Notes from the Methods of Mathematical Physics II lectures of 2013-02-19/05-30 by Prof. Dr. Eugene Trubowitz

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1 *True Lies*: An introduction to distribution theory

Here begins the lecture of 2013-02-28.

In many areas of physics, the Dirac delta "function" is an important tool, e.g., for describing localized phenomena. While it is usually described in vague terms as "only making sense under an integral", physicists tend to still write identities such as

$$\Delta \frac{1}{|\boldsymbol{x}|} = -4\pi\delta(\boldsymbol{x}),$$

and rely on their intuition and previous mistakes to distinguish situations in which they can safely apply this calculus from the ones where that would lead to incorrect results.

How can we formalize this? Note that the operation of taking the integral of the Dirac delta multiplied with another function φ can be considered as a map from that function to a scalar in $\mathbb{C}$. Thus, we could consider

$$\delta_{\boldsymbol{x}}(\varphi) = \int_{\mathbb{R}^n} \delta(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

to be a functional, and more specifically an element of the dual space of whatever class of functions we can allow for φ. But the question is: what restrictions do we need on this space of functions if we want to find objects like the Dirac delta in its dual space?

Actually, we want these functionals to be in the continuous dual of our space, that is, we want them to be functionals U such that $U(\varphi_k) \to U(\varphi)$ for all sequence of functions $(\varphi_k)_{k\in\mathbb{N}}$ that converge to φ in an appropriate norm[1] — we want our space to be a Banach space. Intuition tells us that the smaller, i.e., more restricted, a space of functions is, the stronger the conditions on $\varphi_k \to \varphi$ must be in order for it to be complete, and so the bigger its continuous dual can be, as we only need $U(\varphi_k) \to U(\varphi)$ for the sequences $(\varphi_k)_{k\in\mathbb{N}}$ that converge in that space. Thus, we will investigate a space of functions $\varphi(\boldsymbol{x}) \in C^\infty(\mathbb{R}^n)$ that vanish quickly as $|\boldsymbol{x}| \to \infty$ ("converge with a vengeance"), hoping to find the Dirac delta in its continuous dual.

Definition. *First, a few definitions used throughout this course:*

$C^\infty(\mathbb{R}^n) := \left\{\varphi : \mathbb{R}^n \to \mathbb{C} \,\middle|\, \forall \boldsymbol{\alpha} \in \mathbb{N}^n, \partial^{\boldsymbol{\alpha}}\varphi \in C^0(\mathbb{R}^n)\right\}$ *is the space of* smooth functions *from* $\mathbb{R}^n$ *to* $\mathbb{C}$*, where*

$\mathbb{N} := \{0, 1, 2, \dots\}$, $\quad \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ *is a* multi-index,

$|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$, $\quad \partial_{\boldsymbol{x}}{}^{\boldsymbol{\alpha}} := \dfrac{\partial^{|\boldsymbol{\alpha}|}}{\partial_{x_1}{}^{\alpha_1} \cdots \partial_{x_n}{}^{\alpha_n}}$ *(we use* $\partial^{\boldsymbol{\alpha}}$ *when the variable is implicit),*

and $\boldsymbol{x}^{\boldsymbol{\beta}} := x_1^{\beta_1} \cdots x_n^{\beta_n}$ *for* $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{\beta} \in \mathbb{N}^n$.

Definition (Schwartz space)**.** *Now we define the* Schwartz space

$$\mathscr{S}(\mathbb{R}^n) := \left\{\varphi \in C^\infty(\mathbb{R}^n) \,\middle|\, \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n, \left\|\varphi\right\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} < \infty\right\}, \text{ where } \left\|\varphi\right\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \sup_{\boldsymbol{x}\in\mathbb{R}^n} \left|\boldsymbol{x}^{\boldsymbol{\alpha}} \partial_{\boldsymbol{x}}{}^{\boldsymbol{\beta}} \varphi(\boldsymbol{x})\right|.$$

[1] This simply means that U is a continuous map from the function space endowed with its norm topology to $\mathbb{R}$ endowed with the standard topology.

