

# On the number of crossing-free perfect matchings with designated left and right endpoints

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## Abstract

[TODO main results: the spc bound, the bound dependent on the number of interior points. Do I mention the expressions for the Catalan triangle (are they known)?]

## Introduction

In the study of plane embeddings of planar graphs, also known as plane graphs, embeddings with non-crossing straight edges (called crossing-free geometric graphs) are of particular interest. For instance, it is known [10] that any optimal solution to the travelling salesman problem in the Euclidean plane has non-crossing straight edges. In 1936, K. Wagner proved<sup>1</sup> [CITE] that any planar graph has such an embedding.

A natural enumerative question is then, given a vertex set of  $n$  points in the Euclidean plane, to bound the number of plane graphs of a certain sort (cycles, matchings, etc.) which can be drawn with non-crossing straight edges. For spanning cycles, the question was introduced by M. Newborn and W. O. J. Moser in 1976 [6], and a super-exponential upper bound was given. In 1980, M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi [1] proved that there are only exponentially many crossing-free geometric graphs on a given point set.

Exponential bounds for various sorts of graphs were then improved, in particular for triangulations—the maximal crossing-free geometric graphs. One of the interesting features of triangulations is that finding a bound for them yields a bound for the overall number of crossing-free geometric graphs. [TODO citation] In addition, bounds on the number of triangulations are of interest in geometric modeling [CITE Studies in computational geometry motivated by mesh generation].

Some special configurations of points are better understood; in particular, for point sets in convex position, the numbers of crossing-free perfect matchings, triangulations, and spanning cycles are known. In fact, the study of point sets in convex position vastly predates the general problem, with the number of triangulations given by Euler in 1751 [4].

For perfect matchings, an optimal lower bound of  $C_{\frac{n}{2}}$  was obtained as early as 1995 by A. García, M. Noy, and J. Tejel [5]; upper bounds have been harder to come by, with the best one yet being asymptotically  $10.05^n$ , given by M. Sharir and E. Welzl in 2005 [9] In that same work, an upper bound of  $5.83^n$  is given for the number of crossing-free perfect matchings on a point set whose points are designated as left or right endpoints; moreover, an upper bound of  $4^n$  is given when all the left endpoints are to the left of the right endpoints (matchings across a line).

In this thesis, we start by giving an overview of some of the existing results, in particular a way to classify arrangements of designated left and right endpoints, and we restate the proof of the bound for matchings across a line by M. Sharir and E. Welzl (section 1). We then study in detail the inequalities in that bound and the objects it actually counts, yielding criteria that will allow us to tighten the bound in specific

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<sup>1</sup>This result was independently proved by I. Fáry in 1948 [CITE], and by S. K. Stein in 1951 [CITE]; it is generally known as Fáry's theorem.

cases (section 2). Using these criteria, we give a bound for matchings across a line that depends on the number of points in the interior of the convex hull (section 3). We then give new bounds on the number of crossing-free perfect matchings on a point set with designated left and right endpoints for a specific family of left-to-right orderings of the left and right endpoints (4).

## 1 Background

We first give some definitions, as well as existing proofs whose ideas we will use to derive our results.

### 1.1 Geometric graphs

**Definition (geometric graph).** Given a set of points  $\mathcal{P}$  in the Euclidean plane  $\mathbb{R}^2$ , a *geometric graph* is a collection of straight line segments (edges) whose endpoints are elements of  $\mathcal{P}$ .

It can be described as a simple graph (in the combinatorial sense) on the vertices  $\mathcal{P}$ , where the edge  $\{v, w\}$  corresponds to the segment joining  $v$  and  $w$ .  $\diamond$

**Definition (crossing-free).** A geometric graph is *crossing-free* if no two edges share points other than their endpoints; it is called *crossing* otherwise.  $\diamond$

Note that being crossing-free implies that the corresponding simple graph is planar, and that the geometric graph is a plane embedding.

**Definition (triangulation).** A *triangulation* is a maximal crossing-free geometric graph, that is, geometric graph such that for all  $v$  and  $w$  in  $\mathcal{P}$  that are not joined by a segment, adding the segment joining  $v$  and  $w$  would result in a crossing geometric graph.  $\diamond$

Note that the faces (in the sense of plane graphs) formed by a triangulation are all triangles, with the possible exception of the outer face (thus this definition is *not* equivalent to that of a triangulation of the 2-sphere).

Since a geometric graph corresponds to a simple graph on the underlying point set, we can also look at geometric graphs that belong to special classes of simple graphs. As we will only be focusing on geometric graphs, we simply call *perfect matching* a geometric graph which corresponds to a perfect matching in the combinatorial sense.

As outlined in the introduction, we are interested in statements regarding the number of geometric graphs, possibly restricted to a certain type, on a given point set. Evidently, that number would depend on the choice of the point set  $\mathcal{P}$ , so instead we seek bounds on that number depending on the cardinality  $|\mathcal{P}|$ , and possibly restricting  $\mathcal{P}$  so that it satisfies certain properties.

An important property that  $\mathcal{P}$  can have is general position. We say that a point set in the plane is *in general position* if no three points are collinear. In this we will only study point sets in general position.

In general, if  $g(\mathcal{P})$  is the number of geometric graphs of a certain sort on the point set  $\mathcal{P}$ , we will look for lower bounds  $l$  and upper bounds  $u$  of the form

$$\forall n \in \mathbb{N}, \forall \mathcal{P} \text{ such that } |\mathcal{P}| = n, l(n) \leq g(\mathcal{P}) \leq u(n),$$

where  $\mathcal{P}$  runs over all point sets that satisfy the relevant properties.

In addition, we will want asymptotics on such bounds  $l$  and  $u$ , to compare them with existing results; since the bounds are often exponential, we tend to ignore polynomial factors; we will thus write that

$$f(n) \preccurlyeq u(n)$$

if  $f(n) \leq p(n)u(n)$  for some polynomially-bounded  $p$ .

## 1.2 Triangulations of convex point sets and the Catalan numbers

A set of points is said to be *in convex position* if it is in general position and all points are in the boundary of its convex hull.

For points in convex position, the numbers of triangulations and perfect matchings are known. The following recurrence, given by J. A. Segner in 1758 [8], defines the *Catalan numbers*.

**Theorem (Segner).** *Let  $C_n$  be the number of triangulations of a set of  $n + 2$  points in convex position. Then  $C_0 = 1$ , and for  $n > 0$ ,*

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}. \quad (1.1)$$

**Proof.** Consider a set of  $n + 2$  points in convex position. For  $n = 0$  the result is trivial, the only triangulation is an edge.

