Crossing-Free Perfect Matchings

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blurb

1 Geometric graphs

blurb

Definition (geometric graph). Given a set of points P in the Euclidean plane $\mathbb{R}^2$, a *geometric graph* is a collection of straight line segments (edges) whose endpoints are elements of P.

It can be described as a simple graph (in the combinatorial sense) on the vertices P, where the edge $\{v, w\}$ corresponds to the segment joining v and w.

Definition (crossing-free geometric graph). A geometric graph is *crossing-free* if no two edges share points other than their endpoints; it is called *crossing* otherwise.

Note that this implies that the corresponding simple graph is planar, and that the geometric graph is a plane embedding. ◊

Definition (triangulation). A *triangulation* is a maximal crossing-free geometric graph, that is, a geometric graph such that for all v and w in P that are not joined by a segment, adding the segment joining v and w would result in a crossing geometric graph.

Note that the faces (in the sense of plane graphs) formed by a triangulation are all triangles, with the possible exception of the outer face (thus this definition is *not* equivalent to that of a triangulation of the 2-sphere). ◊

Since a geometric graph corresponds to a simple graph on the underlying point set, we can also look at geometric graphs that belong special classes of simple graphs.

Definition (crossing-free matching). A crossing-free geometric graph is a *crossing-free matching* if it is a matching as a simple graph on the vertices P. ◊

Definition (crossing-free perfect matching). A *crossing-free perfect matching* is a crossing-free geometric graph which is perfect matching as a simple graph on the vertices P. ◊

2 Bounds and asymptotics

There is interest in statements regarding the number of possible geometric graphs in in the aforementioned classes; evidently, that number would depend on the choice of the point set P, so instead one is interested in bounds on that number depending on the cardinality $|P|$, and possibly restricting P so that it satisfies certain properties.

In general, if $g(P)$ is the number of geometric graphs of a certain sort on the point set P, we will look for lower bounds l and upper bounds u of the form

$$\forall n \in \mathbb{N}, \forall P \text{ such that } |P| = n, l(n) \leq g(n) \leq u(n),$$

where the P runs over all point sets that satisfy the relevant properties.

In addition, we may be interested in asymptotics on such ls and us; since these bounds are often exponential, we tend to ignore polynomial factors; we will thus say that

$$f(n) \preccurlyeq u(n)$$

if $f(n) \leq p(n)u(n)$ for some polynomially-bounded p.

3 Convex point sets

[talk about the number of triangulations and crossing-free perfect matchings; this is a good place to introduce the recurrence for Catalan numbers too]

4 Crossing-free perfect matchings

We will call PM_P the set of perfect matchings on the point set P, and CFPM_P the set of crossing-free perfect matchings on the point set P. [TODO cite existing results]

5 Left-right perfect matchings and bracket expressions

A point set in the plane is *in general position* if no three points are aligned.

We say that a point set is *in general position with respect to the horizontal* if it is in general position and no two points lie on a vertical line. Note that any point set in general position can be put in general position with respect to the horizontal by an arbitrarily small rotation. Moreover, note that points in general position with respect to the horizontal are ordered from left to right.

A *bracket expression* of size n is a sequence of n opening brackets $\langle$ or closing brackets $\rangle$. It is a *well-formed prefix* if, when read from left to right, the number of closing brackets encountered never exceeds the number of opening brackets encountered. A *well-formed bracket expression* is a well-formed prefix with the same number of opening and closing brackets.

It is a well-known result [TODO FIND A CITATION] that the number of well-formed bracket expressions of size $2k$ is the Catalan number C_k. In fact, the recurrence can readily be seen from a grammatical definition of well-formed bracket expressions,[1]

$$\begin{aligned} \text{wfbe} ::= & \langle \text{wfbe} \rangle \text{wfbe} \\ & \mid \text{empty}, \end{aligned}$$

where the sum ranges over the sizes of the component bracket expressions.

Note that inserting an opening bracket, followed a closing bracket anywhere after the inserted opening bracket, into a well-formed bracket expression, results in a well-formed bracket expression.

Given a point set P of size n in general position with respect to the horizontal and a perfect matching μ on P, any point in P is either a left or right endpoint of an edge, since it is incident to exactly one edge, and that edge is not vertical.

