector sum of the magnetic fields due to the currents in the two wires. A second **Key Idea** is that we can find the magnetic field due to any current by applying the Biot–Savart law to the current. For points near the current in a long straight wire, that law leads to Eq. 30-6.

In Fig. 30-9*a*, point P is distance R from both currents i_1 and i_2. Thus, Eq. 30-6 tells us that at point P those currents produce magnetic fields $\vec{B}_1$ and $\vec{B}_2$ with magnitudes

$$B_1 = \frac{\mu_0 |i_1|}{2\pi R} \quad \text{and} \quad B_2 = \frac{\mu_0 |i_2|}{2\pi R}.$$

In the right triangle of Fig. 30-9*a*, note that the base angles (between sides R and d) are both 45°. Thus, we may write $\cos 45° = R/d$ and replace R with $d \cos 45°$. Then the field magnitudes B_1 and B_2 become

$$B_1 = \frac{\mu_0 |i_1|}{2\pi d \cos 45°} \quad \text{and} \quad B_2 = \frac{\mu_0 |i_2|}{2\pi d \cos 45°}.$$

We want to combine $\vec{B}_1$ and $\vec{B}_2$ to find their vector sum, which is the net field $\vec{B}^{\text{net}}$ at P. To find the directions of $\vec{B}_1$ and $\vec{B}_2$, we apply the right-hand rule of Fig. 30-4 to each current in Fig. 30-9*a*. For wire 1, with current out of the page, we mentally grasp the wire with the right hand, with the thumb pointing out of the page. Then the curled fingers indicate that the field lines run counterclockwise. In particular, in the region of point P, they are directed upward to the left. Recall that the magnetic field at a point near a long, straight current-carrying wire must be directed perpendicular to a radial line between the point and the current. Thus, $\vec{B}_1$ must be directed upward to the left as drawn in Fig. 30-9*b*. (Note carefully the perpendicular symbol between vector $\vec{B}_1$ and the line connecting point P and wire 1.)

Repeating this analysis for the current in wire 2, we find that $\vec{B}_2$ is directed upward to the right as drawn in Fig. 30-9*b*. (Note the perpendicular symbol between vector $\vec{B}_2$ and the line connecting point P and wire 2.)

We can now vectorially add $\vec{B}_1$ and $\vec{B}_2$ to find the net magnetic field $\vec{B}^{\text{net}}$ at point P, either by using a vector-capable calculator or by resolving the vectors into components and then combining the components of $\vec{B}^{\text{net}}$. However, in Fig. 30-9*b*, there is a third method: Because $\vec{B}_1$ and $\vec{B}_2$ are perpendicular to each other, they form the legs of a right triangle, with $\vec{B}^{\text{net}}$ as the hypotenuse. The Pythagorean theorem then gives us

$$B^{\text{net}} = \sqrt{B_1^2 + B_2^2} = \frac{\mu_0}{2\pi d(\cos 45°)}\sqrt{i_1^2 + i_2^2}$$

$$= \frac{(4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A})\sqrt{(15\,\text{A})^2 + (32\,\text{A})^2}}{(2\pi)(5.3 \times 10^{-2}\,\text{m})(\cos 45°)}$$

$$= 1.89 \times 10^{-4}\,\text{T} \approx 190\,\mu\text{T}. \qquad \text{(Answer)}$$

The angle ϕ between the directions of $\vec{B}^{\text{net}}$ and $\vec{B}_2$ in Fig. 30-9*b* follows from

$$\phi = \tan^{-1}\frac{B_1}{B_2},$$

which, with B_1 and B_2 as given above, yields

$$\phi = \tan^{-1}\frac{i_1}{i_2} = \tan^{-1}\frac{15\,\text{A}}{32\,\text{A}} = 25°.$$

The angle between the direction of $\vec{B}^{\text{net}}$ and the x axis shown in Fig. 30-9*b* is then

$$\phi + 45° = 25° + 45° = 70°. \qquad \text{(Answer)}$$

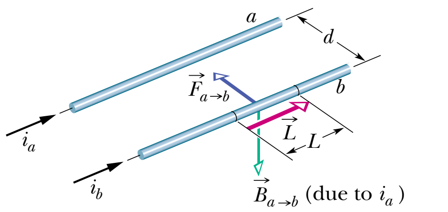


FIGURE 30-10 ■ Two parallel wires carrying currents in the same direction attract each other. $\vec{B}_{a\to b}$ is the magnetic field at wire b produced by the current in wire a. $\vec{F}_{a\to b}$ is the resulting force acting on wire b because it carries current in field $\vec{B}_{a\to b}$.

30-4 Force Between Parallel Currents

Back in 1820, when Ampère was first replicating Oersted's observations, he predicted that two current-carrying wires in parallel would exert forces on each other. This is a logical consequence of the Biot–Savart law, which quantifies the magnetic field surrounding a current-carrying wire, and the magnetic force law, which describes the force on a current in the presence of a magnetic field. Indeed, Ampère observed that there is a mutual interaction between the two wires. In other words, each wire exerts a force on the other. As shown in Fig. 30-10, the application of the right-hand rules that accompany the Biot–Savart law (Eq. 30-3) and the expression for the magnetic force on a current (Eq. 29-22) lead us to predict that wires that carry currents in the same direction will attract, whereas wires that carry currents in opposite directions will repel. It is interesting that the attractions and repulsions are opposite to the electrostatic and magnetic relationships, where unlike charges or poles attract and like charges or poles repel.

We can use the two equations just mentioned to derive a third equation that describes the forces between two parallel current-carrying wires. Why do we want to determine these interaction forces? Three reasons come to mind. First, we can compare the measurement of these forces to the forces predicted by our third equation to verify the Biot–Savart law. Second, these mutual interaction forces enable us to define the ampere as the SI unit of current. Finally, by understanding the nature of these forces we can design an electromagnetic launcher (like that mentioned in the "puzzler" on the first page of this chapter).

Figure 30-10 shows two parallel wires, separated by a distance d and carrying currents i_a and i_b. The first step in analyzing the forces between these wires is to find an expression for the force on wire b due to the current in wire a. The current in wire a produces a magnetic field $\vec{B}_{a\to b}$ at the location of wire b, and it is this magnetic field produced by wire a that actually causes wire b to experience a force denoted as $\vec{F}_{a\to b}$. According to Eq. 30-6, the magnitude of $B_{a\to b}$ at every point along wire b is

$$B_{a\to b} = \frac{\mu_0|i_a|}{2\pi d}. \tag{30-13}$$

The right-hand rule tells us that the direction of $\vec{B}_{a\to b}$ at wire b is down, as shown in Fig. 30-10.

Now that we have determined the magnetic field vector, we can find the force that wire a produces on wire b. The expression for the force on a length of current-carrying wire (Eq. 29-22) tells us that the force on wire b is

$$\vec{F}_{a\to b} = i_b\vec{L} \times \vec{B}_{a\to b}, \tag{30-14}$$

where $\vec{L}$ is the length vector (direction given by the direction of current i) of the wire. In Fig. 30-10 the vectors $\vec{L}$ and $\vec{B}_{a\to b}$ are perpendicular, so using Eqs. 30-13 and 30-14, we can express the magnitude of the force on wire b due to the current in wire a as

$$F_{a\to b} = |i_b|L\,B_{a\to b}\sin 90° = \frac{\mu_0 L|i_a i_b|}{2\pi d}. \tag{30-15}$$

The direction of $\vec{F}_{a\to b}$ is the direction of the cross product $\vec{L} \times \vec{B}_{a\to b}$. Applying the right-hand rule for cross products to $\vec{L}$ and $\vec{B}_{a\to b}$ in Fig. 30-10, we find that $\vec{F}_{a\to b}$ points directly toward wire a, as shown.

The general procedure for finding the force on a current-carrying wire is this:

> To find the force on a current-carrying wire due to a second current-carrying wire, first find the field due to the second wire at the site of the first wire. Then find the force on the first wire due to that field.

We could now use this procedure to compute the force on wire a due to the current in wire b. We would find that the force has the same magnitude but is in the opposite direction. This is true regardless of whether the currents are the same or in opposite directions. Once again, Newton's Third Law holds:

> Parallel currents attract, and antiparallel currents repel.

The forces acting between currents in parallel wires provide us with the basis for defining the ampere, which is one of the seven SI base units. It is appropriately named after André Marie Ampère, who was the first to demonstrate the forces acting between parallel currents. The official SI definition, adopted in 1946, is:

> The **ampere** is that constant current which, if maintained in two straight, parallel conductors of infinite length, of negligible circular cross section, and placed 1 m apart in a vacuum, would produce between each of these conductors a force equal to 2×10^{-7} newton per meter of length.

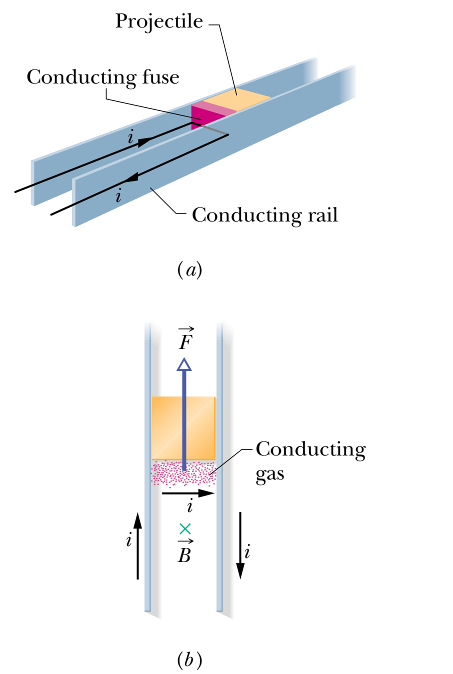


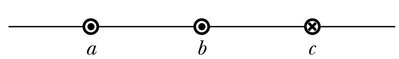
FIGURE 30-11 ■ (a) A rail gun, as a current i is set up in it. The current rapidly causes the conducting fuse to vaporize. (b) The current produces a magnetic field $\vec{B}$ between the rails, and the field causes a force $\vec{F}$ to act on the conducting gas, which is part of the current path. The gas propels the projectile along the rails, launching it.

Rail Gun

A rail gun is a device in which a magnetic force can accelerate a projectile to a high speed in a short time. The basics of a rail gun are shown in Fig. 30-11a. A large current flows in a circuit consisting of two conducting rails joined by a conducting "fuse" (such as a narrow piece of copper) between the rails, and then back to the current source along the second rail. The projectile to be fired lies on the far side of the fuse and fits loosely between the rails. Immediately after the current is established, the fuse element melts and vaporizes, creating a conducting gas between the rails where the fuse had been.

The right-hand rule of Fig. 30-4 shows that the current in the rails of Fig. 30-11a produces a magnetic field that is directed downward between the rails. The net magnetic field $\vec{B}$ exerts a force $\vec{F}$ on the gas due to the current i through the gas (Fig. 30-11b). Using Eq. 30-14 and the right-hand rule for cross products, we find that $\vec{F}$ points outward along the rails. As the gas is forced outward along the rails, it pushes the projectile, accelerating it by as much as $5 \times 10^7\,\text{m/s}^2$ or $(5 \times 10^6\,g)$, and then launches it with a speed of 10 km/s, all within less than one millisecond.

READING EXERCISE 30-3: The figure shows three long, straight, parallel, equally spaced wires with identical amounts of current either into or out of the page. Rank the wires according to the magnitude of the force on each due to the currents in the other two wires, greatest first.



30-5 Ampère's Law

We can find the net electric field due to *any* distribution of charges with the inverse-square law for the differential field $d\vec{E}$ (Eq. 30-5), but if the distribution is complicated, we may have to use a computer. Recall, however, that if the distribution has planar, cylindrical, or spherical symmetry, we can apply Gauss' law to find the net electric field with considerably less effort.