Let  $n > 0$ , and number the points along the boundary of the convex hull, from 0 to  $n + 1$ . The edge  $e$  joining points  $n + 1$  and  $(n)$  is in the boundary of the convex hull, and thus must be in any triangulation, since no other edge can cross it. Let  $i$  be the index of the third point of the triangle containing edge  $e$ , which may be any of the others, from 0 to  $n - 1$ .

Then, restricting the triangulation to the  $i + 2$  points  $\{0, \dots, i, n + 1\}$  yields a triangulation of those points. Similarly, restricting the triangulation of the whole point set to points  $i$  through  $n$  yields a triangulation of  $n - i + 1$  points. It follows that  $C_n \leq \sum_{i=0}^{n-1} C_i C_{n-i-1}$ .

Moreover, given a triangulation of  $\{0, \dots, i, n + 1\}$  and a triangulation of  $\{i, \dots, n\}$ , adding the edge  $e$  forms a triangulation of the whole point set, proving equality.  $\square$

An explicit expression for  $C_n$  was given earlier by Euler [4], along with the generating function. A convenient form is

$$C_n = \frac{1}{n+1} \binom{2n}{n}, \quad (1.2)$$

yielding asymptotically

$$C_n \sim \frac{4^n}{n^{\frac{3}{2}} \sqrt{\pi}} \leq 4^n. \quad (1.3)$$

## 1.3 Crossing-free perfect matchings

We will call  $\text{PM}_{\mathcal{P}}$  the set of perfect matchings on the point set  $\mathcal{P}$ , and  $\text{CFPM}_{\mathcal{P}}$  the set of crossing-free perfect matchings on the point set  $\mathcal{P}$ .

[TODO talk about the number of crossing-free perfect matchings in convex position, give the proof; find the Alfred Errera thing from the octavo of the académie royale de belgique, reference 10 in Sharir–Welzl; this might require a trip to the library?]

For a set  $\mathcal{P}$  of  $n$  points in general position, it was proved by Micha Sharir and Emo Welzl in 2005 [9] that  $|\text{CFPM}_{\mathcal{P}}| \leq 10.05^n$ .

When talking about a perfect matching, we will denote the unique edge incident to point  $p$  by  $e_p$ .

## 1.4 Bracket expressions and their relations to perfect matchings

We say that a point set is *in general position with respect to the horizontal* if it is in general position and no two points lie on a vertical line. Note that any point set in general position can be put in general position with respect to the horizontal by an arbitrarily small rotation. Moreover, note that points in general position with respect to the horizontal are ordered from left to right.

This ordering can be used to classify perfect matchings on the point set; to this end, we introduce bracket expressions. A *bracket expression* of length  $n$  is a sequence



**Theorem (Ruiz-Vargas–Welzl).** *Let  $\mathcal{P}$  be a point set of size  $n = 2k$  in general position with respect to the horizontal, and let  $B$  be a well-formed bracket expression of length  $n$ . Then there exists a crossing-free perfect matching consistent with  $B$ ; in other words,  $v_{\mathcal{P}}(B) \geq 1$ .*  $\square$

**Proof.** Let  $\mu_0$  be a perfect matching on  $\mathcal{P}$  consistent with  $B$ . This is always possible, for instance, parsing the bracket expression, match the point corresponding to an opening bracket and the point corresponding to the matching closing bracket.

Define  $\Lambda(\mu)$  for a perfect matching  $\mu$  on  $\mathcal{P}$  to be the sum of the lengths of the edges of  $\mu$ .

Then, repeat the following procedure, starting at  $i = 0$ . If there is no crossing in  $\mu_i$ , we have found a perfect matching with the desired properties. If there is a crossing, let  $v, v', w$ , and  $w'$  be the points involved, with  $v$  and  $v'$  being the left endpoints and  $w$  and  $w'$  being the right endpoints, so that the edge  $vw$  crosses the edge  $v'w'$ . Remove these edges, and replace them by  $vw'$  and  $v'w$  (thus untangling them). This yields another perfect matching  $\mu_{i+1}$ . By the triangle inequality (see figure 2),  $\Lambda(\mu_{i+1}) < \Lambda(\mu_i)$ .

If this procedure did not terminate, it would yield a sequence  $(\mu_i)_{i \in \mathbb{N}}$  of crossing perfect matchings on  $\mathcal{P}$  on which  $\Lambda$  is strictly decreasing, thus an infinite sequence of distinct graphs on  $\mathcal{P}$ . Since there are only finitely many graphs on  $\mathcal{P}$ , this is a contradiction, so we eventually<sup>4</sup> find a crossing-free perfect matching.  $\square$

This immediately yields a lower bound for the number of crossing-free perfect matchings, since there are  $C_{\frac{n}{2}}$  well-formed bracket expressions of length  $n$ . Note that this lower bound had already been proved by A. García, M. Noy, and J. Tejel in 1995, by a recurrence that split the point set, rather than via well-formed bracket expressions [5].

**Corollary.** *Let  $\mathcal{P}$  be a point set of even size  $n$  in general position. There are at least  $C_{\frac{n}{2}}$  distinct crossing-free perfect matchings on  $\mathcal{P}$ , i.e.,  $|\text{CFPM}_{\mathcal{P}}| \geq C_{\frac{n}{2}}$ .*  $\square$

Moreover, this lower bound is optimal, since it is attained if  $\mathcal{P}$  is in convex position (see section 1.3).

Now that we have a tight uniform lower bound for  $v_{\mathcal{P}}$ , we will start looking at upper bounds dependent on the bracket expression.

## 1.6 Matchings across a line

Again we consider  $n = 2k$  points in general position with respect to the horizontal. The matchings consistent with bracket expressions with  $k$  opening brackets followed by  $k$  closing brackets,  $\langle^k \rangle^k$ , are called *matchings across a line*. Indeed, any edge in such a matching will cross any vertical line that separates the left-points from the right-points.

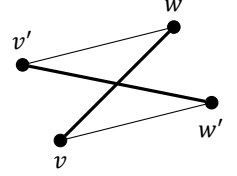
The following result, and its proof, were given by Micha Sharir and Emo Welzl in 2005 [9].

**Theorem (Sharir–Welzl).** *Let  $\mathcal{P}$  be a set of  $n = 2k$  points in general position with respect to the horizontal. Then there are at most  $C_{\frac{n}{2}}^2$  crossing-free perfect matchings across a line on  $\mathcal{P}$ , i.e.,  $v_{\mathcal{P}}(\langle^k \rangle^k) \leq C_k^2$ .*

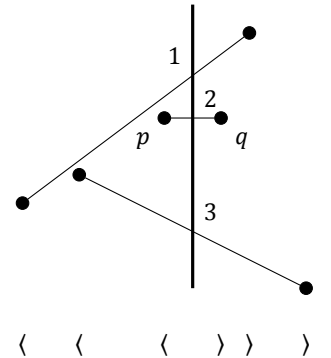
The idea of the proof is as follows. First, pick a vertical line that separates the left-points from the right-points; we will call it *the vertical line*. Further, let us call the set of left-points  $\mathcal{L}$  and the set of right-points  $\mathcal{R}$ .

