On the number of crossing-free perfect matchings with designated left and right endpoints

Robin Leroy

2016-09-24

Abstract

[TODO main results: the spc bound, the bound dependent on the number of interior points. Do I mention the expressions for the Catalan triangle (are they known)?]

Introduction

In the study of plane embeddings of planar graphs, also known as plane graphs, embeddings with non-crossing straight edges (called crossing-free geometric graphs) are of particular interest. For instance, it is known [10] that any optimal solution to the travelling salesman problem in the Euclidean plane has non-crossing straight edges. In 1936, K. Wagner proved[1] [CITE] that any planar graph has such an embedding.

A natural enumerative question is then, given a vertex set of n points in the Euclidean plane, to bound the number of plane graphs of a certain sort (cycles, matchings, *etc.*) which can be drawn with non-crossing straight edges. For spanning cycles, the question was introduced by M. Newborn and W. O. J. Moser in 1976 [6], and a super-exponential upper bound was given. In 1980, M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi [1] proved that there are only exponentially many crossing-free geometric graphs on a given point set.

Exponential bounds for various sorts of graphs were then improved, in particular for triangulations—the maximal crossing-free geometric graphs. One of the interesting features of triangulations is that finding a bound for them yields a bound for the overall number of crossing-free geometric graphs. [TODO citation] In addition, bounds on the number of triangulations are of interest in geometric modeling [CITE Studies in computational geometry motivated by mesh generation].

Some special configurations of points are better understood; in particular, for point sets in convex position, the numbers of crossing-free perfect matchings, triangulations, and spanning cycles are known. In fact, the study of point sets in convex position vastly predates the general problem, with the number of triangulations given by Euler in 1751 [4].

For perfect matchings, an optimal lower bound of $\mathtt{C}_{\frac{n}{2}}$ was obtained as early as 1995 by A. García, M. Noy, and J. Tejel [5]; upper bounds have been harder to come by, with the best one yet being asymptotically 10.05^n, given by M. Sharir and E. Welzl in 2005 [9] In that same work, an upper bound of 5.83^n is given for the number of crossing-free perfect matchings on a point set whose points are designated as left or right endpoints; moreover, an upper bound of 4^n is given when all the left endpoints are to the left of the right endpoints (matchings across a line).

In this thesis, we start by giving an overview of some of the existing results, in particular a way to classify arrangements of designated left and right endpoints, and we restate the proof of the bound for matchings across a line by M. Sharir and E. Welzl (section 1). We then study in detail the inequalities in that bound and the objects it actually counts, yielding criteria that will allow us to tighten the bound in specific

[1] This result was independently proved by I. Fáry in 1948 [CITE], and by S. K. Stein in 1951 [CITE]; it is generally known as Fáry's theorem.

cases (section 2). Using these criteria, we give a bound for matchings across a line that depends on the number of points in the interior of the convex hull (section 3). We then give new bounds on the number of crossing-free perfect matchings on a point set with designated left and right endpoints for a specific family of left-to-right orderings of the left and right endpoints (4).

1 Background

We first give some definitions, as well as existing proofs whose ideas we will use to derive our results.

1.1 Geometric graphs

Definition (geometric graph). Given a set of points $\mathcal{P}$ in the Euclidean plane $\mathbb{R}^2$, a *geometric graph* is a collection of straight line segments (edges) whose endpoints are elements of $\mathcal{P}$.

It can be described as a simple graph (in the combinatorial sense) on the vertices $\mathcal{P}$, where the edge $\{v, w\}$ corresponds to the segment joining v and w. ◊

Definition (crossing-free). A geometric graph is *crossing-free* if no two edges share points other than their endpoints; it is called *crossing* otherwise. ◊

Note that being crossing-free implies that the corresponding simple graph is planar, and that the geometric graph is a plane embedding.

Definition (triangulation). A *triangulation* is a maximal crossing-free geometric graph, that is, geometric graph such that for all v and w in $\mathcal{P}$ that are not joined by a segment, adding the segment joining v and w would result in a crossing geometric graph. ◊

Note that the faces (in the sense of plane graphs) formed by a triangulation are all triangles, with the possible exception of the outer face (thus this definition is *not* equivalent to that of a triangulation of the 2-sphere).

Since a geometric graph corresponds to a simple graph on the underlying point set, we can also look at geometric graphs that belong to special classes of simple graphs. As we will only be focusing on geometric graphs, we simply call *perfect matching* a geometric graph which corresponds to a perfect matching in the combinatorial sense.

As outlined in the introduction, we are interested in statements regarding the number of geometric graphs, possibly restricted to a certain type, on a given point set. Evidently, that number would depend on the choice of the point set $\mathcal{P}$, so instead we seek bounds on that number depending on the cardinality $|P|$, and possibly restricting $\mathcal{P}$ so that it satisfies certain properties.

An important property that $\mathcal{P}$ can have is general position. We say that a point set in the plane is *in general position* if no three points are collinear. In this thesis we will only study point sets in general position.

In general, if $g(\mathcal{P})$ is the number of geometric graphs of a certain sort on the point set $\mathcal{P}$, we will look for lower bounds l and upper bounds u of the form

$$\forall n \in \mathbb{N}, \forall \mathcal{P} \text{ such that } |\mathcal{P}| = n, l(n) \leq g(\mathcal{P}) \leq u(n),$$

where $\mathcal{P}$ runs over all point sets that satisfy the relevant properties.

1.2 Asymptotics

In addition, we will want asymptotics on such bounds l and u, to compare them with existing results. Since the bounds are often exponential, we tend to ignore polynomial factors. We will thus write that

$$u(n) \preccurlyeq_n f(n)$$

if for some polynomial p, $\forall n \in \mathbb{N}, u(n) \leq p(n)f(n)$.

In some cases, we may be interested in classes of functions $g_z(\mathcal{P})$ which depend on some parameter $z \in \Omega_n$. For instance, g_z could count crossing-free graphs on $\mathcal{P}$ that contain a clique of size $z \in [n]$. In this thesis, we will often use classes g_z with z being the left-to-right ordering of designated left and right endpoints.

When giving exponential bounds in n for these parameterized classes of graphs, we will want to keep track of the growth due to z. We will write that

$$u(n,z) \preccurlyeq_n f(n,z) \textit{ uniformly in } z \in \Omega_n$$

if for some polynomial p (that does not depend on z),

$$\forall n \in \mathbb{N}, \forall z \in \Omega_n, u(n,z) \leq p(n)f(n,z).$$

If $f(n,z)$ admits a supremum $\bar{f}(n)$ in $z \in \Omega_n$ for all n, this implies that $u(n,z) \leq p(n)\bar{f}(n)$, giving an upper bound truly independent of z.

Moreover, we will write $g(\mathcal{P}) \preccurlyeq_{|\mathcal{P}|=n} f(n)$ if for some $u(n) \preccurlyeq_n f(n)$,

$$\forall n \in \mathbb{N}, \forall \mathcal{P} \text{ such that } |\mathcal{P}| = n, g(\mathcal{P}) \leq u(n).$$

Similarly, we will write $g_z(\mathcal{P}) \preccurlyeq_{|\mathcal{P}|=n} f(n,z)$ *uniformly in* $z \in \Omega_n$ if for some $u(n,z) \preccurlyeq_n f(n,z)$ uniformly in $z \in \Omega_n$,

$$\forall n \in \mathbb{N}, \forall z \in \Omega_n, \forall \mathcal{P} \text{ such that } |\mathcal{P}| = n, g_z(\mathcal{P}) \leq u(n,z).$$

In both cases $\mathcal{P}$ runs over all point sets that satisfy the relevant properties.