Similarly, we can find the net magnetic field due to any distribution of currents with the inverse-square law for the differential field $d\vec{B}$ (Eq. 30-3), but again we may have to use a computer for a complicated distribution. However, if the distribution has enough symmetry, we can apply *Ampère's law* to find the magnetic field with considerably less effort. This law, which can be derived from the Biot–Savart law, has traditionally been credited to André Marie Ampère (1775–1836), for whom the SI unit of current is named. However, the law actually was advanced by English physicist James Clerk Maxwell.

Ampère's law is

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i^{\text{enc}} \qquad \text{(Ampère's law).} \tag{30-16}$$

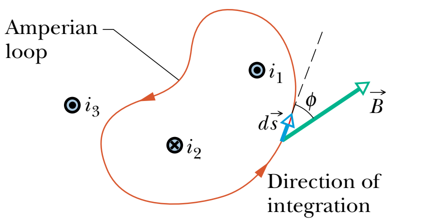


FIGURE 30-12 ■ Ampère's law applied to an arbitrary Ampèrian loop that encircles two long straight wires but excludes a third wire. Note the directions of the currents.

The circle on the integral sign means that the scalar (or dot) product $\vec{B} \cdot d\vec{s}$ is to be integrated around an imaginary *closed* loop, called an *Ampèrian loop*. The current i^{enc} on the right is the *net* current encircled by that loop.

In Gauss' law we choose a closed surface on which to evaluate the integral. The integral flux is proportional to the net charge enclosed by the surface. In Ampère's law, we choose a closed loop on which to evaluate the integral. The integral is proportional to the net current passing through the loop.

To see the meaning of the scalar product $\vec{B} \cdot d\vec{s}$ and its integral, let us first apply Ampère's law to the general situation shown in Fig. 30-12. This figure depicts the cross sections of three long straight wires that carry currents i_1, i_2, and i_3 either directly into or directly out of the page. An arbitrary Ampèrian loop lying in the plane of the page encircles two of the currents but not the third. The counterclockwise direction marked on the loop indicates the arbitrarily chosen direction of integration for Eq. 30-16.

To apply Ampère's law, we mentally divide our imaginary loop into short, nearly straight, directed pieces, $d\vec{s}$. The direction of each of these pieces is tangent to the loop along the direction of integration. Assume that at the location of the element $d\vec{s}$ shown in Fig. 30-12, the net magnetic field due to the three currents is $\vec{B}$. Because the wires are perpendicular to the page, we know that the magnetic field at $d\vec{s}$ due to each current is in the plane of Fig. 30-12; thus, the net magnetic field $\vec{B}$ at $d\vec{s}$ must also be in that plane. However, we do not know the orientation of $\vec{B}$ within the plane. In Fig. 30-12, $\vec{B}$ is arbitrarily drawn at an angle ϕ to the direction of $d\vec{s}$.

The scalar product $\vec{B} \cdot d\vec{s}$ on the left side of Eq. 30-16 is then equal to $(B \cos\phi)ds$. Thus, Ampère's law can be written as

$$\oint \vec{B} \cdot d\vec{s} = \oint (B \cos \phi)\, ds = \mu_0 i^{\text{enc}}. \tag{30-17}$$

We can now interpret the scalar product $\vec{B} \cdot d\vec{s}$ as being the product of a length ds of the Ampèrian loop and the field component $B \cos \phi$ that is tangent to the loop. Then we can interpret the integration as being the summation of all such products around the entire loop.

When we can actually perform this integration, we do not need to know the direction of $\vec{B}$ before integrating. Instead, we arbitrarily assume $\vec{B}$ to be generally in the direction of integration (as in Fig. 30-12). Then we use the following curled fingers-straight thumb right-hand rule to assign a plus sign or a minus sign to each of the currents that make up the net encircled current i^{enc}:

> **CURLED-STRAIGHT RIGHT-HAND RULE FOR AMPÈRE'S LAW:** Curl your right hand around the Ampèrian loop, with the fingers pointing in the direction of integration. A current through the loop in the general direction of your outstretched thumb is assigned a plus sign, and a current generally in the opposite direction is assigned a minus sign.

Finally, we solve Eq. 30-17 for the magnitude of $\vec{B}$. Once we have chosen a coordinate system to describe the system, we can use Ampère's law right-hand rule to decide whether $\vec{B}$ is positive or negative.

In Fig. 30-13 we apply the curled-straight rule for Ampère's law to the situation of Fig. 30-12. With the indicated counterclockwise direction of integration, the net current encircled by the loop is

$$i^{\text{enc}} = i_1 - i_2.$$

(Current i_3 is not encircled by the loop.) We can then rewrite Eq. 30-17 as

$$\left|\oint (B \cos \phi)\, ds\right| = \mu_0 |(i_2 - i_1)|. \tag{30-18}$$

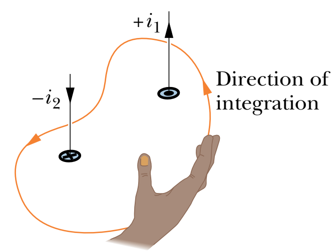


FIGURE 30-13 ■ A right-hand rule for Ampère's law, to determine the signs for currents encircled by an Ampèrian loop. The situation is that of Fig. 30-12.

You might wonder why, since current i_3 contributes to the magnetic-field magnitude B on the left side of Eq. 30-18, it is not needed on the right side. The answer is that the contributions of current i_3 to the magnetic field cancel out because the integration in Eq. 30-18 is made around the full loop. In contrast, the contributions of an encircled current to the magnetic field do not cancel out.

We cannot solve Eq. 30-18 for the magnitude B of the magnetic field, because for the situation of Fig. 30-12 we do not have enough information to solve the integral. However, we do know the magnitude of the integral; it must be equal to the value of $\mu_0 |(i_1 - i_2)|$, which is set by the net current passing through the loop. Next we apply Ampère's law to two situations in which symmetry does allow us to solve the integrals and determine the magnetic fields.

The Magnetic Field Outside a Long Straight Wire with Current

Figure 30-14 shows a long straight wire that carries current i (assumed to be uniformly distributed) that points directly out of the page. The equation for the magnetic field magnitude, B, produced by a long straight wire (Eq. 30-6) tells us that B depends only on the radial distance from the wire. That is, the field $\vec{B}$ has cylindrical symmetry about the wire. We can take advantage of that symmetry to simplify the integral in Ampère's law (Eqs. 30-16 and 30-17) if we encircle the wire with a concentric circular Ampèrian loop of radius r, as in Fig. 30-14. The magnetic field $\vec{B}$ then has the same magnitude B at every point on the loop. We shall integrate counterclockwise, so that $d\vec{s}$ has the direction shown in Fig. 30-14.

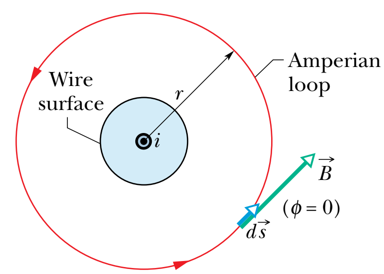


FIGURE 30-14 ■ Using Ampère's law to find the magnetic field produced by a current i in a long straight wire. The Ampèrian loop is a concentric circle that lies outside the wire.

We can further simplify the quantity $B \cos \phi$ in Eq. 30-17 by noting that $\vec{B}$ is tangent to the loop at every point along the loop, as is $d\vec{s}$. Thus, $\vec{B}$ and $d\vec{s}$ are parallel at each point on the loop. Then at every point the angle ϕ between $d\vec{s}$ and $\vec{B}$ is 0° (so $\cos \phi = +1$). The magnitude of the integral in Eq. 30-17 then becomes

$$\left|\oint \vec{B} \cdot d\vec{s}\,\right| = \oint (B \cos \phi)\, ds = B \left|\oint ds\right| = B(2\pi r).$$

Note that $\oint ds$ above is the summation of all the line segment lengths ds around the circular loop; that is, it simply gives the circumference $2\pi r$ of the loop.

The right side of Ampère's law becomes $+\mu_0 |i|$ and we then have

$$B(2\pi r) = \mu_0 i$$

or

$$B = \frac{\mu_0 |i|}{2\pi r}. \tag{30-19}$$

With a slight change in notation, this is Eq. 30-6, which we derived earlier—with considerably more effort—using the Biot-Savart law. We know that the correct direction of $\vec{B}$ must be the counterclockwise one shown in Fig. 30-14 when i is positive. When i is negative, the correct direction for $\vec{B}$ is clockwise.

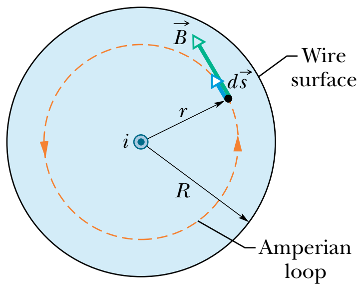


FIGURE 30-15 Using Ampère's law to find the magnetic field that a current i produces inside a long straight wire of circular cross section. The current is uniformly distributed over the cross section of the wire and emerges from the page. An Ampèrian loop is drawn inside the wire.

The Magnetic Field Inside a Long Straight Wire with Current

Figure 30-15 shows the cross section of a long straight wire of radius R that carries a uniformly distributed current i either directly out of the page or directly into the page. Because the current is uniformly distributed over a cross section of the wire, the magnetic field $\vec{B}$ that it produces must have cylindrical symmetry. Thus, to find the magnetic field at points inside the wire, we can again use an Amperian loop of radius r, as shown in Fig. 30-15, where now $r < R$. Symmetry again requires that $\vec{B}$ is tangent to the loop, as shown, so the left side of Ampère's law again yields

$$\oint \vec{B} \cdot d\vec{s} = B\left|\oint d\vec{s}\right| = B(2\pi r). \tag{30-20}$$

To find the right side of Ampère's law, we note that because the current is uniformly distributed, the current i^{enc} encircled by the loop is proportional to the area encircled by the loop; that is,

$$i^{\text{enc}} = i\frac{\pi r^2}{\pi R^2} = i\frac{r^2}{R^2}. \tag{30-21}$$

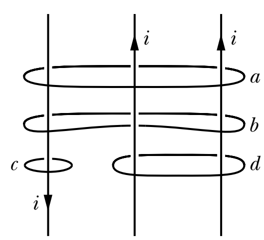
Then Ampère's law gives us

$$B(2\pi r)\mu_0|\,i^{\text{enc}}| = \mu_0|\,i\,|\frac{r^2}{R^2}$$

or

$$B = \left(\frac{\mu_0|\,i\,|}{2\pi R^2}\right)r. \tag{30-22}$$

Thus, inside the wire, the magnitude B of the magnetic field is proportional to r; that magnitude is zero at the center and a maximum at the surface, where $r = R$. Note that Eqs. 30-19 and 30-22 give the same value for B at $r = R$; that is, the expressions for the magnetic field outside the wire and inside the wire yield the same result at the surface of the wire.



READING EXERCISE 30-4: The figure shows three equal currents i (two parallel and one antiparallel) and four Amperian loops. Rank the loops according to the magnitude of $\oint \vec{B} \cdot d\vec{s}$ along each, greatest first.

TOUCHSTONE EXAMPLE 30-3: Hollow Conducting Cylinder

Figure 30-16a shows the cross section of a long hollow conducting cylinder with inner radius $a = 2.0$ cm and outer radius $b = 4.0$ cm. The cylinder carries a current out of the page, and the current density in the cross section is given by $|\vec{J}| = cr^2$, with $c = 3.0 \times 10^6$ A/m^4 and r in meters. What is the magnitude of the magnetic field $\vec{B}$ at a point that is 3.0 cm from the central axis of the cylinder?