A perfect matching consistent with a given well-formed bracket expression (and in particular, a perfect matching across a line) is uniquely defined by a bijection  $\mu : \mathcal{L} \rightarrow \mathcal{R}$  from the left-points to the right-points. Consider such a perfect matching across a line, and let  $E$  be the set of its edges. Now, number the intersections between the edges of the perfect matching and the vertical line from top to bottom. This yields a numbering  $\iota : E \rightarrow [k]$  of the edges.

<sup>4</sup>The untangling procedure itself predates this proof, and was shown to terminate in  $\mathcal{O}(n^3)$  steps by J. van Leeuwen and A. A. Schoone in 1980 [10].



**Figure 2.** Untangling in a perfect matching. Replacing the thick edges by the thin ones reduces the total edge length, while preserving left and right endpoints.



**Figure 3.** Constructing the numberings  $\lambda$  and  $\rho$  of the left- and right-points from a crossing-free perfect matching across a line; here  $\iota(\{p, q\}) = 2$ , and thus  $\lambda(p) = \rho(q) = 2$ .

Define  $\lambda(l) := \iota(e_l)$  mapping a left-point to the intersection number of its edge, and similarly  $\rho(r) := \iota(e_r)$  for the right-points. We have  $\mu = \rho^{-1} \circ \lambda$ .

The bijection  $\lambda$  (respectively  $\rho$ ) determines the order in which the left points (respectively right points) reach the vertical line.

If the matching is crossing-free, we will show that  $\lambda$  and  $\rho$  have to be in sets of size  $C_k$ , thus that there are at most  $C_k^2$  functions  $\mu = \rho^{-1} \circ \lambda$ , i.e., that there can be at most  $C_k^2$  crossing-free perfect matchings across a line.

Since we are going to reuse these concepts in subsequent sections, we will formalize and name the properties of  $\lambda$  and  $\rho$  that we will consider.

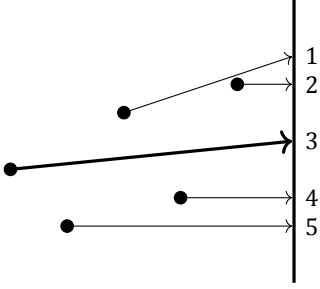
**Definition (crossing-free alignment).** Let  $\mathcal{S}$  be a set of  $k$  points in general position. Let  $V$  be a vertical line to the right of  $\mathcal{S}$ . A bijection  $\lambda$  from  $\mathcal{S}$  to  $[k]$  is a *crossing-free left-alignment* of  $\mathcal{S}$  on  $V$  if there exists a set  $A = \{a_p \mid p \in \mathcal{S}\}$  of  $k$  line segments such that the following hold:

1. for every  $p \in \mathcal{S}$ , the left endpoint of  $a_p$  is  $p$ , and its right endpoint is on  $V$ ;
2. no two segments cross, and no two segments share any endpoints;
3. let  $\iota : A \rightarrow [k]$  the numbering of the right endpoints of the segments in  $A$  from top to bottom, then  $\lambda(p) = \iota(a_p)$ .

Such a set of segments is said to *realize*  $\lambda$ .

Correspondingly, we define a *crossing-free right-alignment*, where “right” and “left” are swapped in the definition.  $\diamond$

A crossing-free left-alignment of five points, with a realizing set of segments—drawn as arrows—is shown in figure 4



**Figure 4.** A crossing-free left-alignment of five points. Once the index of the leftmost point is chosen (thick segment), the rest consists of two crossing-free left-alignments of  $i$  and  $k-i-1$  points each—here  $k = 5$  and  $i = 2$ .

**Lemma.** Let  $\mathcal{S}$  be a set of  $k$  points in general position, and let  $V$  be a vertical line to the right of  $\mathcal{S}$ . There are at most  $C_k$  crossing-free left-alignments of  $\mathcal{S}$  on  $V$ .

**Proof.** Let  $l$  be the leftmost point of  $\mathcal{S}$ , and let  $\lambda$  be a crossing-free left-alignment of  $\mathcal{S}$  on  $V$ . Let  $A = \{a_p \mid p \in \mathcal{S}\}$  be a set of non-crossing line segments realizing  $\lambda$ . For a point  $p \in \mathcal{S}$ , we will call  $a_p$  the segment of  $p$ .

$\lambda(l)$ , being the index on the vertical line  $V$  of the right endpoint of the segment  $a_l$  of the leftmost point, is equal to one plus the number of points of  $\mathcal{S}$  that are above  $a_l$ . Indeed, the segments of points above  $a_l$  must themselves reach the vertical line  $V$  above  $a_l$ , otherwise they would cross  $a_l$ , and correspondingly for points below  $a_l$ , so that there are as many edges reaching the vertical line  $V$  below  $a_l$  as there are points below  $a_l$ .

Moreover, as the oriented angle between  $a_l$  and the horizontal increases, points are only added to the set of points below  $a_l$ , so that choosing the number of points below  $a_l$  determines the sets of points below and above  $a_l$ .

Further, since points above  $a_l$  must reach the vertical line  $V$  above  $a_l$ , and correspondingly for points below, if the point  $p$  is above  $a_l$ , then  $\lambda(p) < \lambda(l)$ , and if it is below,  $\lambda(p) > \lambda(l)$ . Thus  $\lambda$  restricted to the points above  $a_l$  is a crossing-free left-alignment of the  $\lambda(l) - 1$  points above  $a_l$ , and  $\lambda - \lambda(l)$  restricted to the points below  $a_l$  is a crossing-free left-alignment of the  $k - \lambda(l)$  points below  $a_l$ , where the appropriate subsets of  $A$  provide the set of segments required in the definition.

It follows that  $\lambda$  is determined by the choice of  $\lambda(l)$  and crossing-free left-alignments of  $\lambda(l) - 1$  and  $k - \lambda(l)$  points. Letting  $i := \lambda(l) - 1$ , these are crossing-free alignments of  $i$  and  $k - i - 1$  points (see figure 4). Therefore, if  $\varpi_j$  is an upper bound for the number of crossing-free left-alignments of  $j$  points when  $j < k$ , we can give an upper bound on the number of crossing-free left-alignments of  $k$  points,

$$\varpi_k := \sum_{i=0}^{k-1} \varpi_i \varpi_{k-i-1}.$$

We can start the recurrence with  $\varpi_0 = 1$ ; this is the recurrence for the Catalan numbers, thus  $\varpi_k = C_k$ .  $\square$

The same results holds for crossing-free right-alignments, simply exchange “left” and “right” in the above lemma and its proof.