1.3 A remark on the asymptotics of the binomial coefficient

Unsurprisingly, binomial coefficients will be involved in the bounds, so we now give a uniform exponential upper bound for them. It can be shown by writing the logarithm of the factorial as a sum of logarithms, and then bounding that sum by integrals, that

$$\forall k \in \mathbb{N}^*, \mathrm{e}\left(\frac{k}{\mathrm{e}}\right)^k \leq k! \leq k\mathrm{e}\left(\frac{k}{\mathrm{e}}\right)^k.$$

It follows that

$$\begin{aligned}
\forall k \in \mathbb{N}^*, \forall \eta \in]0,1[, \binom{k}{\eta k} &\leq \frac{k\mathrm{e}\left(\frac{k}{\mathrm{e}}\right)^k}{\mathrm{e}\left(\frac{\eta k}{\mathrm{e}}\right)^{\eta k}\mathrm{e}\left(\frac{(1-\eta)k}{\mathrm{e}}\right)^{(1-\eta)k}} \\
&= \frac{k}{\mathrm{e}}\frac{k^k}{(\eta k)^{\eta k}((1-\eta)k)^{(1-\eta)k}} \\
&= \frac{k}{\mathrm{e}}\left(\frac{1}{\eta^\eta(1-\eta)^{1-\eta}}\right)^k \\
&= \frac{k}{\mathrm{e}}\left(\frac{(1-\eta)^{\eta-1}}{\eta^\eta}\right)^k,
\end{aligned}$$

so that, since $\forall k \in \mathbb{N}, \binom{k}{0} = \binom{k}{k} = 1$, we have

$$\binom{k}{\eta k} \preccurlyeq_k \left(\frac{(1-\eta)^{\eta-1}}{\eta^\eta}\right)^k \text{ uniformly in } \eta \in [0,1]. \tag{1.1}$$

1.4 Triangulations of convex point sets and the Catalan numbers

A set of points is said to be *in convex position* if it is in general position and all points are in the boundary of its convex hull.

For points in convex position, the numbers of triangulations and perfect matchings are known. The following recurrence, given by J. A. Segner in 1758 [8], defines the *Catalan numbers*.

Theorem (Segner). *Let C_n be the number of triangulations of a set of $n+2$ points in convex position. Then $\mathrm{C}_0 = 1$, and for $n > 0$,*

$$\mathrm{C}_n = \sum_{i=0}^{n-1} \mathrm{C}_i \mathrm{C}_{n-i-1}. \tag{1.2}$$

Proof. Consider a set of $n+2$ points in convex position. For $n = 0$ the result is trivial, the only triangulation is an edge.

Let $n > 0$, and number the points along the boundary of the convex hull, from 0 to $n+1$, see figure 1. The edge e joining points $n+1$ and n is in the boundary of the convex hull, and thus must be in any triangulation, since no other edge can cross it. Let i be the index of the third point of the triangle containing edge e, which may be any of the others, from 0 to $n-1$.

Then, restricting the triangulation to the $i+2$ points $\{0, \dots, i, n+1\}$ yields a triangulation of those points. Similarly, restricting the triangulation of the whole point set to points i through n yields a triangulation of $n-i+1$ points. It follows that $\mathrm{C}_n \leq \sum_{i=0}^{n-1} \mathrm{C}_i \mathrm{C}_{n-i-1}$.

Moreover, given a triangulation of $\{0, \dots, i, n+1\}$ and a triangulation of $\{i, \dots, n\}$, adding the edge e forms a triangulation of the whole point set, proving equality. □

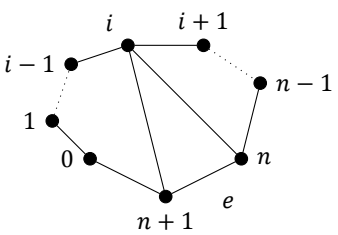

Figure 1. Segner's recurrence.

An explicit expression for C_n was given earlier by Euler [4], along with the generating function. A convenient form is

$$\mathrm{C}_n = \frac{1}{n+1}\binom{2n}{n}, \tag{1.3}$$

yielding asymptotically

$$\mathrm{C}_n \sim \frac{4^n}{n^{\frac{3}{2}}\sqrt{\pi}} \preccurlyeq_n 4^n. \tag{1.4}$$

1.5 Crossing-free perfect matchings

We will call $\mathrm{PM}_{\mathcal{P}}$ the set of perfect matchings on the point set $\mathcal{P}$, and $\mathrm{CFPM}_{\mathcal{P}}$ the set of crossing-free perfect matchings on the point set $\mathcal{P}$.

[TODO talk about the number of crossing-free perfect matchings in convex position, give the proof; find the Alfred Errera thing from the octavo of the académie royale de belgique, reference 10 in Sharir–Welzl; this might require a trip to the library?]

For a set $\mathcal{P}$ of n points in general position, it was proved by Micha Sharir and Emo Welzl in 2005 [9] that $|\mathrm{CFPM}_{\mathcal{P}}| \preccurlyeq_n 10.05^n$.

When talking about a perfect matching, we will denote the unique edge incident to point p by e_p.

1.6 Bracket expressions and their relations to perfect matchings

We say that a point set is *in general position with respect to the horizontal* if it is in general position and no two points lie on a vertical line. Note that any point set in general position can be put in general position with respect to the horizontal by an arbitrarily small rotation. Moreover, note that points in general position with respect to the horizontal are ordered from left to right.

This ordering can be used to classify perfect matchings on the point set; to this end, we introduce bracket expressions. A *bracket expression* of length n is a sequence of n opening brackets $\langle$ or closing brackets $\rangle$. It is a *well-formed prefix* if, when read from left to right, the number of closing brackets encountered never exceeds the number of opening brackets encountered. A *well-formed bracket expression* is a well-formed prefix with the same number of opening and closing brackets.

It is a well-known result, shown[2] by E. C. Catalan in 1838 [3], that the number of well-formed bracket expressions of length $2k$ is the Catalan number C_k. In fact, the

[2]Specifically, Catalan showed that the recurrence (1.2) counts the number of parenthesizings of $n+1$ factors.

recurrence can readily be seen from a grammatical definition of well-formed bracket expressions,[3]

$$\begin{aligned}\text{wfbe} &::= \langle\text{wfbe}\rangle\text{wfbe}\\ &\mid \text{empty},\end{aligned}$$

where the sum in the recurrence ranges over the lengths of the component bracket expressions.

Note that inserting an opening bracket, followed by a closing bracket anywhere after the inserted opening bracket, into a well-formed bracket expression, results in a well-formed bracket expression.

Well-formed bracket expressions can be used to define classes of perfect matchings on a given point set. Given a point set $\mathcal{P}$ of size n in general position with respect to the horizontal and a perfect matching μ on $\mathcal{P}$, any point in $\mathcal{P}$ is either a left or right endpoint of an edge, since it is incident to exactly one edge, and that edge is not vertical.

Construct a bracket expression $\mathfrak{B}_{\mathcal{P}}(\mu)$ of length n as follows: order $\mathcal{P}$ from left to right; the ith bracket is opening if the ith point of $\mathcal{P}$ is a left endpoint of μ, and it is closing otherwise. This bracket expression is well-formed, since it can be constructed by starting from the (well-formed) empty bracket expression, by successively inserting both brackets corresponding to each edge, where the closing bracket will be inserted to the right of the opening bracket.

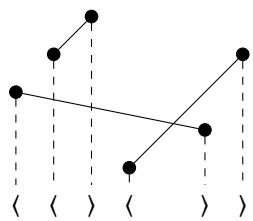

Figure 2. A well-formed bracket expression constructed from a perfect matching.

Given a well-formed bracket expression B and a point set $\mathcal{P}$, we will say that a perfect matching μ is *consistent with* B if $\mathfrak{B}_{\mathcal{P}}(\mu) = B$; moreover, we will refer to the points of $\mathcal{P}$ corresponding to opening brackets of B as *left-points* (since they will be left endpoints of any perfect matching consistent with B), and similarly we will refer to the points of $\mathcal{P}$ corresponding to closing brackets of B as *right-points*.