SOLUTION The point at which we want to evaluate $\vec{B}$ is inside the material of the conducting cylinder, between its inner and outer radii. We note that the current distribution has cylindrical symmetry (it is the same all around the cross section for any given radius). Thus, the **Key Idea** here is that the symmetry allows us to use Ampère's law to find $\vec{B}$ at the point. We first draw the Ampèrian loop shown in Fig. 30-16b. The loop is concentric with the cylinder and has radius $R = 3.0$ cm, because we want to evaluate $\vec{B}$ at that distance from the cylinder's central axis.

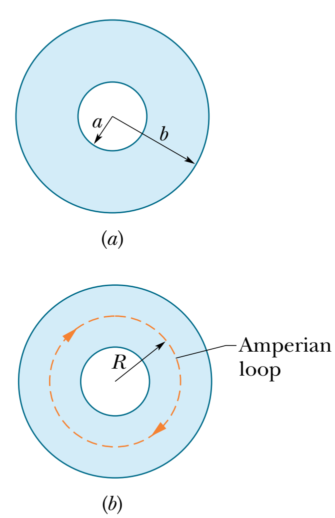


FIGURE 30-16 (a) Cross section of a conducting cylinder of inner radius a and outer radius b. (b) An Ampèrian loop of radius R is added to compute the magnetic field at points that are a distance R from the central axis.

Next, we must compute the current i^{enc} that is encircled by the Ampèrian loop. However, a second **Key Idea** is that we *cannot* set up a proportionality as in Eq. 30-21, because here the current is not uniformly distributed. Instead, we must integrate the current density from the cylinder's inner radius a to the loop radius r. Since $\vec{J}$ and $d\vec{A}$ are parallel, $\vec{J}\cdot d\vec{A} = J\,dA$, so

$$|i^{\text{enc}}| = \left|\int \vec{J}\cdot d\vec{A}\right| = \left|\int_a^R cr^2(2\pi r\,dr)\right|$$

$$= \left|2\pi c\int_a^R r^3\,dr\right| = \left|2\pi c\left[\frac{r^4}{4}\right]_a^R\right|$$

since $$|i^{\text{enc}}| = \frac{\pi c(R^4 - a^4)}{2}, \quad \text{since } R > a.$$

The direction of integration indicated in Fig. 30-16*b* is (arbitrarily) clockwise. Applying the right-hand rule for Ampère's law to that loop, we find that we should take i^{enc} as negative because the current is directed out of the page but our thumb is directed into the page.

We next evaluate the left side of Ampère's law exactly as we did in Fig. 30-15, and we again obtain Eq. 30-20. Then Ampère's law,

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 i^{\text{enc}},$$

gives us

$$(B\cos\phi)(2\pi R) = -\frac{\mu_0\pi c}{2}(R^4 - a^4),$$

where $\cos\phi = \cos 0° = +1$ if $\vec{B}$ is parallel to $d\vec{s}$ and $\cos\phi = \cos 180° = -1$ if $\vec{B}$ is antiparallel to $d\vec{s}$. Solving for $(B\cos\phi)$ for $\phi = 180°$ and substituting known data yield

$$(B\cos\phi) = -\frac{\mu_0 c}{4R}(R^4 - a^4)$$

$$-B = -\frac{(4\pi\times10^{-7}\,\text{T}\cdot\text{m/A})(3.0\times10^6\,\text{A/m}^4)}{4(0.030\,\text{m})}$$

$$\times\,[(0.030\,\text{m})^4 - (0.020\,\text{m})^4]$$

$$-B = -2.0\times10^{-5}\,\text{T}.$$

Thus, the magnetic field $\vec{B}$ at a point 3.0 cm from the central axis is

$$B = 2.0\times10^{-5}\,\text{T} \qquad \text{(Answer)}$$

and forms magnetic field lines that are directed opposite our direction of integration, hence counterclockwise in Fig. 30-16*b*.

30-6 Solenoids and Toroids

Magnetic Field of a Solenoid

We now turn our attention to another situation in which Ampère's law proves useful. It concerns the magnetic field produced by the current in a long, tightly wound helical coil of wire. Such a coil is called a **solenoid** (Fig. 30-17). Solenoids are very common electrical devices that are important in many technological applications.

To make the calculation simpler here, we will assume that the length of the solenoid is much greater than the diameter. Figure 30-18 shows a section through a portion of a "stretched-out" solenoid. The solenoid's magnetic field is the vector sum of the fields produced by the individual turns (loops) that make up the solenoid. For points very close to each turn, the wire behaves magnetically almost like a long straight wire, and the lines of $\vec{B}$ there are almost concentric circles. Figure 30-18 suggests that the field tends to cancel between adjacent turns. It also suggests that, at points inside the solenoid and reasonably far from the wire, $\vec{B}$ is approximately

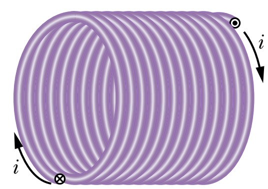


FIGURE 30-17 ■ A solenoid carrying current i.

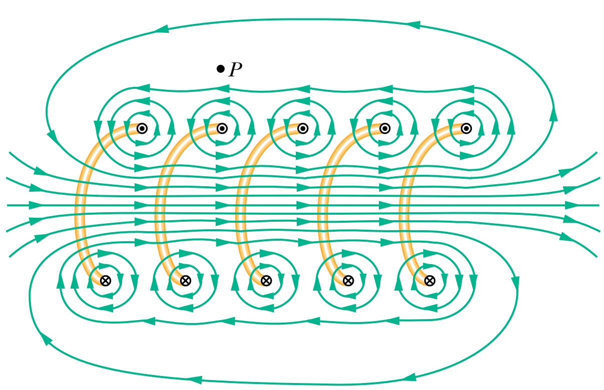


FIGURE 30-18 ■ A vertical cross section through the central axis of a "stretched-out" solenoid. The back portions of five turns are shown, as are the magnetic field lines due to a current through the solenoid. Each turn produces circular magnetic field lines near it. Near the solenoid's axis, the field lines combine into a net magnetic field that is directed along the axis. The closely spaced field lines there indicate a strong magnetic field. Outside the solenoid the field lines are widely spaced; the field there is very weak.

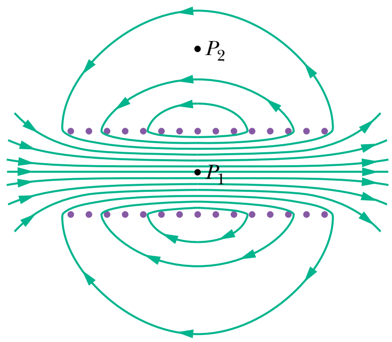


FIGURE 30-19 ■ Magnetic field lines for a real solenoid of finite length. The field is strong and uniform at interior points such as P_1 but relatively weak at external points such as P_2.

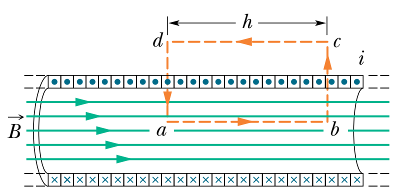


FIGURE 30-20 ■ Application of Ampère's law to a section of a long ideal solenoid carrying a current i. The Ampèrian loop is the rectangle $abcd$.

parallel to the (central) solenoid axis. In the limiting case of an *ideal solenoid,* which is infinitely long and consists of tightly packed (*close-packed*) turns of square wire, the field inside the coil is uniform and parallel to the solenoid axis.

At points above the solenoid, such as P in Fig. 30-18, the field set up by the upper parts of the solenoid turns (marked $\odot$) is directed to the left (as drawn near P) and tends to cancel the field set up by the lower parts of the turns (marked $\otimes$), which is directed to the right (not drawn). In the limiting case of an ideal solenoid, the magnetic field outside the solenoid is zero. Taking the external field to be zero is an excellent assumption for a real solenoid if its length is much greater than its diameter and if we consider external points such as point P that are not near either end of the solenoid. The direction of the magnetic field along the solenoid axis is given by a curled-straight right-hand rule: Grasp the solenoid with your right hand so that your fingers follow the direction of the current in the windings; your extended right thumb then points in the direction of the axial magnetic field.

Figure 30-19 shows the lines of $\vec{B}$ for a real solenoid. The spacing of the lines of $\vec{B}$ in the central region shows that the field inside the coil is fairly strong and uniform over the cross section of the coil. The external field, however, is relatively weak.

Let us now apply Ampère's law,

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i^{\,\text{enc}}, \tag{30-23}$$

to the ideal solenoid of Fig. 30-20, where $\vec{B}$ is uniform within the solenoid and zero outside it, using the rectangular Amperian loop $abcda$. We write $\oint \vec{B} \cdot d\vec{s}$ as the sum of four integrals, one for each loop segment:

$$\oint \vec{B} \cdot d\vec{s} = \int_a^b \vec{B} \cdot d\vec{s} + \int_b^c \vec{B} \cdot d\vec{s} + \int_c^d \vec{B} \cdot d\vec{s} + \int_d^a \vec{B} \cdot d\vec{s}. \tag{30-24}$$

The first integral on the right of Eq. 30-24 is Bh, where B is the magnitude of the uniform field $\vec{B}$ inside the solenoid and h is the (arbitrary) length of the segment from a to b. The second and fourth integrals are zero because for every element ds of these segments, $\vec{B}$ either is perpendicular to ds or is zero, and thus $\vec{B} \cdot d\vec{s}$ is zero. The third integral, which is taken along a segment that lies outside the solenoid, is zero because $\vec{B} = 0$ at all external points. Thus, $\oint \vec{B} \cdot d\vec{s}$ for the entire rectangular loop has the value Bh.

The net current $i^{\,\text{enc}}$ encircled by the rectangular Ampèrian loop in Fig. 30-20 is not the same as the current i in the solenoid windings because the windings pass more than once through this loop. Let n be the number of turns per unit length of the solenoid; then the loop encloses nh turns, so

$$i^{\,\text{enc}} = i(nh).$$

Ampère's law then gives us

$$Bh = \mu_0 |i| nh,$$

or

$$B = n\mu_0 |i| \quad \text{(inside ideal solenoid).} \tag{30-25}$$

Although we derived Eq. 30-25 for an infinitely long ideal solenoid, it holds quite well for actual solenoids if we apply it only at interior points, well away from the solenoid ends. Equation 30-25 is consistent with the experimental fact that the magnetic field magnitude $|\vec{B}| = B$ within a solenoid does not depend on the diameter or the length of the solenoid and that B is uniform over the solenoidal cross section. A solenoid thus

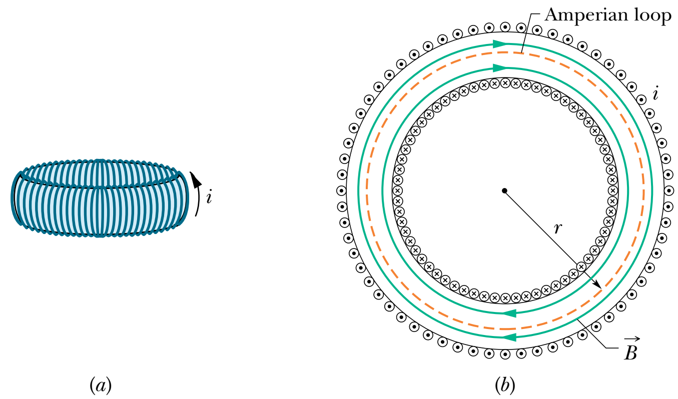


FIGURE 30-21 ■ (*a*) A toroid carrying a current *i*. (*b*) A horizontal cross section of the toroid. The interior magnetic field (inside the doughnut-shaped tube) can be found by applying Ampère's law with the Ampèrian loop shown.

provides a practical way to set up a known uniform magnetic field for experimentation, just as a parallel-plate capacitor provides a practical way to set up a known uniform electric field.