**Proof (of the theorem).** If we have a crossing-free perfect matching across a line, then  $\lambda$  as constructed above is a crossing-free left-alignment of  $\mathcal{L}$  on the vertical line; indeed, letting  $a_p$  be the portion of the edge  $e_p$  to the left of the vertical line yields segments that realize  $\lambda$ . Similarly,  $\rho$  is a crossing-free right-alignment of  $\mathcal{R}$  on the vertical line.  $\square$

## 2 Analysing the overcounting in the upper bound for matchings across a line

The bound given above for the number of matchings across a line is not optimal; an example is given by Sharir and Welzl of a crossing-free left-alignment  $\lambda$  and a crossing-free right-alignment  $\rho$  such that  $\rho^{-1} \circ \lambda$  is not crossing-free (see figure 5).

There is however more to the overcounting than just counting some crossing matchings. To characterize that, it helps to name some functions and sets. Again, are considering a set  $\mathcal{P}$  of  $2k$  points in general position with respect to the horizontal and the bracket expression  $\langle^k \rangle^k$ . We let  $\mathcal{L}$  be the set of left-points, and  $\mathcal{R}$  be the set of right-points. Moreover, we pick a vertical line separating  $\mathcal{L}$  and  $\mathcal{R}$ , and call it  $V$ .

In the previous section, given a crossing-free perfect matching  $\mu$ , we constructed crossing-free left- and right-alignments  $\lambda$  and  $\rho$  (see figure 3). Let us call that construction  $\omega$ , thus  $\omega(\mu) = (\lambda, \rho)$ . Moreover, let us call  $\text{CFLA}_{\mathcal{L},V}$  the set of crossing-free left-alignments of  $\mathcal{L}$  on  $V$ , and  $\text{CFRA}_{\mathcal{R},V}$  the set of crossing-free left-alignments of  $\mathcal{R}$  on  $V$ . Given sets of equal sizes  $\mathcal{L}$  and  $\mathcal{R}$ ,  $\lambda \in \text{CFLA}_{\mathcal{L},V}$ , and  $\rho \in \text{CFRA}_{\mathcal{R},V}$ , let  $c(\lambda, \rho)$  be the (possibly crossing) perfect matching  $\rho^{-1} \circ \lambda$ . Finally, let us name the sets of perfect matchings across a line and of crossing-free perfect matchings across a line,

$$\begin{aligned} \text{PMAL}_{\mathcal{P}} &:= \mathfrak{B}_{\mathcal{P}}^{-1}(\langle^k \rangle^k) \\ \text{CFPMAL}_{\mathcal{P}} &:= \text{PMAL}_{\mathcal{P}} \cap \text{CFPM}_{\mathcal{P}}. \end{aligned}$$

Then the following diagram commutes, *i.e.*, constructing crossing-free alignments from a matching and then composing them yields the initial matching.

$$\begin{array}{ccc} \text{CFPMAL}_{\mathcal{P}} & \xrightarrow{\omega} & \text{CFLA}_{\mathcal{L},V} \times \text{CFRA}_{\mathcal{R},V} \\ \downarrow \subseteq & \swarrow c & \\ \text{PMAL}_{\mathcal{P}} & & \end{array}$$

The result of Sharir and Welzl consists in deducing from the commutative diagram that  $\omega$  must be injective, and so that

$$|\text{CFPMAL}_{\mathcal{P}}| \leq |\text{CFLA}_{\mathcal{L},V} \times \text{CFRA}_{\mathcal{R},V}|, \quad (2.1)$$

and then a bound for the number of crossing-free alignments yields a bound for the number of crossing-free matchings across a line.

The inequality (2.1) is strict, as shown in figure 5. We would thus like to study the pairs of crossing-free alignments which are *not* in the image of  $\omega$ , since they are responsible for that excess; we will call them *overcounted*. Overcounted pairs of crossing-free alignments fall into one of two categories;

1. evidently, if  $c(\lambda, \rho)$  is crossing, then  $(\lambda, \rho)$  is not in the image of  $\omega$ , since the diagram commutes;
2. on the other hand, figure 6 shows a pair of crossing-free alignments  $(\lambda, \rho)$  which composes to a crossing-free alignment  $\mu = c(\lambda, \rho)$ , but which is not equal to  $\omega(\mu)$ .

In addition, we will call a crossing-free left alignment  $\lambda$  overcounted if any pair  $(\lambda, \rho)$  is overcounted.

TODO

Figure 5. Composing crossing-free alignments, only to get a crossing perfect matching.

TODO

Figure 6. Composing crossing-free alignments, and getting a crossing-free perfect matching, but one that does not decompose to the original alignments.

To hunt down overcounted crossing-free alignments, we will examine in more detail the anatomy of a crossing-free left-alignment of  $\mathcal{L}$  and the proof of the Sharir–Welzl bound for the number of crossing-free left-alignments, with some fresh definitions.

First, for a point set  $\mathcal{L}$ , note that we can partition the half-plane to the right of  $\text{lm}(\mathcal{L})$  in  $|\mathcal{L}|$  regions, separated by rays emanating from  $\text{lm}(\mathcal{L})$  and going through other points of  $\mathcal{L}$ . We will call those regions *left  $\mathcal{L}$ -cones*, and number them from top to bottom from 1 to  $|\mathcal{L}|$ , see figure ?? . Note that there are  $i$  points of  $\mathcal{L}$  above the interior of left  $\mathcal{L}$ -cone number  $i + 1$ , and  $|\mathcal{L}| - i - 1$  points below.

Given a crossing-free left-alignment  $\lambda$  of  $\mathcal{L}$  on  $V$ , for any set of segments  $A = \{a_p \mid p \in L\}$  realizing  $\lambda$ , the segment  $a_{\text{lm}(\mathcal{L})}$  of the leftmost point then has to be contained in left  $\mathcal{L}$ -cone number  $\lambda(\text{lm}(\mathcal{L}))$ . As we remarked in the previous section, for a crossing-free perfect matching  $\mu$ , the edges of  $\mu$  restricted to the left side of  $V$  realize the left-alignment in  $\omega(\mu)$ . Thus if  $(\lambda, \rho)$  is in the image of  $\omega$ ,  $\text{lm}(\mathcal{L})$  has to be matched to a right-point in left  $\mathcal{L}$ -cone number  $\lambda(\text{lm}(\mathcal{L}))$ , see figure ?? .