Note that $\varphi \in \mathscr{S}(\mathbb{R}^n)$ implies $\lim_{|\boldsymbol{x}|\to\infty}\left(1+|\boldsymbol{x}|^2\right)^k \partial_{\boldsymbol{x}}{}^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x}) = 0$, as

$$\begin{aligned}\left(1+|\boldsymbol{x}|^2\right)^m \partial_{\boldsymbol{x}}{}^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x}) &= \sum_{j=0}^{m}\binom{m}{j}|\boldsymbol{x}|^{2j}\partial_{\boldsymbol{x}}{}^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x})\\ &= \sum_{j=0}^{m}\binom{m}{j}\left(\sum_{i=1}^{n}x_i^2\right)^j \partial_{\boldsymbol{x}}{}^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x})\\ &= \sum_{j=0}^{m}\binom{m}{j}\sum_{|\boldsymbol{\alpha}|=j}\left(\prod_{i=1}^{n}x_i^{2\alpha_i}\right)\partial_{\boldsymbol{x}}{}^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x})\end{aligned}$$

and $\sup_{\boldsymbol{x}\in\mathbb{R}^n}\left|\boldsymbol{x}^{2\boldsymbol{\alpha}}\varphi(\boldsymbol{x})\right| < \infty$. It immediately follows:

Lemma. *If $\varphi \in \mathscr{S}(\mathbb{R}^n)$, then*

$$\forall k \geq 0 \quad \forall \boldsymbol{\alpha} \in \mathbb{N}^n \quad \exists C_{\boldsymbol{\alpha}} \in \mathbb{R} \quad \left|\partial_{\boldsymbol{x}}{}^{\boldsymbol{\alpha}}\varphi(x)\right| < \frac{C_\alpha}{\left(1+|\boldsymbol{x}|^2\right)^k}.$$

Definition (Convergence in $\mathscr{S}(\mathbb{R}^n)$)**.** *The sequence* $\left(\varphi_k\right)_{k\in\mathbb{N}} \in (\mathscr{S}(\mathbb{R}^n))^{\mathbb{N}}$ converges to $\varphi \in \mathscr{S}(\mathbb{R}^n)$ *if and only if*

$$\forall \boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{N}^N \quad \lim_{k\to\infty}\left\|\varphi_k - \varphi\right\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0.$$

We use the notation $\varphi_k \xrightarrow{\mathscr{S}(\mathbb{R}^n)} \varphi$.

Definition (Tempered distributions)**.** *Note that $\mathscr{S}(\mathbb{R}^n)$ is a complex vector space. We define that U is an element of its continuous dual $\mathscr{S}'(\mathbb{R}^n)$, the space of* tempered distributions, *if and only if:*

1. *U is a linear map from $\mathscr{S}(\mathbb{R}^n)$ to $\mathbb{C}$, i.e., $U \in \mathscr{S}^*(\mathbb{R}^n)$*
2. *U is continuous on $\mathscr{S}(\mathbb{R}^n)$, i.e. if $\varphi_k \xrightarrow{\mathscr{S}(\mathbb{R}^n)} \varphi$ then $\lim_{k\to\infty} U\left(\varphi_k\right) = U\left(\varphi\right)$.*

Let us now consider some examples. First, define the space of locally integrable functions as follows.

Definition (Locally integrable functions)**.**

$$L^1_{\mathrm{loc}}\left(\mathbb{R}^n\right) := \left\{ f : \mathbb{R}^n \to \mathbb{C} \,\middle|\, \forall \boldsymbol{x} \in \mathbb{R}^n, \exists \epsilon > 0, \int_{B_\epsilon \boldsymbol{x}} \left|f\left(\boldsymbol{y}\right)\right| \mathrm{d}\,\boldsymbol{y} < \infty \right\}$$

Now let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If additionally $\sup_{\boldsymbol{x}\in\mathbb{R}^n}\left(1+|\boldsymbol{x}|^2\right)^{-s}\left|f(x)\right| < \infty$ for some $s > 0$ (e.g. $\sum_{|\boldsymbol{\beta}|\leq s} c_{\boldsymbol{\beta}}\, x^{\boldsymbol{\beta}}$), then

$$U_f\left(\varphi\right) := \int_{\mathbb{R}^n} f(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}\,x$$

is a tempered distribution.