Magnetic Field of a Toroid

Figure 30-21*a* shows a **toroid,** which may be described as a solenoid bent into the shape of a hollow doughnut. What magnetic field $\vec{B}$ is set up at its interior points (within the hollow of the doughnut)? We can find out from Ampère's law and the symmetry of the toroid.

From the symmetry, we see that the lines of $\vec{B}$ form concentric circles inside the toroid, directed as shown in Fig. 30-21*b*. Let us choose a concentric circle of radius r as an Ampèrian loop and traverse it in the clockwise direction. Ampère's law (Eq. 30-16) yields

$$B(2\pi r) = N\mu_0|i|,$$

where i is the current in the toroid windings (and is positive for those windings enclosed by the Ampèrian loop) and N is the total number of turns. This gives

$$B = \frac{N\mu_0|i|}{2\pi}\frac{1}{r} \quad \text{(toroid).} \tag{30-26}$$

In contrast to the situation for a solenoid, $\vec{B}$ is not constant over the cross section of a toroid. With Ampère's law, it is easy to show that $\vec{B} = 0$ for points outside an ideal toroid (as if the toroid were made from an ideal solenoid).

The direction of the magnetic field within a toroid follows from our curled-straight right-hand rule: Grasp the toroid with the fingers of your right hand curled in the direction of the current in the windings; your extended right thumb points in the direction of the magnetic field.

30-7 A Current-Carrying Coil as a Magnetic Dipole

So far we have examined the magnetic fields produced by current in a long straight wire, a solenoid, and a toroid. We turn our attention here to the field produced by a coil carrying a current. You saw in Section 29-10 that such a coil behaves as a magnetic dipole in that, if we place it in an external magnetic field $\vec{B}$, a torque $\vec{\tau}$ given by

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{30-27}$$

acts on it. Here $\vec{\mu}$ is the magnetic dipole moment of the coil and has the magnitude NiA, where N is the number of turns (or loops), i is the current in each turn, and A is the area enclosed by each turn.

Recall that the direction of $\vec{\mu}$ is given by a curled-straight right-hand rule: Grasp the coil so that the fingers of your right hand curl around it in the direction of the current; your extended thumb then points in the direction of the dipole moment $\vec{\mu}$.

Magnetic Field of a Coil

We turn now to the other aspect of a current-carrying coil as a magnetic dipole. What magnetic field does *it* produce at a point in the surrounding space? The problem does not have enough symmetry to make Ampère's law useful, so we must turn to the Biot–Savart law. For simplicity, we first consider only a coil with a single circular loop and only points on its central axis, which we take to be a z axis. We shall show that the magnetic field at such points only has a z-component, B_z which is given by

$$\vec{B} = B_z\hat{\mathrm{k}} = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}}\hat{\mathrm{k}}, \tag{30-28}$$

where R is the radius of the circular loop and z is the distance of the point in question from the center of the loop. Furthermore, the direction of the magnetic field $\vec{B}$ is the same as the direction of the magnetic dipole moment $\vec{\mu}$ of the loop.

For axial points far from the loop, we have $z \gg R$ in Eq. 30-28. With that approximation, the equation for the z-component of $\vec{B}$, which is a function of z only, reduces to

$$B_z \approx \frac{\mu_0 i R^2}{2z^3}.$$

Recalling that πR^2 is the area A of the loop and extending our result to include a coil of N turns, we can write this equation as

$$B_z = \frac{\mu_0}{2\pi}\frac{NiA}{z^3}.$$

Further, since $\vec{B}$ and $\vec{\mu}$ have the same direction, we can write the equation in vector form, substituting from $\mu = NiA$ (Eq. 29-32):

$$\vec{B} = B_z\hat{\mathrm{k}} = \frac{\mu_0}{2\pi}\frac{\vec{\mu}}{z^3} \quad \text{(current-carrying coil).} \tag{30-29}$$

Note that the magnetic constant μ_0 and the magnetic moment vector $\vec{\mu}$ are completely different quantities with different units. The choice of the symbol μ to represent both quantities is unfortunate.

In summary, we have two ways in which we can regard a current-carrying coil as a magnetic dipole: (1) it experiences a torque when we place it in an external magnetic field; (2) it generates its own intrinsic magnetic field, given by Eq. 30-29 for distant points along its axis. Figure 30-22 shows some magnetic field lines for a current loop; one side of the loop acts as a north pole (in the direction of $\vec{\mu}$) and the other side as a south pole, as suggested by the lightly drawn magnet in the figure.

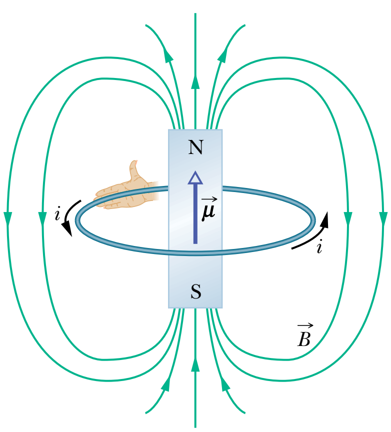


FIGURE 30-22 ■ A current loop produces a magnetic field like that of a bar magnet and thus has associated north and south poles. The magnetic dipole moment $\vec{\mu}$ of the loop, given by a curled-straight right-hand rule, points from the south pole to the north pole, in the direction of the field $\vec{B}$ within the loop.

Proof of Equation 30-28

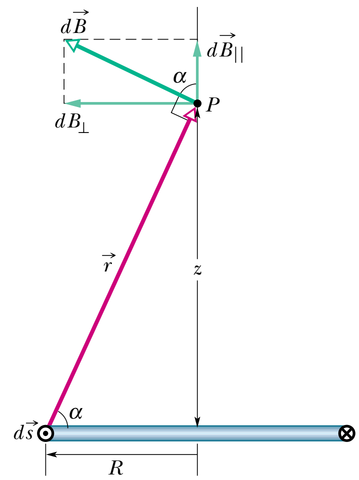


FIGURE 30-23 A current loop of radius R. The plane of the loop is perpendicular to the page and only the back half of the loop is shown. We use the law of Biot and Savart to find the magnetic field at point P on the central axis of the loop.

Figure 30-23 shows the back half of a circular loop of radius R carrying a current i. Consider a point P on the axis of the loop, a distance z from its plane. Let us apply the Biot–Savart law to a differential element $d\vec{s}$ of the loop, located at the left side of the loop. The length vector $d\vec{s}$ for this element points perpendicularly out of the page. The angle θ between $d\vec{s}$ and $\vec{r}$ in Fig. 30-23 is 90°; the plane formed by these two vectors is perpendicular to the plane of the figure and contains both $\vec{r}$ and $d\vec{s}$. Using the Biot–Savart law and the right-hand rule, we see that the differential field $d\vec{B}$ produced at point P by the current in this element is perpendicular to this plane. Thus $d\vec{B}$ lies in the plane of the figure, perpendicular to $\vec{r}$ (as indicated in Fig. 30-23).

Let us resolve $d\vec{B}$ into two components: $d\vec{B}_{\|}$ along the axis of the loop and $d\vec{B}_{\perp}$ perpendicular to this axis. From the symmetry, the vector sum of all the perpendicular components $d\vec{B}_{\perp}$ due to all the loop elements ds is zero. This leaves only the axial components $d\vec{B}_{\|}$ and we have the magnitude of the axial component given by

$$B_{\|} = \int dB_{\|}.$$

For the element $d\vec{s}$ in Fig. 30-23, the Biot–Savart law (Eq. 30-1) tells us that the magnitude of the axial magnetic field component at distance r is

$$dB_{\|} = \frac{\mu_0}{4\pi}\frac{i\,ds\sin 90°}{r^2}.$$

We also have

$$dB_{\|} = dB\cos\alpha.$$

Combining these two relations, we obtain

$$dB_{\|} = \frac{\mu_0 i\cos\alpha\,ds}{4\pi r^2}. \tag{30-30}$$

Figure 30-23 shows that r and α are not independent but are related to each other. Let us express each in terms of the variable z, the distance between point P and the center of the loop. The relations are

$$r = \sqrt{R^2 + z^2} \tag{30-31}$$

and

$$\cos\alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2 + z^2}}. \tag{30-32}$$

Substituting Eqs. 30-31 and 30-32 into Eq. 30-30, we find

$$dB_{\|} = \frac{\mu_0 iR}{4\pi(R^2 + z^2)^{3/2}}\,ds.$$

Note that i, R, and z have the same values for all elements $d\vec{s}$ around the loop, so when we integrate this equation, we find that the magnitude of the axial field component is given as

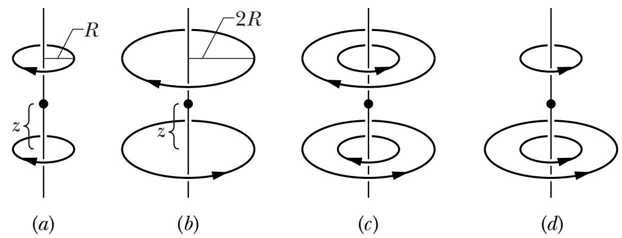
$$\begin{aligned} B_{\|} &= \oint dB_{\|} \\ &= \frac{\mu_0 iR}{4\pi(R^2 + z^2)^{3/2}}\oint ds, \end{aligned}$$

or, since $\oint ds$ is simply the circumference $2\pi R$ of the loop, the axial or z-component of the magnetic field is

$$\vec{B} = B_z\hat{\mathrm{k}} = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}}\hat{\mathrm{k}},$$

which is Eq. 30-28, the relation we sought to prove.

READING EXERCISE 30-5: The figure here shows four arrangements of circular loops of radius r or $2r$, centered on vertical axes (perpendicular to the loops) and carrying identical currents in the directions indicated. Assume the sizes of the loops are exaggerated and that $z \gg R$. Rank the arrangements according to the magnitude of the net magnetic field at the dot, midway between the loops on the central axis, greatest first.



Problems

SEC. 30-3 ■ CALCULATING THE MAGNETIC FIELD DUE TO A CURRENT

1. Surveyor A surveyor is using a magnetic compass 6.1 m below a power line in which there is a steady current of 100 A. (a) What is the magnitude of the magnetic field at the site of the compass due to the power line? (b) Will this interfere seriously with the compass reading? The horizontal component of the Earth's magnetic field at the site is 20 μT.

2. Electron Gun The electron gun in a traditional television tube fires electrons of kinetic energy 25 keV at the screen in a circular beam 0.22 mm in diameter; 5.6×10^{14} electrons arrive each second. Calculate the magnitude of the magnetic field produced by the beam at a point 1.5 mm from the beam axis.

3. Philippines At a certain position in the Philippines, the magnitude of the Earth's magnetic field of 39 μT is horizontal and directed due north. Suppose the net field is zero exactly 8.0 cm above a long, straight, horizontal wire that carries a constant current. What are (a) the size and (b) the direction of the current?

4. Locate Points A long wire carrying a current of 100 A is placed in a uniform external magnetic field of 5.0 mT. The wire is perpendicular to this magnetic field. Locate the points at which the net magnetic field is zero.

5. Particle with Positive Charge A particle with positive charge q is a distance d from a long straight wire that carries a current i; the particle is traveling with speed $|\vec{v}|$ perpendicular to the wire. What are the direction and magnitude of the force on the particle if it is moving (a) toward and (b) away from the wire?

6. Semicircular Arcs A straight conductor carrying a current i splits into identical semicircular arcs as shown in Fig. 30-24. What is the magnitude of the magnetic field at the center C of the resulting circular loop?