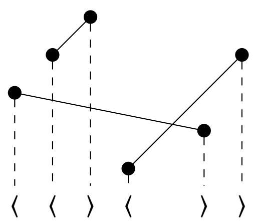

Figure 1. A well-formed bracket expression constructed from a perfect matching

Construct a bracket expression $\beta(\mu)$ of size n as follows: order P from left to right; the ith bracket is opening if the ith point of P is a left endpoint of μ, and it is closing otherwise. This bracket expression is well-formed, since it can be constructed by starting from the (well-formed) empty bracket expression, by successively inserting both brackets corresponding to each edge, where the closing bracket will be inserted to the right of the opening bracket.

Given a well-formed bracket expression B and a point set P, we will say that a perfect matching μ is *consistent with* B if $\beta(\mu) = B$; moreover, we will refer to the points of P corresponding to opening brackets of B as *left-points* (since they will be left endpoints of any perfect matching consistent with B), and similarly we will refer to the points of P corresponding to closing brackets of B as *right-points*.

One approach to bounding the size of CFPM_P is to bound the size of $\beta^{-1}(B) \cap \mathrm{CFPM}_P$ for bracket expressions B of size n, *i.e.* the number of crossing-free perfect matchings on P consistent with B. We thus define

$$\nu_P(B) \coloneqq \beta^{-1}(B) \cap \mathrm{CFPM}_P\,.$$

[1] We have not enclosed nonterminal in angle brackets here, since that would result in hopeless confusion in a grammar describing nothing but sequences of angle brackets.

In order to concisely refer to bracket expressions, we will use the notations $\langle^k$ for k successive opening brackets, and $\rangle^k$ for k successive closing brackets, for instance,

$$\langle^2\rangle\langle^2\rangle^3 = \langle\langle\rangle\langle\langle\rangle\rangle\rangle.$$

[TODO a section or subsection or something here, talking about the trivial upper bounds on $\langle\rangle$, and about the bound on a product of bracket expressions. Maybe about the boring stuff like $\langle\langle\rangle E\rangle$, but frankly that's not very interesting]

[TODO cite the existing upper bound from Sharir—Welzl 2006 (the proof is unrelated to anything here though)]

6 An optimal lower bound for the number of crossing-free perfect matchings

This argument is due to E. Welzl [TODO cite paper to appear, is there a preprint?].

Theorem (Welzl—maybe other people?). *Let P be a point set of size $n = 2k$ in general position with respect to the horizontal, numbered from left to right, and let B be a well-formed bracket expression of size n. Then there exists a crossing-free perfect matching consistent with B; in other words, $\nu_P(B) \geq 1$.*

Proof. Let m_0 be a perfect matching on P consistent with B. This is always possible, for instance, parsing the bracket expression, match the point corresponding with an opening parenthesis to the point corresponding with the matching closing parenthesis.

Define $l(m)$ for a perfect matching m on P to be the sum of the lengths of the edges of m.

Then, repeat the following procedure, starting at $i = 0$. If there is no crossing in m_i, we have found a perfect matching with the desired properties. If there is a crossing, let a, b, c, and d be the points involved, so that the edge ab crosses the edge cd. Remove these edges, and replace them by ad and cb (thus "uncrossing" them). This yields another perfect matching m_{i+1}. By the triangle inequality (see figure 2), $l(m_{i+1}) < l(m_i)$.

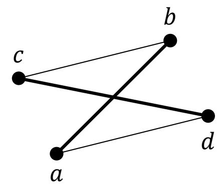

Figure 2. Uncrossing in a left-right perfect matching. Replacing the thick edges by the thin ones reduces the total edge length, while preserving left and right endpoints.

If this did not terminate, it would yield a sequence m of crossing perfect matchings on P on which l is strictly decreasing, thus an infinite sequence of distinct graphs on P. Since there are only finitely many graphs on P, this is a contradiction, so we eventually find a crossing-free perfect matching. □

This immediately yields a lower bound, since there are $C_{\frac{n}{2}}$ well-formed bracket expressions of size n.

Corollary. *Let P be a point set of size n in general position. There are at least $C_{\frac{n}{2}}$ distinct crossing-free perfect matchings on P*, i.e. $\mathrm{CFPM}_P \geq C_{\frac{n}{2}}$. □

Moreover, this lower bound is optimal, since it is attained if P is in convex position.

7 Matchings across a line

Again we consider $n = 2k$ points in general position with respect to the horizontal. The left-right matchings corresponding to brackets expressions with k opening brackets followed by k closing brackets, $\langle^k\rangle^k$, are called *matchings across a line.* Indeed, any segment in such a matching will cross any vertical line that separates the left-points from the right-points.

