# Notes from the Methods of Mathematical Physics II courses of 2013-02-19/05-30 by Prof. Dr. Eugene Trubowitz

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## 1 *True Lies*: An introduction to distribution theory

In many areas of physics, the Dirac delta "function" is an important tool, e.g. for describing localized phenomena. While it is usually described in vague terms as "only making sense under an integral", physicists tend to still write identities such as

$$\Delta \frac{1}{|\boldsymbol{x}|} = -4\pi\delta(\boldsymbol{x}),$$

and rely on their intuition and previous mistakes to distinguish situations in which they can safely apply this calculus from the ones where that would lead to incorrect results.

How can we formalize this? Note that the operation of taking the integral of the Dirac delta multiplied with another function $\varphi$ can be considered as a map from that function to a scalar in $\mathbb{C}$. Thus, we could consider

$$\delta_{\boldsymbol{x}}(\varphi) = \int\limits_{\mathbb{R}^n} \delta(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\,\mathrm{d}\,\boldsymbol{y}$$

to be a functional, and more specifically an element of the dual space of whatever class of functions we can allow for $\varphi$. But the question is: What restrictions do we need on this space of functions functions if we want to find objects like the Dirac delta in its dual space?

Intuition tells us that the smaller, i.e. more restricted, a space of function is, the bigger its dual space should be. Thus, we will investigate a space of functions $\varphi(\boldsymbol{x})$ that vanish quickly as $|\boldsymbol{x}| \to \infty$ ("converge with a vengeance"), hoping to find the Dirac delta in its dual.

First, a few definitions used throughout this course:

$$\begin{gathered} C^\infty(\mathbb{R}^n) \coloneqq \{\varphi : \mathbb{R}^n \to \mathbb{C} | \forall \boldsymbol{\alpha} \in \mathbb{N}^n : \partial^{\boldsymbol{\alpha}}\varphi \in C(\mathbb{R}^n)\} \\ \text{where } \mathbb{N} \coloneqq \{0,1,2,\ldots\}, \quad \boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n, \\ |\boldsymbol{\alpha}| \coloneqq \alpha_1 + \ldots + \alpha_n, \quad \partial^{\boldsymbol{\alpha}} \coloneqq \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial_{x_1}{}^{\alpha_1}\ldots\partial_{x_n}{}^{\alpha_n}}, \\ \text{and } \boldsymbol{x}^{\boldsymbol{\beta}} \coloneqq x_1^{\beta_1}\ldots x_n^{\beta_n} \quad (\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{\beta} \in \mathbb{N}^n) \end{gathered}$$

Now we define the *Schwartz space*

$$\begin{gathered} \mathcal{S}(\mathbb{R}^n) \coloneqq \{\varphi \in C^\infty(\mathbb{R}^n) | \forall \boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{N}^n : \|\varphi\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} < \infty\}, \\ \text{where } \|\varphi\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} \coloneqq \sup_{\boldsymbol{x}\in\mathbb{R}^n} |\boldsymbol{x}^{\boldsymbol{\alpha}}\partial_{\boldsymbol{x}}{}^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x})|. \end{gathered}$$

Note that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ implies $\lim_{|\boldsymbol{x}|\to\infty}(1+|\boldsymbol{x}|^2)^k\partial^{\boldsymbol{\alpha}}\phi(x) = 0$, as

$$\begin{aligned} (1+|\boldsymbol{x}|^2)^m\partial^{\boldsymbol{\alpha}}\phi(x) &= \sum_{j=0}^{m}\binom{m}{j}|\boldsymbol{x}|^{2j}\partial^{\boldsymbol{\alpha}}\varphi \\ &= \sum_{j=0}^{m}\binom{m}{j}\left(\sum_{i=1}^{n}x_i^2\right)^j\partial^{\boldsymbol{\alpha}}\varphi \\ &= \sum_{j=0}^{m}\binom{m}{j}\sum_{|\boldsymbol{\alpha}|=j}\left(\prod_{i=1}^{n}x_i^{2\alpha_i}\right)\partial^{\boldsymbol{\alpha}}\varphi \end{aligned}$$

and $\sup_{\boldsymbol{x}\in\mathbb{R}^n}\left|\boldsymbol{x}^{2\boldsymbol{\alpha}}\varphi(\boldsymbol{x})\right| < \infty$. It immediately follows:

**Lemma.** *If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\forall k \geq 0, \boldsymbol{\alpha} \in \mathbb{N}^n : \exists C_{\boldsymbol{\alpha}} \in \mathbb{R} : |\partial^{\boldsymbol{\alpha}}\varphi(x)| < \frac{C_\alpha}{(1+|\boldsymbol{x}|^2)^k}.$$

We define that the sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ *converges to* $\phi \in \mathcal{S}(\mathbb{R}^n)$ iff $\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^N : \lim_{k\to\infty}\|\varphi_k - \varphi\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$, and note that $\mathcal{S}(\mathbb{R}^n)$ is a complex vector space.

Finally, we define (or rather: state without proof) that $U \in \mathcal{S}^*(\mathbb{R}^n)$, the space of *templered distributions*, iff

1. $U$ is a linear map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathbb{C}$.

2. $U$ is continuous on $\mathcal{S}(\mathbb{R}^n)$, i.e. if $\varphi_k \overset{\mathcal{S}(\mathbb{R}^n)}{\to} \varphi$ then $\lim_{k\to\infty} U(\varphi_k) = U(\varphi)$.

For example, define

$$L^1_{loc}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} | \forall \boldsymbol{x} \in \mathbb{R}^n \exists \epsilon > 0 : \int\limits_{B_\epsilon(\boldsymbol{x})} |f(\boldsymbol{y})| \,\mathrm{d}\, y < \infty\}$$

and let $f \in L^1_{loc}$. If additionally $\sup_{\boldsymbol{x}\in\mathbb{R}^n}\left(1+|\boldsymbol{x}|^2\right)^{-s}|f(x)| < \infty$ for some $s > 0$ (e.g. $\sum\limits_{|\boldsymbol{\beta}|\leq s} c_{\boldsymbol{\beta}} x^{\boldsymbol{\beta}}$), then $U_f(\varphi) := \int\limits_{\mathbb{R}^n} f(\boldsymbol{x})\varphi(\boldsymbol{x}) \,\mathrm{d}\, x$ is a tempered distribution. Another example is the *Dirac delta distribution* as defined above, with $\delta(\varphi) = \varphi(0)$, $\delta_{\boldsymbol{x}}(\varphi) = \varphi(\boldsymbol{x})$.

Distributions have the nice property that we can always differentiate them: Let $U \in \mathcal{S}^*(\mathbb{R}^n)$. Then $\partial^{\boldsymbol{\alpha}} U$ is the distribution defined by $(\partial^{\boldsymbol{\alpha}} U)(\varphi) := (-1)^{|\boldsymbol{\alpha}|} U(\partial^{\boldsymbol{\alpha}}\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$. This is natural, as

$$U_{\partial^{\boldsymbol{\alpha}} f}(\varphi) = \int\limits_{\mathbb{R}^n} (\partial^{\boldsymbol{\alpha}} f(\boldsymbol{x}))\varphi(\boldsymbol{x}) \,\mathrm{d}\, \boldsymbol{x} \underset{\text{int. by parts}}{=} (-1)^{|\boldsymbol{\alpha}|} \int\limits_{\mathbb{R}^n} f(\boldsymbol{x})\partial^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x}) \,\mathrm{d}\, \boldsymbol{x}.$$