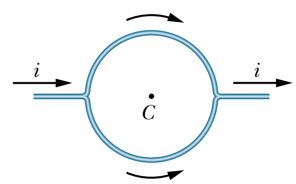


FIGURE 30-24 ■ Problem 6.

7. Two Semi-Infinite A wire carrying current i has the configuration shown in Fig. 30-25. Two semi-infinite straight sections, both tangent to the same circle, are connected by a circular arc, of central angle ϕ, along the circumference of the circle, with all sections lying in the same plane. What must ϕ be in order for $|\vec{B}|$ to be zero at the center of the circle?

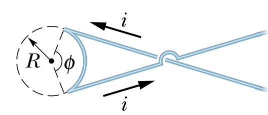


FIGURE 30-25 ■ Problem 7.

8. Use Biot–Savart Use the Biot–Savart law to calculate the magnitude and direction of the magnetic field $\vec{B}$ at C, the common center of the semicircular arcs AD and HJ in Fig. 30-26a. The two arcs, of radii R_2 and R_1, respectively, form part of the circuit $ADJHA$ carrying current i.

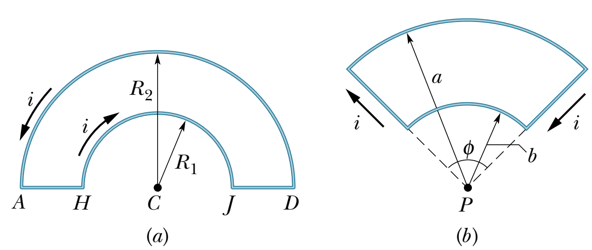


FIGURE 30-26 ■ Problems 8 and 9.

9. Curved Segments In the circuit of Fig. 30-26*b*, the curved segments are arcs of circles of radii a and b with common center P. The straight segments are along radii. Find the magnitude and direction of the magnetic field $\vec{B}$ at point P, assuming a current i in the circuit.

10. Magnitude and Directions The wire shown in Fig. 30-27 carries current i. What are the magnitude and direction of the magnetic field $\vec{B}$ produced at the center C of the semicircle by (a) each straight segment of length L, (b) the semicircular segment of radius R, and (c) the entire wire?

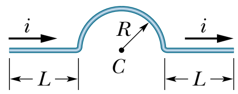


FIGURE 30-27 ■ Problem 10.

11. Straight Wire In Fig. 30-28, a straight wire of length L carries current i. Show that the magnitude of the magnetic field $\vec{B}$ produced by this segment at P_1, a distance R from the segment along a perpendicular bisector, is

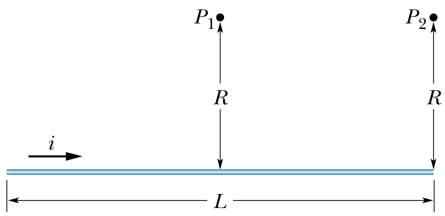


FIGURE 30-28 ■ Problems 11 and 13.

$$B = \frac{\mu_0|i|}{2\pi R}\frac{L}{(L^2 + 4R^2)^{1/2}}.$$

Show that this expression for $|\vec{B}|$ reduces to an expected result as $L \to \infty$.

12. Square Loop A square loop of wire of edge length a carries current i. Using the results of Problem 11, show that, at the center of the loop, the magnitude of the magnetic field produced by the current is

$$B = \frac{2\sqrt{2}\mu_0|i|}{\pi a}.$$

13. Length *L* In Fig. 30-28, a straight wire of length L carries current i. Show that

$$B = \frac{\mu_0|i|}{4\pi R}\frac{L}{(L^2 + R^2)^{1/2}}$$

gives the magnitude of the magnetic field $\vec{B}$ produced by the wire at P_2, a perpendicular distance R from one end of the wire.

14. Rectangular Loop Using the results of Problem 11, show that the magnitude of the magnetic field produced at the center of a rectangular loop of wire of length L and width W, carrying a current i, is

$$B = \frac{2\mu_0|i|}{\pi}\frac{(L^2 + W^2)^{1/2}}{LW}.$$

15. Square Loop Two A square loop of wire of edge length a carries current i. Using the results of Problem 11, show that the magnitude of the magnetic field produced at a point on the axis of the loop and a distance x from its center is

$$B(x) = |\vec{B}(x)| = \frac{4\mu_0|i|a^2}{\pi(4x^2 + a^2)(4x^2 + 2a^2)^{1/2}}.$$

Prove that this result is consistent with the result of Problem 12.

16. Length *a* In Fig. 30-29, a straight wire of length a carries a current i. Show that the magnitude of the magnetic field produced by the current at point P is $B = \sqrt{2}\mu_0|i|/8\pi a$.

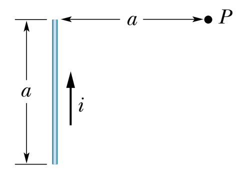


FIGURE 30-29 ■ Problem 16.

17. Two Wires Two wires, both of length L, are formed into a circle and a square, and each carries current i. Show that the square produces a greater magnetic field at its center than the circle produces at its center. (See Problem 12.)

18. Magnetic Field Find the magnitude and direction of the magnetic field $\vec{B}$ at point P in Fig. 30-30. (See Problem 16.)

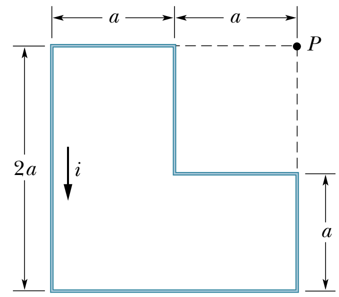


FIGURE 30-30 ■ Problem 18.

19. Long Thin Ribbon Figure 30-31 shows a cross section of a long thin ribbon of width w that is carrying a uniformly distributed total current i into the page. Calculate the magnitude and direction of the magnetic field $\vec{B}$ at a point P in the plane of the ribbon at a distance d from its edge. (*Hint:* Imagine the ribbon to be constructed from many long, thin, parallel wires.)

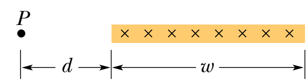


FIGURE 30-31 ■ Problem 19.

20. Find Magnitude and Direction Find the magnitude and direction of the magnetic field $\vec{B}$ at point P in Fig. 30-32, for $|i| = 10$ A and $a = 8.0$ cm. (See Problems 13 and 16.)

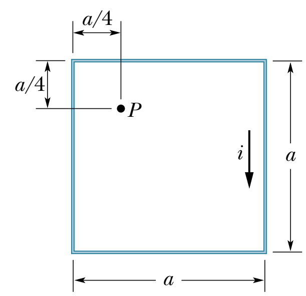


FIGURE 30-32 ■ Problem 20.

21. Perpendicular Bisector Figure 30-33 shows two very long straight wires (in cross section) that each carry currents of 4.00 A directly out of the page. Distance $d_1 = 6.00$ m and distance $d_2 = 4.00$ m. What is the magnitude of the net magnetic field at point P, which lies on a perpendicular bisector to the wires?

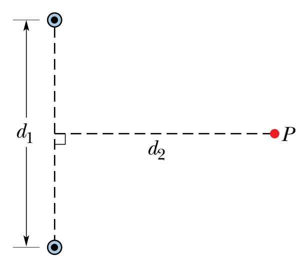


FIGURE 30-33 ■ Problem 21.

22. Greatest and 10% In Fig. 30-34, point P is at perpendicular distance $R = 2.00$ cm from a very long straight wire carrying a current. The magnetic field $\vec{B}$ set up at point P is due to contributions from all the identical current-length elements $i\, d\vec{s}$ along the wire. What is the distance s to the current-length element that makes (a) the greatest contribution to field $\vec{B}$ and (b) 10% of the greatest contribution?

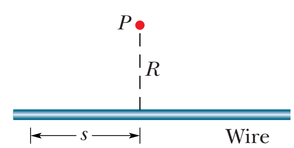


FIGURE 30-34 ■ Problem 22.

SEC. 30-4 ■ FORCE BETWEEN TWO PARALLEL CURRENTS

23. Two Parallel Wires Two long parallel wires are 8.0 cm apart. What equal currents must be in the wires if the magnetic field halfway between them is to have a magnitude of 300 μT? Answer for both (a) parallel and (b) antiparallel currents.

24. *i* and 3*i* Two long parallel wires a distance d apart carry currents of i and $3i$ in the same direction. Locate the point or points at which their magnetic fields cancel.

25. Two Parallel Wires Two Two long, straight, parallel wires, separated by 0.75 cm, are perpendicular to the plane of the page as shown in Fig. 30-35. Wire 1 carries a current of 6.5 A into the page. What must be the current (magnitude and direction) in wire 2 for the resultant magnetic field at point P to be zero?

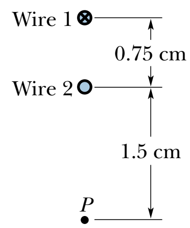


FIGURE 30-35 ■ Problem 25.

26. Five Parallel Wires Figure 30-36 shows five long parallel wires in the xy plane. Each wire carries a current $i = 3.00$ A in the positive x direction. The separation between adjacent wires is $d = 8.00$ cm. In unit-vector notation, what are the magnitude and direction of the magnetic force per meter exerted on each of these five wires by the other wires?

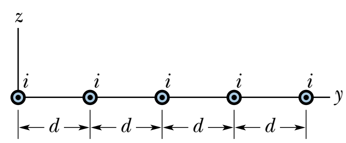


FIGURE 30-36 ■ Problem 26.

27. Four Long Wires Four long copper wires are parallel to each other, their cross sections forming the corners of a square with sides $a = 20$ cm. A 20 A current exists in each wire in the direction shown in Fig. 30-37. What are the magnitude and direction of $\vec{B}$ at the center of the square?

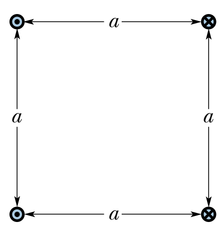


FIGURE 30-37 ■ Problems 27, 28, and 29.

28. Four Currents Form a Square Four identical parallel currents i are arranged to form a square of edge length a as in Fig. 30-37, *except* that they are *all* out of the page. What is the force per unit length (magnitude and direction) on any one wire?

29. Force per Unit Length In Fig. 30-37, what is the force per unit length acting on the lower left wire, in magnitude and direction, with the current directions as shown? The currents are i.

30. Idealized Schematic Figure 30-38 is an idealized schematic drawing of a rail gun. Projectile P sits between two wide rails of circular cross section; a source of current sends current through the rails and through the (conducting) projectile itself (a fuse is not used). (a) Let w be the distance between the rails, R the radius of the rails, and i the current. Show that the magnitude of the force on the projectile is directed to the right along the rails and is given approximately by

$$F = |\vec{F}| = \frac{i^2\mu_0}{\pi}\ln\frac{w+R}{R}.$$

(b) If the projectile starts from the left end of the rails at rest, find the speed v at which it is expelled at the right. Assume that $|i| = 450$ kA, $w = 12$ mm, $R = 6.7$ cm, $L = 4.0$ m, and the mass of the projectile is $m = 10$ g.

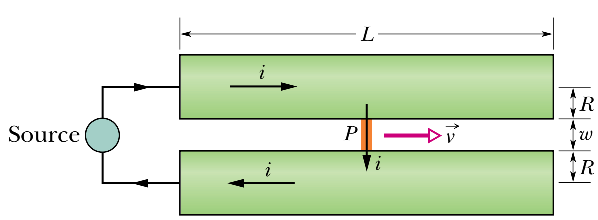


FIGURE 30-38 ■ Problem 30.