One approach to bounding the size of $\mathrm{CFPM}_{\mathcal{P}}$ is to bound the size of $\mathfrak{B}_{\mathcal{P}}^{-1}(B) \cap \mathrm{CFPM}_{\mathcal{P}}$ for bracket expressions B of length n, *i.e.,* the number of crossing-free perfect matchings on $\mathcal{P}$ consistent with B. We thus define

$$\nu_{\mathcal{P}}(B) := |\mathfrak{B}_{\mathcal{P}}^{-1}(B) \cap \mathrm{CFPM}_{\mathcal{P}}|.$$

In order to concisely refer to bracket expressions, we will use the notations $\langle^k$ for k successive opening brackets, and $\rangle^k$ for k successive closing brackets, for instance,

$$\langle^2\rangle\langle^2\rangle^3 = \langle\langle\rangle\langle\langle\rangle\rangle\rangle.$$

[TODO a section or subsection or something here, talking about the trivial upper bounds on $\langle\rangle$, and about the bound on a product of bracket expressions. Maybe about the boring stuff like $\langle\langle\rangle E\rangle$, but frankly that's not very interesting]

[TODO cite the existing upper bound from Sharir–Welzl 2006 (the proof is unrelated to anything here though)]

1.7 An optimal lower bound for the number of crossing-free perfect matchings

The following proof is due to A. J. Ruiz-Vargas and E. Welzl [7].

Theorem (Ruiz-Vargas–Welzl). *Let $\mathcal{P}$ be a point set of size $n = 2k$ in general position with respect to the horizontal, and let B be a well-formed bracket expression of length n. Then there exists a crossing-free perfect matching consistent with B; in other words, $\nu_{\mathcal{P}}(B) \geq 1$.* □

Proof. Let μ_0 be a perfect matching on $\mathcal{P}$ consistent with B. This is always possible, for instance, parsing the bracket expression, match the point corresponding to an opening bracket and the point corresponding to the matching closing bracket.

Define $\Lambda(\mu)$ for a perfect matching μ on $\mathcal{P}$ to be the sum of the lengths of the edges of μ.

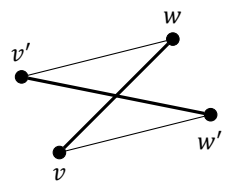

Figure 3. Untangling in a perfect matching. Replacing the thick edges by the thin ones reduces the total edge length, while preserving left and right endpoints.

[3] We have not enclosed nonterminals in angle brackets here, as would be usual in Backus–Naur form, since that would result in hopeless confusion in a grammar describing nothing but sequences of angle brackets.

Then, repeat the following procedure, starting at $i = 0$. If there is no crossing in μ_i, we have found a perfect matching with the desired properties. If there is a crossing, let v, v', w, and w' be the points involved, with v and v' being the left endpoints and w and w' being the right endpoints, so that the edge vw crosses the edge $v'w'$. Remove these edges, and replace them by vw' and $v'w$ (thus untangling them). This yields another perfect matching μ_{i+1}. By the triangle inequality (see figure 3), $\Lambda(\mu_{i+1}) < \Lambda(\mu_i)$.

If this procedure did not terminate, it would yield a sequence $(\mu_i)_{i\in\mathbb{N}}$ of crossing perfect matchings on $\mathcal{P}$ on which Λ is strictly decreasing, thus an infinite sequence of distinct graphs on $\mathcal{P}$. Since there are only finitely many graphs on $\mathcal{P}$, this is a contradiction, so we eventually[4] find a crossing-free perfect matching. □

This immediately yields a lower bound for the number of crossing-free perfect matchings, since there are $\mathrm{C}_{\frac{n}{2}}$ well-formed bracket expressions of length n. Note that this lower bound had already been proved by A. García, M. Noy, and J. Tejel in 1995, by a recurrence that split the point set, rather than via well-formed bracket expressions [5].

Corollary. *Let $\mathcal{P}$ be a point set of even size n in general position. There are at least $\mathrm{C}_{\frac{n}{2}}$ distinct crossing-free perfect matchings on $\mathcal{P}$, i.e., $|\mathrm{CFPM}_{\mathcal{P}}| \geq \mathrm{C}_{\frac{n}{2}}$.* □

Moreover, this lower bound is optimal, since it is attained if $\mathcal{P}$ is in convex position (see section 1.5).

Now that we have a tight uniform lower bound for $\nu_{\mathcal{P}}$, we will start looking at upper bounds dependent on the bracket expression.

1.8 Matchings across a line

Again we consider $n = 2k$ points in general position with respect to the horizontal. The matchings consistent with bracket expressions with k opening brackets followed by k closing brackets, $\langle^k\rangle^k$, are called *matchings across a line.* Indeed, any edge in such a matching will cross any vertical line that separates the left-points from the right-points.

The following result, and its proof, were given by Micha Sharir and Emo Welzl in 2005 [9].

Theorem (Sharir–Welzl). *Let $\mathcal{P}$ be a set of $n = 2k$ points in general position with respect to the horizontal. Then there are at most $\mathrm{C}^2_{\frac{n}{2}}$ crossing-free perfect matchings across a line on $\mathcal{P}$, i.e., $\nu_{\mathcal{P}}(\langle^k\rangle^k) \leq \mathrm{C}^2_k$.*

The idea of the proof is as follows. First, pick a vertical line that separates the left-points from the right-points; we will call it *the vertical line.* Further, let us call the set of left-points $\mathcal{L}$ and the set of right-points $\mathcal{R}$.

A perfect matching consistent with a given well-formed bracket expression (and in particular, a perfect matching across a line) is uniquely defined by a bijection $\mu : pointsetL \to \mathcal{R}$ from the left-points to the right-points. Consider such a perfect matching across a line, and let E be the set of its edges. Now, number the intersections between the edges of the perfect matching and the vertical line from top to bottom. This yield a numbering $\iota : E \to [k]$ of the edges.

Define $\lambda(l) \coloneqq \iota(e_l)$ mapping a left-point to the intersection number of its edge, and similarly $\rho(r) \coloneqq \iota(e_r)$ for the right-points. We have $\mu = \rho^{-1} \circ \lambda$.

The bijection λ (respectively ρ) determines the order in which the left points (respectively right points) reach the vertical line.

If the matching is crossing-free, we will show that λ and ρ have to be in sets of size C_k, thus that there are at most C^2_k functions $\mu = \rho^{-1} \circ \lambda$, *i.e.,* that there can be at most C^2_k crossing-free perfect matchings across a line.

Since we are going to reuse these concepts in subsequent sections, we will formalize and name the properties of λ and ρ that we will consider.

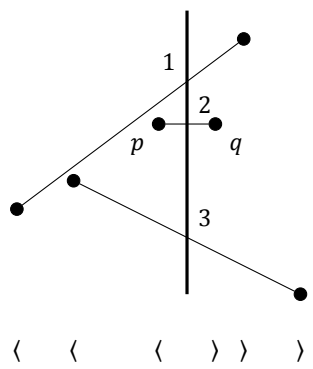

Figure 4. Constructing the numberings λ and ρ of the left- and right-points from a crossing-free perfect matching across a line; here $\iota(\{p, q\}) = 2$, and thus $\lambda(p) = \rho(q) = 2$.

[4]The untangling procedure itself predates this proof, and was shown to terminate in $\mathcal{O}(n^3)$ steps by J. van Leeuwen and A. A. Schoone in 1980 [10].

Definition (crossing-free alignment). Let $\mathcal{S}$ be a set of k points in general position. Let V be a vertical line to the right of $\mathcal{S}$. A bijection λ from $\mathcal{S}$ to $[k]$ is a *crossing-free left-alignment of $\mathcal{S}$ on V* if there exists a set $A = \{\sigma_p \mid p \in \mathcal{S}\}$ of k line segments such that the following hold:

1. for every $p \in \mathcal{S}$, the left endpoint of σ_p is p, and its right endpoint is on V;
2. no two segments cross, and no two segments share any endpoints;
3. let $\iota : A \to [k]$ the numbering of the right endpoints of the segments in A from top to bottom, then $\lambda(p) = \iota(\sigma_p)$.