To extend this reasoning beyond the leftmost point of  $\mathcal{L}$ , we define *left conifers of  $\mathcal{L}$* . A left conifer  $\kappa$  of a point set  $\mathcal{L}$  is a map from points in  $\mathcal{L}$  to left  $\mathcal{L}$ -cones for subsets  $S$  of  $\mathcal{L}$ , satisfying the following properties:

1.  $\kappa(\text{lm}(\mathcal{L}))$  is a left  $\mathcal{L}$ -cone;
2.  $\kappa$  restricted to the points above the interior of  $\kappa(\text{lm}(\mathcal{L}))$  is a left conifer of these points;
3.  $\kappa$  restricted to the points below the interior of  $\kappa(\text{lm}(\mathcal{L}))$  is a left conifer of these points.

See figure ?? for an example. We call  $\text{LC}_{\mathcal{L}}$  the set of left conifers of  $\mathcal{L}$ . From the observations above on the number of cones and the size of the sets of points above and below their interiors, if  $\varpi_k$  is the number of left conifers of  $k$  points, we have

$$\varpi_k = \sum_{i=0}^{k-1} \varpi_i \varpi_{k-i-1} \text{ and } \varpi_0 = 1,$$

so that  $|\text{LC}_{\mathcal{L}}| = C_{|\mathcal{L}|}$ .

Recall that a crossing-free left-alignment  $\lambda$  is determined by the value of  $\lambda(\text{lm}(\mathcal{L}))$  and by crossing-free left alignments of the points above and below any edge realizing  $\lambda$ , *i.e.*, above and below cone number  $\lambda(\text{lm}(\mathcal{L}))$ . This is equivalent to choosing left  $\mathcal{L}$ -cone number  $\lambda(\text{lm}(\mathcal{L}))$ , and recursively doing so in the point subsets above and below the interior of that cone, see figure ?? . This gives an injective map  $\mathfrak{K} : \lambda \mapsto \mathfrak{K}_{\lambda}$  from crossing-free left-alignments of  $\mathcal{L}$  on any line to left conifers. Note that it is not bijective, see figure ?? . In turn, this yields  $|\text{CFLA}_{\mathcal{L},V}| \leq |\text{LC}_{\mathcal{L}}| = C_{|\mathcal{L}|}$ , which is exactly the Sharir–Welzl bound.

Now, note that the map  $\mathfrak{K}$  has the property that for any set of segments  $A = \{a_p \mid p \in L\}$  realizing  $\lambda$ ,  $a_p$  lies inside the cone  $\mathfrak{K}_{\lambda}(p)$ ; since the edges of a matching  $\mu$  realize the crossing-free alignments  $\omega(\mu)$ ,  $\mathfrak{K}_{\lambda}(p)$  must contain the right-point to which  $p$  is matched.

### 3 Highly convex matchings across a line

In particular, this means that if there is no right-point in  $\mathfrak{K}_{\lambda}(p)$ , then  $\lambda$  is overcounted. We will call *vacuous* a left cone that does not contain any right-points, and we will call *vacuous* a left conifer that has vacuous cones; thus  $\lambda$  is overcounted if  $\mathfrak{K}_{\lambda}$  is *vacuous*.

We will now use convexity considerations to give an upper bound on the number of non-vacuous left conifers. Let us first define some concepts related to convex hulls. Given a set of points  $\mathcal{P}$  in general position, the *extreme points of  $\mathcal{P}$*  are the points of  $\mathcal{P}$  that are in the boundary of the convex hull of  $\mathcal{P}$ . Given a set of points  $\mathcal{L}$  in general position with respect to the horizontal and a set of points  $\mathcal{R}$  in general position with respect to the horizontal to the right of  $\mathcal{L}$ , the  *$\mathcal{R}$ -liminary points of  $\mathcal{L}$*  are the points



of  $\mathcal{L}$  which are extreme points of  $L \cup R$ , see figure ?? . We will call  $\text{lm}(\mathcal{L})$  the leftmost point of  $\mathcal{L}$ ; note that it is an  $\mathcal{R}$ -liminary point of  $\mathcal{L}$ .

Note that in polar order around  $\text{lm}(\mathcal{L})$ , the  $\mathcal{R}$ -liminary points of  $\mathcal{L}$  other than  $\text{lm}(\mathcal{L})$  are separated by  $\mathcal{R}$ , that is, in polar order, one subset of the  $\mathcal{R}$ -liminary points of  $\mathcal{L}$  appears first, followed by all points of  $\mathcal{R}$ , followed by the rest of the  $\mathcal{R}$ -liminary points of  $\mathcal{L}$  (except  $\text{lm}(\mathcal{L})$ ). We will call those  $\mathcal{R}$ -liminary points of  $\mathcal{L}$  which appear below  $\mathcal{R}$  the *lower  $\mathcal{R}$ -liminary points of  $\mathcal{L}$* , and those that appear above  $\mathcal{R}$  the *upper  $\mathcal{R}$ -liminary points of  $\mathcal{L}$* .

If there are  $\delta_1$  upper  $\mathcal{R}$ -liminary points of  $\mathcal{L}$ , then the first  $\delta_1$  left  $\mathcal{L}$ -cones are vacuous. Similarly, if there are  $\delta_2$  lower  $\mathcal{R}$ -liminary points of  $\mathcal{L}$ , then the last  $\delta_2$  left  $\mathcal{L}$ -cones are vacuous. Moreover, for any non-vacuous left  $\mathcal{L}$ -cone  $\mathcal{K}$ , any upper  $\mathcal{R}$ -liminary points of  $\mathcal{L}$  are  $\mathcal{R}$ -liminary points of the subset of  $\mathcal{L}$  above  $\mathcal{K}$ , and any lower  $\mathcal{R}$ -liminary points of  $\mathcal{L}$  are  $\mathcal{R}$ -liminary points of the subset of  $\mathcal{L}$  below  $\mathcal{K}$ , so that we get the recurrence

$$\varpi_{k,\delta} := \max_{\delta_1 + \delta_2 = \delta - 1} \sum_{i=\delta_1}^{k-\delta_2-1} \varpi_{i,\delta_1} \varpi_{k-i-1,\delta_2},$$

where  $\varpi_{k,\delta}$  is an upper bound on the number of crossing-free non-vacuous left conifers of  $k$  points of which at least  $\delta$  are  $\mathcal{R}$ -liminary. Moreover, we can start that recurrence with

$$\begin{aligned} \varpi_{k,0} &:= \varpi_{k,1}, \\ \varpi_{k,k} &:= 1, \end{aligned}$$

since there is at least one  $\mathcal{R}$ -liminary point of  $\mathcal{L}$  (the leftmost one), and if all points are  $\mathcal{R}$ -liminary, only one cone is non-vacuous.