31. Rectangular Loop Two In Fig. 30-39, the long straight wire carries a current of 30 A and the rectangular loop carries a current of 20 A. Calculate the resultant force acting on the loop. Assume that $a = 1.0$ cm, $b = 8.0$ cm, and $L = 30$ cm.

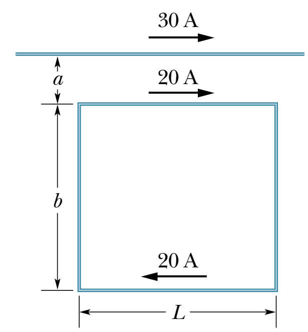


FIGURE 30-39 ■ Problem 31.

SEC. 30-5 ■ AMPÈRE'S LAW

32. Eight Wires Eight wires cut the page perpendicularly at the points shown in Fig. 30-40. A wire labeled with the integer k ($k = 1, 2, \ldots, 8$) carries the current ki. For those with odd k, the current is out of the page; for those with even k, it is into the page. Evaluate $\oint \vec{B}\cdot d\vec{s}$ along the closed path in the direction shown.

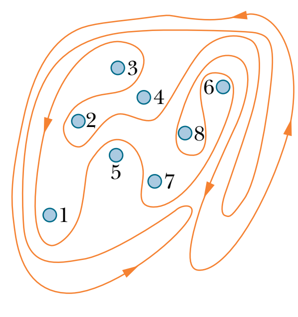


FIGURE 30-40 ■ Problem 32.

33. Eight Conductors Each of the eight conductors in Fig. 30-41 carries 2.0 A of current into or out of the page. Two paths are indicated for the line integral $\oint \vec{B}\cdot d\vec{s}$. What is the value of the integral for the path (a) at the left and (b) at the right?

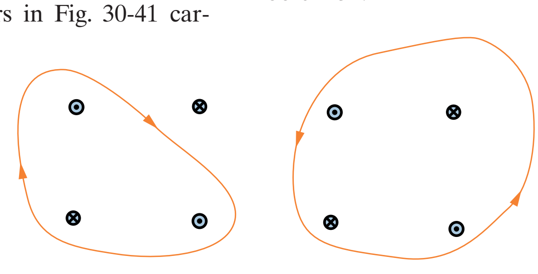


FIGURE 30-41 ■ Problem 33.

34. Cross Section of a Cylindrical Conductor Figure 30-42 shows a cross section of a long cylindrical conductor of radius a, carrying a uniformly distributed current i. Assume that $a = 2.0$ cm and $i = 100$ A, and plot the magnitude of the magnetic field $|\vec{B}(r)| = B(r)$ over the range $0 < r < 6.0$ cm.

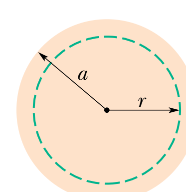


FIGURE 30-42 ■ Problem 34.

35. Cannot Drop to Zero Show that a uniform magnetic field $\vec{B}$ cannot drop abruptly to zero (as is suggested by the lack of field lines to the right of point a in Fig. 30-43) as one moves perpendicular to $\vec{B}$, say along the horizontal arrow in the figure. (*Hint:* Apply Ampère's law to the rectangular path shown by the dashed lines.) In actual magnets "fringing" of the magnetic field lines always occurs, which means that $\vec{B}$ approaches zero in a gradual manner. Modify the field lines in the figure to indicate a more realistic situation.

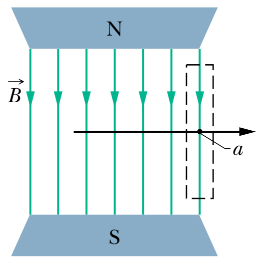


FIGURE 30-43 ■ Problem 35.

36. Two Square Conducting Loops Two square conducting loops carry currents of 5.0 and 3.0 A as shown in Fig. 30-44. What is the value of $\oint \vec{B} \cdot d\vec{s}$ for each of the two closed paths shown?

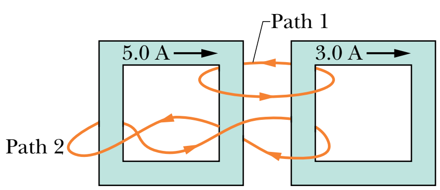


FIGURE 30-44 ■ Problem 36.

37. Current Density The current density inside a long, solid, cylindrical wire of radius a is in the direction of the central axis and varies linearly with radial distance r from the axis according to $|\vec{J}| = |\vec{J}_0|r/a$. Find the magnitude and direction of the magnetic field inside the wire.

38. Uniformly Distributed Current A long straight wire (radius = 3.0 mm) carries a constant current distributed uniformly over a cross section perpendicular to the axis of the wire. If the magnitude of the current density is 100 A/m^2, what are the magnitudes of the magnetic fields (a) 2.0 mm from the axis of the wire and (b) 4.0 mm from the axis of the wire?

39. Cylindrical Hole Figure 30-45 shows a cross section of a long cylindrical conductor of radius a containing a long cylindrical hole of radius b. The axes of the cylinder and hole are parallel and are a distance d apart; a current i is uniformly distributed over the tinted area. (a) Use superposition to show that the magnitude of the magnetic field at the center of the hole is

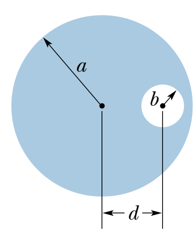


FIGURE 30-45 ■ Problem 39.

$$B = \frac{\mu_0 |i| d}{2\pi(a^2 - b^2)}.$$

(b) Discuss the two special cases $b = 0$ and $d = 0$. (*Hint:* Regard the cylindrical hole as resulting from the superposition of a complete cylinder (no hole) carrying a current in one direction and a cylinder of radius b carrying a current in the opposite direction, both cylinders having the same current density.)

40. Circular Pipe A long circular pipe with outside radius R carries a (uniformly distributed) current i into the page as shown in Fig. 30-46. A wire runs parallel to the pipe at a distance of $3R$ from center to center. Find the amount and direction of the current in the wire such that the net magnetic field at point P has the same magnitude as the net magnetic field at the center of the pipe but is in the opposite direction.

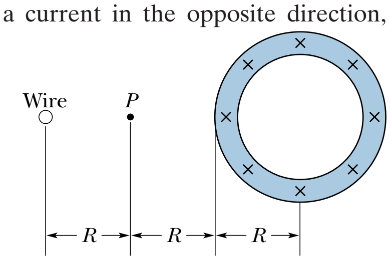


FIGURE 30-46 ■ Problem 40.

41. Conducting Sheet Figure 30-47 shows a cross section of an infinite conducting sheet lying in the x-y plane, carrying a current per unit x-length of λ; the current emerges perpendicularly out of the page. (a) Use the Biot–Savart law and symmetry to show that for all points P above the sheet, and all points P' below it, the magnetic field $\vec{B}$ is parallel to the sheet and directed as shown. (b) Use Ampère's law to prove that $B = \frac{1}{2}\mu_0|\lambda|$ at all points P and P'.

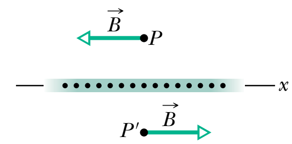


FIGURE 30-47 ■ Problems 41 and 48.

42. Field at *P* is Zero Figure 30-48 shows, in cross section, two long straight wires; the 3.0 A current in the right-hand wire is out of the page. What are the size and direction of the current in the left-hand wire if the net magnetic field at point P is to be zero?

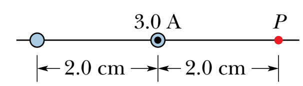


FIGURE 30-48 ■ Problem 42.

SEC. 30-6 ■ SOLENOIDS AND TOROIDS

43. Field Inside Solenoid A 200-turn solenoid having a length of 25 cm and a diameter of 10 cm carries a current of 0.30 A. Calculate the magnitude of the magnetic field $\vec{B}$ inside the solenoid.

44. Field Inside Solenoid Two A solenoid that is 95.0 cm long has a radius of 2.00 cm and a winding of 1200 turns; it carries a current of 3.60 A. Calculate the magnitude of the magnetic field inside the solenoid.

45. Toroid A toroid having a square cross section, 5.00 cm on a side, and an inner radius of 15.0 cm has 500 turns and carries a current of magnitude 0.800 A. (It is made up of a square solenoid—instead of a round one as in Fig. 30-21—bent into a doughnut shape.) What is the magnitude of the magnetic field inside the toroid at (a) the inner radius and (b) the outer radius of the toroid?

46. Length of Wire A solenoid 1.30 m long and 2.60 cm in diameter carries a current of 18.0 A. The magnitude of the magnetic field inside the solenoid is 23.0 mT. Find the length of the wire forming the solenoid.

47. Field Inside Toroid In Section 30-6, we showed that the magnitude of the magnetic field at any radius r *inside* a toroid is given by

$$B = \frac{\mu_0 |i| N}{2\pi r}.$$

Show that as you move from any point just inside a toroid to a point just outside, the magnitude of the *change* in $\vec{B}$ that you encounter is just $\mu_0|\lambda|$. Here $|\lambda|$ is the amount of current per unit length along a circumference of radius r within the toroid. Compare this with the similar result found in Problem 48. Isn't the equality surprising?

48. Solenoid as Cylindrical Conductor Treat an ideal solenoid as a thin cylindrical conductor whose current per unit length, measured parallel to the cylinder axis, is λ. (a) By doing so, show that the magnitude of the magnetic field inside an ideal solenoid can be written as $B = \mu_0|\lambda|$. This is the value of the *change* in $\vec{B}$ that you encounter as you move from inside the solenoid to outside, through the solenoid wall. (b) Show that the same change occurs as you move through an infinite flat current sheet such as that of Fig. 30-47 (see Problem 41). Does this equality surprise you?

49. Direction of Field A long solenoid with 10.0 turns/cm and a radius of 7.00 cm carries a current of 20.0 mA. A current of 6.00 A exists in a straight conductor located along the central axis of the solenoid. (a) At what radial distance from the axis will the direction of the resulting magnetic field be at 45.0° to the axial direction? (b) What is the magnitude of the magnetic field there?

50. Find Current in Solenoid A long solenoid has 100 turns/cm and carries current i. An electron moves within the solenoid in a circle of radius 2.30 cm perpendicular to the solenoid axis. The speed of the electron is $0.0460c$ (c = speed of light). Find the amount of current $|i|$ in the solenoid.

SEC. 30-7 ■ A CURRENT-CARRYING COIL AS A MAGNETIC DIPOLE

51. Magnetic Dipole What is the magnetic dipole moment $\vec{\mu}$ of the solenoid described in Problem 43?

52. One Turn Coil Figure 30-49*a* shows a length of wire carrying a current i and bent into a circular coil of one turn. In Fig. 30-49*b* the same length of wire has been bent more sharply, to give a coil of two turns, each of half the original radius. (a) If B_a and B_b are the magnitudes of the magnetic fields at the centers of the two coils, what is the ratio B_b/B_a? (b) What is the ratio of the magnitude of the dipole moments, μ_b/μ_a of the coils?

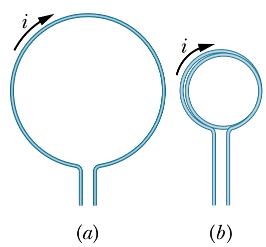


FIGURE 30-49 ■ Problem 52.

53. Student's Electromagnet A student makes a short electromagnet by winding 300 turns of wire around a wooden cylinder of diameter $d = 5.0$ cm. The coil is connected to a battery producing a current of 4.0 A in the wire. (a) What is the magnetic moment of this device? (b) At what axial distance $z \gg d$ will the magnetic field of this dipole have the magnitude 5.0 μT (approximately one-tenth that of the Earth's magnetic field)?