Definition (Dirac delta distribution)**.** *Another example is the* Dirac delta distribution *mentioned above, with*

$$\delta\left(\varphi\right) := \varphi(0), \quad \delta_{\boldsymbol{x}}\left(\varphi\right) := \varphi(\boldsymbol{x}).$$

Distributions have the nice property that we can always differentiate them.

Definition (Derivative of a distribution)**.** *Let $U \in \mathscr{S}(\mathbb{R}^n)$. Then $\partial^{\boldsymbol{\alpha}} U$ is the distribution defined by*

$$\left(\partial^{\boldsymbol{\alpha}} U\right)\left(\varphi\right) := (-1)^{|\boldsymbol{\alpha}|} U\left(\partial^{\boldsymbol{\alpha}}\varphi\right) \quad \text{for } \varphi \in \mathscr{S}\left(\mathbb{R}^n\right).$$

This is natural, as

$$U_{\partial^{\boldsymbol{\alpha}} f}\left(\varphi\right) = \int_{\mathbb{R}^n}\left(\partial^{\boldsymbol{\alpha}} f(\boldsymbol{x})\right)\varphi(\boldsymbol{x})\,\mathrm{d}\,\boldsymbol{x} = (-1)^{|\boldsymbol{\alpha}|}\int_{\mathbb{R}^n} f(\boldsymbol{x})\partial^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x})\,\mathrm{d}\,\boldsymbol{x} \qquad \text{by integrating by parts.}$$

Let's find the derivative of $|\cdot|$ in the distribution world:

$$
\begin{aligned}
\operatorname{Abs}(\varphi) &\coloneqq \int_{-\infty}^{\infty} |x|\varphi(x)\,\mathrm{d}x \\
\left(\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{Abs}\right)(\varphi) &= -\operatorname{Abs}\frac{\mathrm{d}\varphi}{\mathrm{d}x} \\
&= -\int_{0}^{\infty} x\varphi'(x)\,\mathrm{d}x - \int_{-\infty}^{0} (-x)\varphi'(x) \\
&= -\left[x\varphi(x)\right]_0^\infty + \int_{0}^{\infty} \varphi(x)\,\mathrm{d}x + \left[x\varphi(x)\right]_0^\infty - \int_{-\infty}^{0} \varphi(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} h(x)\varphi(x)\,\mathrm{d}x = U_h(\varphi),
\end{aligned}
$$

$$
\text{where } h(x) \coloneqq \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \text{ is the step function.}
$$

We define $H(\varphi) \coloneqq \int_{-\infty}^{\infty} h(x)\varphi(x)\,\mathrm{d}x$ and write $\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{Abs} = H$. This corresponds to $\frac{\mathrm{d}}{\mathrm{d}x}|x| = h(x)$ in physicist-speak.

What about $\Delta \frac{1}{|\boldsymbol{x}|}$? By defining the *Newtonian distribution*

$$
N(\varphi) \coloneqq \int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{x}|}\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},
$$

we get

$$
\Delta N(\varphi) = \sum_{j=1}^{3} \frac{\partial^2}{\partial {x_j}^2} N(\varphi) = \sum_{j=1}^{3} N\left(\frac{\partial^2 \varphi}{\partial {x_j}^2}\right) = \cdots = -4\pi\varphi(0),
$$

in other words,

$$
\Delta N = -4\pi\delta.
$$

Here ends the lecture of 2013-02-28.
Here begins the lecture of 2013-03-05.

Definition (Inner product on $\mathscr{S}(\mathbb{R}^n)$)**.** *Let $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$. We define:*

$$
\langle \psi | \varphi \rangle \coloneqq \int_{\mathbb{R}^n} \overline{\psi}(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.
$$

Here ends the lecture of 2013-03-05.

2 General properties of the Fourier transform

Here begins the lecture of 2013-03-07.

