Crossing-Free Perfect Matchings

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blurb

1 Geometric graphs

blurb

Definition (geometric graph). Given a set of points P in the Euclidean plane $\mathbb{R}^2$, a *geometric graph* is a collection of straight line segments (edges) whose endpoints are elements of P.

It can be described as a simple graph (in the combinatorial sense) on the vertices P, where the edge $\{v, w\}$ corresponds to the segment joining v and w.

Definition (crossing-free geometric graph). A geometric graph is *crossing-free* if no two edges share points other than their endpoints; it is called *crossing* otherwise.

Note that this implies that the corresponding simple graph is planar, and that the geometric graph is a plane embedding. ◊

Definition (triangulation). A *triangulation* is a maximal crossing-free geometric graph, that is, a geometric graph such that for all v and w in P that are not joined by a segment, adding the segment joining v and w would result in a crossing geometric graph.

Note that the faces (in the sense of plane graphs) formed by a triangulation are all triangles, with the possible exception of the outer face (thus this definition is *not* equivalent to that of a triangulation of the 2-sphere). ◊

Since a geometric graph corresponds to a simple graph on the underlying point set, we can also look at geometric graphs that belong special classes of simple graphs.

Definition (crossing-free matching). A crossing-free geometric graph is a *crossing-free matching* if it is a matching as a simple graph on the vertices P. ◊

Definition (crossing-free perfect matching). A *crossing-free perfect matching* is a crossing-free geometric graph which is perfect matching as a simple graph on the vertices P. ◊

2 Bounds and asymptotics

There is interest in statements regarding the number of possible geometric graphs in in the aforementioned classes; evidently, that number would depend on the choice of the point set P, so instead one is interested in bounds on that number depending on the cardinality $|P|$, and possibly restricting P so that it satisfies certain properties.

In general, if $g(P)$ is the number of geometric graphs of a certain sort on the point set P, we will look for lower bounds l and upper bounds u of the form

$$\forall n \in \mathbb{N}, \forall P \text{ such that } |P| = n, l(n) \leq g(n) \leq u(n),$$

where the P runs over all point sets that satisfy the relevant properties.

Alternatively, we may be interested in asymptotics on such ls and us.

3 Convex point sets

4 Crossing-free perfect matchings

[Somewhere, define left-right matching, "general position wrt the horizontal", "numbered from left to right"; prove that a matching yields a WFBE; when talking about matchings we will talk about "the edge of a point"]

5 Brackets expressions and an optimal lower bound

This argument is due to E. Welzl [cite paper to appear, is there a preprint?].

Theorem. *Let P be a point set of size $2n$ in general position with respect to the horizontal, numbered from left to right, and let B be a well-formed bracket expression of size $2n$. Then there exists a crossing-free perfect matching such that the kth point of P is a left endpoint if and only if the kth bracket of B is an opening bracket.*

Proof. Let m_0 be a perfect matching on P consistent with B. This is always possible, for instance, parsing the bracket expression, match the point corresponding with an opening parenthesis to the point corresponding with the matching closing parenthesis.

Define $l(m)$ for a perfect matching m on P to be the sum of the lengths of the edges of m.

Then, repeat the following procedure, starting at $k = 0$. If there is no crossing in m_k, we have found a perfect matching with the desired properties. If there is a crossing, let a, b, c, and d be the points involved, so that the edge ab crosses the edge cd. Remove these edges, and replace them by ad and cb (thus "uncrossing" them). This yields another perfect matching m_{k+1}. By the triangle inequality (see figure [TODO FIGURE]), $l(m_{k+1}) < l(m_k)$.

If this did not terminate, it would yield a sequence m of crossing perfect matchings on P on which l is strictly decreasing, thus an infinite sequence of graphs on P. Since there are only finitely many graphs on P, this is a contradiction, so we eventually find a crossing-free perfect matching. □

This immediately yields a lower bound, since there are C_n well-formed bracket expressions of size $2n$.