Such a set of segments is said to *realize* λ.

Correspondingly, we define a *crossing-free right-alignment*, where "right" and "left" are swapped in the definition. ◊

A crossing-free left-alignment of five points, with a realizing set of segments—drawn as arrows—is shown in figure 5.

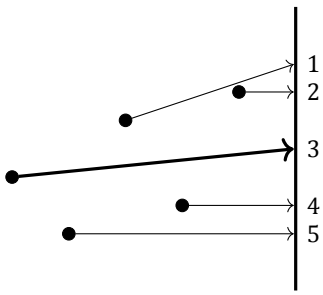

Figure 5. A crossing-free left-alignment of five points. Once the index of the leftmost point is chosen (thick segment), the rest consists of two crossing-free left-alignments of i and $k-i-1$ points each—here $k = 5$ and $i = 2$.

Lemma. *Let $\mathcal{S}$ be a set of k points in general position, and let V be a vertical line to the right of $\mathcal{S}$. There are at most C_k crossing-free left-alignments of $\mathcal{S}$ on V.*

Proof. Let l be the leftmost point of $\mathcal{S}$, and let λ be a crossing-free left-alignment of $\mathcal{S}$ on V. Let $A = \{\sigma_p \mid p \in \mathcal{S}\}$ be a set of non-crossing line segments realizing λ. For a point $p \in \mathcal{S}$, we will call σ_p *the segment of* p.

$\lambda(l)$, being the index on the vertical line V of the right endpoint of the segment σ_l of the leftmost point, is equal to one plus the number of points of $\mathcal{S}$ that are above σ_l. Indeed, the segments of points above σ_l must themselves reach the vertical line V above σ_l, otherwise they would cross σ_l, and correspondingly for points below σ_l, so that there are as many edges reaching the vertical line V below σ_l as there are points below σ_l.

Moreover, as the oriented angle between σ_l and the horizontal increases, points are only added to the set of points below σ_l, so that choosing the number of points below σ_l determines the sets of points below and above σ_l.

Further, since points above σ_l must reach the vertical line V above σ_l, and correspondingly for points below, if the point p is above σ_l, then $\lambda(p) < \lambda(l)$, and if it is below, $\lambda(p) > \lambda(l)$. Thus λ restricted do the points above σ_l is a crossing-free left-alignment of the $\lambda(l) - 1$ points above σ_l, and $\lambda - \lambda(l)$ restricted to the points below σ_l is a crossing-free left-alignment of the $k - \lambda(l)$ points below σ_l, where the appropriate subsets of A provide the set of segments required in the definition.

It follows that λ is determined by the choice of $\lambda(l)$ and crossing-free left-alignments of $\lambda(l) - 1$ and $k - \lambda(l)$ points. Letting $i \coloneqq \lambda(l) - 1$, these are crossing-free alignments of i and $k - i - 1$ points (see figure 5). Therefore, if ϖ_j is an upper bound for the number of crossing-free left-alignments of j points when $j < k$, we can give an upper bound on the number of crossing-free left-alignments of k points,

$$\varpi_k \coloneqq \sum_{i=0}^{k-1} \varpi_i \varpi_{k-i-1}.$$

We can start the recurrence with $\varpi_0 = 1$; this is the recurrence for the Catalan numbers, thus $\varpi_k = \mathrm{C}_k$. □

The same results holds for crossing-free right-alignments, simply exchange "left" and "right" in the above lemma and its proof.

Proof (of the theorem). If we have a crossing-free perfect matching across a line, then λ as constructed above is a crossing-free left-alignment of $\mathcal{L}$ on the vertical line; indeed, letting σ_p be the portion of the edge e_p to the left of the vertical line yields segments that realize λ. Similarly, ρ is a crossing-free right-alignment of $\mathcal{R}$ on the vertical line. □

2 Analysing the overcounting in the upper bound for matchings across a line

The bound given above for the number of matchings across a line is not optimal; an example is given by Sharir and Welzl of a crossing-free left-alignment λ and a crossing-free right-alignment ρ such that $\rho^{-1} \circ \lambda$ is not crossing-free (see figure 6).

TODO

Figure 6. Composing crossing-free alignments, only to get a crossing perfect matching.

There is however more to the overcounting than just counting some crossing matchings. To characterize that, it helps to name some functions and sets. Again, are considering a set $\mathcal{P}$ of $2k$ points in general position with respect to the horizontal and the bracket expression $\langle^k\rangle^k$. We let $\mathcal{L}$ be the set of left-points, and $\mathcal{R}$ be the set of right-points. Moreover, we pick a vertical line separating $\mathcal{L}$ and $\mathcal{R}$, and call it V.

In the previous section, given a crossing-free perfect matching μ, we constructed crossing-free left- and right-alignments λ and ρ (see figure 4). Let us call that construction ω, thus $\omega(\mu) = (\lambda, \rho)$. Moreover, let us call $\mathrm{CFLA}_{\mathcal{L},V}$ the set of crossing-free left-alignments of $\mathcal{L}$ on V, and $\mathrm{CFLA}_{\mathcal{R},V}$ the set of crossing-free left-alignments of $\mathcal{R}$ on V. Given sets of equal sizes $\mathcal{L}$ and $\mathcal{R}$, $\lambda \in \mathrm{CFLA}_{\mathcal{L},V}$, and $\rho \in \mathrm{CFRA}_{\mathcal{R},V}$, let $c(\lambda, \rho)$ be the (possibly crossing) perfect matching $\rho^{-1} \circ \lambda$. Finally, let us name the sets of perfect matchings across a line and of crossing-free perfect matchings across a line,

$$\begin{aligned} \mathrm{PMAL}_{\mathcal{P}} &\coloneqq \mathfrak{B}_{\mathcal{P}}^{-1}(\langle^k\rangle^k) \\ \mathrm{CFPMAL}_{\mathcal{P}} &\coloneqq \mathrm{PMAL}_{\mathcal{P}} \cap \mathrm{CFPM}_{\mathcal{P}} \,. \end{aligned}$$

Then the following diagram commutes, *i.e.,* constructing crossing-free alignments from a matching and then composing them yields the initial matching.

$$\begin{tikzcd} \mathrm{CFPMAL}_{\mathcal{P}} \arrow[r, "\omega"] \arrow[d, hook, "\subseteq"] & \mathrm{CFLA}_{\mathcal{L},V} \times \mathrm{CFRA}_{\mathcal{R},V} \arrow[dl, "c"] \\ \mathrm{PMAL}_{\mathcal{P}} & \end{tikzcd}$$

The result of Sharir and Welzl consists in deducing from the commutative diagram that ω must be injective, and so that

$$|\mathrm{CFPMAL}_{\mathcal{P}}| \leq |\mathrm{CFLA}_{\mathcal{L},V} \times \mathrm{CFRA}_{\mathcal{R},V}|, \tag{2.1}$$

and then a bound for the number of crossing-free alignments yields a bound for the number of crossing-free matchings across a line.

The inequality (2.1) is strict, as shown in figure 6. We would thus like to study the pairs of crossing-free alignments which are *not* in the image of ω, since they are responsible for that excess; we will call them *overcounted.* Overcounted pairs of crossing-free alignments fall into one of two categories;

TODO

Figure 7. Composing crossing-free alignments, and getting a crossing-free perfect matching, but one that does not decompose to the original alignments.

1. evidently, if $c(\lambda, \rho)$ is crossing, then (λ, ρ) is not in the image of ω, since the diagram commutes;

2. on the other hand, figure 7 shows a pair of crossing-free alignments (λ, ρ) which composes to a crossing-free alignment $\mu = c(\lambda, \rho)$, but which is not equal to $\omega(\mu)$.