54. Helmholtz Figure 30-50 shows an arrangement known as a Helmholtz coil. It consists of two circular coaxial coils, each of N turns and radius R, separated by a distance R. The two coils carry equal currents i in the same direction. Find the magnitude of the net magnetic field at P, midway between the coils.

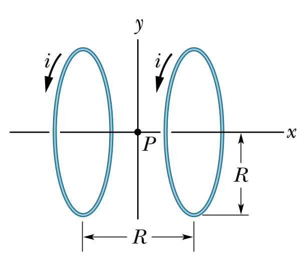


FIGURE 30-50 ■ Problems 54, 55, and 57.

55. Field as a Function of Distance Two 300-turn coils of radius R each carry a current i. They are arranged a distance R apart, as in Fig. 30-50. For $R = 5.0$ cm and $i = 50$ A, plot the magnitude $|B(x)| = B(x)$ of the net magnetic field as a function of distance x along the common x axis over the range $x = -5$ cm to $x = +5$ cm, taking $x = 0$ at the midpoint P. (Such coils provide an especially uniform field $\vec{B}$ near point P.) (*Hint:* See Eq. 30-28).

56. Square Current Loop The magnitude $B(x)$ of the magnetic field at points on the axis of a square current loop of side a is given in Problem 15. (a) Show that the axial magnetic field of this loop, for $x \gg a$, is that of a magnetic dipole (see Eq. 30-29). (b) What is the magnitude of the magnetic dipole moment of this loop?

57. Let the Separation Be In Problem 54 (Fig. 30-50), let the separation of the coils be a variable s (not necessarily equal to the coil radius R). (a) Show that the first derivative of the magnitude of the net magnetic field of the coils (dB/dx) vanishes at the midpoint P regardless of the value of s. Why would you expect this to be true from symmetry? (b) Show that the second derivative (d^2B/dx^2) also vanishes at P, provided $s = R$. This accounts for the uniformity of B near P for this particular coil separation.

58. *abcdefgha* A conductor carries a current of 6.0 A along the closed path *abcdefgha* involving 8 of the 12 edges of a cube of side 10 cm as shown in Fig. 30-51. (a) Why can one regard this as the superposition of three square loops: *bcfgb, abgha,* and *cdefc*? (*Hint:* Draw currents around those square loops.) (b) Use this superposition to find the magnetic dipole moment $\vec{\mu}$ (magnitude and direction) of the closed path. (c) Calculate the magnitude and direction of the magnetic field $\vec{B}$ at the points $(x, y, z) = (0.0\text{ m}, 5.0\text{ m}, 0.0\text{ m})$ and $(5.0\text{ m}, 0.0\text{ m}, 0.0\text{ m})$.

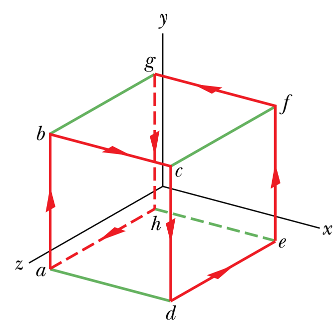


FIGURE 30-51 ■ Problem 58.

59. What Torque A circular loop of radius 12 cm carries a current of 15 A. A flat coil of radius 0.82 cm, having 50 turns and a current of 1.3 A, is concentric with the loop. (a) What magnetic field $\vec{B}$ (magnitude and direction) does the loop produce at its center? (b) What torque acts on the coil? Assume that the planes of the loop and coil are perpendicular and that the magnetic field due to the loop is essentially uniform throughout the volume occupied by the coil.

60. Two Different Arcs A length of wire is formed into a closed circuit with radii a and b, as shown in Fig. 30-52 and carries a current i. (a) What are the magnitude and direction of $\vec{B}$ at point P? (b) Find the magnetic dipole moment of the circuit.

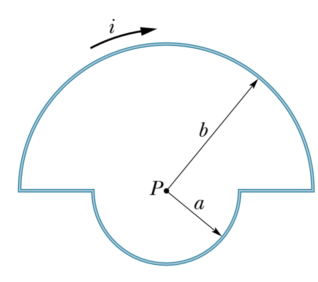


FIGURE 30-52 ■ Problem 60.

Additional Problems

61. Cross Section of a Wire Figure 30-53 shows the cross section of a wire that is perpendicular to the plane of the paper. Suppose a compass is placed at location A, which is a distance r from the wire. The compass points in the direction shown in the diagram. (a) Resketch the diagram and draw arrows to show what direction you expect the compass to point if it were moved to locations B and C. *Note:* Use the symbol $\odot$ if the flow is out of the page and the symbol $\otimes$ if the flow is into the page. (b) Indicate in what direction *positive current* is flowing through the wire and describe the rule you are using to deduce the direction of current in the wire. (c) What is the direction of the flow of *electrons* through the wire?

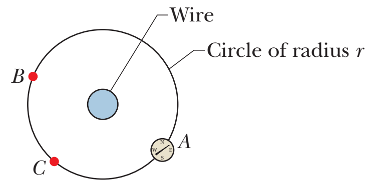


FIGURE 30-53 ■ Problem 61.

62. Wires in a *B*-Field A uniform magnetic field is directed toward the right in the plane of the paper as shown in Fig. 30-54. A wire lying perpendicular to the plane of the paper at location A carries

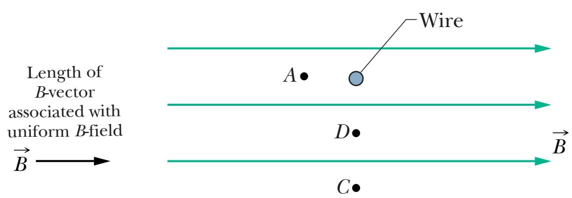


FIGURE 30-54 ■ Problem 62.

a current i. Suppose that the resultant magnetic field at point D due to a superposition of the uniform magnetic field of magnitude B and the magnetic field of the wire of magnitude B_w is zero. (a) Is the direction of the current in the wire into or out of the paper? Explain how you arrived at your conclusion. (b) Assume that point A lies at the same distance from the center of the wire as point D and that the length of the vector assigned to represent the magnitude of the uniform external magnetic field is that shown on the right. Construct a vector diagram showing the net magnetic field vector B_A^{net} at point A. (c) Assume that point C is twice the distance from the center of the wire as point D. Construct a vector diagram showing the net magnetic field vector, B_C^{net}, at point C. (Adapted from A. Arons, Homework and Test Questions for Introductory Physics Teaching, John Wiley and Sons, 1994.)

63. Earth's Field The magnitude of the Earth's magnetic field, B, at either geomagnetic pole, is about 7×10^{-5} T. Using a model in which you assume that this field is produced by a single current loop at the equator, determine the current that would generate such a field ($R_e = 6.37 \times 10^6$ m). *Hint:* The magnitudes of the magnetic field due to a single current loop of radius R at a distance R from its center and perpendicular to the plane of the loop is given by the equation

$$B = \frac{\mu_0 |i|}{2\sqrt{8}R}$$

and see Fig. 30-55.

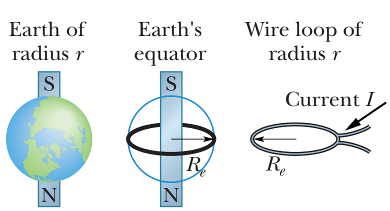


FIGURE 30-55 ■ Problem 63.

64. Comparing Electric and Magnetic Forces One In this problem we consider situations corresponding to three different long thin lines of matter containing charges: 1. A copper wire carrying an electric current from left to right, 2. A long amber rod that has been rubbed with fur and has a uniform excess of negative charge, and 3.

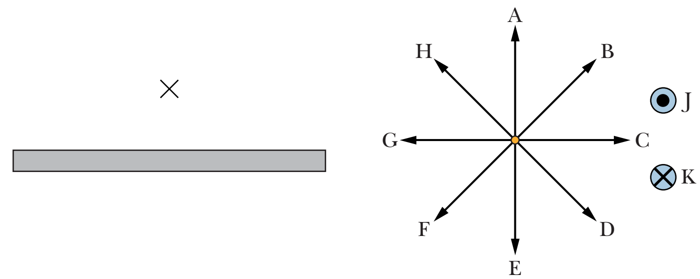


FIGURE 30-56 ■ Problem 64.

A beam of electrons passing from left to right through a vacuum inside a cathode ray tube. The direction of the electric current and of the electron flow are from left to right. Figure 30–56 shows a location marked x and a set of directions with labels on the right.

(a) For each of the three lines of matter, indicate in what direction the electric and magnetic fields at the location x would point. To indicate the direction, use one of the letters associated with a directional arrow on the "compass" in Fig. 30-56. If any of the fields are zero, write 0.

(b) Now consider placing a positive charge at the location x. In one case it is stationary, while in a second case it is moving in the direction C (to the right). Indicate the direction nearest to the total force the charge would feel. (Ignore gravity and air resistance.) Do this for all three lines of matter and for both cases.

65. Comparing Electric and Magnetic Forces Two Figure 30-57 shows a long wire carrying a current i to the right and a long amber rod with a charge density (charge/unit length) of λ. Assume that i and λ are both positive.

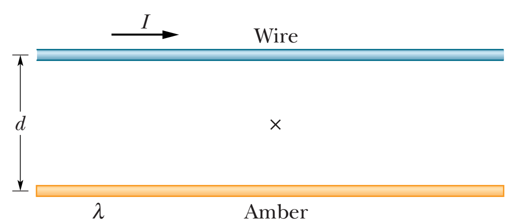


FIGURE 30-57 ■ Problem 65.

(a) The two are separated by a distance d. The point marked x is halfway between them. Copy this figure onto your paper and draw arrows to represent the following. (Be sure to label your arrows clearly to show which one is which.)

i. the direction of the magnetic field at the point marked x
ii. the direction of the electric field at the point marked x
iii. the direction of the electric force that a positive charge q placed at x would feel
iv. the direction of the magnetic force that a positive charge q placed at x would feel if it were moving to the right.

(b) The current, i, is $+10$ A, the charge density, λ, is -1 nC/m ($= 10^{-9}$ C/m) (note that it is negative), and the distance between the wires is 40 cm. At the instant shown, a proton with charge $q = 1.6 \times 10^{-19}$ C is moving into the page with a speed $v = 10^6$ m/s. Ignoring gravity, what is the magnitude and direction of the net force the proton feels at that time?

66. Direction of Magnetic Forces Figure 30-58 shows a cross section of four long parallel wires (labeled A through D) taken in a plane perpendicular to the wires. One or more of the wires may be carrying a current. If a wire carries a current, i_0, it is in the direction indicated and has strength $|i_0|$.

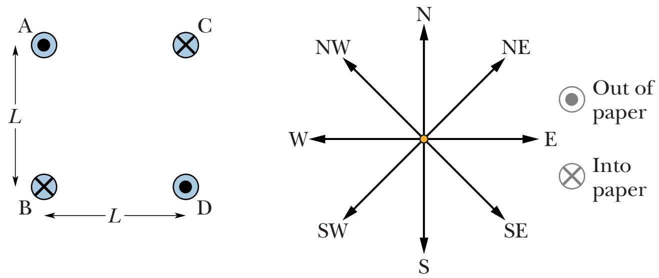


FIGURE 30-58 ■ Problem 66.

For each of the four vector quantities listed in (i) through (iv) below give the direction of the quantity. To indicate the direction, use one of the directions on the "compass" in Fig. 30-58. If the magnitude of the quantity is zero, write "0." If it is nonzero but in none of the indicated directions, write "Other."

i. Only wires B and D are carrying current. The direction of the force on wire D is____.
ii. Only wires B and D are carrying current. The direction of the force felt by an electron traveling in the E direction (on the compass) is____.
iii. Only wires B and D are carrying current. The direction of the force felt by an electron traveling in the N direction (on the compass) is____.
iv. All four wires are carrying current. The direction of the net force felt by wire A is____.