## 4 Three changes of bracket direction

We now consider perfect matchings consistent with bracket expressions which have three changes of bracket directions, *i.e.*, bracket expressions of the form  $\langle^a \rangle^b \langle^c \rangle^d$ ,  $a$  opening brackets,  $b$  closing brackets,  $c$  opening brackets,  $d$  closing brackets, where  $a - b = d - c \geq 0$  and  $a + b + c + d = n$ .

In term of points, this means that four sets can be separated by vertical lines, from left to right,  $a$  left-points forming the set  $\mathcal{A}$ ,  $b$  right-points forming  $\mathcal{B}$ ,  $d$  left-points forming  $\mathcal{D}$ , and  $c$  right-points forming  $\mathcal{C}$ . We pick a vertical line separating  $\mathcal{A}$  and  $\mathcal{B}$  and call it  $V$ , and we pick a vertical line separating  $\mathcal{C}$  and  $\mathcal{D}$  and call it  $W$ , see figure 7.

Given a crossing-free perfect matching on those points, numbering from top to bottom the intersections between  $V$  and edges incident to the points in  $\mathcal{A}$ , we get a crossing-free left-alignment  $\alpha$  of  $\mathcal{A}$  on  $V$ .  $a - b$  of the  $a$  edges intersecting  $V$  are incident to points in  $\mathcal{D}$ ; the other  $b$  are incident to points in  $\mathcal{B}$ . Numbering those  $b$  edges yields a crossing-free right-alignment  $\beta$  of  $\mathcal{B}$  on  $V$ .

Similarly on the right side, we get a crossing-free right-alignment of  $\mathcal{D}$  on  $W$ , and numbering the subset of edges joining  $\mathcal{C}$  and  $\mathcal{D}$ , we get a left-alignment  $\gamma$  of  $\mathcal{C}$  on  $W$ . Call  $\mathcal{S}$  the set of  $a - b$  points of  $\mathcal{A}$  that are matched with points of  $\mathcal{D}$ , and

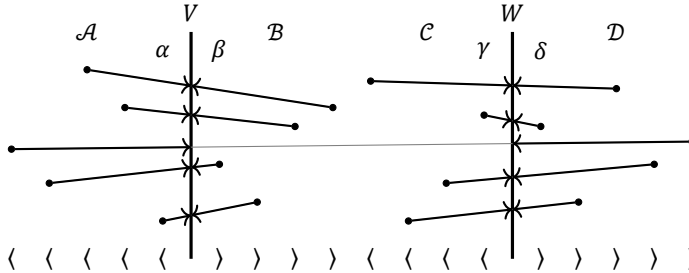


Figure 7. Four crossing-free alignments.

call  $\mathcal{T}$  the set of  $d - c$  points of  $\mathcal{D}$  that are matched to points of  $\mathcal{A}$ .

The matching is uniquely determined by  $\alpha$ ,  $\beta$ , the choice of the subsets  $\mathcal{S} \subseteq \mathcal{A}$  and  $\mathcal{T} \subseteq D$ , and by  $\beta$  and  $\chi$ . This gives the following bound for the number of these matchings:

$$v_{\mathcal{P}}(\langle a \rangle^b \langle c \rangle^d) \leq C_a \binom{a}{b} C_b C_c \binom{c}{d} C_d.$$

Asymptotically, the factor involving  $a$  and  $b$  is

$$C_a \binom{a}{b} C_b \leq 4^{a+b} \binom{a}{b} \leq 4^{a+b} \underbrace{\left( \left( \frac{(1-\eta)^{1-\frac{1}{\eta}}}{\eta} \right)^{\frac{\eta}{\eta+1}} \right)^{a+b}}_{x_{\eta}},$$

where  $a = \eta b$ . The base  $x_{\eta}$  of the exponential bound for the binomial coefficient is maximal when  $\eta = \frac{3-\sqrt{5}}{2}$ , where it is  $\phi = \frac{1+\sqrt{5}}{2}$ . This gives the overall bound

$$v_{\mathcal{P}}(\langle a \rangle^b \langle c \rangle^d) \leq 4^{a+b} \phi^{a+b} 4^{c+d} \phi^{c+d} = (4\phi)^n \approx 6.472^n.$$

Note that this is worse than the existing  $5.3793^n$  bound  $v_{\mathcal{P}}(B)$  for an arbitrary well-formed bracket expression  $B$ , see section 1.4.

#### 4.1 Improving the binomial bound

We can however improve upon that bound. Once  $\alpha$ ,  $\delta$ ,  $\mathcal{S}$ , and  $\mathcal{T}$  are fixed, the edges of the matching that cross both the left line and the right line—let us call these *long edges*—are determined. As a result, the strip  $\Sigma$  between the vertical lines  $V$  and  $W$  is partitioned in trapezoidal cells  $Z_1, \dots, Z_{a-b+1}$  by the long edges, see figure ??.

We will consider the crossing-free left-alignment  $\beta$ ; the same argument will hold for  $\gamma$ . Since  $\mathcal{B}$  lies within the strip  $\Sigma$ , it is partitioned in subsets  $\mathcal{B}_1 \subseteq Z_1, \dots, \mathcal{B}_{a-b+1} \subseteq Z_{a-b+1}$ . Moreover, an edge  $e_p$  emanating from a point  $p \in \mathcal{B}_i$  cannot cross the long edges, and thus the portion  $e_p \cap \Sigma$  of  $e_p$  that lies within the strip  $\Sigma$  is confined to the cell  $Z_i$ .

Let  $\beta_i$  be the crossing-free right-alignment of  $\mathcal{B}_i$  on  $V$  realized by the segments  $\{e_p \cap \Sigma \mid p \in \mathcal{B}_i\}$ . Since the set of segments  $\{e_p \cap \Sigma \mid p \in \mathcal{B}\}$  realizes  $\beta$ ,  $\beta$  is composed of the crossing-free right-alignments  $\beta_i$ , formally,

$$\beta|_{\mathcal{B}_i} = \beta(p) - \sum_{j=1}^{i-1} |\mathcal{B}_j|.$$

Let us look at the edges crossing  $V$ , numbered from top to bottom:  $k-l$  of those are long edges; they are the edges emanating from  $\mathcal{S}$ . Let  $S = \alpha(\mathcal{S}) \subseteq [k]$  be their numbers. In between two long edges, above the first long edge, and below the last one, we have the edges that define the crossing-free right-alignments that make up  $\lambda$ . It follows that a crossing-free right-alignment of  $m$  points that makes up  $\lambda$  corresponds to a maximal sequence of  $m$  consecutive elements of  $[k] \setminus S$ .

We will call the set of maximal sequences of consecutive elements of  $S'$  the *cells* of  $S'$ , written  $\text{cells}(S')$ .