In addition, we will call a crossing-free left alignment λ overcounted if any pair (λ, ρ) is overcounted.

To hunt down overcounted crossing-free alignments, we will examine in more detail the anatomy of a crossing-free left-aligment of $\mathcal{L}$ and the proof of the Sharir–Welzl bound for the number of crossing-free left-alignments, with some fresh definitions.

First, for a point set $\mathcal{L}$, note that we can partition the half-plane to the right of $\mathrm{lm}(L)$ in $|L|$ regions, separated by rays emanating from $\mathrm{lm}(L)$ and going through other points of $\mathcal{L}$. We will call those regions *left $\mathcal{L}$-cones*, and number them from top to bottom from 1 to $|L|$, see figure **??**. Note that there are i points of $\mathcal{L}$ above the interior of left $\mathcal{L}$-cone number $i + 1$, and $|L| - i - 1$ points below.

Given a crossing-free left-alignment λ of $\mathcal{L}$ on V, for any set of segments $A = \{\sigma_p \mid p \in L\}$ realizing λ, the segment $\sigma_{\mathrm{lm}(L)}$ of the leftmost point then has to be contained in left $\mathcal{L}$-cone number $\lambda(\mathrm{lm}(L))$. As we remarked in the previous section, for a

crossing-free perfect matching μ, the edges of μ restricted to the left side of V realize the left-alignment in $\omega(\mu)$. Thus if (λ, ρ) is in the image of ω, $\mathrm{lm}(L)$ has to be matched to a right-point in left $\mathcal{L}$-cone number $\lambda(\mathrm{lm}(L))$, see figure ??.

To extend this reasoning beyond the leftmost point of $\mathcal{L}$, we define *left conifers of* $\mathcal{L}$. A left conifer κ of a point set $\mathcal{L}$ is a map from points in $\mathcal{L}$ to left S-cones for subsets S of $\mathcal{L}$, satisfying the following properties:

1. $\kappa(\mathrm{lm}(L))$ is a left $\mathcal{L}$-cone;
2. κ restricted to the points above the interior of $\kappa(\mathrm{lm}(L))$ is a left conifer of these points;
3. κ restricted to the points below the interior of $\kappa(\mathrm{lm}(L))$ is a left conifer of these points.

See figure ?? for an example. We call $\mathrm{LC}\, L$ the set of left conifers of $\mathcal{L}$. From the observations above on the number of cones and the size of the sets of points above and below their interiors, if ϖ_k is the number of left conifers of k points, we have

$$\varpi_k = \sum_{i=0}^{k-1} \varpi_i \varpi_{k-i-1} \text{ and } \varpi_0 = 1,$$

so that $|\mathrm{LC}\, L| = \mathrm{C}_{|L|}$.

Recall that a crossing-free left-alignment λ is determined by the value of $\lambda(\mathrm{lm}(L))$ and by crossing-free left alignments of the points above and below any edge realizing λ, *i.e.,* above and below cone number $\lambda(\mathrm{lm}(L))$. This is equivalent to choosing left $\mathcal{L}$-cone number $\lambda(\mathrm{lm}(L))$, and recursively doing so in the point subsets above and below the interior of that cone, see figure ??. This gives an injective map $\mathfrak{K} : \lambda \mapsto \mathfrak{K}_\lambda$ from crossing-free left-alignments of $\mathcal{L}$ on any line to left conifers. Note that it is not bijective, see figure ??. In turn, this yields $|\mathrm{CFLA}_{\mathcal{L},V}| \leq |\mathrm{LC}\, L| = \mathrm{C}_{|L|}$, which is exactly the Sharir–Welzl bound.

Now, note that the map $\mathfrak{K}$ has the property that for any set of segments $A = \{\sigma_p \mid p \in L\}$ realizing λ, σ_p lies inside the cone $\mathfrak{K}_\lambda(p)$; since the edges of a matching μ realize the crossing-free alignments $\omega(\mu)$, $\mathfrak{K}_\lambda(p)$ must contain the right-point to which p is matched.

3 Highly convex matchings across a line

In particular, this means that if there is no right-point in $\mathfrak{K}_\lambda(p)$, then λ is overcounted. We will call *vacuous* a left cone that does not contain any right-points, and we will call vacuous a left conifer that has vacuous cones; thus λ is overcounted if $\mathfrak{K}_\lambda$ is vacuous.

We will now use convexity considerations to give an upper bound on the number of non-vacuous left conifers. Let us first define some concepts related to convex hulls. Given a set of points $\mathcal{P}$ in general position, the *extreme points of* $\mathcal{P}$ are the points of $\mathcal{P}$ that are in the boundary of the convex hull of $\mathcal{P}$. Given a set of points $\mathcal{L}$ in general position with respect to the horizontal and a set of points $\mathcal{R}$ in general position with respect to the horizontal to the right of $\mathcal{L}$, the *$\mathcal{R}$-liminary points of* $\mathcal{L}$ are the points of $\mathcal{L}$ which are extreme points of $L \cup R$, see figure ??. We will call $\mathrm{lm}(L)$ the leftmost point of $\mathcal{L}$; note that it is an $\mathcal{R}$-liminary point of $\mathcal{L}$.

Note that in polar order around $\mathrm{lm}(L)$, the $\mathcal{R}$-liminary points of $\mathcal{L}$ other than $\mathrm{lm}(L)$ are separated by $\mathcal{R}$, that is, in polar order, one subset of the $\mathcal{R}$-liminary points of $\mathcal{L}$ appears first, followed by all points of $\mathcal{R}$, followed by the rest of the $\mathcal{R}$-liminary points of $\mathcal{L}$ (except $\mathrm{lm}(L)$). We will call those $\mathcal{R}$-liminary points of $\mathcal{L}$ which appear below $\mathcal{R}$ the *lower $\mathcal{R}$-liminary points of* $\mathcal{L}$, and those that appear above $\mathcal{R}$ the *upper $\mathcal{R}$-liminary points of* $\mathcal{L}$.

If there are δ_1 upper $\mathcal{R}$-liminary points of $\mathcal{L}$, then the first δ_1 left $\mathcal{L}$-cones are vacuous. Similarly, if there are δ_2 lower $\mathcal{R}$-liminary points of $\mathcal{L}$, then the last δ_2 left $\mathcal{L}$-cones are vacuous. Moreover, for any non-vacuous left $\mathcal{L}$-cone $\mathscr{k}$, any upper $\mathcal{R}$-liminary points of $\mathcal{L}$ are $\mathcal{R}$-liminary points of the subset of $\mathcal{L}$ above $\mathscr{k}$, and any lower

$\mathcal{R}$-liminary points of $\mathcal{L}$ are $\mathcal{R}$-liminary points of the subset of $\mathcal{L}$ below $\mathscr{k}$, so that we get the recurrence

$$\varpi_{k,\delta} \coloneqq \max_{\delta_1+\delta_2=\delta-1} \sum_{i=\delta_1}^{k-\delta_2-1} \varpi_{i,\delta_1}\varpi_{k-i-1,\delta_2},$$

where $\varpi_{k,\delta}$ is an upper bound on the number of crossing-free non-vacuous left conifers of k points of which at least δ are $\mathcal{R}$-liminary. Moreover, we can start that recurrence with

$$\varpi_{k,0} \coloneqq \varpi_{k,1},$$
$$\varpi_{k,k} \coloneqq 1,$$

since there is at least one $\mathcal{R}$-liminary point of $\mathcal{L}$ (the leftmost one), and if all points are $\mathcal{R}$-liminary, only one cone is non-vacuous.

4 Three changes of bracket direction

We now consider perfect matchings consistent with bracket expressions which have three changes of bracket directions, *i.e.*, bracket expressions of the form $\langle^a\rangle^b\langle^c\rangle^d$, a opening brackets, b closing brackets, c opening brackets, d closing brackets, where $a-b=d-c\geq 0$ and $a+b+c+d=:n$.