67. Magnetic Forces and Fields Figure 30-59 shows a cross section of three long parallel wires (labeled A through C) taken in a plane perpendicular to the wires. One or more of the wires may be carrying a current. If a wire carries a current, i_0, it is in the direction indicated and has strength $|i_0|$. For each of the five vector quantities (1) through (5) shown, indicate the direction of the quantity on the compass in Fig. 30-59. If the magnitude of the quantity is zero, write "0." If the result is not zero but points in a direction other than one of those indicated, write "other."

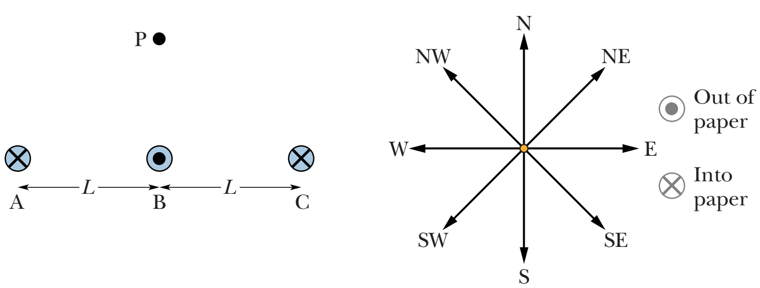


FIGURE 30-59 ■ Problem 67.

1. The magnetic field at point P if only wire A is carrying a current
2. The magnetic field at wire C if only wire A is carrying a current
3. The magnetic force on wire C if only wires A and C carry currents
4. The magnetic force on wire C if only wire A is carrying a current
5. The magnetic force on a proton at P traveling to the right (i.e., in direction E) if only wire B is carrying a current.

68. Right-Hand Rules During our discussions of magnetism and rotation we have encountered a number of different right-hand rules for obtaining the direction or sign of various quantities. Describe three right-hand rules. In your discussion of each one, include a statement of the equation or law in which the rule is applied, and whether the rule is "fundamental" or derived from a more basic principle.

69. Magnetic Forces Figure 30-60 shows parts of two long, current-carrying wires labeled 1 and 2. The wires lie in the same plane and cross at right angles at the point indicated. When carrying a current, each wire carries the same amount of current in the direction shown. At the right is shown a set of coordinate directions for describing the direction of vectors.

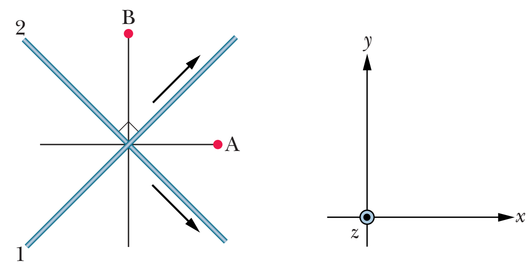


FIGURE 30-60 ■ Problem 69.

For each of the vectors discussed, indicate the direction of the vector using the coordinate system shown. For example, you might specify "the $+x$ direction" or "the $-z$ direction" or "in the x-y plane at 45° between the $+x$ and $+y$ directions." If the magnitude of the vector requested is zero, write "0."

(a) The direction of the force on a positively charged ion at the point B moving in the $+y$ direction if only wire 1 carries current
(b) The direction of the force on a positively charged ion at the point B moving in the $-z$ direction if both wires carry current
(c) The direction of the force on a positively charged ion at the point A moving in the $+x$ direction if only wire 2 carries current

For the next two parts of the problem, select which answer is correct if both wires carry current.

(d) The magnetic force on wire 1 will

i. push it in the $-z$ direction
ii. push it in the $+z$ direction
iii. tend to rotate it clockwise about the joining point
iv. tend to rotate it counterclockwise about the joining point
v. none of the above

(e) The magnetic force on wire 2 will

i. push it in the $-z$ direction
ii. push it in the $+z$ direction
iii. tend to rotate it clockwise about the joining point
iv. tend to rotate it counterclockwise about the joining point
v. none of the above

70. Constrained to a Circle Figure 30-61 shows, in cross section, two long straight wires held against a plastic cylinder of radius 20.0 cm. Wire 1 carries current $i_1 = 60.0$ mA out of the page and is fixed in place at the left side of the cylinder. Wire 2 carries current $i_2 = 40.0$ mA out of the page and can be moved around the cylinder. At what angle θ_2 should wire 2 be positioned such that the net magnetic field at the origin from the two currents has a magnitude of 80.0 nT?

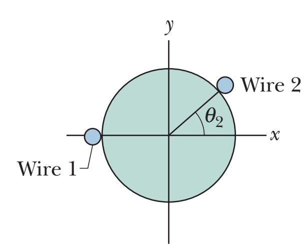


FIGURE 30-61 ■ Problem 70.

71. Element Length Figure 30-62*a* shows an element of length $ds = 1.00\ \mu\text{m}$ in a very long straight wire carrying current. The current in that element sets up a differential magnetic field $d\vec{B}$ at points in the surrounding space. Figure 30-62*b* gives the magnitude dB of the field in pico-Teslas (10^{-12} T) for points 2.5 cm from the element, as a function of angle θ between the wire and a straight line to the point. What is the magnitude of the magnetic field set up by the entire wire at perpendicular distance 2.5 cm from the wire?

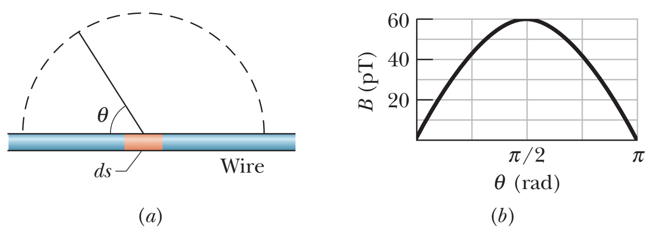


FIGURE 30-62 ■ Problem 71.

72. Where Is Wire 2 Two long straight thin wires with current lie against an equally long plastic cylinder, at radius $R = 20.0$ cm from the cylinder's central axis. Figure 30-63*a* shows, in cross section, the cylinder and wire 1 but not wire 2. With wire 2 fixed in place, wire 1 is moved around the cylinder, from angle $\theta_1 = 0°$ to angle $\theta_1 = 180°$, through the first and second quadrants of the xy coordinate system. The net magnetic field $\vec{B}$ at the center of the cylinder is measured as a function of θ_1. Figure 30-63*b* gives the x-component B_x of that field in micro-Teslas (10^{-6} T) and Fig. 30-63*c* gives the y-component B_y, both as functions of θ_1. (a) At what angle θ_2 is wire 2 located? What are the size and direction of the currents in (b) wire 1 and (c) wire 2?

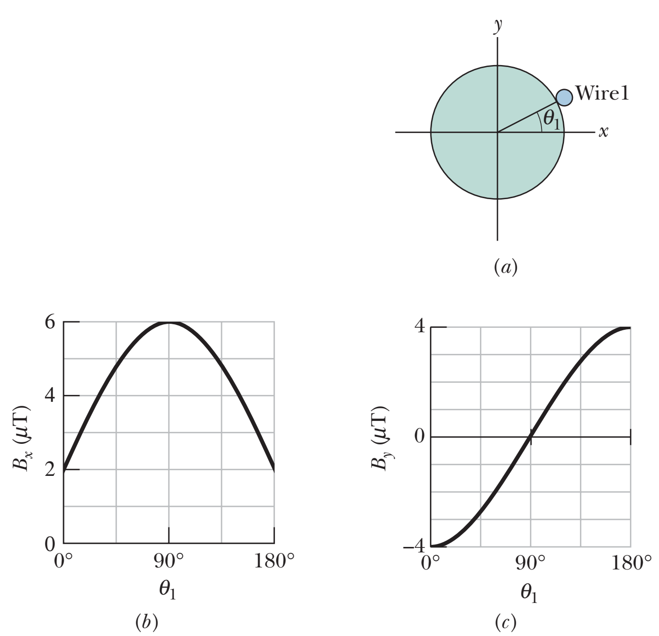


FIGURE 30-63 ■ Problem 72.

73. The Ratio of Currents Figure 30-64*a* shows, in cross section, two long, parallel wires carrying current and separated by distance L. The ratio $|i_1/i_2|$ of their current amounts is 4.00; the directions of the currents are not indicated. Figure 30-64*b* shows the y-component B_y in nano-Teslas (10^{-9} T) of their net magnetic field along the x axis to the right of wire 2. (a) At what value of $x > 0$ is B_y maximum? (b) If $|i_2| = 3$ mA, what is the value of the maximum? What are the directions of (c) current i_1 and (d) current i_2?

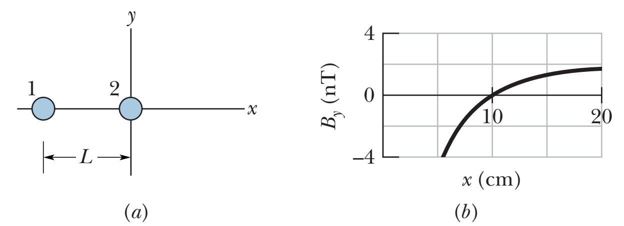


FIGURE 30-64 ■ Problem 73.

74. Same Radius Different Current In Fig. 30-65*a* two circular loops, with different currents but the same radius of 4.0 cm, are centered on a y axis. They are initially separated by distance $L = 3.0$ cm, with loop 2 positioned at the origin of the axis. The currents in the two loops produce a net magnetic field at the origin, with y-component B_y. That component is to be measured as loop 2 is gradually moved in the positive direction of the y axis. Figure 30-65*b* gives B_y in micro-Teslas (10^{-6} T) as a function of the position y of loop 2. The curve approaches an asymptote of $B_y = 7.20\ \mu\text{T}$ as $y \to \infty$. What are (a) current i_1 in loop 1 and (b) current i_2 in loop 2?

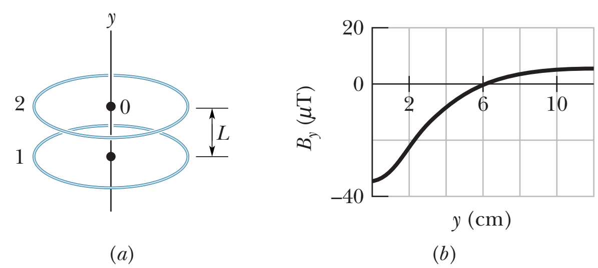


FIGURE 30-65 ■ Problem 74.

75. How Many Revolutions An electron is shot into one end of a solenoid, as it enters the uniform magnetic field within the solenoid, its speed is 800 m/s and its velocity vector makes an angle of 30° with the central axis of the solenoid. The solenoid carries 4.0 A and has 8000 turns along its length. How many revolutions does the electron make along its helical path within the solenoid by the time it emerges from the solenoid's opposite end? (In a real solenoid, where the field is not uniform at the two ends, the number of revolutions would be slightly less than the answer here.)

76. Force per Unit Length Two Figure 30-66 shows wire 1 in cross section; the wire is long and straight, carries a current of 4.00 mA out of the page, and is at distance $d_1 = 2.4$ cm from a surface. Wire 2, which is parallel to wire 1 and also long, is at horizontal distance $d_2 = 5.0$ cm from wire 1 and carries a current of 6.80 mA into the page. What is the x component of the magnetic force per unit length on wire 2 due to the current in wire 1?

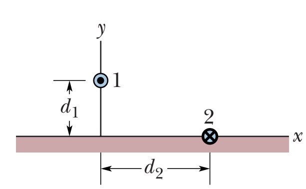


FIGURE 30-66 ■ Problem 76.