Lemma ("Your life in Fourier land depends on it.")**.**

$$\begin{aligned}\widehat{\varphi+\psi} &= \widehat{\varphi}+\widehat{\psi},\\ \widehat{\lambda\varphi} &= \lambda\widehat{\varphi}\end{aligned} \tag{1}$$

$$\left|\widehat{\varphi}(\boldsymbol{k})\right| \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\left\|\varphi\right\|_1 < \infty, \qquad \left\|\varphi\right\|_1 := \int\limits_{\mathbb{R}^n}\left|\varphi(\boldsymbol{x})\right|\mathrm{d}\boldsymbol{x} \tag{2}$$

$$\widehat{\varphi(\lambda\,\cdot\,)}(\boldsymbol{k}) = \frac{1}{|\lambda|^n}\widehat{\varphi}\left(\frac{\boldsymbol{k}}{\lambda}\right), \qquad \lambda\neq 0 \tag{3}$$

$$\widehat{T_{\boldsymbol{y}}\varphi}(\boldsymbol{x}) = \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{y}}\,\widehat{\varphi}(\boldsymbol{k}), \qquad \left(T_{\boldsymbol{y}}\varphi\right)(\boldsymbol{x}) := \varphi\left(\boldsymbol{x}+\boldsymbol{y}\right) \tag{4}$$

$$\begin{aligned}\widehat{\frac{\partial\varphi}{\partial x_j}}(\boldsymbol{k}) &= \mathrm{i}k_j\widehat{\varphi}(\boldsymbol{k})\\ \widehat{x_j\varphi}(\boldsymbol{k}) &= \mathrm{i}\frac{\partial\widehat{\varphi}}{\partial k_j}(\boldsymbol{k})\end{aligned} \tag{5}$$

$$\int\limits_{\mathbb{R}^n}\widehat{\varphi}\left(\boldsymbol{y}\right)\psi\left(\boldsymbol{y}\right)\mathrm{d}\boldsymbol{y} = \int\limits_{\mathbb{R}^n}\varphi\left(\boldsymbol{y}\right)\widehat{\psi}\left(\boldsymbol{y}\right)\mathrm{d}\boldsymbol{y} \tag{6}$$

Proof. Proof of (2):

$$\left|\widehat{\varphi}(\boldsymbol{k})\right| = \left|\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}\right| \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\left|\varphi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\right|\mathrm{d}\boldsymbol{x} = \frac{1}{(2\pi)^{\frac{n}{2}}}\underbrace{\int\limits_{\mathbb{R}^n}\left|\varphi(\boldsymbol{x})\right|}_{\left\|\varphi\right\|_1}\underbrace{\left|\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\right|}_{1}\mathrm{d}\boldsymbol{x}$$

Proof of existence of the integral $\left\|\varphi\right\|_1$:

$$\begin{aligned}\int\limits_{\mathbb{R}^n}\left|\varphi(\boldsymbol{x})\right|\mathrm{d}\boldsymbol{x} &= \int\limits_{\mathbb{R}^n}\left(1+|\boldsymbol{x}|^2\right)^{\frac{s}{2}}\left|\varphi(\boldsymbol{x})\right|\frac{\mathrm{d}\boldsymbol{x}}{\left(1+|\boldsymbol{x}|^2\right)^{\frac{s}{2}}}\\ &\leq \left\|\varphi\right\|_{s,\boldsymbol{0}}\int\limits_{\mathbb{R}^n}\frac{1}{\left(1+|\boldsymbol{x}|^2\right)^{\frac{s}{2}}}\,\mathrm{d}\boldsymbol{x}\\ &= \left\|\varphi\right\|_{s,\boldsymbol{0}}\int\limits_0^\infty\frac{r^{n-1}}{\left(1+r^2\right)^{\frac{s}{2}}}\,\mathrm{d}r < \infty \qquad \text{for } s\geq n+1.\end{aligned}$$

"(1) I will not do!" Proof of (3):

$$\begin{aligned}\widehat{\varphi(\lambda\,\cdot\,)} &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi(\lambda\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}\\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi\left(\boldsymbol{y}\right)\mathrm{e}^{-\mathrm{i}\,\frac{1}{\lambda}\boldsymbol{k}\cdot\boldsymbol{y}}\,\mathrm{d}\left(\frac{\boldsymbol{y}}{\lambda}\right) \qquad \text{where } \boldsymbol{x} = \frac{1}{\lambda}\boldsymbol{y}.\\ &= \frac{1}{|\lambda|^n}\underbrace{\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi\left(\boldsymbol{y}\right)\mathrm{e}^{-\mathrm{i}\,\frac{\boldsymbol{k}}{\lambda}\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}}_{\widehat{\varphi}\left(\frac{\boldsymbol{k}}{\lambda}\right)}\end{aligned}$$