In term of points, this means that four sets can be separated by vertical lines, from left to right, a left-points forming the set $\mathcal{A}$, b right-points forming $\mathcal{B}$, d left-points forming $\mathcal{D}$, and c right-points forming $\mathcal{C}$. We pick a vertical line separating $\mathcal{A}$ and $\mathcal{B}$ and call it V, and we pick a vertical line separating $\mathcal{C}$ and $\mathcal{D}$ and call it W, see figure 8.

Given a crossing-free perfect matching on those points, numbering from top to bottom the intersections between V and edges incident to the points in $\mathcal{A}$, we get a crossing-free left-alignment α of $\mathcal{A}$ on V. $a-b$ of the a edges intersecting V are incident to points in $\mathcal{D}$; the other b are incident to points in $\mathcal{B}$. Numbering those b edges yields a crossing-free right-alignment β of $\mathcal{B}$ on V.

Similarly on the right side, we get a crossing-free right-alignment of δ of $\mathcal{D}$ on W, and numbering the subset of edges joining $\mathcal{C}$ and $\mathcal{D}$, we get a left-alignment γ of $\mathcal{C}$ on W. Call $\mathcal{S}$ the set of $a-b$ points of $\mathcal{A}$ that are matched with points of $\mathcal{D}$, and

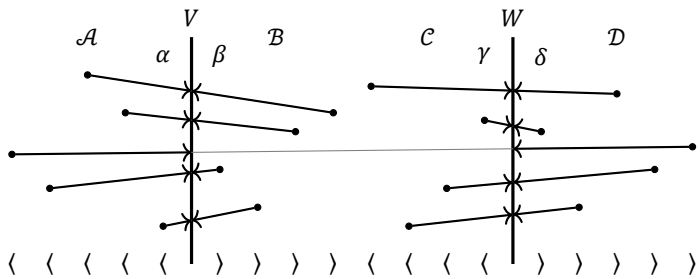

Figure 8. Four crossing-free alignments.

call $\mathcal{T}$ the set of $d-c$ points of $\mathcal{D}$ that are matched to points of $\mathcal{A}$.

The matching is uniquely determined by α, β, the choice of the subsets $\mathcal{S}\subseteq\mathcal{A}$ and $\mathcal{T}\subseteq D$, and by β and χ. This gives the following bound for the number of these matchings:

$$\nu_{\mathcal{P}}(\langle^a\rangle^b\langle^c\rangle^d) \leq \mathrm{C}_a\binom{a}{b}\mathrm{C}_b\mathrm{C}_c\binom{c}{d}\mathrm{C}_d.$$

Asymptotically, the factor involving a and b is, for $b=\eta a$,

$$\mathrm{C}_a\binom{a}{b}\mathrm{C}_b \preccurlyeq_{a+b} 4^{a+b}\binom{a}{b} \preccurlyeq_{a+b} 4^{a+b}\underbrace{\left(\left(\frac{(1-\eta)^{1-\frac{1}{\eta}}}{\eta}\right)^{\left(\frac{\eta}{\eta+1}\right)}\right)}_{x_\eta}^{a+b},$$

uniformly in $\eta \in [0,1]$. The base x_η of the exponential bound for the binomial coefficient is maximal when $\eta = \frac{3-\sqrt{5}}{2}$, where it is $\phi = \frac{1+\sqrt{5}}{2}$. This gives the overall bound

$$\nu_{\mathcal{P}}(\langle^a\rangle^b\langle^c\rangle^d) \preccurlyeq_{|\mathcal{P}|=n} 4^{a+b}\phi^{a+b}4^{c+d}\phi^{c+d} = (4\phi)^n \approx 6.472^n$$

uniformly in a, b, c, and d such that $a+b+c+d=n$. Note that this is worse than the existing 5.3793^n bound $\nu_{\mathcal{P}}(B)$ for an arbitrary well-formed bracket expression B, see section 1.6.

4.1 Improving the binomial bound

We can however improve upon that bound. Once α, δ, $\mathcal{S}$, and $\mathcal{T}$ are fixed, the edges of the matching that cross both the left line and the right line—let us call these *long edges*—are determined. As a result, the strip Σ between the vertical lines V and W is partitioned by the long edges in trapezoidal cells $Z_1, \dots, Z_{a-b+1}$, numbered from top to bottom, where the first and last cells are unbounded, see figure 9.

We will consider the crossing-free right-alignment β; the same argument will hold for γ. Since $\mathcal{B}$ lies within the strip Σ, it is partitioned in $a-b+1$ subsets $\mathcal{B}_i := \mathcal{B} \cap Z_i$. Moreover, an edge e_p emanating from a point $p \in \mathcal{B}_i$ cannot cross the long edges, and thus the portion $e_p \cap \Sigma$ of e_p that lies within the strip Σ is confined to the cell Z_i.

Let β_i be the crossing-free right-alignment of $\mathcal{B}_i$ on V realized by the segments $\{e_p \cap \Sigma \mid p \in \mathcal{B}_i\}$. Since the set of segments $\{e_p \cap \Sigma \mid p \in \mathcal{B}\}$ realizes β, β is composed of the crossing-free right-alignments β_i. Formally, for $p \in \mathcal{B}_i$,

$$\beta(p) = \beta_i(p) + \sum_{j=1}^{i-1} |\mathcal{B}_i|.$$

We now know that β is determined by crossing-free alignments β_i of point sets $\mathcal{B}_i$. In order to turn this into a bound, we need to know the sizes $|\mathcal{B}_i|$ of these point sets.

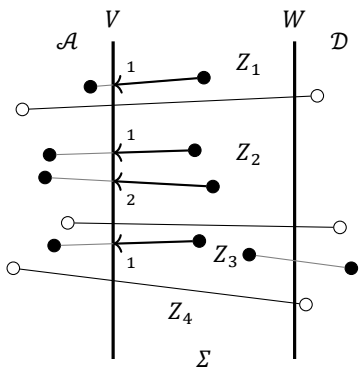

Figure 9. The points of $\mathcal{S} \subseteq \mathcal{A}$ and $\mathcal{T} \subseteq \mathcal{D}$ are shown in white. The vertical lines V and W bound the strip Σ. The long edges join $\mathcal{S}$ and $\mathcal{T}$, partitioning the strip Σ in cells Z_i. The crossing-free right-alignments β_i are shown. Note that here $\mathcal{B}_4$ is empty. In this figure, $a=7$, $b=4$, $c=1$, and $d=4$.

Let us look at the edges crossing V, numbered from top to bottom from 1 to a, see figure 10. Of these a edges, $a-b$ are long edges; they are the edges emanating from $\mathcal{S}$. Their numbers are $S = \alpha(\mathcal{S}) \subseteq [a]$. The other edges are the ones that emanate from points of $\mathcal{B}$. Let s_i be the ith element of S in increasing order, thus $s_1 = \min S$, and $s_{|S|} = s_{a-b} = \max S$.

The numbering of the a edges crossing V, from top to bottom, then starts with the edges emanating from $\mathcal{B}_1$, numbered 1 through $s_1 - 1$, the first long edge, numbered s_1, the edges emanating from $\mathcal{B}_2$, numbered s_1+1 through s_2-1, the second long edge, numbered s_2, the edges emanating from $\mathcal{B}_3$, numbered from s_2+1, *etc.* In other words, the numbers of the edges emanating from any nonempty $\mathcal{B}_i$ forms a maximal set of consecutive elements of $[a] \setminus S$. This motivates the following definition.