The following result was shown by Micha Sharir and Emo Welzl in 2006 [TODO cite].

Theorem (Sharir–Welzl). *There are at most $C^2_{\frac{n}{2}}$ crossing-free perfect matchings across a line on a point set P of size $n = 2k$ points in general position with respect to the horizontal*, i.e. $\nu_P(\langle^k\rangle^k) < C^2_k$.

Proof. Pick a vertical line that separates the left-points from the right-points; we will call it *the vertical line*. Further, let us call set of left-points L and the set of right-points R.

A perfect matching across a line is uniquely defined by a bijection $\mu : L \to R$ from the left-points to the right-points. Now, number the intersections between the edges of the perfect matching and the vertical line from top to bottom. This yield a numbering $\iota : E \to [k]$ of the edges.

Define $\lambda(l) \coloneqq \iota(e_l)$ mapping the number of a left-point to the intersection number of its edge, and similarly $\rho(r) \coloneqq \iota(e_r)$ for the right-points. We have $\mu = \lambda\rho^{-1}$.

The permutations λ (respectively ρ) determine the order in which the left points (respectively right points) reach the vertical line.

The idea of the proof is as follows: if the matching is crossing-free, we will show that λ and ρ have to be in sets of size C_k, thus that $\mu = \lambda\rho^{-1}$ can take at most C_k^2 values, *i.e.* that there can be at most $C_n k^2$ perfect matchings across a line.

Since we are going to reuse these concepts in subsequent proofs, we will formalize and name the properties of λ and ρ that we will consider. A bijection from a set of left-points Λ to $[|\Lambda|]$ that can be constructed by numbering from top to bottom the intersections of the edges incident to Λ with a vertical line to the right of Λ is called a *crossing-free left-alignment of Λ*. Correspondingly, for right-points, we define a *crossing-free right-alignment*.

If we have a crossing-free perfect matching across a line, then λ as constructed above is a crossing-free left-alignment of L, and ρ is a crossing-free right-alignment of R.

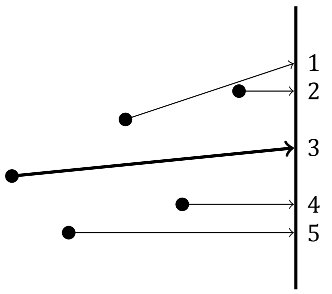

Figure 3. A crossing-free left-alignment of five points. Once the index of the leftmost point is chosen (thick edge), the rest consists in two crossing-free left-alignments of i and $k-i-1$ points each—here $k = 5$ and $i = 2$.

Lemma. *There are at most C_k crossing-free left-alignments of k points.*

Proof. Let l be the leftmost point of a set of L of k left-points, and let λ be a crossing-free left-alignment of L.

$\lambda(l)$, *i.e.* the index of the crossing of the vertical line the edge e_l of the leftmost point, is equal to one plus the number of points that are above e_l. Indeed, the edges of points above e_l must themselves reach the vertical line above e_l, otherwise they would cross e_l, and correspondingly for points below e_l, so that there are as many edges reaching the vertical line below e_l as there are points below e_l.

Moreover, as the oriented angle between e_l and the horizontal increases, points are only added to the set of points below e_l, so that choosing the number of points below e_l determines the sets of points below and above e_l.

Further, since points above e_l must reach the line above e_l and correspondingly for points below, if the point p is above e_l, then $\lambda(p) < \lambda(l)$, and if it is below, $\lambda(p) > \lambda(l)$. Thus, λ restricted do the points above e_l is a crossing-free left-alignment of the $\lambda(l) - 1$ points above e_l, and $\lambda - \lambda(l)$ is a crossing-free left-alignment of the $k - \lambda(l)$ points below e_l.

Thus, λ is determined by the choice of $i \coloneqq \lambda(l) - 1$ and crossing-free left-alignments of i and $k - i - i$ points. It follows that if ϖ_a is a bound for the number of crossing-free left-alignments of a points when $a < k$, we can give a bound on the number of crossing-free left-alignments of k points,

$$\varpi_k \coloneqq \sum_{i=0}^{k} \varpi_i \varpi_{k-i-1}.$$

We can start the recurrence with $\varpi_0 = 1$; this is the recurrence for the Catalan numbers, thus $\varpi_k = C_k$. □

The same result holds for right-alignments, completing the proof of the theorem. □

8 Analysing the overcounting in the upper bound for matchings across a line

9 Highly convex matchings across a line

10 Three changes of bracket direction

We now consider left-right matchings corresponding to bracket expressions which have three changes of bracket directions, *i.e.* bracket expressions of the form $\langle^k\rangle^l\langle^q\rangle^p$, k opening brackets, l closing brackets, q opening brackets, p closing brackets, where $k-l = p-q \geq 0$ and $k+l+p+q =: n$.