Proof of the second part of (5):

$$
\begin{aligned}
\widehat{x_j\varphi}(\boldsymbol{x}) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n} x_j\varphi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{x})\mathrm{i}\frac{\partial}{\partial k_j}\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} && \text{as } \frac{\partial}{\partial k_j}\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}} = \frac{\partial}{\partial k_j}\mathrm{e}^{-\mathrm{i}\sum_{r=0}^{n}k_j x_j} = -\mathrm{i}x_j\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}} \\
&= \mathrm{i}\frac{\partial}{\partial k_j}\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} \\
&= \mathrm{i}\frac{\partial\widehat{\varphi}}{\partial k_j}(\boldsymbol{k})
\end{aligned}
$$

The professor starts whistling some elevator music while waiting for a student to write down the proof before he can wipe the board. When the student is done, the professor notices that there still is some room left on the blackboard and starts writing the rest there, without erasing anything.

Proof of 6:

$$
\begin{aligned}
\int\limits_{\mathbb{R}^n}\widehat{\varphi}(\boldsymbol{y})\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} &= \int\limits_{\mathbb{R}^n}\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{y}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}\,\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \\
&= \int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{x})\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{y}}\,\psi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} && \text{As } \textstyle\int_{\mathbb{R}^{2n}}\left|\varphi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{y}\cdot\boldsymbol{x}}\,\psi(\boldsymbol{y})\right|\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} < \infty\text{, we can use Fubini's theorem.} \\
&= \int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{y})\widehat{\psi}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}
\end{aligned}
$$

□

Proposition. *The Fourier transform* $\widehat{\cdot}$ *is a continuous bijective linear map from the Schwartz space* $\mathscr{S}(\mathbb{R}^n)$ *to itself.* $\widecheck{\widehat{\varphi}} = \varphi$, *where* $\widecheck{\psi}(\boldsymbol{x}) \coloneqq \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\psi(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}$, *read "unhat", "bird", "seagull", or whatever you like.*

Proof.

$$
\begin{aligned}
\varphi \in \mathscr{S}(\mathbb{R}^n) &\Rightarrow \boldsymbol{x}^{\boldsymbol{\alpha}}\partial_{\boldsymbol{x}}{}^{\boldsymbol{\beta}} \in \mathscr{S}(\mathbb{R}^n) \\
\left(\widehat{\boldsymbol{x}^{\boldsymbol{\alpha}}\partial_{\boldsymbol{x}}{}^{\boldsymbol{\beta}}\varphi}\right)(\boldsymbol{k}) &= \mathrm{i}^{|\boldsymbol{\alpha}|}\partial_{\boldsymbol{k}}{}^{\boldsymbol{\alpha}}\left(\widehat{\partial_{\boldsymbol{x}}{}^{\boldsymbol{\beta}}\varphi}\right)(\boldsymbol{k}) && \text{from the second equality in (5), applied } |\boldsymbol{\alpha}| \text{ times.} \\
&= \mathrm{i}^{|\boldsymbol{\alpha}|}\partial_{\boldsymbol{k}}{}^{\boldsymbol{\alpha}}\left(\mathrm{i}^{|\boldsymbol{\beta}|}\boldsymbol{k}^{\boldsymbol{\beta}}\widehat{\varphi}\right)(\boldsymbol{k}) \\
&= \mathrm{i}^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}\partial_{\boldsymbol{k}}{}^{\boldsymbol{\alpha}}\left(\boldsymbol{k}^{\boldsymbol{\beta}}\widehat{\varphi}\right)(\boldsymbol{k})
\end{aligned}
$$

"I should have done it the other way around. [...] Let's try it the other way around." We want:

$$
\forall\boldsymbol{\alpha},\boldsymbol{\beta},\ \sup_{\boldsymbol{k}\in\mathbb{R}^n}\left|\boldsymbol{k}^{\boldsymbol{\alpha}}\partial_{\boldsymbol{k}}{}^{\boldsymbol{\beta}}\widehat{\varphi}(\boldsymbol{k})\right| < \infty
$$

As that implies $\widehat{\varphi} \in \mathscr{S}(\mathbb{R}^n)$.