Definition (cells of a set of integers). The *cells* of $S' \subseteq \mathbb{N}$, written $\mathrm{cells}(S')$, is the set of maximal subsets of consecutive elements S'. ◊

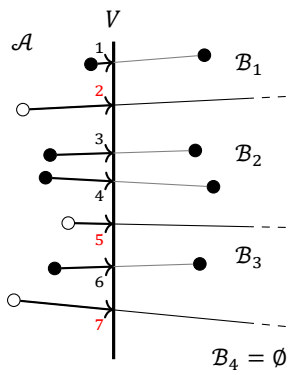

Figure 10. The edges crossing V. The points of $\mathcal{S}$ are shown in white. The crossing-free left-alignment α of $\mathcal{A}$ on V is shown. The elements of $S = \alpha(\mathcal{S})$ are shown in red. We have $\mathrm{cells}([a] \setminus S) = \{\{1\},\{3,4\},\{6\}\}$. Since $\mathcal{B}_4$ is empty, there is no cell of $[a] \setminus S$ corresponding to it.

For instance, $\mathrm{cells}(\{2,3,5,6,7,10,13,14\}) = \{\{2,3\},\{5,6,7\},\{10\},\{13,14\}\}$.

Then, we can improve the $\binom{a}{b}\mathrm{C}_b$ factor in the bound (in which the binomial comes from the choice of $\mathcal{S}$, and the Catalan number comes from the choice of β), summing over the choices of $\mathcal{S}$ (and thus of S above). The improved factor becomes[5]

$$\mathrm{spc}(a,b) := \sum_{S \in \binom{[a]}{a-b}} \prod_{\chi \in \mathrm{cells}([a]\setminus S)} \mathrm{C}_{|\chi|} = \sum_{S' \in \binom{[a]}{b}} \prod_{\chi \in \mathrm{cells}(S')} \mathrm{C}_{|\chi|}, \tag{4.1}$$

and the overall bound becomes

$$\nu_{\mathcal{P}}(\langle^a\rangle^b\langle^c\rangle^d) \le \mathrm{C}_a \,\mathrm{spc}(a,b)\,\mathrm{spc}(c,d)\mathrm{C}_d. \tag{4.2}$$

[5]The name spc was chosen because we define this as a Sum of Products of Catalan numbers.

4.2 A recurrence

In order to compute spc efficiently, and eventually, get its asymptotics, it is useful to get rid of the cells function. We can express spc as a recurrence instead. First, we note that $\mathrm{spc}(a,a) = \mathrm{C}_k$: there is only one summand, S' is the whole set, so it has only one cell, namely S' itself. Otherwise, $a-b \geq 1$; in the sum over the S, consider the greatest element j of S, which is at least $a-b$, and split the sum over that,

$$\mathrm{spc}(a,b) = \sum_{j=a-b}^{a} \sum_{\substack{S \in \binom{[a]}{a-b} \\ j = \max S}} \prod_{\chi \in \mathrm{cells}([a] \setminus S)} \mathrm{C}_{|\chi|}.$$

For fixed j, all summands (of the sum over S) will have a factor with $\chi = \{j+1, \dots, a\}$, and thus a factor of C_{a-j}. Factoring out this C_{a-j}, we get

$$\mathrm{spc}(a,b) = \sum_{j=a-b}^{a} \mathrm{C}_{a-j} \sum_{\substack{S \in \binom{[a]}{a-b} \\ j = \max S}} \prod_{\substack{\chi \in \mathrm{cells}([a] \setminus S) \\ \chi \neq \{j+1,\dots,a\}}} \mathrm{C}_{|\chi|}.$$

Now, note that choosing a subset S of $[a]$ of size $a-b$ whose maximum is j is equivalent to choosing a subset $\tilde{S}$ of $[j-1]$ of size $a-b-1$, where $S = \tilde{S} \cup \{j\}$. Moreover, the cells of $[a] \setminus S$ other than $\{j+1, \dots, a\}$ are exactly the cells of $[j-1] \setminus \tilde{S}$, thus

$$\mathrm{spc}(a,b) = \sum_{j=a-b}^{a} \mathrm{C}_{a-j} \sum_{\tilde{S} \in \binom{[j-1]}{a-b-1}} \prod_{\chi \in \mathrm{cells}([j-1] \setminus \tilde{S})} \mathrm{C}_{|\chi|}.$$

By definition of spc, this means

$$\mathrm{spc}(a,b) = \sum_{j=a-b}^{a} \mathrm{C}_{a-j} \, \mathrm{spc}(j-1, b+j-a).$$

Rewriting this as a sum over $i \coloneqq a-j$, this gives us the following recurrence for spc:

$$\mathrm{spc}(a,b) = \sum_{i=0}^{l} \mathrm{C}_i \, \mathrm{spc}(a-i-1, b-i) \qquad \text{for } b < a, \tag{4.3}$$

$$\mathrm{spc}(a,a) = \mathrm{C}_a. \tag{4.4}$$

1					
1	1				
1	2	2			
1	3	5	5		
1	4	9	14	14	
1	5	14	28	42	42

Figure 11. The first few values of spc; a vertically from 0 to 5, b horizontally from 0 to a. The values appearing in the $b+1$ summands of the recurrence (4.3) with $a = 6$ and $b = 3$ are highlighted.

4.3 A better recurrence

We can now turn this recurrence into a simpler and better-known one, namely

$$\mathrm{spc}(a,b) = \mathrm{spc}(a-1,b) + \mathrm{spc}(a,b-1) \qquad \text{for } a \geq 1,\ 0 \leq b \leq a, \tag{4.5}$$

$$\mathrm{spc}(a,b) = 0 \qquad \text{for } b < 0 \text{ or } b > a, \tag{4.6}$$

$$\mathrm{spc}(0,0) = 1. \tag{4.7}$$

Since spc has not been formally defined for negative arguments, we extend the definition with $\mathrm{spc}(a,b) = 0$ for $b < 0$; this is consistent with original the definition, since it yields a sum over subsets $S \subseteq [a]$ bigger than a, and it yields an empty sum in the recurrence we just derived. In addition to that, we also let $\mathrm{spc}(a,b) = 0$ when $b > a$. This takes care of (4.6). Moreover, we know that $\mathrm{spc}(0,0) = \mathrm{C}_0 = 1$. There thus remains to prove (4.5). We will do so first for $b = a$, then for $b = a-1$, and finally for the $0 \leq b < a-1$.

Note that we have

$$\mathrm{spc}(b,b-1) = \sum_{i=0}^{b-1} \mathrm{C}_i \, \mathrm{spc}(b-i-1, b-i-1) = \sum_{i=0}^{b-1} \mathrm{C}_i \mathrm{C}_{b-i-1} = \mathrm{C}_b, \tag{4.8}$$

and thus, for $a = b > 0$,

$$\operatorname{spc}(a,b) = \mathrm{C}_b = 0 + \mathrm{C}_b = \operatorname{spc}(a-1,b) + \operatorname{spc}(a,b-1). \tag{4.9}$$

Further, for $a - 1 = b$, we get

$$\operatorname{spc}(a,b) = \sum_{i=0}^{b} \mathrm{C}_i \mathrm{C}_{b-i} = \mathrm{C}_b + \sum_{i=0}^{b-1} \mathrm{C}_i \mathrm{C}_{b-i}$$

applying (4.4) on the left and (4.8) on the right,

$$\begin{aligned}
&= \operatorname{spc}(a-1,b) + \sum_{i=0}^{b-1} \mathrm{C}_i \operatorname{spc}(b-i, b-i-1) \\
&= \operatorname{spc}(a-1,b) + \sum_{i=0}^{b-1} \mathrm{C}_i \operatorname{spc}(a-i-1, b-i-1) \\
&= \operatorname{spc}(a-1,b) + \operatorname{spc}(a,b-1).
\end{aligned} \tag{4.10}$$