Let's find the derivative of $|\cdot|$ in the distribution world:

$$\begin{aligned}
\mathrm{Abs}(\varphi) &:= \int\limits_{-\infty}^{\infty} |x|\varphi(x) \,\mathrm{d}\, x \\
\left(\frac{\mathrm{d}}{\mathrm{d}\, x}\mathrm{Abs}\right)(\varphi) &= -\frac{\mathrm{d}}{\mathrm{d\,x}}\mathrm{Abs} \\
&= -\int_0^\infty x\varphi'(x) \,\mathrm{d}\, x - \int_{-\infty}^0 (-x)\varphi'(x) \\
&= -\left. x\varphi(x)\right|_0^\infty + \int_0^\infty \varphi(x) \,\mathrm{d}\, x + \left. x\varphi(x)\right|_{-\infty}^0 - \int_{-\infty}^0 \varphi(x) \,\mathrm{d}\, x \\
&= \int_{-\infty}^\infty h(x)\varphi(x) \,\mathrm{d}\, x = U_h(\varphi),
\end{aligned}$$

$$\text{where } h(x) := \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \text{ is the step function.}$$

We define $H(\varphi) := \int_{-\infty}^\infty h(x)\varphi(x) \,\mathrm{d}\, x$ and write $\frac{\mathrm{d}}{\mathrm{d}\, x}\mathrm{Abs} = H$ (physicist-speak: $\frac{\mathrm{d}}{\mathrm{d}\, x}|x| = h(x)$).

What about $\Delta\frac{1}{|\boldsymbol{x}|}$? By defining the *Newtonian distribution*

$$N(\varphi) := \int\limits_{\mathbb{R}^3} \frac{1}{|\boldsymbol{x}|}\varphi(\boldsymbol{x}) \,\mathrm{d}\, \boldsymbol{x},$$

we get

$$\Delta N(\varphi) = \sum_{j=1}^3 \frac{\partial^2}{\partial {x_j}^2} N(\varphi) = \sum_{j=1}^3 N\left(\frac{\partial^2\varphi}{\partial {x_j}^2}\right) = \ldots = -4\pi\varphi(0),$$

or in short

$$\Delta N = -4\pi\delta.$$

# 2 General properties of the Fourier transform

*Here begins the lecture of 2013-03-06.*

**Lemma** ("Your life in Fourier land depends on it.").

$$\begin{aligned}\widehat{\varphi+\psi} &= \widehat{\varphi}+\widehat{\psi},\\ \widehat{\lambda\varphi} &= \lambda\widehat{\varphi}\end{aligned} \tag{1}$$

$$|\widehat{\varphi}(\boldsymbol{k})| \le \frac{1}{(2\pi)^{\frac{n}{2}}}\|\varphi\|_1 < \infty, \qquad \|\varphi\|_1 \coloneqq \int\limits_{\mathbb{R}^n} |\varphi(\boldsymbol{x})|\,\mathrm{d}\,\boldsymbol{x} \tag{2}$$

$$\widehat{\varphi(\lambda\,\cdot\,)}(\boldsymbol{k}) = \frac{1}{|\lambda|^n}\widehat{\varphi}\left(\frac{\boldsymbol{k}}{\lambda}\right), \qquad \lambda \ne 0 \tag{3}$$

$$\widehat{T_{\boldsymbol{y}}\varphi}(\boldsymbol{x}) = \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{y}}\,\widehat{\varphi}(\boldsymbol{k}), \qquad (T_{\boldsymbol{y}}\varphi)(\boldsymbol{x}) \coloneqq \varphi(\boldsymbol{x}+\boldsymbol{y}) \tag{4}$$

$$\begin{aligned}\widehat{\frac{\partial\varphi}{\partial x_j}}(\boldsymbol{k}) &= \mathrm{i}k_j\widehat{\varphi}(\boldsymbol{k})\\ \widehat{x_j\varphi}(\boldsymbol{k}) &= \mathrm{i}\frac{\partial\widehat{\varphi}}{\partial k_j}(\boldsymbol{k})\end{aligned} \tag{5}$$

$$\int\limits_{\mathbb{R}^n} \widehat{\varphi}(\boldsymbol{y})\psi(\boldsymbol{y})\,\mathrm{d}\,\boldsymbol{y} = \int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{y})\widehat{\psi}(\boldsymbol{y})\,\mathrm{d}\,\boldsymbol{y} \tag{6}$$