$$
\begin{aligned}
\left(\widehat{\partial_{\boldsymbol{x}}{}^{\boldsymbol{\alpha}}\left(\boldsymbol{k}^{\boldsymbol{\beta}}\varphi\right)}\right)(\boldsymbol{k}) &= \mathrm{i}^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}\boldsymbol{k}^{\boldsymbol{\alpha}}\partial_{\boldsymbol{k}}{}^{\boldsymbol{\beta}}\widehat{\varphi}(\boldsymbol{k}) && \text{by applying the first part of (5) } |\boldsymbol{\alpha}| \text{ times and the second part } \left|\boldsymbol{\beta}\right| \text{ times.} \\
\forall\boldsymbol{\alpha},\boldsymbol{\beta},\ \left|\boldsymbol{k}^{\boldsymbol{\alpha}}\partial_{\boldsymbol{k}}{}^{\boldsymbol{\beta}}\widehat{\varphi}(\boldsymbol{k})\right| &\le \frac{1}{(2\pi)^{\frac{n}{2}}}\left\|\partial_{\boldsymbol{x}}{}^{\boldsymbol{\beta}}\left(\boldsymbol{x}^{\boldsymbol{\beta}}\varphi\right)\right\|_1 < \infty && \text{from (2).}
\end{aligned}
$$

We now know $\widehat{\varphi} \in \mathscr{S}(\mathbb{R}^n)$. Let us prove $\check{\widehat{\varphi}} = \varphi$. “When you do the wrong thing I'm gonna scream loudly.”

$$\begin{aligned}\check{\widehat{\varphi}}(\boldsymbol{x}) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \widehat{\varphi}(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}} \,\mathrm{d}\boldsymbol{k} \\ &= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{y}) \mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{y}} \,\mathrm{d}\boldsymbol{y}\, \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}} \,\mathrm{d}\boldsymbol{k}\end{aligned}$$

We don't interchange the integrals here because that would lead to ugly calculations. “If pou paint the walls before you start building, it's not a good idea.” At this point, somebody suggests replacing $\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \,\mathrm{d}\boldsymbol{k}$ by $\delta(\boldsymbol{x}-\boldsymbol{y})$. “It's plausible! — WHY? [...] Ah I said it, so it's plausible.” However, we would need to do some nasty calculations in order to do this. “How are we going to do it so fast that you don't get bored, and yet in enough detail that he's convinced? Be sneaky.”

$$\begin{aligned}\check{\widehat{\varphi}} &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \widehat{\varphi}(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}} \underbrace{\lim_{\epsilon\downarrow 0} \mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}}}_{\substack{\text{1 written in} \\ \text{some other way}}} \,\mathrm{d}\boldsymbol{k} \\ &= \lim_{\epsilon\downarrow 0} \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \widehat{\varphi}(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}} \qquad \text{“Don't worry.” This is actually a one-liner using Lebesgue's dominated convergence theorem.}\end{aligned}$$

The 10-minute break ends with the loud noise of a metallic pointing stick hitting the desk.

$$\begin{aligned}\frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \widehat{\varphi}(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}} \,\mathrm{d}\boldsymbol{k} &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{y}) \mathrm{e}^{-\mathrm{i}\,\boldsymbol{y}\cdot\boldsymbol{k}} \,\mathrm{d}\boldsymbol{y}\, \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}} \,\mathrm{d}\boldsymbol{k} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{y}) \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\, \mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}} \,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{y}\end{aligned}$$

Recall: “every path leads to Rome and every Gaussian is in $\mathscr{S}(\mathbb{R}^n)$.” Also, the last foot of a dactylic hexameter is always a spondee.

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{k}} \,\mathrm{d}\boldsymbol{k} = \frac{\mathrm{e}^{-\frac{1}{2\epsilon}|\boldsymbol{x}-\boldsymbol{y}|^2}}{\epsilon^{\frac{n}{2}}}$$