Now let $a - 1 > b \geq 0$. Assume $\operatorname{spc}(a',b') = \operatorname{spc}(a'-1,b') + \operatorname{spc}(a',b'-1)$ for $0 \leq b' < a' < a$. Then we can apply this assumption to the summands of $\operatorname{spc}(a,b)$:

$$\begin{aligned}
\operatorname{spc}(a,b) &= \sum_{i=0}^{b} \mathrm{C}_i \operatorname{spc}(a-i-1, b-i) \\
&= \sum_{i=0}^{b} \mathrm{C}_i (\operatorname{spc}(a-i-2, b-i) + \operatorname{spc}(a-i-1, b-i-1)) \\
&= \sum_{i=0}^{b} \mathrm{C}_i \operatorname{spc}(a-i-2, b-i) + \sum_{i=0}^{b} \mathrm{C}_i \operatorname{spc}(a-i-1, b-i-1) \\
&= \sum_{i=0}^{b} \mathrm{C}_i \operatorname{spc}((a-1)-i-1, b-i) + \sum_{i=0}^{b-1} \mathrm{C}_i \operatorname{spc}(a-i-1, (b-1)-i) \\
&\qquad + \underbrace{\mathrm{C}_b \operatorname{spc}(a-b-1, -1)}_{=0},
\end{aligned}$$

so, by (4.3),

$$= \operatorname{spc}(a-1,b) + \operatorname{spc}(a,b-1). \tag{4.11}$$

This completes the proof of (4.5).

The recurrence we just proved is the one defining the Catalan triangle[6], and its solution is known, namely

$$\operatorname{spc}(a,b) = \frac{(a+b)!\,(a-b+1)}{b!\,(a+1)!} = \frac{a-b+1}{a+1}\binom{a+b}{a}, \tag{4.12}$$

as shown by L. F. A. Arbogast in 1800 [2, p. 214–217].

$$\begin{array}{llllll}
1 \\
1 & 1 \\
1 & 2 & 2 \\
1 & 3 & 5 & 5 \\
1 & 4 & 9 & 14 & 14 \\
1 & 5 & 14 \longrightarrow & \overset{\downarrow}{28} & 42 & 42
\end{array}$$

Figure 12. The recurrence (4.5).

4.3.1 A combinatorial interpretation of spc

The Catalan triangle counts well-formed prefixes [CITATION]; specifically, $\operatorname{spc}(a,b)$ is the number of well-formed prefixes with a opening brackets and b closing brackets. This can in fact be seen from all the definitions of spc above, thus providing a more combinatorial solution, as we will now see.

In the original definition (4.1) of spc, the sum is over the choices for the positions of the $a - b$ unmatched opening brackets; between those, well-formed bracket expressions (counted by Catalan numbers) are inserted.

[6] [TODO cite something that actually uses that name here]

The first recurrence (4.3) corresponds to the following grammatical definition[7] of well-formed prefixes, where again wfbe denotes a well-formed bracket expression:

$$\begin{aligned}\text{wfprefix} ::= &\text{wfbe}\langle\text{wfprefix}\\ &\mid \text{wfbe}.\end{aligned}$$

In this case the sum ranges over the length of the well-formed bracket expression preceding the first unmatched bracket.

Finally, (4.5) reflects the fact that a well-formed prefix with k opening brackets and l closing brackets either ends with an opening bracket (preceded by a well-formed prefix with $k-1$ opening and l closing brackets), or with a closing bracket (preceded by a well-formed prefix with k opening and $l-1$ closing brackets).

4.3.2 Tying some loose ends

TODO here give a combinatorial proof for the ϖ from the section on highly convex matchings.

4.4 Asymptotics

We can now use the expression (4.12) for spc to study the asymptotics of the bound (4.2). We are interested in the asymptotics as a function of the length $n = a + b + c + d$ of the bracket expression. Since the bound is a product of two identical two-parameter factors, we study one of them,

$$f_{a,b} \coloneqq \mathrm{C}_a \operatorname{spc}(a,b),$$

as a function of $k \coloneqq a+b$. Let $b = \eta a$, thus $0 \le \eta \le 1$, the above expression becomes

$$f_{\frac{k}{\eta+1},\frac{k\eta}{\eta+1}} = \mathrm{C}_{\frac{k}{\eta+1}} \binom{k}{\frac{k}{\eta+1}} \frac{k - k\eta + \eta + 1}{k + \eta + 1} \le \mathrm{C}_{\frac{k}{\eta+1}} \binom{k}{\frac{k}{\eta+1}}.$$

Asymptotically, this yields

$$f_{\frac{k}{\eta+1},\frac{k\eta}{\eta+1}} \preccurlyeq_k 4^{\frac{k}{\eta+1}} \left(\frac{\left(1 - \frac{1}{\eta+1}\right)^{\frac{1}{\eta+1}-1}}{\left(\frac{1}{\eta+1}\right)^{\frac{1}{\eta+1}}} \right)^k = \underbrace{\left(4^{\frac{1}{\eta+1}} \frac{\eta+1}{\eta^{\frac{\eta}{\eta+1}}} \right)}_{y_\eta}^k$$

uniformly in $\eta \in [0,1]$. With some calculus, one can see that as η ranges from 0 to 1, the base y_η of that exponential is maximal for $\eta = \frac{1}{4}$, where its value is 5. We thus have the following asymptotic bound:

$$\nu_{\mathcal{P}}(\langle^a\rangle^b\langle^c\rangle^d) \le \mathrm{C}_a \operatorname{spc}(a,b) \operatorname{spc}(c,d) \mathrm{C}_d \preccurlyeq_{|\mathcal{P}|=n} 5^{a+b} 5^{c+d} = 5^n \tag{4.13}$$

uniformly in a, b, c, and d such that $a+b+c+d = n$.

5 Outlook

[TODO: here a section about spc generalizing to arbitrary WFBEs, and the applicability of overcounting analyses]

[7] This grammar requires infinite look-ahead to parse, but doing a first pass to mark the unmatched opening brackets resolves that.

References

[1] Miklós Ajtai, Václav Chvátal, Monroe M. Newborn, and Endre Szemerédi. Crossing-free subgraphs. In Peter L. Hammer, Alexander Rosa, Gert Sabidussi, and Jean Turgeon, editors, *Theory and Practice of Combinatorics — A collection of articles honoring Anton Kotzig on the occasion of his sixtieth birthday*, volume 60 of *North-Holland Mathematics Studies*, pages 9–12. North-Holland, 1982.

[2] Louis François Antoine Arbogast. *Du Calcul des Dérivations.* Levrault, Frères, 1800.

[3] Eugène Charles Catalan. Note sur une équation aux différences finies. *Journal de Mathématiques Pures et Appliquées*, 3:508–516, 1838.

[4] Leonhard Euler. Letter to Christian Goldbach. `http://eulerarchive.maa.org//correspondence/letters/000868.pdf` (retrieved 2016-09-10), 1751.

[5] Alfredo García, Marc Noy, and Javier Tejel. Lower bounds on the number of crossing-free subgraphs of K_N. *Computational Geometry*, 16(4):211–221, 2000.

[6] Monroe M. Newborn and William Oscar Jules Moser. Optimal crossing-free Hamiltonian circuit drawings of K_n. *Journal of Combinatorial Theory, Series B*, 29(1):13–26, 1980.

[7] Andres J. Ruiz-Vargas and Emo Welzl. Crossing-free perfect matchings in wheel point sets. To appear.

[8] János András Segner. *Enumeratio modorum quibus figurae planae rectilineae per diagonales diuiduntur in triangula.* In *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, volume 7 *pro annis 1758 et 1759*, pages 203–209, 1761.

[9] Micha Sharir and Emo Welzl. On the number of crossing-free matchings, cycles, and partitions. *SIAM Journal on Computing*, 36(3):695–720, 2006.

[10] Jan van Leeuwen and Anneke A. Schoone. Untangling a travelling salesman tour in the plane. In J. R. Mühlbacher, editor, *Proceedings of the 7th Conference on Graph-theoretic Concepts in Computer Science*, pages 87–98, 1981.