*Proof.* Proof of (2):

$$|\widehat{\varphi}(\boldsymbol{k})| = \left|\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\,\boldsymbol{x}\right| \le \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\left|\varphi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\right|\mathrm{d}\,\boldsymbol{x} = \frac{1}{(2\pi)^{\frac{n}{2}}}\underbrace{\int\limits_{\mathbb{R}^n}|\varphi(\boldsymbol{x})|}_{\|\varphi\|_1}\underbrace{\left|\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\right|}_{1}\mathrm{d}\,\boldsymbol{x}$$

Proof of existence of the integral $\|\varphi\|_1$:

$$\begin{aligned}\int\limits_{\mathbb{R}^n}|\varphi(\boldsymbol{x})|\,\mathrm{d}\,\boldsymbol{x} &= \int\limits_{\mathbb{R}^n}\left(1+|\boldsymbol{x}|^2\right)^{\frac{s}{2}}|\varphi(\boldsymbol{x})|\frac{\mathrm{d}\,\boldsymbol{x}}{\left(1+|\boldsymbol{x}|^2\right)^{\frac{s}{2}}}\\ &\le \|\varphi\|_{s,\boldsymbol{0}}\int\limits_{\mathbb{R}^n}\frac{1}{\left(1+|\boldsymbol{x}|^2\right)^{\frac{s}{2}}}\,\mathrm{d}\,\boldsymbol{x}\\ &= \|\varphi\|_{s,\boldsymbol{0}}\int\limits_0^\infty \frac{r^{n-1}}{(1+r^2)^{\frac{s}{2}}}\,\mathrm{d}\,r < \infty \qquad \text{for } s \ge n+1.\end{aligned}$$

"(1) I will not do!" Proof of (3):

$$\begin{aligned}\widehat{\varphi(\lambda\,\cdot\,)} &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi(\lambda\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\,\boldsymbol{x}\\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{y})\mathrm{e}^{-\mathrm{i}\,\frac{1}{\lambda}\boldsymbol{k}\cdot\boldsymbol{y}}\,\mathrm{d}\left(\frac{\boldsymbol{y}}{\lambda}\right) \qquad \text{where } \boldsymbol{x} = \frac{1}{\lambda}\boldsymbol{y}.\\ &= \frac{1}{|\lambda|^n}\underbrace{\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{y})\mathrm{e}^{-\mathrm{i}\,\frac{\boldsymbol{k}}{\lambda}\cdot\boldsymbol{y}}\,\mathrm{d}\,\boldsymbol{y}}_{\widehat{\varphi}\left(\frac{\boldsymbol{k}}{\lambda}\right)}\end{aligned}$$

Proof of the second part of (5):

$$\begin{aligned}
\widehat{x_j\varphi}(\boldsymbol{x}) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n} x_j\varphi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\,\boldsymbol{x} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{x})\mathrm{i}\frac{\partial}{\partial k_j}\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\,\boldsymbol{x} && \text{as } \frac{\partial}{\partial k_j}\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}} = \frac{\partial}{\partial k_j}\mathrm{e}^{-\mathrm{i}\sum_{r=0}^{n} k_j x_j} = -\mathrm{i}x_j\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}} \\
&= \mathrm{i}\frac{\partial}{\partial k_j}\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n} \varphi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\,\boldsymbol{x} \\
&= \mathrm{i}\frac{\partial\widehat{\varphi}}{\partial k_j}(\boldsymbol{k})
\end{aligned}$$

*The professor starts whistling some elevator music while waiting for a student to write down the proof before he can wipe the board. When the student is done, the professor notices that there still is some room left on the blackboard and starts writing the rest there, without erasing anything.*