In term of points, this means that four sets can be separated by vertical lines, from left to right, k left-points forming the set K, l right-points forming L, q left-points forming Q, ad p right-points forming P. We pick a vertical line separating K and L and call it *the left line*, and we pick a vertical line separating P and Q and call it *the right line*.

Given a crossing-free perfect matching on those points, numbering from top to bottom the intersections between edges incident to the points of K and and the left line, we get a crossing-free left-alignment of K. $k-l$ of the k edges on this vertical line are incident to points in P; the other l are incident to points in L. Numbering those l edges yields a crossing-free right-alignment λ of L.

Similarly on the right side, we get a crossing-free right-alignment of π of P, numbering the subset of edges incident to a point in P and a point in Q, we get a left-alignment θ of q.

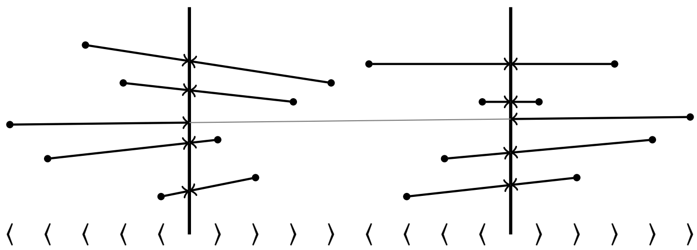

Figure 4. Four crossing-free alignments.

The matching is uniquely determined by κ, π, the choice of the $k-l$ among k points and $k-l = p-q$ among p points that get matched to each other, and by λ and θ, which gives the following bound for the number of these matchings:

$$\nu_P(\langle^k\rangle^l\langle^q\rangle^p) \leq C_k\binom{k}{l}C_lC_q\binom{p}{q}C_p.$$

[TODO give the asymptotics of this here; it's not so good, but it motivates the improvements below]

10.1 Improving the binomial bound

We can however improve upon that bound: indeed, once κ, π, and the set of points of K matched to P are fixed, the edges of the matching that cross both the left line and the right line—let us call these *long edges*—are determined. As a result, the region between the left line and the right line is partitioned in trapezoidal cells, and the portion of any edge from K to L and from P to Q that lies between the left line and the right line is confined to a single of those cells. It follows that λ is composed of crossing-free right-alignments the of subsets of L separated by the long edges, and similarly for θ with subsets of Q.

TODO

Figure 5. Splitting the middle two crossing-free alignments.

Let us look at the edges crossing the left line (the same argument applies to the right line), numbered from top to bottom: $k-l$ of those are long edges, let $S \subseteq [k]$ be their numbers; in between two long edges, above the first long edge, and below the last one, we have the edges that define the crossing-free right-alignments that make up λ. It follows that a crossing-free right-alignment of m points that makes up λ corresponds to a maximal sequence of m consecutive elements of $[k] \setminus S$.

We will call the set of maximal sequences of consecutive elements of S' the *cells* of S', written $\operatorname{cells}(S')$.

Then, we can improve the $\binom{k}{l} C_l$ factor in the bound (in which the binomial comes from the choice of the long edges amongst the k edges on the left line, and the Catalan number comes from the choice of λ), summing over the choices of the long edges (and thus of S above). The improved factor becomes

$$\operatorname{spc}(k,l) \coloneqq \sum_{S \in \binom{[k]}{k-l}} \prod_{c \in \operatorname{cells}([k]\setminus S)} C_{|c|} = \sum_{S' \in \binom{[k]}{l}} \prod_{c \in \operatorname{cells}(S')} C_{|c|},$$

and the overall bound becomes

$$\nu_P(\langle^k\rangle^l \langle^q\rangle^p) \le C_k \operatorname{spc}(k,l) \operatorname{spc}(p,q) C_p. \tag{10.1}$$