We therefore get:

$$\begin{aligned}\frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \widehat{\varphi}(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}} \,\mathrm{d}\boldsymbol{k} &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{y}) \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\, \mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}} \,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{y} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{y}) \frac{\mathrm{e}^{-\frac{1}{2\epsilon}|\boldsymbol{x}-\boldsymbol{y}|^2}}{\epsilon^{\frac{n}{2}}} \,\mathrm{d}\boldsymbol{y} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{y}\sqrt{\epsilon}+\boldsymbol{x}) \frac{\mathrm{e}^{-\frac{1}{2\epsilon}|\boldsymbol{y}\sqrt{\epsilon}|^2}}{\epsilon^{\frac{n}{2}}} \,\mathrm{d}(\boldsymbol{y}\sqrt{\epsilon}) \qquad \text{“Epsilons everywhere!”} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{x}+\boldsymbol{y}\sqrt{\epsilon}) \mathrm{e}^{-\frac{1}{2}|\boldsymbol{y}|^2} \,\mathrm{d}\boldsymbol{y} \qquad \text{“Now we can paint.”} \\ &\xrightarrow{\epsilon\downarrow 0} \frac{1}{(2\pi)^{\frac{n}{2}}} \varphi(\boldsymbol{x}) \int\limits_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}|\boldsymbol{y}|^2} \,\mathrm{d}\boldsymbol{y} = \varphi(\boldsymbol{x}).\end{aligned}$$

□

3 Eigenfunctions of the Fourier transform

"The main ideas of many things are right here." We seek to find the solutions (φ, λ) of

$$\widehat{\varphi} = \lambda\varphi.$$

We shall find all of them. "Not one is going to get away." Actually, we have already got one:

$$\widehat{\mathrm{e}^{-\frac{1}{2}|\boldsymbol{x}|^2}} = \mathrm{e}^{-\frac{1}{2}|\boldsymbol{x}|^2},$$

so we know that 1 is an eigenvalue. How do we find the others? "Here you actually have to have an idea."

Idea Find an $H \in \mathrm{End}(\mathscr{S}(\mathbb{R}^n))$ such that $[H, \widehat{\cdot}\,] = 0$. Then they have the same eigenspaces, so we just need to find the eigenfunctions of H (see *Finite Dimensional Quantum Mechanics*).

Let $n = 1$.

$$\left.\begin{aligned}\widehat{\left(x^2\varphi\right)}(k) &= -\frac{\mathrm{d}^2}{\mathrm{d}\,k^2}\widehat{\varphi}(k)\\ \widehat{\left(\frac{\mathrm{d}^2}{\mathrm{d}\,x^2}\varphi\right)}(k) &= -k^2\widehat{\varphi}(k)\end{aligned}\right\}\quad \text{From the lemma your life depends on.}$$

Subtracting the second equality from the first one above:

$$\mathscr{F}\left(\left(-\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + x^2\right)\varphi\right)(k) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}\,k^2} + k^2\right)\mathscr{F}\,\varphi(k),$$

where $\mathscr{F} = \widehat{\cdot}$ because things are getting a bit too big for the hat. We smell a harmonic potential. "Don't get your fingers too close, they sometimes bite."

Let $H := \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + x^2 - 1\right)$. Then $[H, \widehat{\cdot}\,] = 0$. We multiplied by $\frac{1}{2}$ so that it looks even more like a harmonic potential (it is the Hamiltonian of the quantum harmonic oscillator). Why did we add -1? Define

$$\begin{aligned}A &:= \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}\,x} + x\right),\\ A^\dagger &:= \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}\,x} + x\right).\end{aligned}$$

We then have $H = A^\dagger A$. If a and b commute, then we have $(a-b)(a+b) = a^2 - b^2$, but here x and $\frac{\mathrm{d}}{\mathrm{d}\,x}$ don't quite commute, hence the -1.

Here ends the lecture of 2013-03-07.
Here begins the lecture of 2013-03-12.

Let us look at these operators in more detail: define $\left(Qf\right)(x) := x f(x) \in \mathscr{S}(\mathbb{R})$, $\left(Pf\right)(x) := \frac{\mathrm{d}}{\mathrm{d}\,x} f(x) \in \mathscr{S}(\mathbb{R})$ for $f \in \mathscr{S}(\mathbb{R})$. We saw that $[Q, P] = 1$. We also have:

$$\begin{aligned}\left\langle\varphi\middle|P\psi\right\rangle &= \int_{\mathbb{R}} \overline{\varphi}(x)\frac{\mathrm{d}}{\mathrm{d}\,x}\psi(x)\,\mathrm{d}\,x\\ &= \left[\overline{\varphi}\psi\right]_{-\infty}^{\infty} - \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}\,x}\varphi(x)\psi(x)\,\mathrm{d}\,x\\ &= -\left\langle P\varphi\middle|\psi\right\rangle, \qquad \text{$\left[\overline{\varphi}\psi\right]_{-\infty}^{\infty} = 0$ as $\varphi \in \mathscr{S}(\mathbb{R})$, $\psi \in \mathscr{S}(\mathbb{R})$, and therefore $\overline{\varphi}\psi \in \mathscr{S}(\mathbb{R})$.}\end{aligned}$$

so P is skew symmetric. What is better than skew symmetric? Self adjoint is better. How do we make a skew symmetric operator self adjoint? We add an i. Namely, redefine $\left(Pf\right)(x) := \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\,x} f(x)$, we then have $\left\langle\varphi\middle|P\psi\right\rangle = \left\langle P\varphi\middle|\psi\right\rangle$. Q is obviously self adjoint. We now have $[Q, P] = \mathrm{i}\mathbb{1}$. Multiplying by h, redefine $\left(Pf\right)(x) := \frac{h}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\,x} f(x)$, we get $[Q, P] = h\mathrm{i}\mathbb{1}$.

With $h = \hbar$, the conditions verified by P and Q are the conditions for infinite matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ in the "catechism of matrix mechanics" in *Finite Dimensional Quantum Mechanics*. "Being something a little more, and something a lot less than cockroaches, we are curious." What we have been doing here is not the ideal way of doing this: we have to pick a vector space, in this case $\mathscr{S}(\mathbb{R}^n)$, last year the space of infinite matrices. As we learned in linear algebra, we gain a deeper understanding if we do not use coordinate systems. "We are going to study the pure essence" of this. We are going to replace that by an abstract group, which lives in Plato's cave, and when we need to calculate, we will look at a representation. "The group is as close to God as you can get."

Definition (Heisenberg group)**.** *We define the is the Heisenberg group, introduced by Hermann Weyl.*

$$\left(H := \left\{ \begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \middle| \, r, s, t \in \mathbb{R} \right\}, \textit{ordinary matrix multiplication}\right)$$

You know that if you take $\mathrm{e}^x \mathrm{e}^y$ on reals numbers, you get e^{x+y}, but that doesn't work for matrices. Hermann Weyl's idea is the following: we look at the identities

$$\left(\mathrm{e}^{\mathrm{i}tP} f\right)(x) = f(x+t) \qquad \text{as } \left(\mathrm{e}^{\mathrm{i}t\left(\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}\right)}\right) f(x) \overset{\text{formally}}{=} \sum_{l=0}^{\infty} \frac{t^l}{l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l} f(x), \text{ the Taylor series of } f(x+t) \text{ at } x.$$

$$\mathrm{e}^{\mathrm{i}tP} \mathrm{e}^{\mathrm{i}sQ} = \mathrm{e}^{\mathrm{i}st} \mathrm{e}^{\mathrm{i}sQ} \mathrm{e}^{\mathrm{i}tP}$$

Write $X := \mathrm{e}^{\mathrm{i}Q}$ and $Y := \mathrm{e}^{\mathrm{i}P}$, $X^r Y^s = \mathrm{e}^{\mathrm{i}rs} Y^s X^r$, $Z := \left(\mathrm{e}^{\mathrm{i}}\right)^t \mathbb{1}$. We will generate a group with these three elements. "I'll think up another way of doing this; it's something worth doing twice." How do we preserve all the ideas, and yet get rid of all the infinite dimensional vector spaces? We study a finite dimensional analog, the finite Heisenberg group, in which we use $\mathbb{Z}/n\mathbb{Z}$ instead of $\mathbb{R}$. And we'll use that little group to do all sorts of experiments. No one will complain about what we do to it.

Let us start again. "Those of you who have done the exercises, and those of you who have the determination to go to Prof. Willwacher's — 'Willwacher.' I like that name. — lecture, we are going to do that again, but on a concrete example." We will find all the conjugacy classes. "We're going to hunt them down!"

"I'll see you on Thursday, and I expect there'll be fewer people."

Here ends the lecture of 2013-03-12.