Corollary. *Let P be a point set of size $2n$ in general position. There are at least C_n distinct crossing-free perfect matchings on P.* □

Moreover, this lower bound is optimal, since it is attained if P is in convex position.

[TODO something about the general idea of proving upper bounds for left-right perfect matchings or classes thereof to get an upper bound on perfect matchings]

6 Matchings across a line

Again we consider $2n$ points in general position with respect to the horizontal.

The left-right matchings corresponding to brackets expressions with n opening brackets followed by n closing brackets, $\langle \cdots \langle \rangle \cdots \rangle$, are called *matchings across a line.* Indeed, any segment in such a matching will cross any vertical line that separates the left-points from the right-points. [FIGURE]

The following result was shown by Micha Sharir and Emo Welzl in 2006 [CITATION HERE].

Theorem (Sharir–Welzl). *There are at most C_n^2 crossing-free perfect matchings across a line on $2n$ points in general position with respect to the horizontal.*

Proof. Pick a vertical line that separates the left-points from the right-points; we will call it 'the vertical line'.

Number the left-points from left to right, and the right-points from right to left. A perfect matching across a line is uniquely defined by a bijection from the left-points to the right-points, and thus, via the numbering, by a bijection from $[n]$ to itself*, i.e.* a permutation $\mu \in S_n$. Now, number from bottom to top the intersections between the segments of the perfect matching and the vertical line. This yields two permutations, the left permutation λ mapping the number of a left-point to its intersection number, and the right permutation gr mapping the number of a right-point to its intersection number. [FIGURE] Moreover, we have $\mu = \lambda\rho^{-1}$.

The permutations λ (respectively ρ) determine the order in which the left points (respectively right points) reach the vertical line.

If the matching is crossing-free, we will show that λ and ρ have to be in sets of size C_n, and thus that $\mu = \lambda\rho^{-1}$ can take at most C_n^2 values*, i.e.* that there can be at most C_n^2 perfect matchings across a line.

[REWORD THE PART BELOW, NAME THE SET TO WHICH λ BELONGS]

To constrain λ (the same argument holds for ρ with left and right reversed), we look only at the left-points and the vertical line, and we note that the value of $\lambda(1)$*, i.e.* the index of the crossing of the vertical line the edge e_1 of the leftmost point, is equal to one plus the number of points that are below e_1. Indeed, the edges of points above e_1 must themselves reach the vertical line above e_1, otherwise they would cross e_1, and correspondingly for points below e_1, so that there are as many edges reaching the vertical line below e_1 as there are points below e_1.

Moreover, as the angle of e_1 increases, points are only added to the set of points below e_1, so that choosing the number of points below e_1 determines the sets of points below and above e_1.

Additionally, since points above e_1 must reach the line above e_1 and correspondingly for points below, if the point numbered k is above e_1, then $\lambda(k) > \lambda(1)$, and if it is below, $\lambda(k) < \lambda(1)$. λ restricted do the points above and below e_1 thus yields bijections from the points below e_1 to $[1, \lambda(1) - 1] \cap \mathbb{Z}$, and from the points above to $[\lambda(1) + 1, n]$. Renumbering, those are permutations on $\lambda(1) - 1$ and $n - \lambda(1)$ elements, respectively, and they are defined as λ itself was, by the edges.

Thus, having chosen $i = \lambda(1) - 1$ we can choose independently in the same way the permutation of the $\lambda(1) - 1$, points below and that of the $n - \lambda(1)$ points above e_1; this yields a recurrence for an upper bound on the number ϖ_n of permutations λ on n points that can be obtained without crossings,

$$\varpi_k \leq \sum_{i=0}^{k} \varpi_i \varpi_{n-i-1},$$

where $\varpi_0 = 1$. With equality, this is the recurrence for the Catalan numbers, thus $\varpi_k \leq C_k$. □

7 Overcounting in the upper bounds for matchings across a line

8 Highly convex matchings across a line