10.2 A recurrence

In order to compute spc efficiently, and eventually, get its asymptotics, it is useful to get rid of the cells function. We can express spc as a recurrence instead. First, we note that $\operatorname{spc}(k,k) = C_k$: there is only one summand, S' is the whole set, so it has only one cell, namely itself. Otherwise, $k - l \ge 1$; in the sum over the S, consider the greatest element j of S, which is at least $k - l$, and split the sum over that,

$$\operatorname{spc}(k,l) = \sum_{j=k-l}^{k} \sum_{\substack{S \in \binom{[k]}{k-l} \\ j = \max S}} \prod_{c \in \operatorname{cells}([k]\setminus S)} C_{|c|}.$$

For fixed j, all summands (of the sum over S) will have a factor with $c = \{j+1, \dots, k\}$, and thus a factor of C_{k-j}. Factoring out this C_{k-j}, we get

$$\operatorname{spc}(k,l) = \sum_{j=k-l}^{k} C_{k-j} \sum_{\substack{S \in \binom{[k]}{k-l} \\ j = \max S}} \prod_{\substack{c \in \operatorname{cells}([k]\setminus S) \\ c \ne \{j+1,\dots,k\}}} C_{|c|}.$$

Now, note that choosing a subset S of $[k]$ of size $k-l$ whose maximum is j is equivalent to choosing a subset s of $[j-1]$ of size $k-l-1$, where $S = s \cup \{j\}$. Moreover, the cells of $[k] \setminus S$ other than $\{j+1, \dots, k\}$ are exactly the cells of $[j-1] \setminus s$, thus

$$\operatorname{spc}(k,l) = \sum_{j=k-l}^{k} C_{k-j} \sum_{s \in \binom{[j-1]}{k-l-1}} \prod_{c \in \operatorname{cells}([j-1]\setminus s)} C_{|c|}.$$

By definition of spc, this means

$$\operatorname{spc}(k,l) = \sum_{j=k-l}^{k} C_{k-j} \operatorname{spc}(j-1, l+j-k).$$

Rewriting this as a sum over $i \coloneqq k - j$, this gives us the following recurrence for spc:

$$\operatorname{spc}(k,l) = \sum_{i=0}^{l} C_i \operatorname{spc}(k-i-1, l-i) \qquad \text{for } l < k, \tag{10.2}$$

$$\operatorname{spc}(k,k) = C_k. \tag{10.3}$$

1					
1	1				
1	2	2			
1	3	5	5		
1	4	9	14	14	
1	5	14	28	42	42

Figure 6. The first few values of spc; k vertically from 0 to 5, l horizontally from 0 to k. The $l + 1$ summands in the recurrence (10.2) with $k = 6$ and $l = 3$ are highlighted.

10.3 A better recurrence

We can now turn this recurrence into a simpler recurrence, which we will prove by recurrence.

Since spc has not been formally defined for negative arguments, we extend the definition with $\mathrm{spc}(k,l) = 0$ for $l < 0$; this is consistent with the definition, since it yields a sum over subsets $S \subseteq [k]$ bigger than k, and it yields an empty sum in the recurrence we just derived. In addition to that, we also let $\mathrm{spc}(k,l) = 0$ when $l > k$.

Note that for $k-1 = l \geq 0$, we have

$$\begin{aligned}\mathrm{spc}(k,l) = \sum_{i=0}^{l} C_i \,\mathrm{spc}(k-i-1, l-i) = \sum_{i=0}^{l} C_i \,\mathrm{spc}(l-i, l-i) = \sum_{i=0}^{l} C_i C_{l-i} \\ = C_{l+1},\end{aligned} \tag{10.4}$$

and thus

$$\mathrm{spc}(k,l) = \mathrm{spc}(k-1,l) + \mathrm{spc}(k,l-1) = C_k \text{ for } k = l > 0. \tag{10.5}$$

Further, for $k-1 = l$, we get

$$\mathrm{spc}(k,l) = \sum_{i=0}^{l} C_i C_{l-i} = C_l + \sum_{i=0}^{l-1} C_i C_{((l-1)-i)+1}$$

applying (10.3) on the left and (10.4) on the right,

$$\begin{aligned} &= \mathrm{spc}(k-1,l) + \sum_{i=0}^{l-1} C_i \,\mathrm{spc}(k-i-1, (l-1)-i) \\ &= \mathrm{spc}(k-1,l) + \mathrm{spc}(k,l-1). \end{aligned} \tag{10.6}$$