Then, we can improve the  $\binom{k}{l} C_l$  factor in the bound (in which the binomial comes from the choice of  $\mathcal{S}$ , and the Catalan number comes from the choice of  $\beta$ ), summing over the choices of  $\mathcal{S}$  (and thus of  $S$  above). The improved factor becomes<sup>5</sup>

$$\text{spc}(k, l) := \sum_{S \in \binom{[k]}{k-l}} \prod_{\chi \in \text{cells}([k] \setminus S)} C_{|\chi|} = \sum_{S' \in \binom{[k]}{l}} \prod_{\chi \in \text{cells}(S')} C_{|\chi|}, \quad (4.1)$$

and the overall bound becomes

$$v_{\mathcal{P}}(\langle k \rangle^l \langle q \rangle^p) \leq C_k \text{spc}(k, l) \text{spc}(p, q) C_p. \quad (4.2)$$

<sup>5</sup>The name spc was chosen because we define this as a Sum of Products of Catalan numbers.

## 4.2 A recurrence

In order to compute  $\text{spc}$  efficiently, and eventually, get its asymptotics, it is useful to get rid of the cells function. We can express  $\text{spc}$  as a recurrence instead. First, we note that  $\text{spc}(k, k) = C_k$ : there is only one summand,  $S'$  is the whole set, so it has only one cell, namely itself. Otherwise,  $k - l \geq 1$ ; in the sum over the  $S$ , consider the greatest element  $j$  of  $S$ , which is at least  $k - l$ , and split the sum over that,

$$\text{spc}(k, l) = \sum_{j=k-l}^k \sum_{\substack{S \in \binom{[k]}{k-l} \\ j = \max S}} \prod_{\chi \in \text{cells}([k] \setminus S)} C_{|\chi|}.$$

For fixed  $j$ , all summands (of the sum over  $S$ ) will have a factor with  $\chi = \{j + 1, \dots, k\}$ , and thus a factor of  $C_{k-j}$ . Factoring out this  $C_{k-j}$ , we get

$$\text{spc}(k, l) = \sum_{j=k-l}^k C_{k-j} \sum_{\substack{S \in \binom{[k]}{k-l} \\ j = \max S}} \prod_{\substack{\chi \in \text{cells}([k] \setminus S) \\ \chi \neq \{j+1, \dots, k\}}} C_{|\chi|}.$$

Now, note that choosing a subset  $S$  of  $[k]$  of size  $k-l$  whose maximum is  $j$  is equivalent to choosing a subset  $s$  of  $[j-1]$  of size  $k-l-1$ , where  $S = s \cup \{j\}$ . Moreover, the cells of  $[k] \setminus S$  other than  $\{j+1, \dots, k\}$  are exactly the cells of  $[j-1] \setminus s$ , thus

$$\text{spc}(k, l) = \sum_{j=k-l}^k C_{k-j} \sum_{\substack{s \in \binom{[j-1]}{k-l-1} \\ j = \max s}} \prod_{\chi \in \text{cells}([j-1] \setminus s)} C_{|\chi|}.$$

By definition of  $\text{spc}$ , this means

$$\text{spc}(k, l) = \sum_{j=k-l}^k C_{k-j} \text{spc}(j-1, l+j-k).$$

Rewriting this as a sum over  $i := k-j$ , this gives us the following recurrence for  $\text{spc}$ :

$$\text{spc}(k, l) = \sum_{i=0}^l C_i \text{spc}(k-i-1, l-i) \quad \text{for } l < k, \quad (4.3)$$

$$\text{spc}(k, k) = C_k. \quad (4.4)$$

## 4.3 A better recurrence

We can now turn this recurrence into a simpler recurrence, which we will prove by recurrence.

Since  $\text{spc}$  has not been formally defined for negative arguments, we extend the definition with  $\text{spc}(k, l) = 0$  for  $l < 0$ ; this is consistent with the definition, since it yields a sum over subsets  $S \subseteq [k]$  bigger than  $k$ , and it yields an empty sum in the recurrence we just derived. In addition to that, we also let  $\text{spc}(k, l) = 0$  when  $l > k$ .

Note that for  $k-1 = l \geq 0$ , we have

$$\begin{aligned} \text{spc}(k, l) &= \sum_{i=0}^l C_i \text{spc}(k-i-1, l-i) = \sum_{i=0}^l C_i \text{spc}(l-i, l-i) = \sum_{i=0}^l C_i C_{l-i} \\ &= C_{l+1}, \end{aligned} \quad (4.5)$$

and thus

$$\text{spc}(k, l) = \text{spc}(k-1, l) + \text{spc}(k, l-1) = C_k \text{ for } k = l > 0. \quad (4.6)$$

1					
1	1				
1	2	2			
1	3	5	5		
1	4	9	14	14	
1	5	14	28	42	42

Figure 8. The first few values of  $\text{spc}$ ;  $k$  vertically from 0 to 5,  $l$  horizontally from 0 to  $k$ . The  $l+1$  summands in the recurrence (4.3) with  $k = 6$  and  $l = 3$  are highlighted.

Further, for  $k - 1 = l$ , we get

$$\text{spc}(k, l) = \sum_{i=0}^l C_i C_{l-i} = C_l + \sum_{i=0}^{l-1} C_i C_{((l-1)-i)+1}$$

applying (4.4) on the left and (4.5) on the right,

$$\begin{aligned} &= \text{spc}(k-1, l) + \sum_{i=0}^{l-1} C_i \text{spc}(k-i-1, (l-1)-i) \\ &= \text{spc}(k-1, l) + \text{spc}(k, l-1). \end{aligned} \quad (4.7)$$

Now let  $k-1 > l \geq 0$ . Assume  $\text{spc}(k', l') = \text{spc}(k'-1, l') + \text{spc}(k', l'-1)$  for  $0 \leq l' < k' < k$ . Then we can apply this assumption to the summands of  $\text{spc}(k, l)$ :

$$\begin{aligned} \text{spc}(k, l) &= \sum_{i=0}^l C_i \text{spc}(k-i-1, l-i) \\ &= \sum_{i=0}^l C_i (\text{spc}(k-i-2, l-i) + \text{spc}(k-i-1, l-i-1)) \\ &= \sum_{i=0}^l C_i \text{spc}(k-i-2, l-i) + \sum_{i=0}^l C_i \text{spc}(k-i-1, l-i-1) \\ &= \sum_{i=0}^l C_i \text{spc}((k-1)-i-1, l-i) + \sum_{i=0}^{l-1} C_i \text{spc}(k-i-1, (l-1)-i) \quad (4.8) \\ &\quad + C_l \text{spc}(k-l-1, -1), \end{aligned}$$

so, substituting the recurrence for  $\text{spc}$ ,

$$= \text{spc}(k-1, l) + \text{spc}(k, l-1). \quad (4.9)$$

We thus have the following recurrence for  $\text{spc}$ :

$$\text{spc}(k, l) = \text{spc}(k-1, l) + \text{spc}(k, l-1) \quad \text{for } k \geq 1, 0 \leq l \leq k, \quad (4.10)$$

$$\text{spc}(k, l) = 0 \quad \text{for } l < 0 \text{ or } l > k, \quad (4.11)$$

$$\text{spc}(0, 0) = 1. \quad (4.12)$$

This is the recurrence defining the Catalan triangle<sup>6</sup>, and its solution is known, namely

$$\text{spc}(k, l) = \frac{(k+l)! (k-l+1)}{l! (k+1)!} = \frac{k-l+1}{k+1} \binom{k+l}{k}, \quad (4.13)$$

as shown by L. F. A. Arbogast in 1800 [2, p. 214–217].