Proof of 6:

$$\begin{aligned}
\int\limits_{\mathbb{R}^n}\widehat{\varphi}(\boldsymbol{y})\psi(\boldsymbol{y})\,\mathrm{d}\,\boldsymbol{y} &= \int\limits_{\mathbb{R}^n}\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{y}\cdot\boldsymbol{x}}\,\mathrm{d}\,\boldsymbol{x}\,\psi(\boldsymbol{y})\,\mathrm{d}\,\boldsymbol{y} \\
&= \int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{x})\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{y}}\,\psi(\boldsymbol{y})\,\mathrm{d}\,\boldsymbol{y}\,\mathrm{d}\,\boldsymbol{x} && \text{As } \textstyle\int_{\mathbb{R}^{2n}}\left|\varphi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{y}\cdot\boldsymbol{x}}\,\psi(\boldsymbol{y})\right|\mathrm{d}\,\boldsymbol{x}\,\mathrm{d}\,\boldsymbol{y} < \infty\text{, we can use Fubini's theorem.} \\
&= \int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{y})\widehat{\psi}(\boldsymbol{y})\,\mathrm{d}\,\boldsymbol{y}
\end{aligned}$$

□

**Proposition.** *The Fourier transform* $\widehat{\cdot}$ *is a continuous bijective linear map from the Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ *to itself.* $\check{\widehat{\varphi}} = \varphi$*, where* $\check{\psi}(\boldsymbol{x}) \coloneqq \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\psi(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\,\boldsymbol{x}$*, read "unhat", "bird", "seagull", or whatever you like.*

*Proof.*

$$\begin{aligned}
\varphi \in \mathcal{S}(\mathbb{R}^n) &\Rightarrow \boldsymbol{x}^{\boldsymbol{\alpha}}\partial_{\boldsymbol{x}}{}^{\boldsymbol{\beta}} \in \mathcal{S}(\mathbb{R}^n) \\
\left(\widehat{\boldsymbol{x}^{\boldsymbol{\alpha}}\partial_{\boldsymbol{x}}{}^{\boldsymbol{\beta}}\varphi}\right)(\boldsymbol{k}) &= \mathrm{i}^{|\boldsymbol{\alpha}|}\partial_{\boldsymbol{k}}{}^{\boldsymbol{\alpha}}\left(\widehat{\partial_{\boldsymbol{x}}{}^{\boldsymbol{\beta}}\varphi}\right)(\boldsymbol{k}) && \text{from the second equality in (5), applied } |\boldsymbol{\alpha}| \text{ times.} \\
&= \mathrm{i}^{|\boldsymbol{\alpha}|}\partial_{\boldsymbol{k}}{}^{\boldsymbol{\alpha}}\left(\mathrm{i}^{|\boldsymbol{\beta}|}\boldsymbol{k}^{\boldsymbol{\beta}}\widehat{\varphi}\right)(\boldsymbol{k}) \\
&= \mathrm{i}^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}\partial_{\boldsymbol{k}}{}^{\boldsymbol{\alpha}}\left(\boldsymbol{k}^{\boldsymbol{\beta}}\widehat{\varphi}\right)(\boldsymbol{k})
\end{aligned}$$

"I should have done it the other way around. [...] Let's try it the other way around." We want:

$$\forall\boldsymbol{\alpha},\boldsymbol{\beta},\ \sup_{\boldsymbol{k}\in\mathbb{R}^n}\left|\boldsymbol{k}^{\boldsymbol{\alpha}}\partial_{\boldsymbol{k}}{}^{\boldsymbol{\beta}}\widehat{\varphi}(\boldsymbol{k})\right| < \infty$$

As that implies $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$.