Now let $k-1 > l \geq 0$. Assume $\mathrm{spc}(k',l') = \mathrm{spc}(k'-1,l') + \mathrm{spc}(k',l'-1)$ for $0 \leq l' < k' < k$. Then we can apply this assumption to the summands of $\mathrm{spc}(k,l)$:

$$\begin{aligned}\mathrm{spc}(k,l) &= \sum_{i=0}^{l} C_i \,\mathrm{spc}(k-i-1, l-i) \\ &= \sum_{i=0}^{l} C_i (\mathrm{spc}(k-i-2, l-i) + \mathrm{spc}(k-i-1, l-i-1)) \\ &= \sum_{i=0}^{l} C_i \,\mathrm{spc}(k-i-2, l-i) + \sum_{i=0}^{l} C_i \,\mathrm{spc}(k-i-1, l-i-1) \\ &= \sum_{i=0}^{l} C_i \,\mathrm{spc}((k-1)-i-1, l-i) + \sum_{i=0}^{l-1} C_i \,\mathrm{spc}(k-i-1, (l-1)-i) \qquad (10.7) \\ &\quad + C_l \,\mathrm{spc}(k-l-1, -1),\end{aligned}$$

so, substituting the recurrence for spc,

$$= \mathrm{spc}(k-1,l) + \mathrm{spc}(k,l-1). \tag{10.8}$$

TODO

Figure 7. The recurrence applied to the sum, yielding two sums, proving the recurrence.

We thus have the following recurrence for spc:

$$\mathrm{spc}(k,l) = \mathrm{spc}(k-1,l) + \mathrm{spc}(k,l-1) \qquad \text{for } k \geq 1,\ 0 \leq l \leq k, \tag{10.9}$$

$$\mathrm{spc}(k,l) = 0 \qquad \text{for } l < 0 \text{ or } l > k, \tag{10.10}$$

$$\mathrm{spc}(0,0) = 1. \tag{10.11}$$

This is the recurrence defining the Catalan triangle [CITATIONS], and its solution is known, namely [CITATION]

$$\mathrm{spc}(k,l) = \frac{(k+l)!\,(k-l+1)}{l!\,(k+1)!} = \frac{k-l+1}{k+1}\binom{k+l}{k}. \tag{10.12}$$

10.3.1 A combinatorial interpretation of the first recurrence

10.4 Asymptotics

We can now use the expression (10.12) for spc to study the asymptotics of the bound (10.1). We are interested in the asymptotics as a function of the size of of the bracket expression $n = k+l+p+q$. Since the bound is a product of two identical two-parameter factors, we study one of them,

$$C_k \operatorname{spc}(k, l),$$

as a function of $n_1 \coloneqq k + l$. Let $l = \alpha k$, thus $0 \leq \alpha \leq 1$, the above expression becomes

$$C_{\frac{n_1}{\alpha+1}} \binom{n_1}{\frac{n_1}{\alpha+1}} \frac{(1-\alpha)n + \alpha + 1}{n + \alpha + 1}.$$

Asymptotically as $n_1 \longrightarrow \infty$, this yields

$$\frac{1}{n_1\sqrt{\pi n_1}} 4^{\frac{n_1}{\alpha+1}} \frac{\sqrt{1+\alpha}}{\sqrt{2\pi n_1 \alpha}} \left(\frac{1+\alpha}{\alpha^{\frac{\alpha}{\alpha+1}}}\right)^{n_1} (1-\alpha),$$

or, up to a polynomially-bounded factor,

$$4^{\frac{n_1}{\alpha+1}} \left(\frac{1+\alpha}{\alpha^{\frac{\alpha}{\alpha+1}}}\right)^{n_1} = \left(4^{\alpha+1} \frac{1+\alpha}{\alpha^{\frac{\alpha}{\alpha+1}}}\right)^{n_1}.$$

As α ranges from 0 to 1, the base of that exponential reaches a maximum of 5 at $\alpha = \frac{1}{4}$. We thus have the following asymptotic bound.

$$\nu_P(\langle^k\rangle^l \langle^q\rangle^p) \leq C_k \operatorname{spc}(k, l) \operatorname{spc}(p, q) C_p \preccurlyeq 5^{k+l} 5^{p+q} = 5^n. \tag{10.13}$$

[TODO: would it be interesting or feasible to average over all $\langle^k\rangle^l \langle^q\rangle^p$, for fixed n?]