#### 4.3.1 A combinatorial interpretation of $\text{spc}$

The Catalan triangle counts well-formed prefix bracket expressions [CITATION]. This can in fact be seen from all the definitions of  $\text{spc}$  above, thus providing a more combinatorial solution.

In the original definition (4.1) of  $\text{spc}$ , the sum is over the choices for the positions of the unmatched  $k-l$  opening brackets; between those, well-formed bracket expressions (counted by Catalan numbers) are inserted.

The first recurrence (4.3) corresponds to the following grammatical definition<sup>7</sup> of well-formed prefixes, where again  $\text{wfbe}$  denotes a well-formed bracket expression:

$$\begin{aligned} \text{wfprefix} &::= \text{wfbe}(\text{wfprefix} \\ &\quad | \text{wfbe}, \end{aligned}$$

<sup>6</sup>[TODO cite something that actually uses that name here]

<sup>7</sup>This grammar requires infinite look-ahead to parse, but doing a first pass to mark the unmatched opening brackets resolves that.

1					
1	1				
1	2	2			
1	3	5	5		
1	4	9	14	14	
1	5	14	28	42	42

Figure 9. The recurrence (4.10).

where the sum ranges over the length of the well-formed bracket expression preceding the first unmatched bracket.

Finally, (4.10) reflects the fact that a well-formed prefix with  $k$  opening brackets and  $l$  closing brackets either ends with an opening bracket (preceded by a well-formed prefix with  $k - 1$  opening and  $l$  closing brackets), or with a closing bracket (preceded by a well-formed prefix with  $k$  opening and  $l - 1$  closing brackets).

#### 4.4 Asymptotics

We can now use the expression (4.13) for  $\text{spc}$  to study the asymptotics of the bound (4.2). We are interested in the asymptotics as a function of the length of the bracket expression  $n = k + l + p + q$ . Since the bound is a product of two identical two-parameter factors, we study one of them,

$$C_k \text{spc}(k, l),$$

as a function of  $n_1 := k + l$ . Let  $l = \eta k$ , thus  $0 \leq \eta \leq 1$ , the above expression becomes

$$C_{\frac{n_1}{\eta+1}} \left( \frac{n_1}{\eta+1} \right) \frac{(1-\eta)n + \eta + 1}{n + \eta + 1}.$$

Asymptotically as  $n_1 \rightarrow \infty$ , this yields

$$\frac{1}{n_1 \sqrt{\pi n_1}} 4^{\frac{n_1}{\eta+1}} \frac{\sqrt{1+\eta}}{\sqrt{2\pi n_1 \eta}} \left( \frac{1+\eta}{\eta^{\frac{\eta}{\eta+1}}} \right)^{n_1} (1-\eta),$$

or, up to a polynomially-bounded factor,

$$4^{\frac{n_1}{\eta+1}} \left( \frac{1+\eta}{\eta^{\frac{\eta}{\eta+1}}} \right)^{n_1} = \left( 4^{\eta+1} \frac{1+\eta}{\eta^{\frac{\eta}{\eta+1}}} \right)^{n_1}.$$

As  $\eta$  ranges from 0 to 1, the base of that exponential reaches a maximum of 5 at  $\eta = \frac{1}{4}$ . We thus have the following asymptotic bound.

$$\nu_{\mathcal{P}}(\langle^k \rangle^l \langle^q \rangle^p) \leq C_k \text{spc}(k, l) \text{spc}(p, q) C_p \leq 5^{k+l} 5^{p+q} = 5^n. \quad (4.14)$$

[TODO: would it be interesting or feasible to average over all  $\langle^k \rangle^l \langle^q \rangle^p$ , for fixed  $n$ ?]  
 [TODO: here a section about  $\text{spc}$  generalizing to arbitrary WFBs, and the applicability of overcounting analyses]

## References

- [1] M. Ajtai, V. Chvátal, M.M. Newborn, and E. Szemerédi. Crossing-free subgraphs. In Peter L. Hammer, Alexander Rosa, Gert Sabidussi, and Jean Turgeon, editors, *Theory and Practice of Combinatorics — A collection of articles honoring Anton Kotzig on the occasion of his sixtieth birthday*, volume 60 of *North-Holland Mathematics Studies*, pages 9–12. North-Holland, 1982.
- [2] Louis François Antoine Arbogast. *Du Calcul des Dérivations*. Levrault, Frères, 1800.
- [3] Eugène Charles Catalan. Note sur une équation aux différences finies. *Journal de Mathématiques Pures et Appliquées*, 3:508–516, 1838.
- [4] Leonhard Euler. Letter to Christian Goldbach. <http://eulerarchive.maa.org/correspondence/letters/000868.pdf> (retrieved 2016-09-10), 1751.
- [5] Alfredo García, Marc Noy, and Javier Tejel. Lower bounds on the number of crossing-free subgraphs of  $K_N$ . *Computational Geometry*, 16(4):211–221, 2000.
- [6] Monroe Newborn and W. O. J. Moser. Optimal crossing-free Hamiltonian circuit drawings of  $K_n$ . *Journal of Combinatorial Theory, Series B*, 29(1):13–26, 1980.
- [7] Andres J. Ruiz-Vargas and Emo Welzl. Crossing-free perfect matchings in wheel point sets. To appear.
- [8] János András Segner. *Enumeratio modorum quibus figurae planae rectilineae per diagonales diuiduntur in triangula*. In *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, volume 7 *pro annis 1758 et 1759*, pages 203–209, 1761.
- [9] Micha Sharir and Emo Welzl. On the number of crossing-free matchings, cycles, and partitions. *SIAM Journal on Computing*, 36(3):695–720, 2006.
- [10] Jan van Leeuwen and Anneke A. Schoone. Untangling a travelling salesman tour in the plane. In J. R. Mühlbacher, editor, *Proceedings of the 7th Conference on Graph-theoretic Concepts in Computer Science*, pages 87–98, 1981.