$$\begin{aligned}
(\widehat{\partial_{\boldsymbol{x}}{}^{\boldsymbol{\alpha}}(\boldsymbol{k}^{\boldsymbol{\beta}}\varphi)})(\boldsymbol{k}) &= \mathrm{i}^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}\boldsymbol{k}^{\boldsymbol{\alpha}}\partial_{\boldsymbol{k}}{}^{\boldsymbol{\beta}}\widehat{\varphi}(\boldsymbol{k}) && \text{by applying the first part of (5) } |\boldsymbol{\alpha}| \text{ times and the second part } |\boldsymbol{\beta}| \text{ times.} \\
\forall\boldsymbol{\alpha},\boldsymbol{\beta},\left|\boldsymbol{k}^{\boldsymbol{\alpha}}\partial_{\boldsymbol{k}}{}^{\boldsymbol{\beta}}\widehat{\varphi}(\boldsymbol{k})\right| &\le \frac{1}{(2\pi)^{\frac{n}{2}}}\left\|\partial_{\boldsymbol{x}}{}^{\boldsymbol{\beta}}\left(\boldsymbol{x}^{\boldsymbol{\beta}}\varphi\right)\right\|_1 < \infty && \text{from (2).}
\end{aligned}$$

We now know $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$. Let us prove $\check{\widehat{\varphi}} = \varphi$. "When you do the wrong thing I'm gonna scream loudly."

$$\begin{aligned}
\check{\widehat{\varphi}}(\boldsymbol{x}) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\widehat{\varphi}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\,\boldsymbol{k} \\
&= \frac{1}{(2\pi)^{n}}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{y})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{y}}\,\mathrm{d}\,\boldsymbol{y}\,\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\,\boldsymbol{k}
\end{aligned}$$

We don't interchange the integrals here because that would lead to ugly calculations. "If pou paint the walls before you start building, it's not a good idea." At this point, somebody suggests replacing $\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,\mathrm{d}\,\boldsymbol{k}$ by $\delta(*)\boldsymbol{x}-\boldsymbol{y}$. "It's plausible! — WHY? [...] Ah I said it, so it's plausible." However, we would need to do some nasty calculations in order to do this. "How are we going to do it so fast that you don't get bored, and yet in enough detail that he's convinced? Be sneaky."

$$\begin{aligned}\check{\hat{\varphi}} &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n} \widehat{\varphi}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\underbrace{\lim_{\epsilon\downarrow 0}\mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}}}_{\substack{\text{1 written in}\\ \text{some other way}}}\,\mathrm{d}\,\boldsymbol{k} \\ &= \lim_{\epsilon\downarrow 0}\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\widehat{\varphi}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}}\end{aligned}$$

"Don't worry." This is actually a one-liner using Lebesgue's dominated convergence theorem.

*The 10-minute break ends with the loud noise of a metallic pointing stick hitting the desk.*

$$\begin{aligned}\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\widehat{\varphi}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}}\,\mathrm{d}\,\boldsymbol{k} &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{y})\mathrm{e}^{-\mathrm{i}\,\boldsymbol{y}\cdot\boldsymbol{k}}\,\mathrm{d}\,\boldsymbol{y}\,\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}}\,\mathrm{d}\,\boldsymbol{k} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{y})\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,\mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}}\,\mathrm{d}\,\boldsymbol{k}\,\mathrm{d}\,\boldsymbol{y}\end{aligned}$$

Recall: "every path leads to Rome and every Gaussian is in $\mathcal{S}(\mathbb{R}^n)$." Also, the last foot of a dactylic hexameter is always a spondee.

$$\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}}\,\mathrm{e}^{\mathrm{i}\,(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{k}}\,\mathrm{d}\,\boldsymbol{k} = \frac{\mathrm{e}^{-\frac{1}{2\epsilon}|\boldsymbol{x}-\boldsymbol{y}|^2}}{\epsilon^{\frac{n}{2}}}$$

We therefore get:

$$\begin{aligned}\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\widehat{\varphi}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}}\,\mathrm{d}\,\boldsymbol{k} &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{y})\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,\mathrm{e}^{-\epsilon\frac{|\boldsymbol{k}|^2}{2}}\,\mathrm{d}\,\boldsymbol{k}\,\mathrm{d}\,\boldsymbol{y} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi(\boldsymbol{y})\frac{\mathrm{e}^{-\frac{1}{2\epsilon}|\boldsymbol{x}-\boldsymbol{y}|^2}}{\epsilon^{\frac{n}{2}}}\,\mathrm{d}\,\boldsymbol{y} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi\big(\boldsymbol{y}\sqrt{\epsilon}+\boldsymbol{x}\big)\frac{\mathrm{e}^{-\frac{1}{2\epsilon}\left|\boldsymbol{y}\sqrt{\epsilon}\right|^2}}{\epsilon^{\frac{n}{2}}}\,\mathrm{d}\big(\boldsymbol{y}\sqrt{\epsilon}\big) && \text{“Epsilons everywhere!”} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n}\varphi\big(\boldsymbol{x}+\boldsymbol{y}\sqrt{\epsilon}\big)\mathrm{e}^{-\frac{1}{2}|\boldsymbol{y}|^2}\,\mathrm{d}\,\boldsymbol{y} && \text{“Now we can paint.”} \\ &\overset{\epsilon\downarrow 0}{\to} \frac{1}{(2\pi)^{\frac{n}{2}}}\varphi(\boldsymbol{x})\int\limits_{\mathbb{R}^n}\mathrm{e}^{-\frac{1}{2}|\boldsymbol{y}|^2}\,\mathrm{d}\,\boldsymbol{y} = \varphi(\boldsymbol{x}).\end{aligned}$$

□

# 3 Eigenfunctions of the Fourier transform

"The main ideas of many things are right here." We seek to find the solutions $(\varphi,\lambda)$ of

$$\widehat{\varphi} = \lambda\varphi.$$

We shall find all of them. "Not one is going to get away." Actually, we have already got one:

$$\widehat{\mathrm{e}^{-\frac{1}{2}|\boldsymbol{x}|^2}} = \mathrm{e}^{-\frac{1}{2}|\boldsymbol{x}|^2},$$

so we know that 1 is an eigenvalue. How do we find the others? "Here you actually have to have an idea."

**Idea** Find an $H \in \operatorname{End}(\mathcal{S}(\mathbb{R}^n))$ such that $[H, \widehat{\cdot}\,] = 0$. Then they have the same eigenspaces, so we just need to find the eigenfunctions of $H$ (see *Finite Dimensional Quantum Mechanics*).

Let $n = 1$.

$$\left.\begin{aligned} \widehat{(x^2\varphi)}(k) &= -\frac{\mathrm{d}^2}{\mathrm{d}\,k^2}\widehat{\varphi}(k) \\ \left(\widehat{\frac{\mathrm{d}^2}{\mathrm{d}\,x^2}\varphi}\right)(k) &= -k^2\widehat{\varphi}(k) \end{aligned}\right\} \qquad \text{From the lemma your life depends on.}$$

Subtracting the second equality from the first one above:

$$\mathcal{F}\left(\left(-\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + x^2\right)\varphi\right)(k) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}\,k^2} + k^2\right)\mathcal{F}\,\varphi(k),$$

where $\mathcal{F} = \widehat{\cdot}$ because things are getting a bit too big for the hat. We smell a harmonic potential. "Don't get your fingers too close, they sometimes bite."

Let $H \coloneqq \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}\,x^2} + x^2 - 1\right)$. Then $[H, \widehat{\cdot}\,] = 0$. We multiplied by $\frac{1}{2}$ so that it looks even more like a harmonic potential (it is the Hamiltonian of the quantum harmonic oscillator). Why did we add $-1$? Define

$$\begin{aligned} A &\coloneqq \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}\,x} + x\right), \\ A^\dagger &\coloneqq \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}\,x} + x\right). \end{aligned}$$

We then have $H = A^\dagger A$. If $a$ and $b$ commute, then we have $(a-b)(a+b) = a^2 - b^2$, but here $x$ and $\frac{\mathrm{d}}{\mathrm{d}\,x}$ don't quite commute, hence the $-1$.

*Here ends the lecture of 2013-03-06.*